Hierarchies of total functionals over the reals

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Abstract

We compare two natural constructions, the $A$-hierarchy and the $R$-hierarchy, of hereditarily total, continuous and extensional functionals of finite types over the reals. The $A$-hierarchy is based on the closed interval domain representation of the reals while the $R$-hierarchy is based on the binary negative digit representation. We show that the two hierarchies share a common maximal core. To this end, we construct an alternative to the $R$-hierarchy and prove a density theorem for this alternative hierarchy.

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1. Introduction

One motivation for studying objects of finite types is that they may be used to give denotational interpretations of programs in some real or ideal typed programming languages. This was the motivation when Scott [13,14] introduced the partial continuous functionals of finite types.

Since programs do not always terminate on a given input, partiality is essential in order to give a denotational semantics for programs. However, what we are interested in, at least under some circumstances, are programs terminating on each relevant input. Thus, given a hierarchy of partial continuous functionals, the hereditarily total ones will be of a special interest. Moreover, if we restrict our interest to the hereditarily total functionals, it is natural to identify functionals that act the same way on each total input. This technically will mean that we consider the extensional collapse of the hierarchy of hereditarily total functionals.

If we want to know who the continuous functionals of type 17 are, when the set of natural numbers is the base type, it seems that the answer is robust in the sense that all reasonable approaches have led to the same class, the Kleene–Kreisel continuous functionals.
functionals. In this paper we will consider hierarchies of continuous functionals over the reals, and then the answer to what continuous functionals of higher types are is not so clear. We will consider two approaches based on different ways of representing the reals via algebraic domains. One, using approximations via closed, rational intervals, is suitable for languages where the reals are considered as basic data objects, while the other, using digital representations for approximations to reals, are suitable for languages where a real is not a basic data-object but may be represented in several inconsistent ways via binary data.

The question is if the choice of how to represent the reals as a datatype will influence on the set of total, extensional functionals that may actually exist of a certain type. We will not answer this question, but show in a precise sense that these hierarchies share a common core that is dense in both hierarchies.

Our inspiration was the paper by Bauer et al. [1], where they considered several natural ways to construct a hierarchy of total ‘continuous’ functionals over the reals. One way is to use the closed interval domain representation of the reals. Equivalently, we may start with the algebraic domain representation of the reals based on closed rational intervals. The TTE-approach of Weihrauch [16] leads to an isomorphic hierarchy of quotient spaces. This is so because both hierarchies can be characterized as the one obtained in the category of Kuratowski limit spaces, see Normann [8,9] and Schröder [11,12].

Bauer et al. [1] suggest an alternative hierarchy, essentially based on a TTE-approach at ground level, but working inside the category of algebraic domains for higher types. In [4–6] Di Gianantonio use the binary negative digit representation of the reals in order to give a denotational semantics for his calculus for exact real valued computations. This approach is equivalent to the one taken in [1].

In this paper we will investigate the hierarchy of hereditarily total functionals based on Di Gianantonio’s approach more closely. For trivial reasons, the hereditarily total objects will not form a dense subset of the underlying domain even at type 1 in this hierarchy. Instead of characterising the set of compacts that may be extended to total objects, we construct an alternative hierarchy, the S-hierarchy, designed to satisfy density. We then prove that the S-hierarchy leads to the same type-hierarchy of total functionals as the original one. Finally we use the proof of the density theorem for the S-hierarchy to establish connections with the hierarchy based on closed rational intervals.

2. Preliminaries

For notational simplicity, we will restrict ourselves to the pure types

\[ 0, 1, 2, \ldots \]

but will occasionally consider the spaces \( \mathbb{R}^n \to \mathbb{R} \) and the corresponding types, all considered to be types at level 1.
A **type-structure** will be a sequence

\[ \{ T_n \}_{n \in \mathbb{N}} , \]

where \( T_0 \) is any set and \( T_{n+1} \) is some set of functions \( f : T_n \to T_0 \).

If \( \sim_0 \) is a partial equivalence relation on \( T_0 \), it will induce a partial equivalence relation \( \sim_n \) on \( T_n \) by

\[ f \sim_{n+1} g \iff \forall x, y \in T_n ( x \sim_n y \to f(x) \sim_0 g(y) ) . \]

Since we will be interested in representations of type structures over the reals, we will in general assume that we have a surjective, partial map \( \rho_0^T : T_0 \to \mathbb{R} \) defined on

\[ \{ x \in T_0 \mid x \sim_0 x \} \]

such that

\[ x_1 \sim_0 x_2 \iff \rho_0^T(x_1) = \rho_0^T(x_2) . \]

This will induce a hierarchy \( \{ \mathbb{R}_n^T \}_{n \in \mathbb{N}} \) of functionals with partial, surjective maps \( \rho_n^T : T(n) \to \mathbb{R}_n^T \) defined on \( \{ x \in T_n \mid x \sim_n x \} \) by recursion on \( n \) as follows:

- \( \mathbb{R}_0^T = \mathbb{R} \) with \( \rho_0^T \) as given,
- \( \rho_{n+1}^T(f)(\rho_n^T(a)) = \rho_0^T(f(a)) \),
- \( \mathbb{R}_{n+1}^T = \{ \rho_{n+1}^T(f) \mid f \sim_{n+1} f \} \).

It is easy to verify by induction on \( n \) that \( \rho_n^T \) is well defined, and that

\[ x_1 \sim_n x_2 \Leftrightarrow \rho_n^T(x_1) = \rho_n^T(x_2) \]

when the two latter values are defined.

In this paper, and in a context as above, an element \( x \in T_n \) will be called **hereditarily total** or just **total** if \( x \sim_n x \). The hierarchy \( \{ \mathbb{R}_n^T \}_{n \in \mathbb{N}} \) will be called the **extensional collapse** of the hereditarily total functionals induced by \( \{ T_n \}_{n \in \mathbb{N}} \), \( \sim_0^T \) and \( \rho_0^T \).

Whenever we need to use the term ‘total’ in a different way, we will be explicit about it.

**Example 1.** We use nonstandard analysis. Let \( c \) be a nonstandard natural number. Let \( n_{0} = \{ k/c \mid c^2 < k < c^3 \} \).

Let \( n_{n+1} \) be the set of internal maps \( f : n_n \to n_0 \).

Let \( k_1/c \sim_0^T k_2/c \) if they are both infinitesimally close to the same real number, which we denote \( \rho_0^T(k_i/c) \) for \( i = 1, 2 \).

In Normann [10] we investigate this example in more detail. Note that all elements in \( n_n \) will be **hyperfinite**, and that \( n_0 \) is a commonly used internal discretisation of the real line.
This first example is somewhat outside the scope of the paper. From now on we will stay within the theory of domains. The next example is the standard approach via continuous domains:

**Example 2.** Let \( C_0 = \{ [a, b] \mid a \leq b \} \cup \{ \mathbb{R} \} \) ordered by reversed inclusion, where \( a \) and \( b \) are reals.

\( C_0 \) is a *continuous domain*, and inductively, we let \( C_{n+1} \) be the function-space \( C_n \rightarrow C_0 \) in the cartesian closed category of continuous domains.

We let
\[
[x, y] \sim^C_0 [a, b] \iff x = y = a = b
\]
and we let \( \rho_0^C([x, x]) = x \).

We then get \( \mathbb{R}_n^C \) and \( \rho_n^C \) from the general construction.

Our third example is essentially the same as the second one, using algebraic domains.

**Example 3.** Let \( A_0 \) be the algebraic domain based on the set of closed and bounded rational intervals together with the real line, ordered by the reversed inclusion ordering (i.e. the ideal completion).

An ideal \( I \) represents a real \( x \) if
\[
\{ x \} = \bigcap_{I \in \mathcal{J}} I.
\]

Two ideals will be \( \sim^A_0 \)-equivalent if they represent the same real in this way, and if \( \mathcal{J} \) represents \( x \) we let \( \rho_0^A(\mathcal{J}) = x \). We let \( A_n \) be the canonical interpretation of the type \( n \) in the category of algebraic domains and \( A_n^* \) be the set of compacts in \( A_n \).

**Theorem 1.** For all \( n \), \( \mathbb{R}_n^C = \mathbb{R}_n^A = \mathbb{R}_n^{ns} \).

The first equality is well known, and the second equality is the main result of Normann [10].

Our next example is a minor adjustment of the type-structure studied in Bauer et al. [1]. It is based on the binary negative representation used e.g. by Di Gianantonio [4–6].

**Example 4.** Let \( R_0^* \) consist of the empty sequence \( e \) together with all finite sequences
\[
a, b_1, \ldots, b_k,
\]
where \( k \geq 0 \), \( a \in \mathbb{Z} \) and each \( b_i \in \{-1, 0, 1\} \).

This set is ordered by sequence extension, and thus forms the basis of an algebraic domain \( R_0 \).
We identify a maximal ideal with the infinite sequence \( a, b_1, b_2, \ldots \), which will represent the real

\[
p_0^R(a, b_1, \ldots) = a + \sum_{i=1}^{\infty} 2^{-i} b_i,
\]

\( \sim_0^R \) will be an equivalence relation on the set of all maximal ideals.

Let \( R_n \) be the corresponding interpretation of type \( n \) in the category of algebraic domains.

One motivation for considering this hierarchy, as pointed out in [1], is that the base type can be identified with a retraction of the domain interpretation of \( \mathbb{N}^\mathbb{N} \). Thus higher type functionals may be represented by functionals in the domain hierarchy over the natural numbers. For this hierarchy we have a well developed theory of computability, and e.g. a clear distinction between sequential and nonsequential algorithms. However, the hierarchy, as we have defined it, does not satisfy density. It is not known if \( R_n^C = R_n^R \) in general. Bauer et al. [1] proved:

**Theorem 2.** For \( n \leq 2 \) we have that \( R_n^C = R_n^R \).

A topological space is zero-dimensional if there is a basis of closed-open sets. Bauer et al. [1] proved in addition that if the Kleene-Kreisel continuous functionals (see e.g. Normann [7]) of type 2 is zero-dimensional, then \( R_3^C = R_3^R \).

### 3. A density theorem

The \( R \)-hierarchy will not satisfy density, in the sense that the hereditarily total objects do not form a dense subset of the underlying domain for any type but the ground type.

In this section we will describe an alternative hierarchy, the \( S \)-hierarchy. We will prove a density theorem for the \( S \)-hierarchy, and in the next section we will show that for all \( n \), \( R_n^R = R_n^S \).

The domain \( S_{n+1} \) will be a subset of \( S_n \to S_0 \), and the ordered set of finitary or compact elements in \( S_{n+1} \) will be a subordering of the ordered set of finitary elements of \( S_n \to S_0 \).

**Definition 1.** For each number \( n \), we define the set \( S_n^C \) of compacts in the domain \( S_n \), together with the binary coherence-relation \( \approx_n \) on \( S_n^C \) as follows:

- \( S_0^C = R_0^C \) with the same ordering.
- Let \( v_0(e) = R \) and let

\[
v_0(a, b_1, \ldots, b_k) = \left[ a + \sum_{i=1}^{k} 2^{-i} b_i - 2^{-k}, a + \sum_{i=1}^{k} 2^{-i} b_i + 2^{-k} \right].
\]

For \( p \) and \( q \) in \( S_0^C \) we let \( p \approx_0 q \) if \( v_0(p) \cap v_0(q) \neq \emptyset \).
• Let $S_{n+1}^c$ be the set of functions
  \[ F_{\{(q_1,r_1),\ldots,(q_k,r_k)\}} \in S_n \rightarrow S_0 \]
defined by
  \[ F_{\{(q_1,r_1),\ldots,(q_k,r_k)\}}(x) = \bigcup \{ r_i \mid q_i \sqsubseteq_n x \}, \]
where
  • each $q_i$ is in $S_n^c$,
  • each $r_i$ is in $S_0^c$,
  • if $\{q_i,q_j\}$ is bounded in $S_n^c$, then $\{r_i,r_j\}$ is bounded in $S_0^c$,
  • if $q_i \approx_n q_j$, then $r_i \approx_0 r_j$.

We let
  \[ F_{\{(q_1,r_1),\ldots,(q_k,r_k)\}} \approx_{n+1} F_{\{(q'_1,r'_1),\ldots,(q'_k,r'_k)\}} \]
if for each $i \leq k$ and $j \leq k'$ we have that
  \[ q_i \approx_n q'_j \Rightarrow r_i \approx_0 r'_j. \]

• We let $S_{n+1}$ be the closure of $S_{n+1}^c$ in $S_n \rightarrow S_0$, with the inherited ordering $\sqsubseteq_{n+1}$.

In this definition we used the ‘least upper bound’ $\bigsqcup$, and we defined $F_X \approx_n F_Y$ referring to properties of the sets $X$ and $Y$. To see that the definition is sound, we have to establish the following

**Lemma 1.** (a) The definitions of $S_n^c$, $\approx_n$ and $S_n$ are sound.
(b) If $n = m + 1$ and $F_X$ and $F_Y$ are in $S_n^c$ as described, then $F_X \sqsubseteq F_Y$ if and only if
  \[ \forall (p,q) \in X (q \sqsubseteq_0 \bigsqcup \{ q' \mid \exists p' \sqsubseteq_m p ((p',q') \in Y) \}). \]
(c) If $x$, $y$, $z$ and $w$ are in $S_n^c$, $x \approx_n y$, $z \sqsubseteq_n x$ and $w \sqsubseteq_n y$, then $z \approx_n w$.
(d) If $\mathcal{F}$ is a finite, pairwise bounded set in $S_n^c$, then $\mathcal{F}$ has a least upper bound $\bigsqcup \mathcal{F}$ in $S_n^c$.
  If $n = m + 1$ and $\mathcal{F} = \{ F_{X_1},\ldots,F_{X_k} \}$ is pairwise bounded in $S_m^c$, then $\bigsqcup \mathcal{F} = F_{X_1 \cup \cdots \cup X_k}$.
(e) If $n = m + 1$, $F_X \in S_n^c$ and $G \sqsubseteq F_X$ is a compact in $S_m \rightarrow S_0$, then $G \in S_n^c$.

**Proof.** We will use induction on $n$. For $n = 0$ this is trivial.

For $n = m + 1$ we use (d) of the induction hypothesis to prove that the definition of $S_n^c$ is sound. (b) is a standard fact of domain theory.

In order to prove (c) for explicitly defined $F_X$, $F_Y$, $F_Z$ and $F_W$, we use (b) and we use (c) of the induction hypothesis. Then the soundness of the definition of $\approx_n$ follows.

We use (b) to prove (d).

(e) follows from the characterisation of the ordering of the compacts in $S_m \rightarrow S_0$ together with (c) and (d) of the induction hypothesis. The soundness of the definition of $S_n$ follows from (e).
All arguments are easy, and are left for the reader.

We let the total objects in $S_0$, $\sim^S_0$ and $p_0^S$ be as for the $R$-hierarchy.

Since $S_{n+1} \subseteq S_n \to S_0$, this defines the hereditarily total objects, the relation $\sim^S_n$, the function $p_n^S$ and the set $R_n^S$ for all types $n$. □

Lemma 2. For each $n$ there is a monotone map $v_n : S_n^c \to A_n^c$ such that whenever $p$ and $q$ are in $S_n^c$, then $p \approx_n q$ if and only if $v_n(p)$ and $v_n(q)$ are consistent in $A_n$.

Proof. We let $v_0$ be as in the first part of Definition 1. Clearly the lemma holds for $v_0$.

Now assume that $v_n$ is defined satisfying the lemma. For $y \in A_n$ let

$$v_{n+1}(F_{\left((p_1, q_1), \ldots, (p_k, q_k)\right)}(y)) = \bigcup \{ v_0(q_i) \mid v_n(p_i) \sqsubseteq^A_n y\}.$$ 

By the induction hypothesis, $\{ p_i \mid v_n(p_i) \sqsubseteq^A_n y\}$ is pairwise coherent. By the definition of $S^c_{n+1}$ then $\{ q_i \mid v_n(p_i) \sqsubseteq^A_n y\}$ is pairwise coherent, and by the definition of $v_0$ it follows that $\{ v_0(q_i) \mid v_n(p_i) \sqsubseteq^A_n y\}$ has a nonempty intersection.

Thus $v_{n+1}$ is well defined.

Monotonicity for $v_{n+1}$ follows from monotonicity for $v_n$ using Lemma 1 (b). Finally, using a similar argument, we see that $F_X \approx_{n+1} F_Y$ if and only if $v_{n+1}(F_X)$ and $v_{n+1}(F_Y)$ are consistent.

From now on we will take the liberty to write the denotation

$$\{(p_1, q_1), \ldots, (p_k, q_k)\}$$

instead of

$$F_{\left((p_1, q_1), \ldots, (p_k, q_k)\right)},$$

and consider $\sqsubseteq_n$ as a preordering of such denotations, defined as in Lemma 1 (b).

We will also consider domains of the form $S_0^n \to S_0$. When we say that $f \in S_0^n \to S_0$ is total, we mean that $f$ also will respect equivalence. We extend $v_0$ to $v_0^n : (S_0^n)^n \to (A_0^n)^n$ coordinatewise. We let $f[X]$ denote the image of the set $X$ under the function $f$. □

Lemma 3. Let $p = \{(\tilde{u}_1, v_1), \ldots, (\tilde{u}_k, v_k)\}$ be a compact in $S_0^n \to S_0$ (respecting coherence).

Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous such that

$$f[v_0^n(\tilde{u}_i)] \subseteq v_0(v_i)$$

for each $i \leq k$.

Then $f$ may be represented by a total object in $S_0^n \to S_0$ extending $p$.

The construction is a simple adjustment of the general construction of a representation of $f$ in the $R$-hierarchy.

Lemma 4. Let $p$ be a compact in $S_0^n \to S_0$ respecting coherence.
Then $p$ can be extended to a total function $f : S_0^n \rightarrow S_0$.

**Proof.** The corresponding result for the $A$-hierarchy is proved in Normann [8,9]. Combine this with Lemmas 2 and 3.

**Theorem 3.** Let $p$ and $p'$ be coherent elements of $S_n^c$.

Then there are equivalent, total extensions $x$ and $x'$ of $p$ and $p'$ resp. in $S_n$.

**Proof.** We use induction on $n$. For $n = 0$ this follows from the definition of coherence, and for $n = 1$ we use Lemma 2, Lemma 3 and the density theorem for $A_0 \rightarrow A_0$.

Now we consider the case $n + 2$, where we will assume that the theorem holds for all $m < n + 2$.

**Claim 1.** If $x$ and $y$ are total elements of $S_m$ for some $m \leq n + 2$, then $x$ and $y$ are equivalent if and only if all $p \subseteq x$ and $q \subseteq y$ with $p, q \in S_m^n$ are coherent.

This is an easy, but tedious, consequence of the induction hypothesis, and the proof is left for the reader.

**Claim 2.** Let $q$ and $q'$ be elements in $S_{n+1}^c$ such that $\{q, q'\}$ is unbounded. Then there is a total map

$B : S_{n+1} \rightarrow \mathbb{B}_\perp$

such that $B(q) = \mathbf{tt}$ and $B(q') = \mathbf{ff}$.

Here we do not require that $B$ identifies equivalent functions.

**Proof.** Since $\{q, q'\}$ is unbounded, there will either be

(*) bounded $\{u_i, u'_j\}$ such that $v_i$ and $v'_j$ are inconsistent, or

(**) coherent $u_i$ and $u'_j$ such that $v_i$ and $v'_j$ are not coherent.

In case (*), use the induction hypothesis and let $x$ be a total extension of $u_i$ and $u'_j$. Let $B(f) = \mathbf{tt}$ if $f(x)$ extends $v_i$, $B(f) = \mathbf{ff}$ if $f(x)$ is inconsistent with $v_i$ and $\perp$ otherwise.

In case (**) we let $x$ and $x'$ be equivalent total extensions of $u_i$ and $u'_j$ resp.

Let $B(f) = \mathbf{tt}$ if $f(x)$ extends $v_i$, let $B(x) = \mathbf{ff}$ if $f(x)$ is inconsistent with $v_i$ or if $f(x')$ extends $v'_j$ and let $B(f) = \perp$ otherwise.

If $f$ is total, clearly $B(f) \neq \perp$, and clearly $B(q) = \mathbf{tt}$ while $B(q') = \mathbf{ff}$. It remains to verify that we cannot have both $B(f) = \mathbf{tt}$ and $B(f) = \mathbf{ff}$ in case (**) (see [9]). But since $x$ and $x'$ are equivalent, approximations $u$ and $u'$ to $x$ and $x'$ will be coherent. Then approximations $v$ and $v'$ to $f(u)$ and $f(u')$ will be coherent. If $f(u)$ extends $v_i$ we can neither have that $f(u)$ is inconsistent with $v_i$ nor that $f(u')$ extends $v'_j$, since $v_i$ and $v'_j$ are incoherent. This ends the proof of the claim.

We are now ready to handle the induction step.
Let
\[ p = \{(q_1, r_1), \ldots, (q_k, r_k)\} \]
and
\[ p' = \{(q'_1, r'_1), \ldots, (q'_k, r'_k)\} \]
be coherent elements of \( S_{n+2}^c \).

Let
\[ q_i = \{(u_{i,1}, v_{i,1}), \ldots, (u_{i,m_i}, v_{i,m_i})\} \]
and
\[ q'_i = \{(u'_{i,1}, v'_{i,1}), \ldots, (u'_{i,m'_i}, v'_{i,m'_i})\} \]

We let \( C \) be the disjoint union of \( C_1, C_2 \) and \( C_3 \) defined as follows:

\( C_1 \) consists of all unordered pairs \( c = \{i, j\} \) such that \( r_i \) and \( r_j \) are inconsistent.

\( C_2 \) consists of all unordered pairs \( c = \{i, j\} \) such that \( r'_i \) and \( r'_j \) are inconsistent.

\( C_3 \) consists of all ordered pairs \( c = (i, j) \) such that \( r_i \) and \( r'_j \) are incoherent.

For each \( c \in C_1 \oplus C_2 \oplus C_3 \), we will define a total function \( G_c \) (for \( c \in C_1 \oplus C_2 \)) or two total, equivalent functions \( G_c \) and \( G'_c \) (for \( c \in C_3 \)) as follows:

Let \( c = \{i, j\} \in C_1 \). There will be two cases.

1. \( r_i \) and \( r_j \) are coherent. Then \( q_i \) and \( q_j \) will not be bounded. Then either there are \( l \) and \( l' \) such that \( \{u_{i,l}, u_{j,l'}\} \) is bounded but \( \{v_{i,l}, v_{j,l'}\} \) is not bounded, or such that \( u_{i,l} \) and \( u_{j,l'} \) are coherent but \( v_{i,l} \) and \( v_{j,l'} \) are not coherent. In any case we will have that \( u_{i,l} \) and \( u_{j,l'} \) are coherent while \( \{v_{i,l}, v_{j,l'}\} \) is not bounded.

By the induction hypothesis and Claim 1, let \( x \) and \( x' \) be equivalent, total extensions of \( u_{i,l} \) and \( u_{j,l'} \) resp. By Claim 2, let \( B : S_{n+1} \rightarrow B_\bot \) be total such that \( B(q_i) = tt \) and \( B(q_j) = ff \).

Let
\[ G_c(f) = f(x) \text{ if } B(f) = tt \]
\[ G_c(f) = f(x') \text{ if } B(f) = ff. \]

Then \( G_c \) is total, also respecting equivalence, since \( x \) and \( x' \) are equivalent.

By construction \( v_{i,l} \subseteq G_c(q_i) \) and \( v_{j,l'} \subseteq G_c(q_j) \), so \( G_c(q_i) \) and \( G_c(q_j) \) are inconsistent.

2. \( r_i \) and \( r_j \) are incoherent. Then \( q_i \) and \( q_j \) are incoherent, and there will be \( l \) and \( l' \) such that \( u_{i,l} \) and \( u_{j,l'} \) are coherent while \( v_{i,l} \) and \( v_{j,l'} \) are incoherent.

By the same method as in Case 1, we construct \( G_c \) such that \( G_c(q_i) \) and \( G_c(q_j) \) are incoherent.

For \( c \in C_2 \), we construct \( G_c \) in the same way.

Now, let \( c = (i, j) \in C_3 \).

Then \( r_i \) and \( r'_j \) are incoherent, and, since \( p \) and \( p' \) are coherent, it follows that \( q_i \) and \( q'_j \) are incoherent. Then there are \( l \) and \( l' \) such that \( u_{i,l} \) and \( u'_{j,l'} \) are coherent while \( v_{i,l} \) and \( v'_{j,l'} \) are incoherent.
Let \( x \) and \( x' \) be equivalent extensions of \( u_{i,l} \) and \( u'_{j,l'} \). Let \( G_c(f) = f(x) \) and \( G'_c(f) = f(x') \). Clearly \( G_c \) and \( G'_c \) are equivalent.

This ends the construction of \( G_c \) and \( G'_c \). For the sake of notational simplicity, we define \( G'_c \) as \( G_c \) for \( c \in C_1 \oplus C_2 \).

Let \( A_i = (G_c(q_i))_{c \in C} \) and \( A'_i = (G'_c(q'_i))_{c \in C} \).

Then \( \{(A_1, r_1), \ldots, (A_k, r_k)\} \) and \( \{(A'_1, r'_1), \ldots, (A'_k, r'_k)\} \) are coherent sets of compacts in \( S^C_0 \rightarrow S_0 \).

By Lemma 2 and the proof of Lemma 4 we see that these compacts may be extended to equivalent total objects \( F \) and \( F' \).

Then \( H \) and \( H' \) defined by \( H(f) = F(G(f)) \) and \( H'(f) = F'(G'(f)) \) will be equivalent, and by construction, extensions of \( p \) and \( p' \) resp.

This ends the proof of the theorem.

**Corollary 1.** Each \( p \in S^C_0 \) may be extended to a total object.

### 4. Equivalence of \( S \) and \( R \)

One important consequence of the main result of Section 3 is that for total elements \( x_1 \) and \( x_2 \) in \( S_n \), \( x_1 \sim_n x_2 \) if and only if \( p_1 \approx_n p_2 \) whenever \( p_1 \) and \( p_2 \) are compacts such that \( p_1 \subseteq x_1 \) and \( p_2 \subseteq x_2 \).

We will use this to prove

**Theorem 4. For all** \( n \in \mathbb{N} \), \( R^S_n = R^R_n \).

**Proof.** By recursion on \( n \) we will construct total, continuous maps \( \phi_n : S_n \rightarrow R_n \) and \( \psi_n : R_n \rightarrow S_n \) respecting equivalence such that \( \phi_n \circ \psi_n \) and \( \psi_n \circ \phi_n \) are equivalent to the respective identity functions. Moreover, the maps will up to equivalence, commute with application.

Let \( \phi_0 = \psi_0 = \) the identity map on \( S_0 = R_0 \).

Assume that \( \phi_n \) and \( \psi_n \) are constructed with the properties required.

We let \( \phi_{n+1}(f)(x) = f(\psi_n(x)) \) for \( f \in S_{n+1} \) and \( x \in R_n \). Clearly, \( \phi_{n+1} \) is both total, continuous and respects equivalence.
The construction of $\psi_{n+1}$ is not quite that simple. Let $\psi_{n+1}(g)(y) = g(p_n(y))$ for $g \in R_{n+1}$ and $y \in S_n$.

If $g$ is total, $\psi_{n+1}(g) \in S_n \rightarrow S_0$ is total and continuous and will respect equivalence. So we have

(*) $\psi_{n+1}(g) \in S_{n+1}$ for total $g \in R_{n+1}$.

For $g$'s that cannot be extended to total objects we do however have a problem, then $\psi_{n+1}(g)$ is not necessarily in $S_{n+1}$. We will construct a total $\psi_{n+1} \subseteq \psi_{n+1}$ that maps all $g$'s into $S_{n+1}$.

Let $\{(p_k, q_k, r_k)\}_{k \in \mathbb{N}}$ be an enumeration of all triples $(p, q, r)$ where $p \in R_{n+1}$ can be extended to a total object, $q \in S_n$, $r \in S_0$ and $r \subseteq \psi_{n+1}(p)(q)$.

We define $\Gamma$ as follows: $(p, q, r) \in \Gamma$ if for some $k$, $p_k \subseteq p$, $q_k \subseteq q$, $r \subseteq r_k$ and for all $l < k$, if $r$ and $r_l$ are incoherent, then either $p$ is inconsistent with $p_l$ or $q$ is inconsistent with $q_l$. The elements of $\Gamma$ can be seen as compacts approximating $\psi_{n+1}$, and then $\Gamma$ is bounded in $R_{n+1} \rightarrow (S_n \rightarrow S_0)$.

With this convention, we let $\psi_{n+1} = \bigcup \Gamma$. □

**Claim 3.** $\psi_{n+1}(g) \in S_{n+1}$ for all $g \in R_{n+1}$.

**Proof.** Let $q$ and $q'$ be coherent and let $r \subseteq \psi_{n+1}(g)(q)$, $r' \subseteq \psi_{n+1}(g)(q')$.

Then for some $p, p' \subseteq g$ we have that $(p, q, r) \in \Gamma$ and $(p', q', r') \in \Gamma$.

We will show that $r$ and $r'$ are coherent. Let $k$ and $k'$ be witnesses to $(p, q, r) \in \Gamma$ and $(p', q', r') \in \Gamma$ resp. Assume that $k \leq k'$.

If $k = k'$ then clearly $r$ and $r'$ are coherent.

If $k < k'$ then either $r'$ and $r_k$ are coherent, $q'$ and $q_k$ are inconsistent or $p'$ and $p_k$ are inconsistent.

But $p'$ and $p_k$ are both approximations to $g$ so they are consistent. $q_k \subseteq q$ which is coherent with $q'$, so $q_k$ is coherent with $q'$. Thus $r'$ is coherent with $r_k$, and then with $r \subseteq r_k$.

This ends the proof of the claim. □

**Claim 4.** $\psi_{n+1}$ is total.

**Proof.** Let $g \in R_{n+1}$ be total, $y \in S_n$ be total and $r \subseteq \psi_{n+1}(g)(y)$.

Then for some $k$, $p_k \subseteq g$, $q_k \subseteq y$ and $r \subseteq r_k$.

We will show that there will be $p \subseteq g$ such that $(p, q_k, r) \in \Gamma$ via $(p_k, q_k, r_k)$.

Let $l < k$. If $q_l$ is incoherent with $q_k$ the requirement is satisfied. Assume that $q_l$ and $q_k$ are coherent. □

**Subclaim.** $p_l$ is inconsistent with $g$ and consequently with a finitary approximation to $g$.

**Proof of subclaim.** If not, $g \sqcup p_l$ is total in $R_{n+1}$ since any extension of a total object will be total. As observed in (*) then, $\psi_{n+1}(g \sqcup p_l) \in S_{n+1}$, while $\psi'(g \sqcup p_l)$ will extend $\{(q_l, r_l), (q_k, r_k)\}$, which will not be in $S_{n+1}^c$.

This ends the proof of the subclaim.
Let $p \sqsubseteq g$ be inconsistent with $p_1$ for all $l<k$ such that $p_1$ is inconsistent with $g$. Then $(p, q_k, r) \in \Gamma$, i.e. $r \sqsubseteq \psi_{n+1}(p)(q_k)$.

It follows that $r \sqsubseteq \psi_{n+1}(g)(y)$ and that $\psi_{n+1}$ is total and below $\psi'_{n+1}$. This ends the proof of the claim. □

It is easy to see from the definition of $\psi'_{n+1}$ that $\psi_{n+1} \sqsubseteq \psi'_{n+1}$ respects equivalence, that the compositions are equivalent to the respective identities and that they commute with application up to equivalence. This ends the proof of the theorem.

Remark 1. We conjecture that the set of compacts $p \in R^c_n$ that may be extended to a total object, is decidable. This is however not known. If the conjecture holds, the maps $\phi_n$ and $\psi_n$ will be effective, otherwise it is unlikely that such effective maps exist.

5. A common subhierarchy

It is still an open problem if $\mathbb{R}^S_n = \mathbb{R}^A_n$ for all $n$ or not. In this section we will show that the constructions of the countable dense subsets can be “translated” from one hierarchy to the other, and see that there will be maximal subhierarchies of the two that are isomorphic as type structures.

In the proof of the density theorem for the $A$-hierarchy (Normann [8,9]) we first produce a dense countable set $\Gamma_0$ of total objects in $A_0$ and dense countable sets $\Gamma^k_0$ of total objects in each $A^k_0 \to A_0$, $(\Gamma_1 = \Gamma^1_1)$ and then define $\Gamma_{n+2}$ as follows:

If $F \in \Gamma^k_1$ and $a_1, \ldots, a_k \in \Gamma_n$, then $G \in \Gamma_{n+2}$ where

$$G(f) = F(f(a_1), \ldots, f(a_k)).$$

We then prove that $\Gamma_n$ is dense in $A_n$ for all $n$.

So the elements of the countable dense sets of type $n$ are defined from base elements of pure type 0 and mixed type 1 by composition and application. As a consequence we observe

Lemma 5. There is a base consisting of a countable dense subset $\tilde{\Gamma}_0$ of $\mathbb{R}$ and countable dense subsets $\tilde{\Gamma}^k_1$ of $\mathbb{R}^k \to \mathbb{R}$ such that the set $\tilde{\Gamma}_n$ of elements in $\mathbb{R}^n$ definable from this base by composition and application is topologically dense in $\mathbb{R}^n$.

Of course, if a base is sufficient for this, and we extend the base, the extended set of objects definable from the elements of the extended base by application and composition will still be dense.

If we let $\tilde{\Lambda}_0 = \tilde{\Gamma}_0$ and $\tilde{\Lambda}^k_1 = \tilde{\Gamma}^k_1$ we may define the corresponding set $\tilde{\Lambda}_n$ in $\mathbb{R}^n$, and this hierarchy will be isomorphic to the $\tilde{\Gamma}$-hierarchy. It is not obvious that these sets are topologically dense at each level.

Now we will make a similar analysis of the density theorem for the $S$-hierarchy. There our base will consist of a pair of equivalent total extensions in $S_0$ for each coherent pair of compacts in $S_0$ and a pair $(F_1, F_2)$ of equivalent total extensions in $S^k_0 \to S_0$ for each coherent pair of compacts in $S^k_0 \to S_0$. If $q_1$ and $q_2$ are coherent
compacts in $S_{n+2}$ we construct a pair $G_1, G_2$ of total, equivalent extensions by the following pattern:

- We find total objects $a_{1,1}, a_{1,2}, a_{2,1}, \ldots, a_{k,1}, a_{k,2}$ such that $a_{i,1}$ and $a_{i,2}$ are equivalent for all $i$.
- We find a(n equivalent) pair $F_1, F_2$ from the base.
- We find a total function $B : S_{n+2} \to (\{1, 2\}_\perp)^k$.
- We let

$$\begin{align*}
G_1(f) &= F_1(f(a_{1, R(f)(1)}), \ldots, f(a_{k, R(f)(k)})) \\
G_2(f) &= F_2(f(a_{1, R(f)(1)}), \ldots, f(a_{k, R(f)(k)})).
\end{align*}$$

Now, let $a_i = \rho_n^S(a_{i,1}) = \rho_n^S(a_{i,2})$, $F = \rho_n^{S_k}(F_1) = \rho_n^{S_k}(F_2)$ and $G = \rho_n^S(G_1) = \rho_n^S(G_2)$. Since $B$ is total and is only used to select between equivalent objects identified by $\rho_n^S$, we will have that

$$G(f) = F(f(a_1), \ldots, f(a_k)).$$

Thus the elements of the topologically dense subsets of $\mathbb{R}^S_n$ are definable from a countable set of base elements by application and composition.

We have now established

**Lemma 6.** For each $n$ there is a countable dense set $\{\xi^n_i\}_{i \in \mathbb{N}}$ in $\mathbb{R}^S_n$ and a countable dense set $\{\eta^n_i\}_{i \in \mathbb{N}}$ in $\mathbb{R}^A_n$ such that:

- $\xi^n_i = \eta^n_i$ for all $i \in \mathbb{N}$.
- For all $n$, $i$, and $j$:

$$\xi^{n+1}_i(\xi^n_j) = \eta^{n+1}_i(\eta^n_j).$$

From now on we will let the $\xi$’s and the $\eta$’s be as constructed in the proof of Lemma 6.

**Definition 2.** (a) For $F \in \mathbb{R}^S_{n+1}$, let $h^S_F(i) = F(\xi^n_i)$ and for $F \in \mathbb{R}^A_{n+1}$, let $h^A_F(i) = F(\eta^n_i)$.

If $x \in \mathbb{R}$, let $h^S_x = h^A_x = x$.

(b) If $F_1 \in \mathbb{R}^S_n$ and $F_2 \in \mathbb{R}^A_n$, we let $F_1 \approx F_2$ if $h^S_{F_1} = h^A_{F_2}$.

**Lemma 7.** Let $F_1 \approx F_2$ of type $n+1$ and $a_1 \approx a_2$ of type $n$ and let $G_1 \approx G_2$ of type $n+1$. Then $F_1(a_1) = F_2(a_2)$.

**Proof.** Let $G_1$ be total in $S_{n+1}$ with $F_1 = \rho^S_{n+1}(G_1)$ and let $G_2, b_1$ and $b_2$ be related to $F_2, a_1$ and $a_2$ in a similar way.

For each $k$, let $v_k$ be the map from $S^C_k$ to $A^C_k$ defined in the proof of Lemma 2 and let $\tilde{v}_k$ be the continuous extension of $v_k$ to $\tilde{v}_k : S_k \to A_k$.

Then $\tilde{v}_{n+1}(G_1)$ is consistent with $G_2$ (since $h^S_{G_1} = h^A_{G_2}$), and $\tilde{v}_n(b_1)$ is consistent with $b_2$.

Thus $\tilde{v}_{n+1}(G_1)(\tilde{v}_n(b_1))$ is consistent with $G_2(b_2)$.

But $\tilde{v}_0(G_1(b_1)) \subseteq \tilde{v}_{n+1}(G_1)(\tilde{v}_n(b_1))$ and $\tilde{v}_0(G_1(b_1))$ is total and equivalent to $G_2(b_2)$.
The lemma follows.

The relation \( \approx \) can be seen as the graph of a partial isomorphism \( /BS \approx /BS \) from a subhierarchy of \( \{ R^n_S \}_{n \in \mathbb{N}} \) to a subhierarchy of \( \{ R^n_A \}_{n \in \mathbb{N}} \) respecting application. Any partial morphism \( \Phi \) respecting application such that \( \Phi(\xi^n_i) = \eta^n_i \) will satisfy \( h^A_{\Phi(F)} = \eta^A_i \) whenever \( \Phi(F) \) is defined. Thus \( \Phi \approx \) is a maximal partial isomorphism, and we have proved. \( \square \)

**Theorem 5.** There are maximal subhierarchies of \( \{ R^n_S \}_{n \in \mathbb{N}} \) and \( \{ R^n_A \}_{n \in \mathbb{N}} \) that are isomorphic.

It is clear that if \( R^n_S \neq R^n_A \) for some \( n \), then \( R^n_S \subset R^n_A \) for the least such \( n_0 \). If this is the case, let \( G \in R^n_A \setminus R^n_S \).

Let \( d \) be a metric for \( R^N \) with the product topology. For \( F \in R^n_S \), let

\[
\Phi(F) = \frac{1}{d(h^A_G, h^S_F)}.
\]

Since \( \lambda F(\xi^n_i) \) is representable in the \( S \)-hierarchy for each \( i \), the map \( F \mapsto h^S_F \) is representable. It follows that \( \Phi \in R^n_{S+1} \). If for some \( \Phi \in R^n_A \) we have that \( \Phi \approx \Psi \), then we obtain a contradiction since, by continuity, \( \Psi(G) = \infty \).

Thus either the hierarchies are equal or they will be incomparable above some level.

6. Epilogue

In this paper we have made a comparison between essentially two approaches to the hereditarily total continuous functionals of finite types over the reals, without discussing the advantages or disadvantages of either of them. This is discussed at more depth in [1].

The \( C \)-hierarchy has been used e.g. by Escardó [2,3] for the denotational semantics of Real PCF. The \( R \)-hierarchy was likewise used by Di Giantantonio [4–6] for his calculus for exact real valued computations.

The \( S \)-hierarchy has so far only one advantage, it can be used to understand the quotient spaces of total objects in the \( R \)-hierarchy.

The prime advantage of the \( R \)-hierarchy is that it combines a data-stream representation of the reals with the possible use of domains for the denotational semantics of \( \lambda \)-calculi. Another important aspect is that we have three layers at each type, the underlying domain, the quotient space of total objects under consistency and the quotient space of total objects under equivalence.

Tucker and Zucker [15] takes a completely different approach to the computability of real valued functions. One of their points is that in the real world there are algorithms that are non-deterministic in the sense that you may choose one of several paths to the final result, and they model such algorithms using multivaluedness. One example is the algorithm for inverting a matrix of reals. The \( R \)-hierarchy may be suitable for combining this idea with PCF-like algorithms, since the source for multivaluedness in [15] is that different representations of the input leads to different paths in the execution
of a program. If we try to interpret such multivaluedness in the $S$-hierarchy, much will be lost since we insist on respecting coherence.

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**References**