ON FIBRE SPACE STRUCTURES OF A PROJECTIVE IRREDUCIBLE SYMPLECTIC MANIFOLD

DAISUKE MATSUSHITA

(Received 21 October 1997; in revised form 20 January 1998)

In this note, we investigate fibre space structures of a projective irreducible symplectic manifold. We prove that a 2n-dimensional projective irreducible symplectic manifold admits only an n-dimensional fibration over a Fano variety which has only Q-factorial log-terminal singularities and whose Picard number is one. Moreover we prove that a general fibre is an abelian variety up to finite unramified cover, especially, for 4-fold, a general fibre is an abelian surface and all fibres are equidimensional. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

We first define an irreducible symplectic manifold.

Definition 1. A complex manifold $X$ is called irreducible symplectic if $X$ satisfies the following three conditions:

1. $X$ is compact and Kähler.
2. $X$ is simply connected.
3. $H^0(X, \Omega^2_X)$ is spanned by an everywhere non-degenerate two-form $\omega$.

Such a manifold can be considered as a building block of all compact Kähler manifolds $X$ with $c_1(X) = 0$ due to the following Bogomolov decomposition theorem.

Theorem 1 (Bogomolov decomposition theorem [2]). A compact Kähler manifold $X$ with $c_1(X) = 0$ admits a finite unramified covering of $\mathcal{X}$ which is isomorphic to a product $T \times X_1 \times \cdots \times X_r \times A$ where $T$ is a complex torus, $X_i$ are irreducible symplectic manifolds and $A$ is a projective manifold with $h^0(A, \Omega^2_A) = 0$.

In dimension 2, K3 surfaces are the only irreducible symplectic manifold, and irreducible symplectic manifolds are considered as higher-dimensional analogies of K3 surfaces. In this note, we investigate fibre space structures of a projective irreducible symplectic manifolds.

Definition 2. For an algebraic variety $X$, a fibre space structure of $X$ is a proper surjective morphism $f : X \to S$ which satisfies the following two conditions:

1. $X$ and $S$ are normal varieties such that $0 < \dim S < \dim X$.
2. A general fibre of $f$ is connected.

Certain K3 surfaces $S$ admit a fibre space structure $f : S \to \mathbb{P}^1$ whose general fibre is an elliptic curve. As a higher-dimensional analogy, we obtain the following results.
**Theorem 2.** Let \( f : X \to B \) be a fibre space structure of a projective irreducible symplectic \( 2n \)-fold \( X \) with projective base \( B \). Then a general fibre \( F \) of \( f \) and \( B \) satisfy the following four conditions:

1. \( f \) is equidimensional in codimension 2 points of \( B \).
2. \( F \) is an abelian variety up to finite unramified cover and \( K_F \sim \mathcal{O}_F \).
3. \( \dim B = n \) and \( B \) has only \( \mathbb{Q} \)-factorial log-terminal singularities.
4. \( K_B \) is ample and Picard number \( \rho(B) \) is one.

Especially, if \( X \) is 4-dimensional, a general fibre of \( f \) is an abelian surface and \( f \) is equi-dimensional.

**Example.** Let \( S \) be a \( K3 \) surface with an elliptic fibration \( g : S \to \mathbb{P}^1 \) and \( S^{[n]} \) the \( n \)-pointed Hilbert scheme of \( S \). It is known that \( S^{[n]} \) is an irreducible symplectic \( 2n \)-fold and there exists a birational morphism \( \pi : S^{[n]} \to S^{(n)} \) where \( S^{(n)} \) is the symmetric \( n \)-product of \( S \) (cf. [1]). We can consider the \( n \)-dimensional abelian fibration \( g^{(n)} : S^{(n)} \to \mathbb{P}^n \) for the symmetric \( n \)-product of \( S^{(n)} \). Then the composition morphism \( g^{(n)} \circ \pi : S^{[n]} \to \mathbb{P}^n \) gives an example of a fibre space structure of an irreducible symplectic manifold.

**Remark.** Markushevich obtained some result of Theorem 2 in [6, Theorem 1, Proposition 1] under the assumption \( \dim X = 4 \) and \( f : X \to B \) is a Lagrangian fibration. The author does not know whether there exists a fibre space structure of an irreducible symplectic manifold such that general fibres are not lagrangian.

### 2. Proof of Theorem

First we introduce the following theorem due to Fujiki [3] and Beauville [1].

**Theorem 3** (Fujiki [3, Theorem 4.7, Lemma 4.11, Remark 4.12] and Beauville [1, Théorème 5]). Let \( X \) be an irreducible symplectic \( 2n \)-fold. Then there exists a nondegenerate quadratic form \( q_X \) of signature \( (3, b_2(X) - 3) \) on \( H^2(X, \mathbb{Z}) \) which satisfies

\[
\alpha^{2n} = a_0 q_X(\alpha, \alpha)^n,
\]

\[
c_{2i}(X)\alpha^{2n-2i} = a_i q_X(\alpha, \alpha)^{n-i} \quad (i \geq 1)
\]

where \( \alpha \in H^2(X, \mathbb{Z}) \) and \( a_i \)’s are constants depending on \( X \).

We shall prove Theorem 2 in six steps.

1. \( \dim B = n \) and \( B \) has only log-terminal singularities;
2. A general fibre \( F \) of \( f \) is an abelian variety up to unramified finite cover and \( K_F \sim \mathcal{O}_F \);
3. \( \rho(B) = 1 \);
4. \( B \) is \( \mathbb{Q} \)-factorial;
5. \( f \) is equidimensional in codimension 2 points of \( B \);
6. \( -K_B \) is ample.

**Step 1.** \( \dim B = n \) and \( B \) has only log-terminal singularities.
LEMMA 1. Let $X$ be an irreducible symplectic projective $2n$-fold and $E$ a divisor on $X$ such that $E^{2n} = 0$. Then,

1. If $E \cdot A^{2n-1} = 0$ for some ample divisor $A$, $E \sim 0$.
2. If $E \cdot A^{2n-1} > 0$ for an ample divisor $A$ on $X$, then $q_X(E, A) > 0$ and

$$\begin{cases} E_m A^{2n-m} = 0 & (m > n) \\ > 0 & (m \leq n). \end{cases}$$

Proof of Lemma. Let $V = \{E \in H^2(X, \mathbb{Z}) | E \cdot A^{2n-1} = 0\}$. By [3, Lemma 4.13], $q_X$ is negative definite on $W$ where $V = H^{2,0} \oplus H^{0,2} \oplus W$. Thus, if $E \cdot A^{2n-1} = 0$ and $E^{2n} = 0$, $E \equiv 0$. Since $\pi_1(X) = \{1\}$, $E \sim 0$. Next we prove (2). From Theorem 3, for every integer $t$,

$$\begin{aligned} (tE + A)^{2n} &= a_0(q_X(tE + A, tE + A))^n. \end{aligned}$$

Because $E^{2n} = a_0(q_X(E, E))^n = 0$,

$$q_X(tE + A, tE + A) = 2tq_X(E, A) + q_X(A, A).$$

Thus equation (1) has order at most $n$. Comparing both the sides of equation (1), we can obtain $E_m A^{2n-m} = 0$ for $m > n$. Moreover, comparing the $t$ term on both sides of equation (1), we can obtain $q_X(E, A) q_X(A, A)^{m-1} = c E \cdot A^{2n-1}$ where $c$ is a positive constant. Since $E \cdot A^{2n-1} > 0$, $q_X(E, A) > 0$. The coefficients of other terms on the left-hand side of (1) can be written as polynomial expressions in $q_X(E, A)$ and $q_X(A, A)$ with positive coefficients. Therefore, we can obtain $E_m A^{2n-m} > 0$ for $0 < m \leq n$. 

Let $H$ be a very ample divisor on $B$. Then $f^*H$ is a nef divisor such that $(f^*H)^{2n} = 0$, $(f^*H) \cdot A^{2n-1} > 0$ for an ample divisor $A$ on $X$. Thus $\dim B = n$. From [7, Theorem 2], $B$ has only log-terminal singularities.

Step 2. A general fibre $F$ of $f$ is an abelian variety up to unramified finite cover and $K_F \sim \mathcal{O}_F$.

From adjunction formula, $K_F \sim 0$. By [8], in order to prove that a certain étale cover of $F$ is abelian it suffices to prove that $c_2(F) (A|_p)^{n-2} = 0$ where $A$ is an ample divisor on $X$. Up to a scalar multiple $[F] = (f^*H)^p$ where $H$ is an ample divisor on $B$. Thus $c_2(F) (A|_p)^{n-2} = c_2(X) A^{n-2} (f^*H)^n$. Now consider

$$c_2(X)(tf^*H + A)^{2n-2} = a_1 q_X(tf^*H + A, tf^*H + A)^{n-1} = a_1(2tq_X(f^*H, A) + q_X(A, A))^{n-1}.$$ 

The latter is a polynomial in $t$ of degree less than $n$. Hence, the coefficient of $t^n$ on the left-hand side is trivial, that is, $c_2(F)(A|_p)^{n-2} = 0$.

Step 3. $\rho(B) = 1$.

LEMMA 2. Let $E$ be a divisor of $X$ such that $E^{2n} = 0$ and $E^n (f^*H)^n = 0$. Then $E \sim \mathcal{O} \lambda f^*H$ for some rational number $\lambda$.

Proof of Lemma. Considering $t^n$ term on both the sides of the following equation,

$$\begin{aligned} (E - tf^*H)^{2n} &= a_0 q_X(E - tf^*H, E - tf^*H)^n \\ &= -a_0 (2tq_X(E, f^*H))^n \\ &= -a_0 (2tq_X(E, f^*H))^n. \end{aligned}$$
we can obtain $q_x(E, f^*H)^n = cE^n(f^*H)^n = 0$ where $c$ is a constant. Thus $(E - tf^*H)^{2n} = 0$. Because $f^*H \cdot A^{2n-1} > 0$ for every ample divisor $A$ on $X$, we can choose a rational number $\lambda$ such that $(E - \lambda f^*H)A^{2n-1} = 0$. Then $E - \lambda f^*H \sim 0$ by Lemma 1.

Let $D$ be a Cartier divisor on $B$. Then $(f^*D)^{2n} = 0$ and $(f^*D)^n \cdot (f^*H)^n = 0$, thus $D \sim \lambda f^*H$ and $\rho(B) = 1$.

**Step 4.** $B$ is $\mathbb{Q}$-factorial.

Let $D$ be an irreducible and reduced Weil divisor on $B$ and $D_i (1 \leq i \leq k)$ divisors on $X$ whose supports are contained in $f^{-1}(D)$. We shall construct a divisor $\bar{D} := \sum\lambda_i D_i$ such that $\bar{D}^2 = 0$, $\bar{D}^n(f^*H)^n = 0$ and $f(\bar{D}) = D$. If such a divisor $\bar{D}$ exists, we can conclude that $\bar{D} \sim \lambda f^*H$ by Lemma 1, and that $D \sim \lambda f^*H$ because $f(\bar{D}) = D$. Therefore $B$ is $\mathbb{Q}$-factorial.

Let $A$ be a very ample divisor on $X$, $H$ a very ample divisor on $B$, $S := A^{n-1}(f^*H)^{n-1}$ and $C := H^{n-1}$. Then there exists a surjective morphism $f^* : S \rightarrow C$. If we choose $H$ and $A$ general, we may assume that $S$ and $C$ are smooth and $C \cap D$ is contained in the smooth locus of $B$. Because $D$ is a Cartier divisor in a neighborhood of $C \cap D$, we can define $f^*D := \sum\lambda_i D_i | U_i$, $(\lambda_i \geq 0)$ in a neighborhood $U$ of $S$. Let $\bar{D} := \sum\lambda_i D_i$. Note that if $\lambda_i > 0$, $f(D_i) = D$ because we choose $C$ generally. Since $f(\bar{D}) = D$, $\bar{D}^n(f^*H)^n = 0$. We prove that $\bar{D}^2 = 0$. Comparing the $t^n$ term of the both hand sides of the following equation:

$$(\bar{D} + tf^*H)^2 = a_0q_x(\bar{D} + tf^*H, \bar{D} + tf^*H)$$

we can see that $\bar{D}^n(f^*H)^n = cq_x(\bar{D}, f^*H)^n$ where $c$ is a nonzero constant. Thus $q_x(\bar{D}, f^*H) = 0$. Moreover, comparing the $s^2t^{n-1}$ term on both sides of the following equation,

$$(s\bar{D} + A + tf^*H)^2 = a_0q_x(s\bar{D} + A + tf^*H, s\bar{D} + A + tf^*H)^n$$

we can obtain $q_x(\bar{D}, \bar{D})q_x(A, f^*H)^{n-1} = c\bar{D}^2 \cdot A^{n-1}(f^*H)^{n-1}$ where $c$ is a nonzero constant. Since $\bar{D}^2 \cdot A^{n-1}(f^*H)^{n-1}$ is a multiple of a fibre class of $f'$, $\bar{D}^2 \cdot A^{n-1}(f^*H)^{n-1} = 0$. By Lemma 1, $q_x(A, f^*H) > 0$. Therefore $a_0q_x(\bar{D}, \bar{D}) = \bar{D}^2 = 0$.

**Step 5.** $f$ is equidimensional in codimension 2 points of $B$.

It is enough to prove that there exist no divisors $E$ on $X$ such that $\dim f(E) < n - 1$. Assume the contrary. Then there exists an effective divisor $E$ on $X$ such that $E(f^*H)^{n-1} \cdot A^n = 0$ for an ample divisor $A$ on $X$ and an ample divisor $H$ on $B$. Since $E^n(f^*H)^n = 0$, comparing the coefficient of $t^n$ of the following equation,

$$(E + tf^*H)^2 = a_0q_x(E + tf^*H, E + tf^*H)^n$$

we can obtain $q_x(f^*H, E) = 0$. Considering the following equation:

$$(sE + tf^*H + A)^2 = a_0q_x(sE + tf^*H + A, sE + tf^*H + A)^n$$

we can obtain $q_x(f^*H, E) = 0$. Considering the following equation:

$$(sE + tf^*H + A)^2 = a_0q_x(sE + tf^*H + A, sE + tf^*H + A)^n$$
we can obtain \( q_X(E, A)q_X(f^*H, A)^{n-1} = cE(f^*H)^{n-1} \cdot A^n \) where \( c \) is a nonzero constant. Thus \( q_X(E, A) = 0 \). However, from the following equation,

\[
(tE + A)^{2n} = a_0q_X(tE + A, tE + A)^n
\]

we can obtain \( E \cdot A^{2n-1} = cq_X(E, A)q_X(A, A)^{n-1} = 0 \). Since \( E \) is an effective divisor, a contradiction.

**Step 6.** \(-K_B\) is ample.

From Steps 3 and 4, we can write \(-K_B \sim \partial H\). It is enough to prove \( t > 0 \). Because \( K_X \sim \mathcal{O}_X \) and a general fibre of \( f : X \to B \) is a minimal model, \( \kappa(B) \leq 0 \) by [5, Theorem 1.1] and \( t \geq 0 \). Assume that \( t = 0 \). If \( K_B \not\sim \mathcal{O}_B \), we consider the following diagram:

\[
\begin{array}{c}
X \\
\alpha \downarrow \\
\tilde{X} := X \times_B \tilde{B} \rightarrow \tilde{B},
\end{array}
\]

where \( \beta \) is an unramified finite cover outside non Gorenstein point of \( B \) and \( K_B \sim \mathcal{O}_B \). Let \( Z := \{ x \in B \mid K_B \) is not Cartier divisor at \( x \} \). Since \( B \) is normal, codimension of \( Z \) is greater than two. Moreover, \( f \) is equidimensional in codimension 2 points of \( B \), the real codimension of \( f^{-1}(Z) \) is greater than four. Therefore \( \pi_1(X - f^{-1}(Z)) = \pi_1(X) = \{1\} \). Thus, there exists a morphism from \( X - f^{-1}(Z) \) to \( \tilde{B} - Z \). Since \( \dim f^{-1}(Z) < 2n - 1 \), there exists a global holomorphic \( n \)-form \( \omega' \) on \( X \) coming from \( \tilde{B} \). However, there does not exist any global holomorphic form on \( X \) which comes from a lower dimension manifold, it is a contradiction. Thus \( t > 0 \) and we completed the proof of Theorem 2.

Acknowledgements—The author expresses his thanks to Professors Y. Miyaoka, S. Mori and N. Nakayama for their advice and encouragement. He also expresses his thanks to referee for his comments relevant to this paper. This paper owes much to the paper [4], which is a survey article of irreducible symplectic manifolds.

REFERENCES


4. Huybrechts, D., Compact hyperkähler manifolds: basic results. alg-geom9705025.


Research Institute of Mathematical Science
Kyoto University
Kitashirakawa, Oiwake-cho
Kyoto 606-01, Japan