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ON REGULAR RINGS WITH STABLE RANGE 2

Pere MENAL
Secció de Matemàtiques, Universitat Autònoma de Barcelona, Barcelona, Bellaterra, Spain
Euskal Herriko Unibertsitatea, Bilbao, Spain

Jaume MONCASI*
Secció de Matemàtiques, Universitat Autònoma de Barcelona, Barcelona, Bellaterra, Spain

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0. Introduction

In [2] Henriksen proves that if R is a unit regular ring, then every matrix over R admits diagonal reduction. On the other hand it is well known, cf. [1, 4.12; 4.13; 4.15], that a regular ring R is unit regular if and only if $R\oplus A\cong R\oplus B$ implies $A\cong B$, for all right R-modules A, B. In Theorem 5 below we complete Henriksen's work [2] by proving that every matrix (possibly rectangular) over a regular ring R admits diagonal reduction if and only if $R^2\oplus A\cong R\oplus B$ implies $R\oplus A\cong B$, for all right R-modules A, B; it is also shown that this holds if and only if R is a regular Hermite ring. As we shall see any regular right Hermite ring is left Hermite, this will follow from the fact, cf. [5], that the stable range of a ring coincides with the stable range of its opposite ring. In Section 1 we also extend some results from unit regular rings to regular rings with finite stable range.

In Section 2 we construct some regular rings with stable range 2, thus answering a question of Handelman [1, Problem 49] and a question of Vasershtein [5, Remarks on Theorem 4]. G. Bergman, cf. [1, 4.26], constructs a regular ring R such that perspectivity is transitive in the lattice, $L(R)$, of principal right ideals of R, but R is not unit regular. The construction of our examples of rings of stable range 2 was inspired by that example, in fact we offer a regular ring R with stable range 2 such that $L(R)\cong L(S)$, where S is a subring of R which is unit regular. We see then that R is not unit regular, but it has the same 'lattice' properties as a unit regular ring, in particular perspectivity is transitive in $L(R)$. It can be shown, by using the methods we shall develop here, that Bergman's example has stable range 2, but it appears to us that our examples are simpler than Bergman's ones.

We prove that, unlike the case of unit regular rings, finitely generated projective

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modules over regular rings with stable range 2 need not have stable range 2.

Goodearl [1, 6.13] constructs a regular ring R satisfying comparability axiom for which the natural pre-order on $K_0(R)$ is not a partial order (as can be seen by a simple inspection of that ring). We shall note that such an example having, in addition, stable range 2 can be constructed, moreover its corresponding K_0 is isomorphic to a group of the type $Z \oplus Z/nZ$. This shows that $K_0(R)$ need not be torsion-free whenever R is a regular ring with stable range 2. The analogue for unit regular rings is an open question, cf. [1, Problem 27].

1. Regular rings with finite stable range and diagonal reduction

Throughout this paper R will denote an associative ring with 1.

We denote by $M_n(R)$ the ring of all $n \times n$ matrices over R and by $\mathrm{GL}_n(R)$ its group of unities. We write $\mathrm{GE}_n(R)$ for the subgroup of $\mathrm{GL}_n(R)$ generated by elementary matrices. R is said to be a GE_n*-ring* if $\mathrm{GL}_n(R) = \mathrm{GE}_n(R)$.

R is said to be *regular* if for every $a \in R$ there exists an $x \in R$ such that $axa = a$.

An n-row $x \in R^n$ is said to be *unimodular* if there exists an n-column $y \in {}^nR$ such that $xy = 1$. If $x = (x_1, \ldots, x_n) \in R^n$ is unimodular, then we say that x is *reducible* if there exists $y = (y_1, \ldots, y_{n-1}) \in R^{n-1}$ such that the $(n-1)$-row $(x_1 + x_n y_1, \ldots, x_{n-1} + x_n y_{n-1})$ is unimodular, in other words, the n-row

$$x\begin{pmatrix} 1_{n-1} & 0 \\ y & 0 \end{pmatrix}$$

is unimodular. R is said to have *stable range* $n \geq 1$ if n is the least positive integer such that every unimodular $(n+1)$-row is reducible. It is a well-known fact (and easy to prove) that if R has stable range 1, then R is a GE_n-ring for all $n \geq 2$. A regular ring is said to be *unit regular* if for every $a \in R$ there exists a unit $u \in R$ such that $aua = a$, the unit regular rings are precisely those regular rings which have stable range 1 [1, 4.12]. In particular, since any commutative regular ring is unit regular, we see that any commutative regular ring is a GE_n-ring, for all $n \geq 2$.

The $n \times m$ matrix $A = (a_{ij})$, $1 \leq i \leq n$, $1 \leq j \leq m$, is said to be diagonal if $a_{ij} = 0$ for all $i \neq j$. The $n \times m$ matrix A *admits diagonal reduction* if there exist $P \in \mathrm{GL}_n(R)$, $Q \in \mathrm{GL}_m(R)$ such that PAQ is a diagonal matrix. We recall [3, p. 465] that R is said to be *right* (*left*) *Hermite* if every 1×2 (2×1) matrix admits diagonal reduction, and if both, R is an Hermite ring.

The following proposition is a natural extension of results of Evans [1, 4.13] and it also follows from Theorem 1.3 [6], however, we give an independent proof.

Proposition 1. *Let M be a right R-module such that $S = \mathrm{End}_R(M)$ has stable range $\leq n$. If $M^n \oplus B \cong M \oplus C$ then*

(a) *B is isomorphic to a direct summand of C*

(b) *If in addition $S^n \cong S \oplus X$ implies $S^{n-1} \cong X$, then $M^{n-1} \oplus B \cong C$ (here X and S are viewed as left S-modules).*

Proof. Set $N=M^n\oplus B$ and $P=M\oplus C$. Let $\varphi:N\to P$ and $\delta:P\to N$ be R-homomorphisms such that $\varphi\delta=1_P$, $\delta\varphi=1_N$. In matricial form (relative to the above decompositions) we have

$$\varphi=\begin{pmatrix}\varphi_{11} & \cdots & \varphi_{n+1,1}\\ \varphi_{12} & \cdots & \varphi_{n+1,2}\end{pmatrix},\qquad \delta=\begin{bmatrix}\delta_{11} & \delta_{21}\\ \vdots & \vdots\\ \delta_{1,n+1} & \delta_{2,n+1}\end{bmatrix},$$

then $\varphi_{11}\delta_{11}+\cdots+\varphi_{n+1,1}\delta_{1,n+1}=1_M$. Since S has stable range $\leq n$ there exist $\alpha_1,\ldots,\alpha_n\in S$ such that

$$(\varphi_{11}+\varphi_{n+1,1}\delta_{1,n+1}\alpha_1)S+\cdots+(\varphi_{n,1}+\varphi_{n+1,1}\delta_{1,n+1}\alpha_n)S=S.$$

Now consider the automorphism η of N defined by the matrix

$$\eta=\begin{bmatrix}1 & & & & \\ 0 & 1 & & 0 & \\ \vdots & & \ddots & & \\ 0 & & & & \\ \delta_{1,n+1}\alpha_1 & \cdots & \delta_{1,n+1}\alpha_n & & 1\end{bmatrix}.$$

Thus by replacing φ,δ by $\varphi\eta,\eta^{-1}\delta$ respectively we may assume that $(\varphi_{11},\ldots,\varphi_{n,1})$ is S-unimodular, say $\varphi_{11}\beta_1+\cdots+\varphi_{n,1}\beta_n=1$ where $\beta_i\in S, i=1,\ldots,n$. Let γ be the automorphism of N defined by

$$\gamma=\begin{bmatrix}1 & 0 & \beta_1\varphi_{n+1,1}\\ & \ddots & \vdots\\ & & \beta_n\varphi_{n+1,1}\\ 0 & & -1\end{bmatrix},$$

again, by replacing φ,δ by $\varphi\gamma,\gamma^{-1}\delta$ respectively, we may assume $\varphi_{n+1,1}=0$. But then $\delta_{2,n+1}\varphi_{n+1,2}=1_B$. This shows that B is isomorphic to $\varphi_{n+1,2}(B)$ which is a direct summand of C.

Suppose now that S satisfies the additional assumption of (b). This is to say that every n-row S-unimodular is a row of an element of $GL_n(S)$. Then, since $(\varphi_{11},\ldots,\varphi_{n,1})$ is S-unimodular, there exists $U\in GL_n(S)$ such that the first row of φU is $(1,0,\ldots,0)$. It is then clear that we may assume

$$\varphi=\begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & \varphi_{22} & \cdots & \varphi_{n+1,2}\end{pmatrix}.$$

But now it is easily seen that $(\varphi_{22},\ldots,\varphi_{n+1,2}):M^{n-1}\oplus B\to C$ is an isomorphism. The result follows. □

The following result is well known and easily proved.

Lemma 2. *Let M be a right R-module. Then* $\mathrm{End}_R(M)$ *is a regular ring if and only if for each* $x\in\mathrm{End}_R(M)$, $\mathrm{Ker}\,x$ *and* $\mathrm{Im}\,x$ *are direct summands of M.* □

Now we are ready to prove a result which characterizes those regular rings which have stable range $\leq n$.

Theorem 3. (a) *A ring R is a regular ring with stable range $\leq n$ if and only if for a given $x \in R^n$ there exists an unimodular column $y \in {}^nR$ such that $xyx = x$.*

(b) *Let M be a right R-module such that $S = \mathrm{End}_R(M)$ is a regular ring, then the following are equivalent:*

(i) *S has stable range $\leq n$.*

(ii) *$M^n \oplus B \cong M \oplus C$ implies that B is isomorphic to a direct summand of C, for all right R-modules B, C.*

(iii) *For every $x \in S$, $M^n \oplus x(M) \cong M \oplus C$ implies that $x(M)$ is isomorphic to a direct summand of C, for all right modules C.*

Proof. (a) Suppose first that R is a regular ring which has stable range $\leq n$. Let $x \in R^n$, since R is regular there exists an idempotent $e \in R$ such that $x({}^nR) = eR$ and so $x({}^nR) + (1-e)R = R$. By the stable range condition there exists $\gamma \in R^n$ such that $x + (1-e)\gamma$ is unimodular. Hence there exists $y \in {}^nR$ such that $(x + (1-e)\gamma)y = 1$. By multiplying left and right by e and x respectively, we obtain $xyx = x$. Because y is unimodular the result follows.

Conversely, it is clear that R is a regular ring. In order to prove that R has stable range $\leq n$ suppose we are given an unimodular row $x \in R^{n+1}$. Write $x = (x', x_{n+1})$ where $x' \in R^n, x_{n+1} \in R$. By hypothesis there exists an unimodular column $y \in {}^nR$ with $x'yx' = x'$, if we set $e = x'y$ then e is an idempotent such that $eR + x_{n+1}R = R$. By [2, Remark D] there exists $t \in R$ such that $e + x_{n+1}t$ is a unit. Since y is unimodular there exists $\gamma \in R^n$ such that $t = \gamma y$. Now we obtain that $(x' + x_{n+1}\gamma)y$ is a unit. Therefore x is reducible and the proof is complete.

(b) By Proposition 1 (i)$\Rightarrow$(ii). Obviously (ii)$\Rightarrow$(iii).

(iii)$\Rightarrow$(i). By (a) it suffices to find, for each $x \in S^n$, an unimodular $y \in {}^nS$ with $xyx = x$. Since S is a regular ring there exists an $s \in {}^nS$ such that $xsx = x$, hence $e = xs$ is an idempotent. Thus we have an onto R-homomorphism $x: {}^nM \to e(M)$. It is easily seen that ${}^nM = \mathrm{Ker}\, x \oplus (sx)({}^nM)$ and clearly the left multiplication by s gives an isomorphism $s: e(M) \to (sx)({}^nM) = T$. On the other hand we have $M = (1-e)(M) \oplus e(M)$ and so we get $M^n \oplus (1-e)(M) \cong M \oplus \mathrm{Ker}\, x$. By hypothesis we conclude that there is an epimorphism $f: \mathrm{Ker}\, x \to (1-e)(M)$ and by the above there is also an isomorphism $s^{-1}: T \to e(M)$, that is we have an R-homomorphism

$$\begin{pmatrix} f & 0 \\ 0 & s^{-1} \end{pmatrix} : \begin{matrix} \mathrm{Ker}\, x \\ \oplus \\ T \end{matrix} \to \begin{matrix} (1-e)(M) \\ \oplus \\ e(M) \end{matrix} .$$

It is easily seen, from Lemma 2 applied to $\mathrm{End}_R(M^n)$, that f has a right inverse, f', so that

$$\begin{pmatrix} f' & 0 \\ 0 & s \end{pmatrix}$$

is a right inverse of

$$\begin{pmatrix} f & 0 \\ 0 & s^{-1} \end{pmatrix}.$$

These homomorphisms can be written in matricial form $\varepsilon : {}^nM \to M$, $y : M \to {}^nM$ with $\varepsilon y = 1_M$. By construction we have $xy(1-e) = 0$. Since when y is restricted to $e(M)$ coincides with s we have $ye = se$. From these relations we obtain $xyx = x$. By noting that y is unimodular the result follows. □

We say the right R-module M has the *n-weak cancellation property* if it satisfies the condition (ii) in Theorem 3.

Theorem 3(b) can be extended to those endomorphism rings S such that $S/J(S)$ (where $J(S)$ is the Jacobson radical of S) is regular and idempotents modulo $J(S)$ can be lifted. The stable range 1 case is [6, 2.4] and the general case will follow similarly by using Theorem 3(b). More precisely we have

Corollary 4. *Let S be a ring such that $S/J(S)$ is regular and idempotents modulo $J(S)$ can be lifted. Then the following are equivalent*

(i) *S has stable range $\leq n$.*

(ii) *Every right module whose endomorphism ring is isomorphic to S has the n-weak cancellation property.*

(iii) *There exists a ring R and a right R-module M such that* $\mathrm{End}_R(M) \cong S$ *and M has the n-weak cancellation property.* □

The following result is useful for checking examples, the $n = 1$ case is due to Bergman, cf. [1, 4.16; 4.17].

Proposition 5. (i) *Let I be an ideal in a regular ring S with stable range $\leq n$, and let R be a subring of S which contains I. If R/I is a regular ring with stable range $\leq n$, then so is R.*

(ii) *Any finite subdirect product of regular rings with stable range $\leq n$ has stable range $\leq n$.*

For simplicity, the notation $A \lesssim B$ means that A is isomorphic to a submodule of B.

Proof. (i) Suppose $R^n \oplus A \cong R \oplus B$ where $A \lesssim R$ and B is a right R-module, by Theorem 3(iii) we need only to prove that $A \lesssim B$ (notice that A and B are finitely generated projective modules over the regular ring R, so $A \lesssim B$ implies that A is isomorphic to a direct summand of B). Since R/I has stable range $\leq n$ we know that $A/AI \lesssim B/BI$ and, by [1, 2.20], we have decompositions

$$A = A_1 \oplus A_2, \qquad B = B_1 \oplus B_2$$

such that $A_1 \cong B_1$ and $A_2 I = A_2$.

Because $A_1 \lesssim R$ we see that $R^{n+1} \oplus A_2 \cong R^2 \oplus B_2$. On the other hand A_2 is a right S-module (since $A_2 I = A_2$) and it follows from the fact that A_2 is right R-flat and $I^2 = I$ that $A_2 \otimes_R S \cong A_2$ as right S-modules. Hence we have $S^{n+1} \oplus A_2 \cong S^2 \oplus (B_2 \otimes_R S)$. By hypothesis S has stable range $\leq n$ which implies that every unimodular $(n+1)$-row of elements of S is a row of some element of $GL_{n+1}(S)$ (in fact of $GE_{n+1}(S)$) so Proposition 1(b) yields $S^n \oplus A_2 \cong S \oplus (B_2 \otimes_R S)$ again by Proposition 1(a) we obtain $A_2 \lesssim B_2 \otimes_R S$ that is, there is an S-module epimorphism $f: B_2 \otimes_R S \to A_2$. Let $\bar{f}$ be the restriction of f over B_2 so that $\bar{f}: B_2 \to A_2$ is an R-homomorphism. We claim that $\bar{f}$ is onto. Since A_2 is cyclic let $a \in A_2$ with $aR = A_2$, then $a = f(b)$ for some $b = \sum b_i \otimes s_i \in B_2 \otimes_R S$. We have $a = \sum \bar{f}(b_i) s_i$ and, since $A_2 = A_2 I$, $\bar{f}(b_i) = a\alpha_i$ for some $\alpha_i \in I$. Since R is regular we can write $\alpha_i = \alpha_i \eta_i$ for suitable $\eta_i \in I$, but then we have $a = \sum \bar{f}(b_i) s_i = \sum \bar{f}(b_i) \eta_i s_i = \bar{f}(\sum b_i \eta_i s_i)$ so $\bar{f}$ is onto as claimed. Because A_2 is projective we have shown that $A_2 \lesssim B_2$ and then $A = A_1 \oplus A_2 \lesssim B_1 \oplus B_2 = B$ as desired.

(ii) It is an immediate consequence of (i). □

Next we study the stable range condition in regular rings satisfying general comparability axiom. Recall first some definitions. A regular ring is said to satisfy the *comparability axiom* provided that, for any $x, y \in R$, either $xR \lesssim yR$ or $yR \lesssim xR$. It can be shown that if A and B are finitely generated projective right R-modules and R satisfies the comparability axiom, then either $A \lesssim B$ or $B \lesssim A$, cf. [1, 8.2]. A regular ring satisfies *general comparability* provided that for any $x, y \in R$ there exists a central idempotent e in R such that $exR \lesssim eyR$ and $(1-e)yR \lesssim (1-e)xR$.

Proposition 6. *Let $R \neq 0$ be a regular ring satisfying the comparability axiom. Then the following are equivalent*

(i) *R has finite stable range.*
(ii) *R^n does not contain a submodule isomorphic to R^{n+1}, for all $n \geq 1$.*
(iii) *R^2 does not contain a submodule isomorphic to R^3.*
(iv) *R has stable range ≤ 2.*

Proof. For arbitrary regular rings (i) ⇒ (ii). Suppose $R^{n+1} \lesssim R^n$ for some $n \geq 1$, then $R^{n+k} \lesssim R^n$ for all $k \geq 1$. By Proposition 1(a) we get $R = 0$.

Obviously (ii) ⇒ (iii).

(iii) ⇒ (iv). If $R^2 \oplus A \cong R \oplus B$ where $A \lesssim R$ then we will obtain that $A \lesssim B$ and thus the result will follow from Theorem 3. If this is not the case we may assume, since R satisfies the comparability axiom, that $B \lesssim A$, but then we have decompositions $A \cong B \oplus C$ and $R \cong B \oplus D$ for suitable right R-modules C, D. Hence $R^3 \oplus C \cong R^2 \oplus B \oplus D \oplus C \cong R^2 \oplus A \oplus D \cong R \oplus B \oplus D \cong R^2$ and so $R^3 \lesssim R^2$. □

Corollary 7. *Let R be a regular ring satisfying general comparability axiom, then its stable range is either 1, 2 or ∞.*

Proof. Suppose that R has finite stable range. Let $\bar{R}$ be an indecomposable factor ring of R, then $\bar{R}$ satisfies the comparability axiom. It follows from Proposition 6 that $\bar{R}$ has stable range ≤ 2, so every indecomposable factor ring of R has stable range ≤ 2. If R does not have stable range ≤ 2 there exists an unimodular row $x \in R^3$ which is not reducible, but then by a simple application of Zorn's lemma we can choose an ideal I of R such that $\bar{R} = R/I$ is indecomposable and $\bar{x} \in \bar{R}^3$ is not reducible, and this is a contradiction. □

In order to obtain our characterization of regular Hermite rings we need some previous results. A ring R is said to be *right* (*left*) *Bezout* if every finitely generated right (left) ideal is principal. For example, any right Hermite ring is right Bezout and any regular ring is both right and left Bezout. Note that the definition of right Hermite is not left-right symmetric, for if it suffices to consider any right principal ideal domain which is not left principal. However we shall see that a right Hermite ring is left Hermite provided that it is left Bezout. This yields, in particular, that any regular right Hermite ring is left Hermite.

Proposition 8. (i) *If R is a right or left Hermite ring then R has stable range ≤ 2.*
(ii) *If R is left Hermite and right Bezout then R is right Hermite.*

Proof. (i) Suppose $x \in R^3$ is reducible, then we claim that

$$x' = x\begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix},$$

where $U \in \mathrm{GL}_2(R)$, is also reducible. Since x is reducible there exists $\gamma \in R^2$ such that

$$A = \begin{pmatrix} 1 & 0 \\ \gamma & 0 \end{pmatrix} \in M_3(R) \tag{$*$}$$

and xA is unimodular. It is immediately seen that

$$B = \begin{pmatrix} U^{-1} & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$$

is a matrix of the form ($*$). But then $x'B$ is unimodular and so x' is reducible as claimed.

Suppose now that R is a right Hermite ring and let $x = (x_1, x_2, x_3)$ an unimodular row. Then there exists $U \in \mathrm{GL}_2(R)$ such that $(x_1, x_2)U$ is of the form $(d, 0)$. Hence

$$x\begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$$

is an unimodular row with 0 as an entry, therefore it is clearly reducible. Now the result follows from the above paragraph.

Because the stable range of a ring coincides with the stable range of its opposite ring, cf. [5, Theorem 2], the result also follows if R is left Hermite.

(ii) Assume R is right Bezout and left Hermite. It follows immediately from the definition of left Hermite that every unimodular 2-column is a column of an invertible 2×2 matrix over R. Under our hypotheses we first prove the analogous result for rows. For if $(x,y)\in R^2$ is an unimodular 2-row, say $xx'+yy'=1$, there exists an invertible 2×2 matrix of the form

$$U=\begin{pmatrix} x' & * \\ y' & * \end{pmatrix}.$$

Clearly $(x,y)U=(1,z)$ for some $z\in R$. If

$$V=U\begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix}$$

then $V\in \mathrm{GL}_2(R)$ and $(x,y)V=(1,0)$, that is (x,y) is the first row of the matrix V^{-1}.

Now we prove that R is right Hermite. Suppose we are given $x,y\in R$, then, since R is right Bezout, $xR+yR=dR$ say $x=dx'$, $y=dy'$, $d=x\alpha+y\beta$. From these relations we get $d(x'\alpha+y'\beta-1)=0$ so that $x'R+y'R+zR=R$ for some $z\in R$ such that $dz=0$. By (i) R has stable range ≤ 2, thus $(x'+zt_1)R+(y'+zt_2)R=R$, where $t_1,t_2\in R$. By the above we can find an invertible matrix of the form

$$U=\begin{pmatrix} x'+zt_1 & y'+zt_2 \\ * & * \end{pmatrix}.$$

Clearly $(x,y)U^{-1}=(d,0)$ so R is right Hermite. □

In [2, p. 134] it is claimed that if each 2×2 matrix over a ring R admits diagonal reduction then each $n\times n$ matrix admits diagonal reduction. This is said to be an application of Kaplansky's result [3, 5.1], but it is unjustified because the term 'diagonal reduction' is used by Kaplansky, cf. [3, p. 465], in a different meaning. If each 2×2 matrix over an arbitrary ring admits diagonal reduction then we have been unable to prove by induction on n that each $n\times n$ matrix admits diagonal reduction. Fortunately, if R is an Hermite regular ring this induction works since the peculiar diagonal reduction of the 2×2 matrices. More precisely, we have the following.

Theorem 9. *Let M be a right R-module such that $S=\mathrm{End}_R(M)$ is a regular ring. Then the following statements are equivalent*

(i) *S is left Hermite.*

(ii) *S is right Hermite.*

(iii) *$M^2\oplus B\cong M\oplus C$ implies $M\oplus B\cong C$, for all right R-modules B,C.*

(iv) *For every $x\in S$, $M^2\oplus x(M)\cong M\oplus C$ implies $M\oplus x(M)\cong C$ for all right R-module C.*

(v) *Every matrix over S admits diagonal reduction.*

Proof. By Proposition 8(ii), (i)$\Leftrightarrow$(ii).

(ii) ⇒ (iii). If S is an Hermite ring then, by Proposition 8(i), it has stable range ≤ 2. Since every unimodular 2-row is a column of an invertible 2×2 matrix the result follows from Proposition 1(b).

Trivially (iii) ⇒ (iv).

(iv) ⇒ (ii). First we shall show that every unimodular 2-column $x \in {}^2S$ is a column of some element of $\mathrm{GL}_2(S)$. Since x is unimodular there exists $y \in S^2$ such that $yx = 1_M$. Clearly ${}^2M = \mathrm{Ker}\, y \oplus x(M) \cong \mathrm{Ker}\, y \oplus M$ and by hypothesis there is an isomorphism $M \to \mathrm{Ker}\, y$. Therefore we can construct an automorphism of ${}^2M$, say U, such that $U\binom{0}{m} = x(m)$ for all $m \in M$. That is $U^{-1}x = \binom{0}{1}$ as required.

Now we prove that S is right Hermite. Let $x \in S^2$, since S is regular it follows from Theorem 3(a) that there exists an unimodular $y \in {}^2S$ such that $xyx = x$. By the above paragraph y is the first column of some $U \in \mathrm{GL}_2(S)$ such that $xU = (e, et)$, where $e = xy$ and $t \in S$. Then

$$V = U\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(S)$$

and xV is of the form $(*, 0)$. The result follows.

(i) ⇔ (v). Obviously we need only to prove (i) ⇒ (v). Let A be an $n \times m$ matrix over S. In order to prove that A admits diagonal reduction we proceed by induction on the minimum of n and m. If either $n = 1$ or $m = 1$ the result follows from [3, 3.5]. So assume $n, m \geq 2$, by symmetry we may suppose without loss of generality, that $m \leq n$. Since S is left Hermite we need only to consider the case of a matrix A of the form

$$\begin{pmatrix} B & 0 \\ a & b \end{pmatrix}$$

where B is an $(n-1)\times(m-1)$ matrix, $a \in S^{n-1}, b \in S$. By induction there exist $U \in \mathrm{GL}_{n-1}(S)$, $V \in \mathrm{GL}_{m-1}(S)$ such that UBV is a diagonal matrix. Thus A is equivalent to

$$C = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} B & 0 \\ a & b \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix},$$

C is of the form

$$C = \begin{bmatrix} x & \cdots & 0 & \cdots & 0 \\ 0 & & X & & 0 \\ y & y_2 & \cdots & y_{m-1} & z \end{bmatrix}$$

where X is a diagonal $(n-2)\times(m-2)$ matrix (the case $m = 2$ is also included if we think of X as an $(n-2)\times 0$ matrix, that is

$$C = \begin{bmatrix} x & 0 \\ 0 & 0 \\ y & z \end{bmatrix} \;).$$

Let $e \in S$ be an idempotent such that $xS = eS$, then $xt = e, x = eu$ for some $t, u \in S$. Set

$$W = \begin{pmatrix} t & 0 & 1-tu \\ 0 & 1_{m-2} & 0 \\ 1 & 0 & -u \end{pmatrix} \in \mathrm{GL}_m(S),$$

then CW is of the same form than C, but its $(1,1)$-entry is e, an idempotent. Hence we may assume that $x = e$.

Let $z' \in S$ such that $zz'z = z$. By adding to the first column of C its m-th column right multiplied by $-z'y$, we may assume that $zz'y = 0$. Set

$$Z = \begin{pmatrix} z' & 0 & zz'-1 \\ 0 & 1_{n-2} & 0 \\ 1+z'z & 0 & z'zz' \end{pmatrix} \in \mathrm{GL}_n(S),$$

then

$$ZC = \begin{pmatrix} z'e-y & * & 0 \\ 0 & X & 0 \\ (1+z'z)e & z'zz'y_2 \cdots z'zz'y_{m-1} & z'z \end{pmatrix}.$$

By elementary column transformations we see that ZC, and so A, can be reduced to a matrix of the form

$$\begin{pmatrix} c & * & 0 \\ 0 & X & 0 \\ e & 0 & z'z \end{pmatrix}.$$

Hence we may identify A with the above matrix. Set $f = z'z$, so that f is an idempotent. Let $\delta \in S$ such that $(1-f)e\delta(1-f)e = (1-f)e$, then $g = (1-f)e\delta(1-f)$ is an idempotent satisfying

$$(f+g)S = fS + eS,$$

$$gf = fg = 0, \qquad ge = (1-f)e.$$

If we add to the first row of A the n-th row left multiplied by $-c\delta(1-f)$ we obtain a matrix

$$A = \begin{pmatrix} c(1-\delta(1-f)) & * & 0 \\ 0 & X & 0 \\ e & 0 & f \end{pmatrix}.$$

On the other hand we have:

$$\begin{pmatrix} c(1-\delta(1-f)) & 0 \\ e & f \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e & 1 \end{pmatrix} \begin{pmatrix} 1 & \delta(1-f) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -(1-f)e & 1 \end{pmatrix} \begin{pmatrix} 1 & -\delta(1-f) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c(1-\delta(1-f)) & 0 \\ 0 & f+g \end{pmatrix}.$$

This shows that

$$\begin{pmatrix} c(1-\delta(1-f)) & 0 \\ e & f \end{pmatrix}$$

can be reduced to diagonal form by using column transformations *only*. Hence it is clear that A is equivalent to

$$\begin{bmatrix} c(1-\delta(1-f)) & * & 0 \\ 0 & X & 0 \\ 0 & 0 & f+g \end{bmatrix}.$$

Since

$$\begin{pmatrix} c(1-\delta(1-f)) & * \\ 0 & X \end{pmatrix}$$

is an $(n-1)\times(m-1)$ matrix the result follows by induction.

Notice that the above arguments show that any triangular $n\times n$ matrix, over a regular ring, admits diagonal reduction. □

Corollary 10. *Let $M=M_1\oplus\cdots\oplus M_k$ be an R-module such that $S=\mathrm{End}_R(M)$ is a regular ring. Then*

(i) *If $\mathrm{End}_R(M_i)$ is an Hermite ring, $i=1,\dots,k$, then S is an Hermite ring.*

(ii) *If $\mathrm{End}_R(M_i)$ has stable range $\le n$, $i=1,\dots,k$, then S has stable range $\le n$.*

Proof. (i) By Theorem 9 it suffices to show that $M^2\oplus B\cong M\oplus C$ implies $M\oplus B\cong C$. We have $M_1^2\oplus\cdots\oplus M_k^2\oplus B\cong M_1\oplus\cdots\oplus M_k\oplus C$ and since $\mathrm{End}_R(M_i)$ is a regular Hermite ring for $i=1,\dots,k$ the result follows by repeated application of Theorem 9 (iii).

(ii) It follows similarly as (i) by using Theorem 3 instead of Theorem 9. □

2. Examples

In this section we shall offer several examples of regular rings with stable range 2 some of which are of independent interest. We begin by proving some useful results to this end.

Lemma 11. *Let R be a regular ring*

(i) *If $x\in R^2$ has an idempotent entry, then there exists $U\in \mathrm{GE}_2(R)$ such that xU is of the form $(*\ 0)$.*

(ii) *If for each $A\in \mathrm{GL}_2(R)$ we may obtain, by elementary transformations from A, a matrix which has an idempotent entry, then R is a GE_2-ring.*

Proof. (i) Set $x=(e,a)$ with $e^2=e$ and $a\in R$. Since

$$(e,a)\begin{pmatrix}1 & -a\\ 0 & 1\end{pmatrix}=(e,(1-e)a),$$

there is no restriction in assuming $ea=0$. Now choose an idempotent $f\in R$ such that $eR+aR=fR$, so there exist $t,v\in R$ such that $et+av=f$. Because $ea=0$ we get $et=ef$. Then we see that

$$(e,a)\begin{pmatrix}1 & 0\\ v & 1\end{pmatrix}\begin{pmatrix}f & 1-f\\ 1-f & f\end{pmatrix}=(f+a(1-f),b)$$

for some $b\in fR$. It is easily seen that $(f+a(1-f))R=fR$. But then $b=(f+a(1-f))x', x'\in R$. Thus

$$(f+a(1-f),b)\begin{pmatrix}1 & -x'\\ 0 & 1\end{pmatrix}=(f+a(1-f),0).$$

So (i) follows.

(ii) If $A\in \mathrm{GL}_2(R)$ satisfies our hypotheses it follows from (i) that VAU is a triangular matrix for suitable $U,V\in \mathrm{GE}_2(R)$. Since any invertible triangular matrix belongs to $\mathrm{GE}_2(R)$, the result follows. □

Lemma 12. *Let I be an ideal of a regular ring R contained in a unit regular subring of R. If R/I is an Hermite* GE_2*-ring, then R is an Hermite* GE_2*-ring.*

Proof. We set $\bar{R}=R/I$ and we write $\bar{a}$ for $a+I$, $a\in R$. Let $(x,y)\in R^2$. By hypotheses there exists $U\in \mathrm{GE}_2(\bar{R})$ such that $(\bar{x},\bar{y})U$ is a diagonal matrix. Note that $\mathrm{GE}_2(\bar{R})$ is generated, as group, by its diagonal matrices and the subgroup, $E_2(\bar{R})$, generated by all matrices of the form

$$\begin{pmatrix}1 & 0\\ * & 1\end{pmatrix},\qquad \begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix},$$

moreover $E_2(\bar{R})$ is a normal subgroup of $\mathrm{GE}_2(\bar{R})$. From this remark we can write $U=VD$ where $V\in E_2(\bar{R})$ and $D\in \mathrm{GE}_2(\bar{R})$ is diagonal. Clearly $(\bar{x},\bar{y})V$ is a diagonal matrix, thus we may assume $U\in E_2(\bar{R})$. Clearly there exists $W\in \mathrm{GL}_2(R)$ such that $\bar{W}=U$ and so $(x,y)W$ is of the form (z,i) where $z\in R$ and $i\in I$. Because I is contained in a unit regular subring of R there exists a unit $u\in R$ such that $e=iu$ is an idempotent. Therefore

$$(z,i)\begin{pmatrix}1 & 0\\ 0 & u\end{pmatrix}=(z,e)$$

and the result follows from Lemma 11. □

The next result is an abstract point of view of the methods used in Bergman's example [1, 5.10].

Lemma 13. *Let R be a regular ring and let K be a commutative ring. If $\varphi : R \to K$ is a ring homomorphism then:*

(i) *If* $\operatorname{Ker} \varphi$ *is contained in a directly finite subring of R, then R is a homomorphic image of a directly finite regular ring.*

(ii) *If* $\operatorname{Ker} \varphi$ *is contained in a unit regular subring of R, then R is a homomorphic image of a directly finite regular Hermite ring.*

Proof. (i) Define $S = \{(x, y) \in R \times R^0 : \varphi(x) = \varphi(y)\}$, where R^0 denotes the opposite ring of R. Since K is commutative, S is a subring of $R \times R^0$. Then S is regular because it is a subdirect product of R and R^0 which are both regular. We claim that S is directly finite. For this, let $(x, y)(x', y') = 1$ in S. On the other hand we have the following relations $y \in x + \operatorname{Ker} \varphi$, $y' \in x' + \operatorname{Ker} \varphi$ so that $xy' \in 1 + \operatorname{Ker} \varphi$ and $yx' \in 1 + \operatorname{Ker} \varphi$. By hypothesis there exists a directly finite subring, T, of R containing $\operatorname{Ker} \varphi$. Hence $xy', yx' \in T$ and clearly $(xy')(yx') = 1$. But T is directly finite so $(yx')(xy') = 1$. From this it is easily seen that $(x', y')(x, y) = 1$. Thus S is directly finite.

(ii) Construct S as in (i) so that S is directly finite. We need only to show that S is Hermite. Since K is commutative we see that $R/\operatorname{Ker} \varphi$ is a regular commutative ring and thus it is an Hermite GE_2-ring. Now it follows from Lemma 12 that R is an Hermite GE_2-ring. Through the map $x \mapsto (x, 0)$ from $\operatorname{Ker} \varphi$ to S we can think of $\operatorname{Ker} \varphi$ as an ideal of S. Then $S/\operatorname{Ker} \varphi \cong R^0$, by Lemma 12 it suffices to prove that $\operatorname{Ker} \varphi$ is contained in a unit regular subring of S. By hypothesis there exists a unit regular subring T of R which contains $\operatorname{Ker} \varphi$ (as ideal of R). The subring of S, $(T \times T^0) \cap S$, contains $\operatorname{Ker} \varphi$ (as ideal of S) and it is a subdirect product of unit regular rings and so unit regular, cf. [1, 4.17], or Proposition 5(ii). □

Example 1. *There exists a regular ring R satisfying:*

(i) *R is directly infinite;*

(ii) *R satisfies the comparability axiom;*

(iii) *R contains a unit regular subring S such that $L(R) \cong L(S)$ (in particular perspectivity is transitive in $L(R)$);*

(iv) *the endomorphism ring of every finitely generated projective R-module has stable range ≤ 2.*

Let T be the endomorphism ring of an infinite-countable-dimensional K-vector space. Let M be the maximal ideal of T, that is M consists of those endomorphisms of V whose images are of finite dimension. Set $A = T/M$ and write $\bar{x}$ for $x + M$. Let $(e_i)_{i \geq 0}$ a K-basis for V and define $a, b \in T$ by

$$a(e_0) = 0, \quad a(e_{i+1}) = e_i \qquad \text{all } i \geq 0,$$

$$b(e_i) = e_{i+1} \quad \text{all } i \geq 0.$$

Clearly $ab = 1$ and $(ba - 1)(V) = Ke_0$, consequently $\bar{a}$ is a unit of A. We claim that, for every nonzero polynomial $p(x)\in K[X]$, the element $p(\bar{a})$ is a unit of A. Since $\bar{a}$ is a unit we may assume that $p(x)=\alpha_0+\cdots+\alpha_n x^n$, $\alpha_i\in K$ and $\alpha_0\neq 0$. But then $p(x)$ has an inverse $\sum_{i\geq 0}\beta_i x^i\in K[[X]]$. Note that $a^m(e_i)=0$ for all $m>i$ so that $\sum_{i\geq 0}\beta_i a^i$ defines an element of T which is a two sided inverse of $p(a)$. Thus we have shown that F, the field of quotients of $K[\bar{a}]$, is contained in A. Define R to be the subring of T such that $R/M=F$. Then R is a directly infinite regular ring. Since T satisfies the comparability axiom, it follows from [1, 8.4] that R satisfies the comparability axiom. On the other hand $S=M+K$ is a unit regular subring of R which contains all the idempotents of R, hence the map $L(R)\to L(S)$ defined by $I\mapsto I\cap S$ is a lattice isomorphism, cf. [1, 3.15(b)].

We shall prove that the endomorphism ring of every finitely generated projective right R-module is an Hermite ring (and so has stable range ≤ 2). As is well known every finitely generated projective right R-module over a regular ring is isomorphic to a direct sum of principal right ideals of R. By Corollary 10(i) we may assume that the projective module is of the form eR, for some idempotent $e\in R$. Then we must prove that eRe is Hermite. Since $e\in S$ we see that eMe is contained in the unit regular subring eSe of eRe and either $eRe=eSe$ or $eRe/eMe\cong F$. By Lemma 12 we see that eRe is Hermite. □

Example 2. *There exists a regular ring R which has stable range 2 and $M_n(R)$ is directly finite for all $n\geq 1$.*

Construct R as in Example 1 and define S as in the proof of Lemma 13(ii). Then we know that S is a directly finite regular Hermite ring and so its stable range is ≤ 2. Since R is a homomorphic image of S we see that the stable range of S is 2. By a theorem of Kaplansky [3, 3.9] the full matrix rings over a directly finite Hermite ring are also directly finite. □

In [2, Theorem 7] Henriksen notes that the full matrix rings over a unit regular ring are unit regular. Later Handelman, cf. [1, 4.7], shows that $\mathrm{End}_R(M)$ is unit regular if M is a finitely generated projective right module over the unit regular ring R. In the following example we shall see that this result does not extend to regular rings with stable range 2. In fact we prove that *any* regular ring can be embedded, as a subring, in the endomorphism ring of a cyclic projective right module over a regular ring with stable range 2.

Example 3. *If R is a regular ring there is a regular ring S which has stable range 2 and an idempotent $e\in S$ such that R is isomorphic to a subring of eSe.*

Since any regular ring is contained in a direct product of full linear rings over commutative fields we may assume without loss of generality that $R=\mathrm{End}_K(V)$, where V is a K-vector space of dimension τ. Choose a cardinal $\pi>\tau$ and consider a K-vector space W of dimension π in such a way that V is a direct summand of W.

Now let M be the ideal of $\mathrm{End}_K(W)$ consisting of those endomorphisms φ with $\dim \mathrm{Im}\, \varphi < \pi$. Define $S = M + K$, then we claim that S is a regular Hermite ring (this result has been also announced in [4]). We are indebted to Ken Goodearl for the following proof. If $\pi = \aleph_0$, then S is unit regular and so it is an Hermite ring, so assume that $\pi > \aleph_0$. It is then easily seen that $(1-e)S \cong S$ for all idempotent $e \in M$, from this we see that $S \oplus eS \cong S$ for all idempotents $e \in M$. It follows that any finitely generated projective right S-module is either free or else isomorphic to eS for an idempotent $e \in M$. In order to prove that S is Hermite we will use Theorem 9(iv), so assume $S^2 \oplus B \cong S \oplus C$ where B and C are finitely generated projective right S-modules. Then $S \oplus B$ must be free. Since $(S/M)^2 \oplus (B/BM) \cong (S/M) \oplus (C/CM)$, $C/CM \neq 0$, so C must be free. Then $S \oplus B \cong S^n$ and $C \cong S^k$, for some $n, k \in \mathbb{N}$. Hence

$$(S/M)^{n+1} \cong (S^2 \oplus B)/(S^2 \oplus B)M \cong (S \oplus C)/(S \oplus C)M \cong (S/M)^{k+1},$$

which implies $n+1 = k+1$ so that $S \oplus B \cong C$.

Choose V' such that $V \oplus V' = W$ and define $e \in \mathrm{End}_K(W)$ by $e \mid V = 1$, $e \mid V' = 0$. Clearly $e^2 = e \in S$, but now we have that $eSe \cong \mathrm{End}_K(V)$. □

Let $P(R)$ denote the class of all finitely generated projective right R-modules. Let $K_0(R)$ denote the Grotendieck group of the ring R, that is the abelian group generated by $K_0(R)^+ = \{[A] : A \in P(R)\}$ subject to the relations $[A] + [B] = [C]$ whenever $A \oplus B \cong C$. Note that $K_0(R)^+$ is a submonoid of $K_0(R)$ so that it defines a pre-order (the natural pre-order) on $K_0(R)$, more precisely: $x \leq y$ if and only if $y - x \in K_0(R)^+$. If R is unit regular then the natural pre-order on $K_0(R)$ is a partial order, cf. [1, 15.2(d)]. In the following example we shall see that this property is not shared for regular rings R with stable range 2, moreover the same example shows that $K_0(R)$ is not necessarily torsion-free whenever R has stable range 2.

Example 4. *For each integer $n > 1$ there exists a regular ring S satisfying*

(i) *S has stable range 2;*
(ii) *S satisfies the comparability axiom;*
(iii) *the natural pre-order on $K_0(S)$ is not a partial order;*
(iv) *$K_0(S) \cong Z \oplus Z/nZ$.*

Let $K^* < K$ be commutative fields such that $\dim_{K^*}(K) = n$ and let V be an infinite-countable-dimensional K-vector space. Set $Q = \mathrm{End}_K(V)$, $Q^* = \mathrm{End}_{K^*}(V)$ and denote by M and M^* their maximal ideals. Let R be the ring in example 1, so that R/M is a field F. Define $S = R + M^*$, clearly S is a subring of Q^*. Since $M^* \cap Q = M$ we see that $S/M^* \cong R/M \cong F$, so S is a regular ring and by Lemma 12, S is an Hermite ring and hence has stable range ≤ 2. Because S is not unit regular its stable range is 2.

It follows from [1, 8.4] that S satisfies the comparability axiom.

Let a, b the elements in R (so in S) defined in Example 1, then $f = 1 - ba$ is an idempotent and we have $S \cong S \oplus fS$. Since $\dim_K f(V) = 1$, $\dim_{K^*} f(V) = n$, thus there exist orthogonal idempotents $f_1, \ldots, f_n \in M^*$ such that $\bigoplus_{i=1}^n f_i S = fS$. Clearly $f_1 Q^* \cong f_i Q^*$, $i = 1, \ldots, n$, and, since $f_i \in M^*$, we see that $e = f_1, \ldots, f_n$ are isomorphic idempotents in S. Hence $S \cong S \oplus (eS)^n$. Likewise [1, 6.13] we can deduce that $S \oplus (eS)^k$ is not isomorphic to S, for $k = 1, \ldots, n-1$. Now we prove that the natural pre-order on $K_0(S)$ is not a partial order. Obviously we have $[S] \leq [S \oplus eS]$ and by the above also $[S \oplus eS] \leq [S]$. If $[S] = [S \oplus eS]$, this would imply $S \oplus S^m \cong S \oplus eS \oplus S^m$, for some $m \geq 1$. But S is an Hermite ring so $S \cong S \oplus eS$, which is a contradiction.

In order to compute $K_0(S)$ set $\varphi : S \to S/M^*$ the natural projection. Since S/M^* is a field we get, by using [1, 15.15], an exact sequence

$$0 \to \mathrm{Ker}(K_0(\varphi)) \to K_0(S) \to Z \to 0,$$

where $\mathrm{Ker}(K_0(\varphi)) = \langle [xS] : x \in M^* \rangle$. For a given $x \in M^*$, $\dim_{K^*} x(V)$ is finite, say m. We have $x(V) \cong e(V) \oplus \cdots \oplus e(V)$ and from this we deduce that $xS \cong (eS)^m$. This proves that $[eS]$ generates $\mathrm{Ker}(K_0(\varphi))$. From $S \cong S \oplus (eS)^n$ we see that $n[eS] = [0]$. Suppose $m[eS] = [0]$, that is $S^h \oplus (eS)^m \cong S^h$ for some $m, h \in \mathbb{N}$. Using again that S is an Hermite ring we see that $m \in n\mathbb{N}$. Hence $[eS]$ has order n and so $\mathrm{Ker}(K_0(\varphi)) \cong Z/nZ$. Therefore $K_0(S) \cong Z \oplus Z/nZ$ as required.

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