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Representations of automorphism groups of finite $\mathfrak{o}$-modules of rank two ☆

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Abstract

Let $\mathfrak{o}$ be a complete discrete valuation domain with finite residue field. In this paper we describe the irreducible representations of the groups $\mathrm{Aut}(M)$ for any finite $\mathfrak{o}$-module M of rank two. The main emphasis is on the interaction between the different groups and their representations. An induction scheme is developed in order to study the whole family of these groups coherently. The results obtained depend on the ring $\mathfrak{o}$ in a very weak manner, mainly through the degree of the residue field. In particular, a uniform description of the irreducible representations of $\mathrm{GL}_2(\mathfrak{o}/\mathfrak{p}^\ell)$ is obtained, where $\mathfrak{p}$ is the maximal ideal of $\mathfrak{o}$.

MSC: 20G05; 22E50

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1. Introduction

1.1. Overview

Let F be a non-Archimedean local field with ring of integers $\mathfrak{o} = \mathfrak{o}_F$, maximal ideal $\mathfrak{p} = \mathfrak{p}_F$, a fixed uniformizer $\pi = \pi_F$ and residue field of cardinality q. Let $\tilde{\mathfrak{o}} \supset \mathfrak{o}$ be the ring of integers of

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a degree n non-ramified extension $\tilde{F} \supset F$ and let $\tilde{\mathfrak{o}}_\ell \supset \mathfrak{o}_\ell$ be the reductions modulo $\mathfrak{p}^\ell$ ($\ell \in \mathbb{N}$). Let Λ_n denote the set of partitions of length n and $\Lambda = \bigcup \Lambda_n$. For any $\lambda = (\ell_1, \dots, \ell_n) \in \Lambda$ let M_λ denote the $\mathfrak{o}$-module $\bigoplus_i \mathfrak{o}_{\ell_i}$ of type λ. Endow Λ with a partial order where $\mu \leqslant \lambda$ if and only if a module of type μ can be embedded in a module of type λ. Let

$$G_\lambda = \mathrm{Aut}_{\mathfrak{o}}(M_\lambda).$$

In this paper the complex irreducible representations of the groups G_λ ($\lambda \in \Lambda_2$) are classified and discussed in detail. The Harish-Chandra philosophy of cusp forms [7] is adopted and implemented in order to interconnect the representation theories of these groups. Representations of G_λ are built from representations of G_μ ($\mu < \lambda$) using various induction functors. Representations which cannot be obtained from groups of lower type are called cuspidal and are constructed as well.

This approach was carried out by Green [6] who classified the representations of the groups $G_{1^n} = \mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_q^n)$ ($n \in \mathbb{N}$). The cuspidal representations of G_{1^n}, i.e., those which are not contained in parabolically induced representations from G_{1^m} ($m < n$), are parameterized by an object of arithmetic flavor: orbits of primitive characters of $\mathbb{F}_{q^n}^\times$ under the Galois action. On the other hand, the induction part of the theory is of combinatorial flavor, is independent of the field, and essentially coincides with the representation theory of the symmetric group, see [34]. Such a picture is the model we seek in general. In this paper we achieve it for automorphism groups of rank two $\mathfrak{o}$-modules.

The special types of rectangular shape $\ell^n = (\ell, \dots, \ell)$ draw much of attention since the representations of the groups G_{ℓ^n} ($\ell \in \mathbb{N}$) exhaust the complex continuous representations of $\mathrm{GL}_n(\mathfrak{o}_F)$, the maximal compact subgroup of $\mathrm{GL}_n(F)$. One of the main goals of this paper is to provide evidence that the groups G_λ ($\lambda < \ell^n$) play indispensable role in the classification of irreducible representations of G_{ℓ^n}.

1.2. The main results

We begin with a short description of the subgroups of G_λ which play the role of maximal parabolics for the groups G_{1^n} ($n > 1$). The first family of these subgroups is a straightforward generalization. Given a direct sum decomposition $M_\lambda = M_\mu \oplus M_{\lambda/\mu}$ (λ/μ denotes the type of the quotient M_λ/M_μ), let $P_{\mu,\lambda}$ be the stabilizer of M_μ in M_λ. Then $P_{\mu,\lambda}$ surjects on $G_\mu \times G_{\lambda/\mu}$, and one can construct representations of G_λ by pulling back representations from the product $G_\mu \times G_{\lambda/\mu}$ to $P_{\mu,\lambda}$ and inducing them to G_λ. Since the term 'parabolic' is quite loaded we shall call the groups $P_{\mu,\lambda}$ (*maximal*) *geometric stabilizers* in G_λ.

The second family has no analogue in the case of the groups G_{1^n}. Given types $\mu \leqslant \lambda$, call μ *symmetric* in λ if the embedding of M_μ in M_λ is unique up to automorphism. In such case let λ/μ denote the well defined type of the quotient M_λ/M_μ. As will be explained later on, it follows that there is a unique embedding (up to automorphism) of a module $M_{\lambda/\mu}$ of type λ/μ such that $M_\lambda/M_{\lambda/\mu}$ is of type μ. Let μ be a symmetric type in λ. Assume also that as partitions μ and λ are of same length, or equivalently, that the corresponding modules have the same rank. This assumption guarantees that there is no overlap with the previous family. Set

$$P_{\mu\hookrightarrow\lambda} = \mathrm{Stab}_{G_\lambda}(M_\mu \subset M_\lambda),$$
$$P_{\lambda\twoheadrightarrow\mu} = \mathrm{Stab}_{G_\lambda}(M_{\lambda/\mu} \subset M_\lambda).$$

The fact that μ is symmetric in λ translates into canonical epimorphisms from $P_{\mu\hookrightarrow\lambda}$ and $P_{\lambda\twoheadrightarrow\mu}$ to G_μ. The first is defined by restriction of elements of $P_{\mu\hookrightarrow\lambda}$ to M_μ, and the second by identifying elements of $P_{\lambda\twoheadrightarrow\mu}$ as automorphisms of $M_\lambda/M_{\lambda/\mu}$. Again, we construct representations of G_λ by pulling back representations from G_μ to either $P_{\mu\hookrightarrow\lambda}$ or $P_{\lambda\twoheadrightarrow\mu}$, and then inducing them to G_λ. We shall call the groups $P_{\lambda\twoheadrightarrow\mu}$ and $P_{\mu\hookrightarrow\lambda}$ *infinitesimal stabilizers* in G_λ. The types which eventually contribute to the infinitesimal induction are types μ which are symmetric in λ and of same length and same height (the latter stands for the largest part of the partition). We denote the set of these types by $\mathcal{I}_\lambda$.

By a *twist* of a representation we mean its tensor product with a one-dimensional representation. If $\flat\lambda \geqslant 1^n$, there is a natural reduction map

$$G_\lambda = \mathrm{Aut}_{\mathfrak{o}}(M_\lambda) \twoheadrightarrow G_{\flat\lambda} = \mathrm{Aut}_{\mathfrak{o}}(\pi M_\lambda)$$

defined by restriction of automorphisms of M_λ to πM_λ (this generalizes the usual reduction modulo $\mathfrak{p}^{\ell-1}$ in the case of $\lambda = \ell^n$). Call a representation $\rho \in \hat{G}_\lambda$ *primitive* if none of its twists is a pullback from $\hat{G}_{\flat\lambda}$. By using an inductive approach one can focus only on primitive representations as the non-primitive representations are twists of representations of groups of lower type.

Definition 1.1. A primitive irreducible representation of G_λ is *cuspidal* if none of its twists is contained in some geometrically or infinitesimally induced representation. The set of cuspidal representations in $\hat{G}_\lambda$ is denoted by $\hat{C}_\lambda$.

Theorem 1.2. *For any $\lambda \in \Lambda_2$ the primitive irreducible representations of G_λ fall precisely into one of the following classes.*

(1) *Contained in a geometrically induced representation.*
(2) *Infinitesimally induced from a unique cuspidal representation $\rho \in \hat{C}_\mu$ ($\mu \in \mathcal{I}_\lambda$) up to twist.*
(3) *Cuspidal.*

The cuspidal representations $\hat{C}_\lambda$ ($\lambda \in \Lambda_2$) are explicitly constructed in Section 6. In [1] it is further shown that there is a canonical bijection between cuspidal representations of G_{ℓ^n} and Galois orbits of strongly primitive characters of $\tilde{\mathfrak{o}}_\ell^\times$ whenever n is prime.

Let $f_m = f_m^{\lambda,\mathfrak{o}}$ denote the number of irreducible representations of G_λ ($\lambda \in \Lambda$) of dimension m.

Conjecture 1.3. *For any $\lambda \in \Lambda$,*

(Weak version) the isomorphism type of the group algebra $\mathbb{C}G_\lambda$ depends only on λ and $q = |\mathfrak{o}/\mathfrak{p}|$.
(Strong version) $f_m = f_m^\lambda \in \mathbb{Q}[q]$.

Theorem 1.4. *Let $R_\lambda(\mathcal{D}) = \sum_m f_m \mathcal{D}^m \in \mathbb{Z}[\mathcal{D}]$. Then*

(1) $R_\lambda(\mathcal{D}) = q^{\ell-2}(q-1)^2\mathcal{D} + q^{\ell-2}(q^2-1)\mathcal{D}^{q-1} + q^{\ell-2}(q-1)^3\mathcal{D}^q$, $\lambda = (\ell, 1)$, $\ell > 1$.
(2) $R_\lambda(\mathcal{D}) = qR_{\flat\lambda}(\mathcal{D}) + q^{\ell_1+\ell_2-3}(q^2-1)\mathcal{D}^{q^{\ell_2-1}(q-1)} + q^{\ell_1+\ell_2-3}(q-1)^3\mathcal{D}^{q^{\ell_2}}$, $\lambda = (\ell_1, \ell_2)$, $\ell_1 > \ell_2 > 1$.

(3) $R_{\ell^2}(\mathcal{D}) = q R_{(\ell-1)^2}(\mathcal{D}) + \frac{1}{2}(q-1)(q^2-1)q^{2\ell-3}\mathcal{D}^{q^{\ell-1}(q-1)} + q^{2\ell-2}(q-1)\mathcal{D}^{q^{\ell-2}(q^2-1)} + \frac{1}{2}q^{2\ell-3}(q-1)^3\mathcal{D}^{q^{\ell-1}(q+1)}$, $\ell > 1$.

In particular, the strong version of Conjecture 1.3 *holds for all* $\lambda \in \Lambda_2$.

The dimensions of the irreducible representations seem to be polynomials in q as well. Moreover, we conjecture that for $\lambda = \ell^n$ the following holds.

Conjecture 1.5. *The dimensions of primitive irreducible representations of G_{ℓ^n} are polynomials in $\mathbb{Z}[q]$ of degree d with $d \leqslant \binom{n}{2}\ell$.*

This holds for example when $\lambda = 1^n$ by Green's work (in this case we treat all the representations as primitive) or when $\lambda = \ell^2$ by Theorem 1.4.

1.3. History of the problem

A special case of the problem under consideration, namely when the group is specialized to the general linear group $\mathrm{GL}_2(\mathfrak{o}_\ell) = \mathrm{Aut}_{\mathfrak{o}}(\mathfrak{o}_\ell^2)$ or to its closely related subgroup, the special linear group $\mathrm{SL}_2(\mathfrak{o}_\ell)$, has been intensively studied by several authors. First results concerning the representation theory of the group $\mathrm{SL}_2(\mathbb{Z}/p^\ell\mathbb{Z})$ were obtained by Kloosterman in his Annals papers [14,15]. Kloosterman constructed some of the representations of $\mathrm{SL}_2(\mathbb{Z}/p^\ell\mathbb{Z})$ using the Weil representation (see also [30] for a description of Kloosterman's work). Tanaka [32] generalized Kloosterman's construction and obtained all the irreducible representations of $\mathrm{SL}_2(\mathbb{Z}/p^\ell\mathbb{Z})$. All these results assume that p is odd. In a series of papers by Nobs, and Nobs and Wolfart [21,22,24–26], these constructions are generalized in several directions: some of the constructions are applied for $p = 2$, some of them are valid when $\mathbb{Z}_p$ is replaced by any ring of integers of a local field, and some are applied to GL_2. Other publications which do not use the Weil representation include: Kutzko [16] and Nagornyĭ [20] for $\mathrm{SL}_2(\mathbb{Z}/p^\ell\mathbb{Z})$, Silberger [29] for $\mathrm{PGL}_2(\mathfrak{o}_\ell)$ and Shalika [28] for $\mathrm{SL}_2(\mathfrak{o}_\ell)$ when $\mathfrak{o}$ has characteristic zero. In a more recent paper [13], Jaikin-Zapirain computes the representation zeta function for $\mathrm{SL}_2(\mathfrak{o})$ in the case of odd characteristic. Nevertheless, only in a very recent preprint [31] by Stasinski all the irreducible representations of $\mathrm{GL}_2(\mathfrak{o}_\ell)$ are constructed for general $\mathfrak{o}$. For higher dimensions there are some partial results of a more general nature by Hill [8–11], Lusztig [18] and Aubert, Onn, Prasad and Stasinski [1] however, the general case still remains out of reach.

1.4. Contents of the paper

The paper is organized as follows. In Section 2 the various induction and restriction functors are discussed in complete generality and the concept of cuspidality is introduced. In Section 3 we specialize to the rank two case and explain the structure of the various automorphism groups involved and their stabilizing subgroups. In Section 4 we settle the case of the groups $G_{(\ell,1)}$ ($\ell > 1$) due to their special structure and role. The classification of their irreducible representations is simpler comparing to the other automorphism groups. In Section 5 we explain the inductive scheme in detail. In Section 6 we construct the cuspidal representations $\hat{C}_\lambda \subset \hat{G}_\lambda$ for λ with $\ell_1 > \ell_2 > 1$ and briefly review the construction of $\hat{C}_{\ell^2}$ which appeared in [22] (but we shall follow the route taken in [1]). Section 7 is devoted to geometric and infinitesimal inductions. In Section 8 we collect all the results to obtain the classification of irreducible representations of G_λ

($\lambda \in \Lambda_2$), in particular, Theorems 1.2 and 1.4 are proved. In Section 9 we classify the conjugacy classes in G_λ ($\lambda \in \Lambda_2$).

2. Operations on representations and cuspidality

For a type $\lambda \in \Lambda$ let $\mathcal{C}_\lambda$ be the category of finite-dimensional complex representations of G_λ. We shall now define a family of functors between the categories $\{\mathcal{C}_\lambda\}$ that generalize the usual functors of parabolic induction and their adjoints which were defined by Green in [6] for the family $\{\mathcal{C}_{1^n}\}$ (see also [34]).

2.1. Geometric induction and restriction

The first family is the natural generalization of [6] which comes from direct sum decompositions. Let $M_\lambda = M_\mu \oplus M_{\mu'}$ be a decomposition of a finite $\mathfrak{o}$-module M_λ into a direct sum of $\mathfrak{o}$-modules of types μ and μ'. Let $P_{M_\mu, M_\lambda} \subset \mathrm{Aut}_\mathfrak{o}(M_\lambda)$ be the stabilizing subgroup of M_μ in M_λ. Then we have a surjection

$$\mathrm{Aut}_\mathfrak{o}(M_\lambda) \supset P_{M_\mu, M_\lambda} \xrightarrow{\iota} \mathrm{Aut}_\mathfrak{o}(M_\mu) \times \mathrm{Aut}_\mathfrak{o}(M_\lambda/M_\mu), \tag{2.1}$$

where the left part of ι is restriction and the right part comes from the action of P_{M_μ, M_λ} on M_λ/M_μ. The functors of geometric induction and geometric restriction are defined by

$$\begin{aligned}
&\mathbf{i}^\lambda_{\mu,\mu'} : \mathcal{C}_\mu \times \mathcal{C}_{\mu'} \to \mathcal{C}_\lambda, \qquad && (\xi, \xi') \mapsto \mathrm{Ind}^{\mathrm{Aut}_\mathfrak{o}(M_\lambda)}_{P_{M_\mu, M_\lambda}}\big(\iota^*(\xi \boxtimes \xi')\big), \qquad && (\xi, \xi') \in \mathcal{C}_\mu \times \mathcal{C}_{\mu'},\\
&\mathbf{r}^\lambda_{\mu,\mu'} : \mathcal{C}_\lambda \to \mathcal{C}_\mu \times \mathcal{C}_{\mu'}, \qquad && \eta \mapsto \iota_* \mathrm{Res}^{\mathrm{Aut}_\mathfrak{o}(M_\lambda)}_{P_{M_\mu, M_\lambda}}(\eta), \qquad && \eta \in \mathcal{C}_\lambda,
\end{aligned}$$

where ι^* stands for pullback of functions and ι_* is averaging along fibers, i.e. taking invariants with respect to $\mathrm{Ker}(\iota)$.

2.2. Infinitesimal induction and restriction

The second family of functors is an exclusive feature of higher level groups which disappears in the case of the categories $\{\mathcal{C}_{1^n}\}$. It stems from the fact that an $\mathfrak{o}$-module which is not annihilated by $\mathfrak{p}$ can have proper submodules of same rank.

Proposition–Definition 2.1. *Let $\mu \leqslant \lambda$ be isomorphism types of finite $\mathfrak{o}$-modules. The following are equivalent.*

(1) *(Unique embedding) For every pair of embeddings $M_\mu \hookrightarrow M_\lambda$ and $M'_\mu \hookrightarrow M'_\lambda$ of modules of type μ in modules of type λ and an isomorphism $h : M_\mu \simeq M'_\mu$ there exists an isomorphism $\tilde{h} : M_\lambda \simeq M'_\lambda$ which extends h.*
(2) *(Unique quotient) For every pair of surjections $M_\lambda \twoheadrightarrow M_\mu$ and $M'_\lambda \twoheadrightarrow M'_\mu$ of modules of type λ onto modules of type μ and an isomorphism $h : M_\mu \simeq M'_\mu$ there exists an isomorphism $\tilde{h} : M_\lambda \simeq M'_\lambda$ which lifts h.*
(3) *($\mathcal{G}(\mu, M_\lambda)$ is a homogeneous G_λ-space) The group G_λ acts transitively on $\mathcal{G}(\mu, M_\lambda)$, the Grassmannian of submodules of type μ in M_λ.*

In such case say that μ is symmetric in λ.

Proof. If $m \leqslant n$, multiplication by π^{n-m} is an injective $\mathfrak{o}$-homomorphism of $\mathfrak{o}_m$ into $\mathfrak{o}_n$. Let $E = \varinjlim \mathfrak{o}_n$ denote the direct limit. Then E is an injective $\mathfrak{o}$-module and for any finite $\mathfrak{o}$-module M, its *dual* is defined to be $\check{M} = \mathrm{Hom}_{\mathfrak{o}}(M, E)$. In [19, Chapter 2] it is shown that $M \mapsto \check{M}$ is an exact contravariant functor, that $M \simeq \check{\check{M}}$, and that submodules of type μ and cotype ν in M are in bijective correspondence with submodules of type ν and cotype μ in $\check{M}$. It follows that any diagram of the form

$$\begin{array}{ccc} M_\mu & \hookrightarrow & M_\lambda \\ \uparrow\wr & & \\ M'_\mu & \hookrightarrow & M'_\lambda \end{array} \tag{2.2}$$

could be completed to a commutative diagram with vertical isomorphisms if and only if any dual diagram of the form

$$\begin{array}{ccc} \check{M}_\lambda & \twoheadrightarrow & \check{M}_\mu \\ & & \wr\downarrow \\ \check{M}'_\lambda & \twoheadrightarrow & \check{M}'_\mu \end{array} \tag{2.3}$$

could be completed as well. This establishes the equivalence of (1) and (2). Part (3) is a simple reformulation of (1). □

Remark 2.2.

(a) The rectangular partitions of the form $\ell^n = (\ell, \ldots, \ell)$ enjoy the property that every $\mu \leqslant \ell^n$ is symmetric in them (in [2] such types are called symmetric themselves). The reader is referred to [2] for more details on $\mathcal{G}(\mu, M_{\ell^n})$ and the representations of G_{ℓ^n} arising from it.
(b) Though the definition makes perfect sense for any length of the type μ, we shall assume in the sequel that $\mathrm{rank}(M_\mu) = \mathrm{rank}(M_\lambda)$, or equivalently $\mathrm{length}(\mu) = \mathrm{length}(\lambda)$, in order to avoid overlap with geometric induction and restriction.

Let μ be symmetric in λ with equal length. We define four functors

$$\mathbf{i}^\lambda_{\mu,\hookrightarrow}, \mathbf{i}^\lambda_{\mu,\twoheadrightarrow} : \mathcal{C}_\mu \to \mathcal{C}_\lambda,$$
$$\mathbf{r}^\lambda_{\mu,\hookrightarrow}, \mathbf{r}^\lambda_{\mu,\twoheadrightarrow} : \mathcal{C}_\lambda \to \mathcal{C}_\mu,$$

as follows. Choose a submodule M_μ of type μ in a module M_λ of type λ. Let $P_{M_\mu, M_\lambda} \subset \mathrm{Aut}_{\mathfrak{o}}(M_\lambda)$ be the stabilizing group of M_μ. Then by the unique embedding property the restriction map gives an epimorphism

$$\mathrm{Aut}_{\mathfrak{o}}(M_\lambda) \supset P_{M_\mu, M_\lambda} \xrightarrow{\varphi} \mathrm{Aut}_{\mathfrak{o}}(M_\mu). \tag{2.4}$$

Let ξ be a representation of $\mathrm{Aut}_{\mathfrak{o}}(M_\mu)$ and η be a representation of $\mathrm{Aut}_{\mathfrak{o}}(M_\lambda)$. Define

$$\mathbf{i}^\lambda_{\mu,\hookrightarrow} : \mathcal{C}_\mu \to \mathcal{C}_\lambda, \qquad \xi \mapsto \mathrm{Ind}_{P_{M_\mu, M_\lambda}}^{\mathrm{Aut}_{\mathfrak{o}}(M_\lambda)}(\varphi^*\xi),$$
$$\mathbf{r}^\lambda_{\mu,\hookrightarrow} : \mathcal{C}_\lambda \to \mathcal{C}_\mu, \qquad \eta \mapsto \varphi_* \mathrm{Res}_{P_{M_\mu, M_\lambda}}^{\mathrm{Aut}_{\mathfrak{o}}(M_\lambda)}(\eta).$$

Let $M_{\lambda/\mu}$ be a submodule of M_λ with $M_\lambda/M_{\lambda/\mu}$ of type μ. Let $P_{M_{\lambda/\mu},M_\lambda} \subset \mathrm{Aut}_{\mathfrak{o}}(M_\lambda)$ be the stabilizing group of $M_{\lambda/\mu}$. Then the group $P_{M_{\lambda/\mu},M_\lambda}$ acts on the quotient $M_\lambda/M_{\lambda/\mu}$, and hence maps into $\mathrm{Aut}_{\mathfrak{o}}(M_\lambda/M_{\lambda/\mu})$. This map is in fact onto since μ has the unique quotient property. Thus, similarly to (2.4) we have

$$\mathrm{Aut}_{\mathfrak{o}}(M_\lambda) \supset P_{M_{\lambda/\mu},M_\lambda} \xrightarrow{\varepsilon} \mathrm{Aut}_{\mathfrak{o}}(M_\lambda/M_{\lambda/\mu}), \tag{2.5}$$

and can define

$$\mathbf{i}^{\lambda}_{\mu,\twoheadrightarrow} : \mathcal{C}_\mu \to \mathcal{C}_\lambda, \qquad \xi \mapsto \mathrm{Ind}^{\mathrm{Aut}_{\mathfrak{o}}(M_\lambda)}_{P_{M_{\lambda/\mu},M_\lambda}}(\varepsilon^*\xi),$$
$$\mathbf{r}^{\lambda}_{\mu,\twoheadrightarrow} : \mathcal{C}_\lambda \to \mathcal{C}_\mu, \qquad \eta \mapsto \varepsilon_* \mathrm{Res}^{\mathrm{Aut}_{\mathfrak{o}}(M_\lambda)}_{P_{M_{\lambda/\mu},M_\lambda}}(\eta).$$

2.3. Cuspidality

The point of view taken in this paper is that the geometric induction functors defined above, though enabling the construction of many representations of the group G_λ, fall too short of leaving out a manageable part of the irreducible representations to be constructed by other means. However, together with infinitesimal inductions, one obtains a wealth of irreducible representations, hopefully leaving out a set which has a natural parametrization and can be constructed in a uniform manner.

Definition 2.3. A primitive irreducible representation of G_λ is *cuspidal* if the image of any of its twists by one-dimensional character under the geometric restriction functors $\mathbf{r}^{\lambda}_{\mu',\mu''}$ (with $\lambda = \mu' \cup \mu''$) and the infinitesimal restriction functors $\mathbf{r}^{\lambda}_{\mu,\hookrightarrow}$, $\mathbf{r}^{\lambda}_{\mu,\twoheadrightarrow}$ (with μ symmetric in λ with same height and length) vanishes.

Remark 2.4.

(a) For the special case $\lambda = \ell^n$, a rectangular partition, there are related notions of cuspidality for the group G_{ℓ^n} which are oriented towards the construction of supercuspidal representations of $\mathrm{GL}_n(F)$, hence their name. See e.g., Kutzko [17, §2], Carayol [3, §4] (*très cuspidale*), Hill [11, §4], Casselman [4, p. 316] (*strongly cuspidal*) and Prasad [27, §2] (*very cuspidal*). Most of them, and in fact up to slight variants which will not be discussed here all of them, agree with the following definition. A representation of G_{ℓ^n} is *strongly cuspidal* if its restriction to $\mathrm{Ker}\{G_{\ell^n} \to G_{(\ell-1)^n}\} \simeq M_n(\mathbb{F}_q)$ consists of matrices with irreducible characteristic polynomial under the identification $M_n(\mathbb{F}_q) \to M_n(\mathbb{F}_q)^\wedge$, given by $A \mapsto \psi(\mathrm{trace}(A\cdot))$, where $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ is a non-trivial character. In [1] it is shown that strongly cuspidal representations are cuspidal, with a reverse implication when n is prime.

(b) Since the groups P_{M_μ,M_λ} and $P_{M_{\lambda/\mu},M_\lambda}$ above depend only on μ and λ up to conjugation we shall denote them $P_{\mu\hookrightarrow\lambda}$ and $P_{\lambda\twoheadrightarrow\mu}$ (respectively) whenever chances for confusion are slim.

3. Specializing to the rank two case

From this point onwards we specialize the discussion to automorphism groups of finite $\mathfrak{o}$-modules of rank two. We shall now write down concrete matrix realization of the groups under study. We denote $\mathfrak{p}^m_\ell$ for the image of $\mathfrak{p}^m$ in $\mathfrak{o}_\ell$ ($m \leqslant \ell$). Let $\lambda = (\ell_1, \ell_2)$ be a type of length two, i.e. a partition with two positive parts written in a descending order and let $M_\lambda = \mathfrak{o}_{\ell_1} e_1 \oplus \mathfrak{o}_{\ell_2} e_2$ denote the $\mathfrak{o}$-module of type λ with basis $\{e_1, e_2\}$. The endomorphism ring $E_\lambda = \mathrm{End}_{\mathfrak{o}}(M_\lambda)$ consists of elements

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in \begin{bmatrix} \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{o}_{\ell_1} e_1, \mathfrak{o}_{\ell_1} e_1) & \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{o}_{\ell_2} e_2, \mathfrak{o}_{\ell_1} e_1) \\ \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{o}_{\ell_1} e_1, \mathfrak{o}_{\ell_2} e_2) & \mathrm{Hom}_{\mathfrak{o}}(\mathfrak{o}_{\ell_2} e_2, \mathfrak{o}_{\ell_2} e_2) \end{bmatrix}$$

acting by

$$f(x_1, x_2) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_{11}(x_1) + f_{12}(x_2) \\ f_{21}(x_1) + f_{22}(x_2) \end{pmatrix}.$$

Writing $f_{ij}(e_j) = a_{ij} e_i$ with $a_{ij} \in \mathfrak{o}_{\ell_i}$, observe that composition of maps $f \circ f'$ is regular multiplication of the corresponding matrices $g_f \cdot g_{f'}$, and we have

$$E_\lambda \simeq \begin{bmatrix} \mathfrak{o}_{\ell_1} & \mathfrak{p}_{\ell_1}^{\ell_1 - \ell_2} \\ \mathfrak{o}_{\ell_2} & \mathfrak{o}_{\ell_2} \end{bmatrix} \supset \begin{bmatrix} \mathfrak{o}_{\ell_1}^\times & \mathfrak{p}_{\ell_1}^{\ell_1 - \ell_2} \\ \mathfrak{o}_{\ell_2} & \mathfrak{o}_{\ell_2}^\times \end{bmatrix} \simeq G_\lambda \tag{3.1}$$

if $\ell_1 > \ell_2$, and

$$E_{\ell^2} \simeq M_2(\mathfrak{o}_\ell) \supset \mathrm{GL}_2(\mathfrak{o}_\ell) \simeq G_{\ell^2} \tag{3.2}$$

if $\ell_1 = \ell_2 = \ell$.

The ‘parabolic’ subgroups which are of interest arise from pairs $M_\mu \subset M_\lambda$ possessing the unique embedding property. These are classified in the next lemma.

Lemma 3.1. *Let $\lambda = (\ell_1, \ell_2)$ be an element in Λ_2.*

(1) *If $\ell_1 = \ell_2$ then every $\mu \leqslant \lambda$ is symmetric in λ.*
(2) *If $\ell_1 > \ell_2$ then $\mu = (m_1, m_2)$ is symmetric in λ if and only if $\ell_1 = m_1$.*

Proof. The ring $\mathfrak{o}_\ell$ is self injective, therefore, any diagram of the form (2.2) with $\lambda = \ell^n$ can be completed to a commutative diagram with vertical isomorphisms. In particular (1) follows.

To prove part (2), first assume that $m_1 = \ell_1$. Given inclusions $M_\mu \subset M_\lambda$, $M'_\mu \subset M'_\lambda$ and isomorphism $\varphi : M_\mu \to M'_\mu$, let $\{e_1, e_2\}$ be a basis of M_μ such that $\mathfrak{o}_{m_i} e_i \simeq \mathfrak{o}_{m_i}$ for $i = 1, 2$. As e_2 belongs to the π^{m_2}-torsion of M_λ, there exist $f_2 \in M_\lambda$ such that $e_2 = \pi^{\ell_2 - m_2} f_2$ and $\{e_1, f_2\}$ is a basis of M_λ. Similarly, $\{e'_1 = \varphi(e_1), e'_2 = \varphi(e_2)\}$ is a basis of M'_μ, and there exist $f'_2 \in M'_\lambda$ such that $e'_2 = \pi^{\ell_2 - m_2} f'_2$ and $\{e'_1, f'_2\}$ is a basis of M'_λ. Then, $\tilde{\varphi} : M_\lambda \to M'_\lambda$ defined by $\tilde{\varphi}(e_1) = \varphi(e_1) = e'_1$ and $\tilde{\varphi}(f_2) = f'_2$ is the desired extension of φ.

Conversely, if $m_1 < \ell_1$, we exhibit two inequivalent embeddings; let $\{f_1, f_2\}$ be a basis of M_λ such that $\mathfrak{o}_{\ell_i} f_i \simeq \mathfrak{o}_{\ell_i}$ $(i = 1, 2)$ and let $e_i = \pi^{\ell_i - m_i} f_i$ be a basis of a submodule of type $\mu = (m_1, m_2)$. We separate into two cases: if $\ell_2 \leqslant m_1$ the map φ defined by $\varphi(e_1) = e_1 + f_2$ and $\varphi(e_2) = e_2$ is an isomorphism of M_μ onto $\varphi(M_\mu)$ which has no extension, since $e_1 \in \pi M_\lambda$ but $\varphi(e_i) \notin \pi M_\lambda$; if $\ell_2 > m_1$ the map φ defined by $\varphi(e_1) = \pi^{\ell_2 - m_1} f_2$ and $\varphi(e_2) = \pi^{\ell_1 - m_2} f_1$ is an isomorphism of M_μ onto $\varphi(M_\mu)$ which has no extension, since $e_1 \in \pi^{\ell_1 - m_1} M_\lambda$ but $\varphi(e_1) \notin \pi^{\ell_1 - m_1} M_\lambda$. $\square$

The case $\mu = (\ell_1, 0)$ corresponds to the direct sum decomposition which contributes via the geometric induction functor. The other types with $0 < m_2 < \ell_2$ and $\ell_1 = m_1$, which are symmetric in λ and have the same length and height, are denoted $\mathcal{I}_\lambda$, and contribute via the infinitesimal induction functors. Unraveling definitions and using the matrix representation of the groups, the infinitesimal stabilizers are given by

$$P_{\mu\hookrightarrow\lambda} = \begin{bmatrix} \mathfrak{o}_{\ell_1}^\times & \mathfrak{p}_{\ell_1}^{\ell_1-\ell_2} \\ \mathfrak{p}_{\ell_2}^{\ell_2-m_2} & \mathfrak{o}_{\ell_2}^\times \end{bmatrix} \twoheadrightarrow \begin{bmatrix} \mathfrak{o}_{\ell_1}^\times & \mathfrak{p}_{\ell_1-\ell_2+m_2}^{\ell_1-\ell_2} \\ \mathfrak{p}_{\ell_2}^{\ell_2-m_2} & \mathfrak{o}_{m_2}^\times \end{bmatrix} \simeq G_\mu, \tag{3.3}$$

$$P_{\lambda\twoheadrightarrow\mu} = \begin{bmatrix} \mathfrak{o}_{\ell_1}^\times & \mathfrak{p}_{\ell_1}^{\ell_1-m_2} \\ \mathfrak{o}_{\ell_2} & \mathfrak{o}_{\ell_2}^\times \end{bmatrix} \twoheadrightarrow \begin{bmatrix} \mathfrak{o}_{\ell_1}^\times & \mathfrak{p}_{\ell_1}^{\ell_1-m_2} \\ \mathfrak{o}_{m_2} & \mathfrak{o}_{m_2}^\times \end{bmatrix} \simeq G_\mu. \tag{3.4}$$

The subgroup $P_{\mu\hookrightarrow\lambda}$ consists of elements in G_λ which stabilize $M_\mu = \mathfrak{o}_{\ell_1}e_1 \oplus \mathfrak{p}_{\ell_2}^{\ell_2-m_2}e_2$, and the epimorphism is restriction to M_μ. The subgroup $P_{\lambda\twoheadrightarrow\mu}$ consists of elements in G_λ which stabilize the unique embedding of a module of type $\lambda/\mu = (\ell_2 - m_2)$ in M_λ (up to automorphism) with cotype μ, namely $M_{\lambda/\mu} = \mathfrak{p}_{\ell_2}^{m_2}e_2 \subset M_\lambda$, and the epimorphism is the action of these elements on $M_\lambda/M_{\lambda/\mu} = \mathfrak{o}_{\ell_1}e_1 \oplus \mathfrak{o}_{m_2}e_2$.

If $m_2 = 0$ the groups $P_{\mu\hookrightarrow\lambda}$ and $P_{\lambda\twoheadrightarrow\mu}$ become the geometric stabilizers of $\mathfrak{o}_{\ell_1}e_1$ and $\mathfrak{o}_{\ell_2}e_2$ (respectively) in M_λ:

$$P_{\mu\hookrightarrow\lambda} = P_{(\ell_1),(\ell_1,\ell_2)} = \begin{bmatrix} \mathfrak{o}_{\ell_1}^\times & \mathfrak{p}_{\ell_1}^{\ell_1-\ell_2} \\ & \mathfrak{o}_{\ell_2}^\times \end{bmatrix}, \tag{3.5}$$

$$P_{\lambda\twoheadrightarrow\mu} = P_{(\ell_2),(\ell_1,\ell_2)} = \begin{bmatrix} \mathfrak{o}_{\ell_1}^\times & \\ \mathfrak{o}_{\ell_2} & \mathfrak{o}_{\ell_2}^\times \end{bmatrix}. \tag{3.6}$$

4. Representations of $G_{(\ell,1)}$

The group $G_{(\ell,1)}$ is somewhat different from the groups $G_{(\ell_1,\ell_2)}$ with $\ell_2 > 1$. The fundamental difference from the point of view taken here is that the geometric induction suffices to build the induced part. We treat this group separately since its analysis is easier than the other automorphism groups. It plays an important role, being the first step in the inductive scheme which is developed later on. The aim of this section is to classify the irreducible representations of $G_{(\ell,1)}$ and to identify the cuspidal representations. We assume that $\ell > 1$, as the case $\ell = 1$ is well known and also does not fall into the present setup. We have

$$\begin{array}{ccccccc} (\mathfrak{o}_1,+) & \hookrightarrow & H(\mathfrak{o}_1) & \hookrightarrow & G_{(\ell,1)} & \twoheadrightarrow & G_{(\ell-1)} \times G_{(1)} \\ \wr\| & & \wr\| & & \| & & \| \\ Z = \begin{bmatrix} 1+\mathfrak{p}_\ell^{\ell-1} & \\ & 1 \end{bmatrix} & \hookrightarrow & \begin{bmatrix} 1+\mathfrak{p}_\ell^{\ell-1} & \mathfrak{p}_\ell^{\ell-1} \\ \mathfrak{o}_1 & 1 \end{bmatrix} & \hookrightarrow & \begin{bmatrix} \mathfrak{o}_\ell^\times & \mathfrak{p}_\ell^{\ell-1} \\ \mathfrak{o}_1 & \mathfrak{o}_1^\times \end{bmatrix} & \overset{p}{\twoheadrightarrow} & \mathfrak{o}_{\ell-1}^\times \times \mathfrak{o}_1^\times \end{array}$$

where $H = H(\mathfrak{o}_1)$ stands for the $\mathfrak{o}_1$-points of the Heisenberg group. One verifies directly that indeed $\mathrm{Ker}(p) \xrightarrow{\sim} H(\mathfrak{o}_1)$ by

$$\begin{pmatrix} 1+\pi^{\ell-1}u & \pi^{\ell-1}v \\ w & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & v & u \\ & 1 & w \\ & & 1 \end{pmatrix}, \quad u, v, w \in \mathfrak{o}_1.$$

We shall freely identify H with $\mathrm{Ker}(p)$ and $Z(H)$ with Z.

By the Stone–von Neumann theorem, for each non-trivial character of the center $\chi : Z \to \mathbb{C}^\times$, there is a unique irreducible representation $\rho_{H,\chi}$ of H of dimension $q = [H : Z]^{1/2}$. The representation $\rho_{H,\chi}$ is constructed by choosing any extension of χ to a maximal abelian subgroup of H and then inducing it to H.

The other irreducible representations of H correspond to the trivial character and hence factor through the quotient $H/Z \simeq \mathfrak{o}_1 \oplus \mathfrak{o}_1$. Summarizing, the irreducible representations of H are

$$\hat{H} = \widehat{H/Z} \sqcup \{\rho_{H,\chi} \mid 1 \neq \chi \in \hat{Z}\}.$$

By Clifford's theorem (cf. [12, Theorem 6.2]) the restriction of any irreducible representation of a finite group to a normal subgroup is a multiple of a sum over a full orbit of representations. Applying the theorem to $H \lhd G_{(\ell,1)}$ we need to analyze the action of $G_{(\ell,1)}$ on $\hat{H}$ and then classify the irreducible representations of $G_{(\ell,1)}$ which lie above each orbit.

Case 1: The q-dimensional representations of H. Each of the representations $\rho_{H,\chi}$ is in fact stabilized by $G_{(\ell,1)}$. Indeed, the subgroup Z is central in $G_{(\ell,1)}$, hence the action of $G_{(\ell,1)}$ on its characters is trivial. Since each of the representations $\rho_{H,\chi}$ is determined by χ, it must be stable under the $G_{(\ell,1)}$-action as well. We claim that $\rho_{H,\chi}$ can be extended to a representation of $G_{(\ell,1)}$. Indeed, let $T \simeq \mathfrak{o}_\ell^\times \times \mathfrak{o}_1^\times$ be the subgroup of diagonal elements in $G_{(\ell,1)}$, and let H_1 be a maximal abelian subgroup of H. Then χ can be extended from $Z = T \cap H_1$ to TH_1, and inducing the extension from TH_1 to $G_{(\ell,1)}$ gives a q-dimensional representation which must extend $\rho_{H,\chi}$. Thus, there are $|G_{(\ell,1)}/H| = |T/Z| = q^{\ell-2}(q-1)^2$ different extensions of $\rho_{H,\chi}$, and altogether $q^{\ell-2}(q-1)^3$ representations of $G_{(\ell,1)}$ counting for all $1 \neq \chi \in \hat{Z}$.

Case 2: The one-dimensional representations of H. Since these representations are in fact representations of the quotient $H/Z \simeq V = \mathfrak{o}_1 \oplus \mathfrak{o}_1$ we need to understand the action of $G_{(\ell,1)}/Z$ on $\hat{V}$. Fix a non-trivial character $\psi : \mathfrak{o}_1 \to \mathbb{C}^\times$, and identify $\hat{V}$ with V by $\langle(\hat{v}, \hat{w}), (v, w)\rangle = \psi(\hat{v}v + \hat{w}w)$ for $(\hat{v}, \hat{w}), (v, w) \in V$. The action of $G_{(\ell,1)}/Z$ on V by conjugation gives

$$g : (v, w) \mapsto \left(d^{-1}av, a^{-1}dw\right), \quad (v, w) \in V, \qquad g = \begin{pmatrix} a & b\pi^{\ell-1} \\ c & d \end{pmatrix} Z \in G_{(\ell,1)}/Z,$$

and then, using $\langle g \cdot (\hat{v}, \hat{w}), (v, w)\rangle = \langle(\hat{v}, \hat{w}), g^{-1} \cdot (v, w)\rangle$, we have

$$g : (\hat{v}, \hat{w}) \mapsto \left(a^{-1}d\hat{v}, d^{-1}a\hat{w}\right), \quad (\hat{v}, \hat{w}) \in \hat{V}.$$

There are three sporadic orbits and one family of orbits parameterized by $\mathfrak{o}_1^\times$:

$$\begin{array}{cccc} \mathrm{A} & \mathrm{B}_+ & \mathrm{B}_- & \mathrm{C} = \left\{\mathrm{C}_{\hat{w}} \mid \hat{w} \in \mathfrak{o}_1^\times\right\} \\ [\hat{0}, \hat{0}] & \left[\hat{0}, \mathfrak{o}_1^\times\right] & \left[\mathfrak{o}_1^\times, \hat{0}\right] & \left\{\mathfrak{o}_1^\times \cdot [1, \hat{w}] \mid \hat{w} \in \mathfrak{o}_1^\times\right\} \end{array}$$

The representations of $G_{(\ell,1)}$ which lie above the character A are one-dimensional and are pullbacks from characters of $G_{(\ell-1)} \times G_{(1)}$ along p. The stabilizer in $G_{(\ell,1)}$ of any of the characters in $\mathrm{B}_\pm$ and C is easily seen to be the subgroup DH with $D = \{d \cdot \mathrm{Id} \mid d \in \mathfrak{o}_\ell^\times\} \subset G_{(\ell,1)}$. As in the previous case, they can be extended to DH, in $[D:Z]$ many ways, each of them induce irreducibly to $G_{(\ell,1)}$, giving altogether

$$\big(|\mathrm{B}_+|+|\mathrm{B}_-|+|C|\big)[D:Z]=(q+1)(q-1)q^{\ell-2}$$

distinct irreducible representations of $G_{(\ell,1)}$ of dimension $[G_{(\ell,1)}:DH]=q-1$.

Theorem 4.1.

(1) $\mathcal{R}_{(\ell,1)}(\mathcal{D})=q^{\ell-2}(q-1)^2\mathcal{D}^1+q^{\ell-2}(q^2-1)\mathcal{D}^{q-1}+q^{\ell-2}(q-1)^3\mathcal{D}^q$.
(2) *The $q^{\ell-2}(q-1)^2$ representations of dimension $(q-1)$ which lie above the orbit* C *are cuspidal. All the other representations, namely those lying above types* A, $\mathrm{B}_\pm$ *and the q-dimensional representations, are induced.*

Proof. Part (1) follows from the discussion above and supplies a proof of Theorem 1.4 part (1) from the introduction. As for part (2), the representations of types A and $\mathrm{B}_\pm$ contain the trivial character of either

$$U_+=\begin{bmatrix}1 & \mathfrak{p}_\ell^{\ell-1}\\ & 1\end{bmatrix}\quad\text{or}\quad U_-=\begin{bmatrix}1 & \\ \mathfrak{o}_1 & 1\end{bmatrix}. \tag{4.1}$$

It follows that applying one of the functors $\mathbf{r}^{(\ell,1)}_{(\ell),(1)}$ or $\mathbf{r}^{(\ell,1)}_{(1),(\ell)}$ to these representations gives a non-zero representation of $G_{(\ell)}\times G_1$, hence they are geometrically induced. The representations of type C do not contain the trivial character of the subgroups in (4.1) and are hence annihilated by both functors. Since these are the only restriction functors for the type $(\ell,1)$ we conclude that these representations are cuspidal. As for the q-dimensional representations, we know that each of them is an extension of $\rho_{H,\chi}$. One of the ways to construct the latter is by inducing a character of $ZU_\pm$ which is trivial on $U_\pm$ and non-trivial on Z, in particular $\rho_{H,\chi}$ and any of its extensions contains the trivial representation upon restriction to $U_\pm$. □

5. The induction scheme

To set up the induction scheme we start by analyzing the orbits of G_λ on the reduction map kernel. For any $\lambda\in\Lambda_n$ which has the property that $\flat\lambda$ is also in Λ_n let

$$K_\lambda\hookrightarrow G_\lambda\twoheadrightarrow G_{\flat\lambda},$$

be defined as follows. The type of πM_λ is $\flat\lambda$, and the embedding $\pi M_\lambda\subset M_\lambda$ has the property that any automorphism of πM_λ can be extended to an automorphism of M_λ. It follows that restriction maps G_λ onto $G_{\flat\lambda}$. The kernel of the map is $K_\lambda=\mathrm{Id}+\mathrm{Hom}_{\mathfrak{o}}(M_\lambda,M_\lambda[\pi]\subset M_\lambda)$, where $M_\lambda[\pi]$ stands for the π-torsion points in M_λ, and can be identified with $(M_n(\mathfrak{o}_1),+)$. Appealing to Clifford theory, the irreducibles of G_λ can hence be naturally partitioned according to the orbits they mark on $\hat{K}_\lambda$.

Let $(2,2)\leqslant\lambda\in\Lambda_2$. We shall now describe the action of G_λ on $\hat{K}_\lambda$ explicitly. Using the matricial form (3.1) the identification $K_\lambda\xrightarrow{\sim}M_2(\mathfrak{o}_1)$ is given by

$$I+\begin{pmatrix}\pi^{\ell_1-1}u & \pi^{\ell_1-1}v\\ \pi^{\ell_2-1}w & \pi^{\ell_2-1}z\end{pmatrix}\mapsto\begin{pmatrix}u & v\\ w & z\end{pmatrix}.$$

Identify the dual $M_2(\mathfrak{o}_1)\hat{}$ with $M_2(\mathfrak{o}_1)$ by $x \mapsto \psi(\mathrm{trace}({}^{\mathrm{t}}x\cdot))$, $\psi : \mathfrak{o}_1 \to \mathbb{C}^\times$ a non-trivial character. Using these identifications, the action of G_λ on $\hat{K}_\lambda$ can be written explicitly in terms of $M_2(\mathfrak{o}_1)\hat{}$ as

$$g = \begin{pmatrix} a & b\delta \\ c & d \end{pmatrix} : \begin{pmatrix} \hat{u} & \hat{v} \\ \hat{w} & \hat{z} \end{pmatrix} \mapsto \begin{pmatrix} \hat{u} & \frac{d}{a}\hat{v} + \frac{c}{a}\hat{u} \\ \frac{a}{d}\hat{w} - \frac{b}{d}\hat{u} & \hat{z} - \frac{b}{a}\hat{v} + \frac{c}{d}\hat{w} - \frac{bc}{ad}\hat{u} \end{pmatrix}, \quad \ell_1 > \ell_2, \tag{5.1}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} \hat{u} & \hat{v} \\ \hat{w} & \hat{z} \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \hat{u} & \hat{v} \\ \hat{w} & \hat{z} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \ell_1 = \ell_2 = \ell, \tag{5.2}$$

where $\delta = \pi^{\ell_1 - \ell_2}$. The action (5.1) is a particular instance of the action (6.1) which we use here without proof, postponing the detailed discussion to Section 6, whereas the action (5.2) of G_{ℓ^2} is simply the action of $G_{1^2} = \mathrm{GL}_2(\mathfrak{o}_1)$ on $M_2(\mathfrak{o}_1)$ by conjugation via the reduction map $G_{\ell^2} \twoheadrightarrow G_{1^2}$.

The first step in the induction scheme is to identify and isolate from the discussion the non-primitive irreducible representations of G_λ. These representations are twists of pullbacks of representations of $G_{\flat\lambda}$. Essentially, there are $q = |\mathfrak{o}_1|$ such twists which are described as follows. Let 'det' denote the determinant map of G_λ

$$\det : G_\lambda \to \mathfrak{o}_{\ell_2}^\times,$$
$$\begin{pmatrix} a & b\delta \\ c & d \end{pmatrix} \mapsto ad - bc\delta \pmod{\pi^{\ell_2}}.$$

Any character $\chi : \mathfrak{o}_{\ell_2}^\times \to \mathbb{C}^\times$ gives rise to a one-dimensional character $\det^* \chi$ of G_λ. However, our interest lies in characters of $\mathfrak{o}_{\ell_2}^\times$ modulo the characters of $\mathfrak{o}_{\ell_2-1}^\times$. These are represented by an arbitrary extension of each of the characters $\psi(\hat{z}\cdot) : 1 + \mathfrak{p}_{\ell_2}^{\ell_2 - 1} \simeq \mathfrak{o}_1 \to \mathbb{C}^\times$ ($\hat{z} \in \mathfrak{o}_1$), from $1 + \mathfrak{p}_{\ell_2}^{\ell_2-1}$ to $\mathfrak{o}_{\ell_2}^\times$. Denote these representatives by $\chi_{\hat{z}}$ ($\hat{z} \in \mathfrak{o}_1$).

5.1. Non-rectangular case

Let $\lambda \in \Lambda_2$ be a partition with unequal parts greater than 1. Using (5.1), representatives for the orbits of G_λ on $\hat{K}_\lambda$, their number, and the total number of elements they contain, are given in Table 1.[1]

Define the following subgroups of G_λ

$$U_+ = \begin{bmatrix} 1 & \mathfrak{p}_{\ell_1}^{\ell_1 - \ell_2} \\ & 1 \end{bmatrix} \quad \text{and} \quad U_- = \begin{bmatrix} 1 & \\ \mathfrak{o}_{\ell_2} & 1 \end{bmatrix},$$

which play the role of 'upper/lower unipotent' subgroups. Note that they are both isomorphic to $(\mathfrak{o}_{\ell_2}, +)$ but not conjugate in G_λ since λ is non-rectangular. Let $V_\pm = U_\pm \cap K_\lambda$ and let $V_1 < K_\lambda$ and $V_2 < K_\lambda$ denote the embeddings of $1 + \mathfrak{p}_{\ell_1}^{\ell_1 - 1}$ and $1 + \mathfrak{p}_{\ell_2}^{\ell_2 - 1}$ in the $(1,1)$ and $(2,2)$ entries respectively. Thus, $K_\lambda = V_1 V_2 V_+ V_-$.

[1] The '$\hat{1}$' in the matrices' entries stands for $x \mapsto \psi(1 \cdot x)$ and is a convenient non-canonical choice of a non-trivial character of $\mathfrak{o}_1$. '$\hat{0}$' stand for $x \mapsto \psi(0 \cdot x) \equiv 1$, the trivial character of $\mathfrak{o}_1$.

Table 1
Orbits in the non-rectangular case

	Representatives for $\rho_{\mid K}$	Number of orbits	Total number of elements
(i)	$\begin{pmatrix} \hat{0} & \hat{0} \\ \hat{0} & \hat{z} \end{pmatrix}$, $\hat{z} \in \mathfrak{o}_1$	q	q
(ii)	$\begin{pmatrix} \hat{u} & \hat{0} \\ \hat{0} & \hat{z} \end{pmatrix}$, $\hat{z} \in \mathfrak{o}_1$, $\hat{u} \in \mathfrak{o}_1^\times$	$q(q-1)$	$q^3(q-1)$
(iii)	$\begin{pmatrix} \hat{0} & \hat{0} \\ \hat{1} & \hat{0} \end{pmatrix}$	1	$q(q-1)$
(iv)	$\begin{pmatrix} \hat{0} & \hat{1} \\ \hat{0} & \hat{0} \end{pmatrix}$	1	$q(q-1)$
(v)	$\begin{pmatrix} \hat{0} & \hat{1} \\ \hat{w} & \hat{0} \end{pmatrix}$, $\hat{w} \in \mathfrak{o}_1^\times$	$q-1$	$q(q-1)^2$

Proposition 5.1. *Let ρ be an irreducible representation of G_λ, then*

(1) *ρ is a twist of a pullback from $\hat{G}_{\flat\lambda} \iff \langle \rho_{\mid (V_1 V_- V_+)}, \hat{0}_{(V_1 V_- V_+)} \rangle \neq 0$ ($\rho_{\mid K_\lambda}$ is of type* (i)*).*
(2) *ρ is a cuspidal $\iff \langle \rho_{\mid V_-}, \hat{0}_{V_-} \rangle = \langle \rho_{\mid V_+}, \hat{0}_{V_+} \rangle = 0 \neq \langle \rho_{\mid V_1}, \hat{0}_{V_1} \rangle$ ($\rho_{\mid K_\lambda}$ is of type* (v)*).*
(3) *ρ is induced (up to twist) $\iff \rho_{\mid K_\lambda}$ is of type* (ii)*,* (iii) *or* (iv).

Proof. (1) We observe that $\det_{\mid K_\lambda}$ maps V_2 isomorphically on $1 + \mathfrak{p}_{\ell_2}^{\ell_2 - 1}$, while $V_1 V_+ V_- = \mathrm{Ker}(\det)$. We conclude that the effect of twisting a representation ρ of G_λ with a one-dimensional character $\det^* \chi_{\hat{z}}$ is expressed on $\rho_{\mid K_\lambda}$ by an additive shift of the $(2,2)$-entry. It follows that the orbits of type (i) in Table 1 occur precisely for twists of pullbacks of representations of $G_{\flat\lambda}$.

(2) Let $\rho \in \hat{C}_\lambda$. In order that the image of ρ will be zero under each of the restriction functors it must satisfy $\langle \rho_{\mid V_+}, \hat{0}_{V_+} \rangle = 0 = \langle \rho_{\mid V_-}, \hat{0}_{V_-} \rangle$ (use the explicit form in Section 3). This implies, by using the action (5.1), that $\hat{u} = 0$ and $\hat{w}, \hat{v} \in \mathfrak{o}_1^\times$. Then, we may choose $\hat{v} = \hat{1}$ and $\hat{z} = 0$. Conversely, we shall see in Section 6 how each of the representatives

$$\begin{pmatrix} \hat{0} & \hat{1} \\ \hat{w} & \hat{0} \end{pmatrix}, \quad \hat{w} \in \mathfrak{o}_1^\times, \tag{5.3}$$

occur as the restriction of a cuspidal representation.

(3) The character $\left(\begin{smallmatrix} \hat{0} & \hat{0} \\ \hat{1} & \hat{0} \end{smallmatrix}\right)$ is trivial on $V_+ V_2$ and is thus not annihilated by the functor $\mathbf{r}^{(\ell_1,\ell_2)}_{(\ell_1,\ell_2-1),\hookrightarrow}$. This means that any representation of $G_{(\ell_1,\ell_2)}$ lying above orbit of type (iii) is contained in an infinitesimally induced representation of $G_{(\ell_1,\ell_2-1)}$ using $\mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_1,\ell_2-1),\hookrightarrow}$. Similarly, representations lying above type (iv) are contained in an infinitesimally induced representation of $G_{(\ell_1,\ell_2-1)}$ using $\mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_1,\ell_2-1),\twoheadrightarrow}$. Same argument holds for representations lying above orbits of type (ii) after a possible twist. $\square$

Remark 5.2. The third part of Proposition 5.1 is given here in order to motivate and explain the strategy. We shall later on obtain more precise results as in Theorem 1.2. In particular, all the representations of type (ii) are geometrically induced and no twist is required in (3).

Table 2
Orbits in the rectangular case

	Representatives for $\rho_{\|K}$	Number of orbits	Total number of elements
(i)	$\begin{pmatrix} \hat{u} & \hat{0} \\ \hat{0} & \hat{u} \end{pmatrix}$, $\hat{u} \in \mathfrak{o}_1$	q	q
(ii)	$\begin{pmatrix} \hat{u} & \hat{0} \\ \hat{0} & \hat{z} \end{pmatrix}$, $S_2 \backslash \{\hat{u}, \hat{z} \in \mathfrak{o}_1 \mid \hat{u} \neq \hat{z}\}$	$\frac{1}{2}q(q-1)$	$\frac{1}{2}q^2(q^2-1)$
(iii)	$\begin{pmatrix} \hat{u} & \hat{1} \\ \hat{0} & \hat{u} \end{pmatrix}$, $\hat{u} \in \mathfrak{o}_1$	q	$q(q^2-1)$
(iv)	$\begin{pmatrix} \hat{u} & \hat{v} \\ \hat{1} & \hat{u} \end{pmatrix}$, $\hat{u} \in \mathfrak{o}_1,\ \hat{v} \in \mathfrak{o}_1^\times \smallsetminus (\mathfrak{o}_1^\times)^2$	$\frac{1}{2}q(q-1)$	$\frac{1}{2}q^2(q-1)^2$

5.2. *Rectangular case*

Let $\lambda = \ell^2$ with $\ell > 1$. The orbits of the action (5.2) are well known, given in Table 2.

In complete analogy with Proposition 5.1, representations lying over orbits of type (i) are twists of pullbacks from lower level. Representations lying over type (iv) are cuspidal. The main differences comparing to the non-rectangular case are the following.

(1) A twist with a primitive character manifests itself on $\hat{K}_{\ell^2}$ as a shift with a scalar matrix. As a result, primitive representations of G_{ℓ^2} are infinitesimally induced from cuspidal representations of G_μ ($\mu \in \mathcal{I}_{\ell^2}$) only up to twist.
(2) There is an extra symmetry in the infinitesimal induction which results in equivalences between $\mathbf{i}^{\ell^2}_{\mu,\hookrightarrow}$ and $\mathbf{i}^{\ell^2}_{\mu,\twoheadrightarrow}$, which is simply denoted $\mathbf{i}^{\ell^2}_{\mu}$ in this case. In terms of orbits, this can be seen in the difference between the orbit types: types (iii)–(iv) in Table 1 degenerate to one type, (iii), in Table 2.

6. Construction of the cuspidal representations

6.1. *Non-rectangular case*

In this section the cuspidal representations of the groups G_λ are constructed for $\lambda = (\ell_1, \ell_2)$ with $\ell_1 > \ell_2 > 1$. Let

$$K_\lambda^{i,\sigma} = I + \begin{bmatrix} \mathfrak{p}_{\ell_1}^{\ell_1-i} & \mathfrak{p}_{\ell_1}^{\ell_1-i} \\ \mathfrak{p}_{\ell_2}^{\ell_2-i+\sigma} & \mathfrak{p}_{\ell_2}^{\ell_2-i+\sigma} \end{bmatrix}, \quad 1 \leqslant i \leqslant \ell_2,\ \sigma \in \{0, 1\}.$$

Lemma 6.1. *$K_\lambda^{i,\sigma}$ is a normal subgroup of G_λ. If $i \leqslant (\ell_2 + \sigma)/2$ then $K_\lambda^{i,\sigma}$ is abelian, and*

$$K_\lambda^{i,\sigma} \simeq L_{i,\sigma} = \left(\begin{bmatrix} \mathfrak{o}_i & \mathfrak{o}_i \\ \mathfrak{o}_{i-\sigma} & \mathfrak{o}_{i-\sigma} \end{bmatrix}, + \right), \qquad I + \begin{pmatrix} \pi^{\ell_1-i}u & \pi^{\ell_1-i}v \\ \pi^{\ell_2-i+\sigma}w & \pi^{\ell_2-i+\sigma}z \end{pmatrix} \mapsto \begin{pmatrix} u & v \\ w & z \end{pmatrix}.$$

Proof. Straightforward. $\square$

We shall proceed in the following steps.

(1) Analysis of the action of G_λ on $\hat{K}_\lambda^{i,\sigma}$.
(2) Identification of the possible orbits of characters of a 'large' normal abelian subgroup $K_\lambda^{\ell,\epsilon}$ which can give rise to cuspidal representations.
(3) Construction of the cuspidal representations by extending these characters from $K_\lambda^{l,\epsilon}$ to their normalizer and then inducing them to G_λ.

Step 1. Assume from now on that $i \leqslant (\ell_2+\sigma)/2$, so that Lemma 6.1 is satisfied. The action of G_λ on $K_\lambda^{i,\sigma}$ by conjugation translates by the above identification to an action on $L_{i,\sigma}$, explicitly given by

$$g:\begin{pmatrix} u & v \\ w & z \end{pmatrix} \mapsto e^{-1}\begin{pmatrix} u-\frac{c}{d}v+\frac{b\pi^\sigma}{a}w-\frac{bc\pi^\sigma}{ad}z & \frac{a}{d}v+\frac{b\pi^\sigma}{d}z-\frac{b\delta}{d}u-\frac{b^2\delta\pi^\sigma}{ad}w \\ \frac{d}{a}w-\frac{c}{a}z+\frac{c\delta}{a\pi^\sigma}u-\frac{c^2\delta}{ad\pi^\sigma}v & z-\frac{b\delta}{a}w+\frac{c\delta}{d\pi^\sigma}v-\frac{bc\delta^2}{ad\pi^\sigma}u \end{pmatrix},$$

where $g=\left(\begin{smallmatrix} a & b\delta \\ c & d \end{smallmatrix}\right)$, $e=1-a^{-1}d^{-1}bc\delta$ and $\delta=\pi^{\ell_1-\ell_2}$. Note that the inverse of g is given by

$$g^{-1}=e^{-1}\begin{pmatrix} a^{-1} & -a^{-1}d^{-1}b\delta \\ -a^{-1}d^{-1}c & d^{-1} \end{pmatrix}.$$

In the sequel we shall need to analyze the action of G_λ also on the characters of $L_{i,\sigma}$. Let $\psi:\mathfrak{o}_i \to \mathbb{C}^\times$ be a fixed primitive additive character. We identify $L_{i,\sigma}$ with its dual by

$$\left\langle \begin{pmatrix} \hat{u} & \hat{v} \\ \hat{w} & \hat{z} \end{pmatrix}, \begin{pmatrix} u & v \\ w & z \end{pmatrix} \right\rangle = \psi\big(\hat{u}u+\hat{v}v+\pi^\sigma(\hat{w}w+\hat{z}z)\big).$$

The action of $g\in G_\lambda$ on the dual is defined by $\langle g\cdot\theta, t\rangle = \langle \theta, g^{-1}\cdot t\rangle$, and is given explicitly by

$$g:\begin{pmatrix} \hat{u} & \hat{v} \\ \hat{w} & \hat{z} \end{pmatrix} \mapsto e^{-1}\begin{pmatrix} \hat{u}+\frac{b\delta}{a}\hat{v}-\frac{c\delta}{d}\hat{w}-\frac{bc\delta^2}{ad}\hat{z} & \frac{d}{a}\hat{v}-\frac{c\delta}{a}\hat{z}+\frac{c}{a}\hat{u}-\frac{c^2\delta}{ad}\hat{w} \\ \frac{a}{d}\hat{w}+\frac{b\delta}{d}\hat{z}-\frac{b}{d}\hat{u}-\frac{b^2\delta}{ad}\hat{v} & \hat{z}-\frac{b}{a}\hat{v}+\frac{c}{d}\hat{w}-\frac{bc}{ad}\hat{u} \end{pmatrix}. \tag{6.1}$$

Equipped with the action of G_λ on the various groups $\hat{K}_\lambda^{i,\sigma}$ we proceed to the construction of the cuspidal representations in several steps. We repeatedly make use of Clifford theorem with respect to the normal subgroups $K_\lambda^{i,\sigma}$.

Step 2. Let $\epsilon=\ell_2 \pmod 2$ be the parity of ℓ_2 and let $\ell=(\ell_2+\epsilon)/2$. We wish to extend characters of the type (5.3) from $K_\lambda^{1,0}$ to $K_\lambda^{\ell,\epsilon}$ ($K_\lambda^{1,0}$ is denoted by K in Section 5). Let $\eta_{\hat{u},\hat{w}}$ be the following characters of $L_{\ell,\epsilon}$

$$\begin{pmatrix} \hat{u} & \hat{1} \\ \hat{w} & \hat{0} \end{pmatrix}, \quad \hat{w}\in\mathfrak{o}_{\ell-\epsilon}^\times,\ \hat{u}\in\mathfrak{p}_\ell. \tag{6.2}$$

Proposition 6.2. *The characters $\eta_{\hat{u},\hat{w}}$ ($\hat{w}\in\mathfrak{o}_{\ell-\epsilon}^\times$, $\hat{u}\in\mathfrak{p}_\ell$) form a complete set of representatives for the orbits of G_λ on characters of $\hat{L}_{\ell,\epsilon}$ which lie above the characters* (5.3) *of $\hat{L}_{1,0}$.*

Proof. Define the following functions on $\hat{L}_{\ell,\epsilon}$

$$\mathrm{Tr}_\delta \begin{pmatrix} \hat{u} & \hat{v} \\ \hat{w} & \hat{z} \end{pmatrix} = \hat{u} + \delta\hat{z} \quad \left(\mathrm{mod}\ \pi^{\ell}\right),$$

$$\mathrm{Det} \begin{pmatrix} \hat{u} & \hat{v} \\ \hat{w} & \hat{z} \end{pmatrix} = \hat{u}\hat{z} - \hat{w}\hat{v} \quad \left(\mathrm{mod}\ \pi^{\ell-\epsilon}\right).$$

Using the action (6.1) one easily checks that Tr_δ and Det are invariants of the G_λ-orbits. It follows that the characters $\eta_{\hat{u},\hat{w}}$ lie in distinct G_λ-orbits since $\mathrm{Tr}_\delta(\eta_{\hat{u},\hat{w}}) = \hat{u}$ and $\mathrm{Det}_\delta(\eta_{\hat{u},\hat{w}}) = -\hat{w}$. Conversely, one easily verifies that any element which lie above the characters (5.3) can be brought to an element of the form $\eta_{\hat{u},\hat{w}}$ the assertion follows. □

Step 3. Let $N_\lambda = N_\lambda(\hat{u},\hat{w}) = \mathrm{Stab}_{G_\lambda}(\eta_{\hat{u},\hat{w}})$ be the stabilizer of $\eta_{\hat{u},\hat{w}}$ in G_λ. The following theorem completes the construction of the cuspidal representations of G_λ.

Theorem 6.3.

(1) *The normalizer of $\eta_{\hat{u},\hat{w}}$ in G_λ is given by*

$$N_\lambda(\hat{u},\hat{w}) = \left\{ \begin{pmatrix} a & b\delta \\ c & d \end{pmatrix} \;\middle|\; b \equiv c\hat{w} \left(\mathrm{mod}\ \pi^{\ell-\epsilon}\right),\ d \equiv a - c\hat{u} \left(\mathrm{mod}\ \pi^{\ell}\right) \right\}.$$

(2) *The character $\eta_{\hat{u},\hat{w}}$ can be extended to $N_\lambda(\hat{u},\hat{w})$.*

(3) *For any extension $\tilde{\eta}$ of $\eta_{\hat{u},\hat{w}}$ the induced representation $\rho_{\tilde{\eta}} = \mathrm{Ind}_{N_\lambda}^{G_\lambda}(\tilde{\eta}_{\hat{u},\hat{w}})$ is irreducible and cuspidal and distinct $\tilde{\eta}$'s give rise to inequivalent representations. Any cuspidal representation of G_λ is of this form. There exist $q^{\ell_1+\ell_2-3}(q-1)^2$ such representations and their dimension is $q^{\ell_2-1}(q-1)$.*

Proof. (1) Using the action (6.1) we see that if $g \cdot \eta_{\hat{u},\hat{w}} = \eta_{\hat{u},\hat{w}}$, we get

$$\begin{aligned}
&(1,1) \qquad \hat{u} = \frac{\hat{u}}{e} + \frac{b\delta}{ae}\hat{1} - \frac{c\delta}{de}\hat{w} \quad (\text{in } \mathfrak{o}_\ell),\\
&(1,2) \qquad \hat{1} = \frac{d}{ae}\hat{1} + \frac{c}{ae}\hat{u} - \frac{c^2\delta}{ead}\hat{w} \quad (\text{in } \mathfrak{o}_\ell),\\
&(2,2) \qquad \hat{0} = -\frac{b}{a}\hat{1} + \frac{c}{d}\hat{w} - \frac{bc}{ad}\hat{u} \quad (\text{in } \mathfrak{o}_{\ell-\epsilon}).
\end{aligned}$$

Adding up $[(2,2) - d^{-1}be \cdot (1,2)]$ gives

$$b \equiv c\hat{w} \quad \left(\mathrm{mod}\ \pi^{\ell-\epsilon}\right). \tag{6.3}$$

Equation $(1,1)$ is equivalent to

$$\frac{c\delta}{d}\hat{w} = \frac{b\delta}{a}\hat{1} + \frac{bc\delta}{ad}\hat{u} \quad (\text{in } \mathfrak{o}_l)$$

and the latter gives after substitution in (2, 1)

$$d \equiv a - c\hat{u} \pmod{\pi^{\ell}}. \tag{6.4}$$

Conversely, if relations (6.3) and (6.4) hold then $g \cdot \eta_{\hat{u},\hat{w}} = \eta_{\hat{u},\hat{w}}$.

(2) Let

$$A_\lambda = A_\lambda(\hat{u}', \hat{w}') = \left\{ \begin{pmatrix} a & c\hat{w}'\delta \\ c & a - c\hat{u}' \end{pmatrix} \,\middle|\, a \in \mathfrak{o}_{\ell_1}^{\times},\ c \in \mathfrak{o}_{\ell_2} \right\},$$

where $\hat{u}', \hat{w}' \in \mathfrak{o}_{\ell_2}$ are lifts of $\hat{u}$, $\hat{w}$ respectively. We have the following diagram

$$\begin{array}{ccccc} & & G_\lambda & & \\ & & | & & \\ & & K_\lambda^{\ell,\epsilon} A_\lambda & & \\ & / & & \backslash & \\ K_\lambda^{\ell,\epsilon} & & & & A_\lambda. \\ & \backslash & & / & \\ & & K_\lambda^{\ell,\epsilon} \cap A_\lambda & & \end{array}$$

Note that $N_\lambda = K_\lambda^{\ell,\epsilon} A_\lambda$, and since A_λ is abelian, we can extend $\eta_{\hat{u},\hat{w}}$ to N_λ in $[A_\lambda : K_\lambda^{\ell,\epsilon} \cap A_\lambda] = [K_\lambda^{\ell,\epsilon} A_\lambda : K_\lambda^{\ell,\epsilon}]$ different ways.

(3) Going over all the possible choices of $\hat{u}$ and $\hat{w}$ and all the possible extensions to the stabilizers $N_\lambda(\hat{u}, \hat{w})$, we get

$$|\mathfrak{p}_\ell| \left|\mathfrak{o}_{\ell-\epsilon}^{\times}\right| \left|A_\lambda / K_\lambda^{\ell,\epsilon} \cap A_\lambda\right| = q^{\ell_1+\ell_2-3}(q-1)^2$$

different characters. By [12, Theorem 6.11] all these characters induce irreducibly to G_λ. Since all the possible candidates of orbits in $\hat{K}_\lambda^{\ell,\epsilon}$ which might lie below a cuspidal representation of G_λ are covered, these induced representations exhaust the cuspidal representations of G_λ. Their dimension is $|G_\lambda / N_\lambda| = q^{\ell_2 - 1}(q-1)$. □

6.2. Cuspidal representations of G_{ℓ^2}

The cuspidal representations of G_{ℓ^2} were constructed by Nobs in [22, §4] using the Weil representation under the name *la série non ramifiée* (see also the more general treatment in [5]). There are $\frac{1}{2}(q^2-1)(q-1)q^{2\ell-3}$ such representations and their dimension is $q^{\ell-1}(q-1)$. The number of these representations matches the number of Galois orbits of strongly primitive characters of $\tilde{\mathfrak{o}}_\ell^{\times}$ (the latter is the units in the non-ramified extension of $\mathfrak{o}_\ell$ of degree 2). This is of no coincidence. In fact, there exist a canonical bijective correspondence between such orbits of characters and the cuspidal representations of G_{ℓ^2}. This is proved in general in [1] for types $\lambda = \ell^n$ where n is prime. The reader is referred to [1] for another construction of the cuspidal representation of G_{ℓ^2} which is more in the spirit of this paper and avoids the use of the Weil representation. In short, the correspondence is established as follows. A character of $\theta : \tilde{\mathfrak{o}}_\ell^{\times} \to \mathbb{C}^{\times}$ is *strongly primitive* if its restriction to $\mathrm{Ker}\{\mathfrak{o}_\ell^{\times} \to \mathfrak{o}_{\ell-1}^{\times}\} \simeq (\tilde{\mathfrak{o}}_1, +)$ gives a primitive additive character of $\tilde{\mathfrak{o}}_1$. Using the natural embedding $\tilde{\mathfrak{o}}_\ell^{\times} \hookrightarrow G_{\ell^2}$, it is shown that the centralizer of a strongly primitive character θ is $N_{\lfloor \ell/2 \rfloor} \cdot \tilde{\mathfrak{o}}_\ell^{\times}$, where $N_i = \mathrm{Ker}\{G_{\ell^2} \to G_{i^2}\}$, and that there exists a unique representation ρ_θ of the centralizer which restricts to θ or to one of its Galois conjugates. Inducing the ρ_θ's to G_{ℓ^2} gives the cuspidal representations.

7. Infinitesimal and geometric induction

In this section we take a closer look at the infinitesimal and geometric inductions. The first subsection describes some basic properties of geometric and infinitesimal inductions in the general setting. We then specialize to the rank two case and in particular obtain finer results regarding the cases in which such inductions are irreducible. Throughout we use the notation $K_\lambda = \operatorname{Ker}\{G_\lambda \to G_{\flat\lambda}\}$.

7.1. Basic properties

All the induction functors defined in Section 2 are compositions of inflation and usual induction: Ind $\circ$ Inf. Similarly, all the restriction functors are compositions of usual restriction and a functor of fixed points: Inv $\circ$ Res. Since (Ind, Res) form an adjoint pair, and so do (Inf, Inv), it follows that

$$\left(\mathbf{i}^\lambda_{\mu,\mu'}, \mathbf{r}^\lambda_{\mu,\mu'}\right), \qquad \left(\mathbf{i}^\lambda_{\mu,\hookrightarrow}, \mathbf{r}^\lambda_{\mu,\hookrightarrow}\right), \quad \text{and} \quad \left(\mathbf{i}^\lambda_{\mu,\twoheadrightarrow}, \mathbf{r}^\lambda_{\mu,\twoheadrightarrow}\right),$$

are adjoint pairs as well. In order to prove some associativity properties of these functors we shall need the following lemma.

Lemma 7.1. *Let G be a finite group and $G \supset P_1 \supset P_2 \supset U_2 \supset U_1$ a chain of subgroups with $U_i \lhd P_i$. Then*

$$\operatorname{Ind}_{P_2}^{G} \circ \operatorname{Inf}_{P_2/U_2}^{P_2} = \operatorname{Ind}_{P_1}^{G} \circ \operatorname{Inf}_{P_1/U_1}^{P_1} \circ \operatorname{Ind}_{P_2/U_1}^{P_1/U_1} \circ \operatorname{Inf}_{P_2/U_2}^{P_2/U_1}.$$

Proof. By [33, Lemma 3.4] we have $\operatorname{Inf}_{P_1/U_1}^{P_1} \circ \operatorname{Ind}_{P_2/U_1}^{P_1/U_1} = \operatorname{Ind}_{P_2}^{P_1} \circ \operatorname{Inf}_{P_2/U_1}^{P_2}$, which together with the associativity of induction and inflation proves the assertion. $\square$

Claim 7.2. *Let $\nu \leqslant \mu \leqslant \lambda$ be types in Λ with ν symmetric in μ and μ symmetric in λ. Assume that any module of type ν is contained in a unique module of type μ. Then*

(1) $\mathbf{i}^\lambda_{\mu,\hookrightarrow} \circ \mathbf{i}^\mu_{\nu,\hookrightarrow} = \mathbf{i}^\lambda_{\nu,\hookrightarrow}$. *Dually,* $\mathbf{r}^\mu_{\nu,\hookrightarrow} \circ \mathbf{r}^\lambda_{\mu,\hookrightarrow} = \mathbf{r}^\lambda_{\nu,\hookrightarrow}$.
(2) $\mathbf{i}^\lambda_{\mu,\twoheadrightarrow} \circ \mathbf{i}^\mu_{\nu,\twoheadrightarrow} = \mathbf{i}^\lambda_{\nu,\twoheadrightarrow}$. *Dually,* $\mathbf{i}^\mu_{\nu,\twoheadrightarrow} \circ \mathbf{i}^\lambda_{\mu,\twoheadrightarrow} = \mathbf{i}^\lambda_{\nu,\twoheadrightarrow}$.

Proof. Let $M_\nu \subset M_\mu \subset M_\lambda$ be of types ν, μ and λ. We should show that

$$\operatorname{Ind}_{P_{M_\nu,M_\lambda}}^{G_\lambda} \circ \operatorname{Inf}_{G_\nu}^{P_{M_\nu,M_\lambda}} = \operatorname{Ind}_{P_{M_\mu,M_\lambda}}^{G_\lambda} \circ \operatorname{Inf}_{G_\mu}^{P_{M_\mu,M_\lambda}} \circ \operatorname{Ind}_{P_{M_\nu,M_\mu}}^{G_\mu} \circ \operatorname{Inf}_{G_\nu}^{P_{M_\nu,M_\mu}}.$$

By assumption we have the inclusion $P_{M_\nu,M_\lambda} \subset P_{M_\mu,M_\lambda}$, and

$$\begin{array}{ccccc} U_{M_\nu,M_\lambda} & \hookrightarrow & P_{M_\nu,M_\lambda} & \twoheadrightarrow & G_\nu \\ \cup & & \cap & & \\ U_{M_\mu,M_\lambda} & \hookrightarrow & P_{M_\mu,M_\lambda} & \twoheadrightarrow & G_\mu. \end{array}$$

The first part of assertion (1) follows now from Lemma 7.1 using the identifications

$$\begin{aligned} P_{M_\nu,M_\mu} &= P_{M_\nu,M_\lambda}/U_{M_\mu,M_\lambda}, \\ G_\mu &= P_{M_\mu,M_\lambda}/U_{M_\mu,M_\lambda}, \\ G_\nu &= P_{M_\nu,M_\lambda}/U_{M_\nu,M_\lambda} = P_{M_\nu,M_\mu}/U_{M_\nu,M_\mu}, \end{aligned}$$

and the second part by uniqueness of the adjoint. Part (2) follows along the same lines by replacing the modules $M_\nu \subset M_\mu$ by modules $M_{\lambda/\mu} \subset M_{\lambda/\nu}$ of cotypes μ and ν respectively. $\square$

7.2. Infinitesimal induction in rank two

Proposition 7.3. *Let $\lambda = (\ell_1, \ell_2) \in \Lambda_2$ and let ρ be an irreducible representation of G_λ.*

(1) *If $\ell_1 > \ell_2 > 1$ and either $\rho_{|K_\lambda} \geqslant \left(\begin{smallmatrix} \hat{0} & \hat{0} \\ \hat{1} & \hat{0} \end{smallmatrix}\right)$ or $\rho_{|K_\lambda} \geqslant \left(\begin{smallmatrix} \hat{0} & \hat{1} \\ \hat{0} & \hat{0} \end{smallmatrix}\right)$, then* $\dim(\rho) = q^{\ell_2 - 1}(q-1)$.
(2) *If $\ell_1 = \ell_2 = \ell > 1$ and $\rho_{|K_{\ell^2}} \geqslant \left(\begin{smallmatrix} \hat{0} & \hat{1} \\ \hat{0} & \hat{0} \end{smallmatrix}\right)$, then* $\dim(\rho) = q^{\ell-2}(q^2-1)$.

Proof. (1) We argue by induction on ℓ_2. Let ρ be an irreducible representation of G_λ which contains the character $\left(\begin{smallmatrix} \hat{0} & \hat{0} \\ \hat{1} & \hat{0} \end{smallmatrix}\right) \in \hat{K}_\lambda$. Using the action (5.1) we verify that this orbit contains the $q(q-1)$ characters $\{\left(\begin{smallmatrix} \hat{0} & \hat{0} \\ a\hat{1} & c\hat{1} \end{smallmatrix}\right) \mid a \in \mathfrak{o}_1^\times,\ c \in \mathfrak{o}_1\}$, out of which $(q-1)$ survive the application of the functor $\mathbf{r}^{(\ell_1,\ell_2)}_{(\ell_1,\ell_2-1),\hookrightarrow}$, namely those having $c = 0$. It follows that if $\xi = \mathbf{r}^{(\ell_1,\ell_2)}_{(\ell_1,\ell_2-1),\hookrightarrow}(\rho)$ then $\dim(\xi) = \dim(\rho)/q$. Using the explicit form (3.3) we observe that $\langle \xi_{|V_-}, \hat{0}_{V_-}\rangle = 0$, which means that $\xi_{|K_{(\ell_1,\ell_2-1)}}$ contains the orbit of $\left(\begin{smallmatrix} \hat{0} & \hat{0} \\ \hat{1} & \hat{0} \end{smallmatrix}\right)$ or a cuspidal orbit (that is, orbits (iii) or (iv) in Table 1). In either case, using the induction hypothesis or the dimension of cuspidal representations, we get that ξ is irreducible and that $\dim(\xi) = q^{\ell_2-2}(q-1)$, from which the assertion follows. The basis of the induction is $\ell_2 = 2$ in which case the role of $K_{(\ell_1,1)}$ is taken over by H of Section 4 and ξ lies above the orbits B_+ or C. If ρ lies above the orbit of $\left(\begin{smallmatrix} \hat{0} & \hat{1} \\ \hat{0} & \hat{0} \end{smallmatrix}\right)$ we argue similarly using the functor $\mathbf{r}^{(\ell_1,\ell_2)}_{(\ell_1,\ell_2-1),\twoheadrightarrow}$.

(2) Assume $\rho \in \hat{G}_{\ell^2}$ such that $\rho_{|K_{\ell^2}} \geqslant \left(\begin{smallmatrix} \hat{0} & \hat{1} \\ \hat{0} & \hat{0} \end{smallmatrix}\right)$. This orbit contains the (q^2-1) characters $\{\left(\begin{smallmatrix} ab\hat{1} & -a^2\hat{1} \\ b^2\hat{1} & -ab\hat{1} \end{smallmatrix}\right) \mid a, b \in \mathfrak{o}_1,\ (a,b) \neq (0,0)\}$ (using the action (5.2)), out of which $(q-1)$ survive the application of the functor $\mathbf{r}^{(\ell,\ell)}_{(\ell,\ell-1),\hookrightarrow}$, namely those having $a = 0$. It follows that if $\xi = \mathbf{r}^{(\ell,\ell)}_{(\ell,\ell-1),\hookrightarrow}(\rho)$ then $\dim(\xi) = \dim(\rho)/(q+1)$. Observing that $\xi_{|K_{(\ell,\ell-1)}} \geqslant \left(\begin{smallmatrix} \hat{0} & \hat{0} \\ \hat{1} & \hat{0} \end{smallmatrix}\right)$ we can use part (1) and the result follows. $\square$

Theorem 7.4. *Let $\lambda = (\ell_1, \ell_2) \in \Lambda_2$ and $\mu = (\ell_1, m_2) \in \mathcal{I}_\lambda$. Then for any $\rho \in \hat{C}_\mu$ the infinitesimal inductions $\mathbf{i}^\lambda_{\mu,\hookrightarrow}(\rho)$ and $\mathbf{i}^\lambda_{\mu,\twoheadrightarrow}(\rho)$ are irreducible of dimension*

$$i_\lambda = \begin{cases} q^{\ell_2-1}(q-1), & \text{if } \ell_1 > \ell_2; \\ q^{\ell-2}(q^2-1), & \text{if } \ell_1 = \ell_2 = \ell. \end{cases}$$

Non-equivalent representations induce to non-equivalent representations.

Proof. We argue for $\mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho)$, the assertion for $\mathbf{i}^{\lambda}_{\mu,\twoheadrightarrow}(\rho)$ is similar. We claim that

(a) $\dim \mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho) = i_\lambda$.
(b) $\mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho)_{|K_\lambda} \geqslant \left(\begin{smallmatrix}\hat{0} & \hat{0}\\ \hat{1} & \hat{0}\end{smallmatrix}\right)$.
(c) $\rho = \mathbf{r}^{\lambda}_{\mu,\hookrightarrow}\mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho)$.

Before proving these assertions note that they imply the theorem since for any $\rho, \rho' \in \hat{C}_\mu$

$$\left\langle \mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho), \mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho')\right\rangle = \left\langle \rho, \mathbf{r}^{\lambda}_{\mu,\hookrightarrow}\mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho')\right\rangle = \langle \rho, \rho'\rangle = \begin{cases} 1, & \text{if } \rho \simeq \rho';\\ 0, & \text{otherwise.}\end{cases}$$

To prove (a), we use the fact that $\dim(\rho) = q^{m_2-1}(q-1)$ (by Theorem 6.3) together with

$$[G_\lambda : P_{\mu\hookrightarrow\lambda}] = \begin{cases} q^{\ell_2-m_2}, & \text{if } \ell_1 > \ell_2;\\ q^{\ell-m_2-1}(q+1), & \text{if } \ell_1 = \ell_2 = \ell.\end{cases}$$

As for (b), the representation $\mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho)$ must contain an irreducible component which lies above the orbit of $\left(\begin{smallmatrix}\hat{0} & \hat{0}\\ \hat{1} & \hat{0}\end{smallmatrix}\right) \in \hat{K}_\lambda$. This follows by recalling (2.4) and the explicit form (3.3), which implies that $\left(\begin{smallmatrix}\hat{0} & \hat{0}\\ \hat{1} & \hat{0}\end{smallmatrix}\right) \leqslant \varphi^*\rho_{|K_\lambda}$, and since $\varphi^*\rho \leqslant \mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho)_{|P_{\mu\hookrightarrow\lambda}}$, we have $\left(\begin{smallmatrix}\hat{0} & \hat{0}\\ \hat{1} & \hat{0}\end{smallmatrix}\right) \leqslant \mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho)_{|K_\lambda}$. Using Proposition 7.3 and (a), we conclude that $\mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho)$ is irreducible.

To prove (c), observe that $\rho \leqslant \mathbf{r}^{\lambda}_{\mu,\hookrightarrow}\mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho)$ by adjointness of $\mathbf{r}_{\hookrightarrow}$ and $\mathbf{i}_{\hookrightarrow}$. The assertion would then follow once we show that $\dim\rho = \dim \mathbf{r}^{\lambda}_{\mu,\hookrightarrow}\mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho)$. To show that we apply the functors

$$\mathbf{r}^{(\ell_1,\ell_2)}_{(\ell_1,\ell_2-1),\hookrightarrow}, \mathbf{r}^{(\ell_1,\ell_2-1)}_{(\ell_1,\ell_2-2),\hookrightarrow}, \ldots, \mathbf{r}^{(\ell_1,m_2+1)}_{(\ell_1,m_2),\hookrightarrow}$$

successively to $\mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\rho)$. Observe that for any $\mu = (\ell_1, m) \leqslant \mu' = (\ell_1, m') \in \mathcal{I}_\lambda$, and any submodule $M_\mu \subset M_\lambda$ of type μ, there is a unique module $M_{\mu'} \supset M_\mu$ of type μ', namely $M_{\mu'} = M_\mu + M_\lambda[\pi^{m'}]$. We can therefore use Claim 7.2 and write

$$\mathbf{r}^{(\ell_1,m_2+1)}_{(\ell_1,m_2),\hookrightarrow} \circ \cdots \circ \mathbf{r}^{(\ell_1,\ell_2)}_{(\ell_1,\ell_2-1),\hookrightarrow} = \mathbf{r}^{\lambda}_{\mu,\hookrightarrow}.$$

As we know by the proof of Proposition 7.3 how the dimension drops after each successive application of the functors $\mathbf{r}_{\hookrightarrow}$, the assertion follows. □

7.3. Geometric induction in rank two

Let $\theta = (\theta_1, \theta_2)$ be a character of $G_{(\ell_1)} \times G_{(\ell_2)} \simeq \mathfrak{o}^\times_{\ell_1} \times \mathfrak{o}^\times_{\ell_2}$. In this subsection we shall study the representations

$$\xi_\theta = \mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_1),(\ell_2)}(\theta) = \operatorname{Ind}^{G_{(\ell_1,\ell_2)}}_{P_{(\ell_1),(\ell_1,\ell_2)}}(\iota^*\theta),$$

$$\xi^\vee_\theta = \mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_2),(\ell_1)}(\theta) = \operatorname{Ind}^{G_{(\ell_1,\ell_2)}}_{P_{(\ell_2),(\ell_1,\ell_2)}}(\iota^*\theta)$$

(recall the notations (3.5), (3.6) and (2.1)). The representations ξ_θ and $\xi_\theta^\vee$ are of dimension

$$[G_{(\ell_1,\ell_2)} : P_{(\ell_1),(\ell_1,\ell_2)}] = [G_{(\ell_1,\ell_2)} : P_{(\ell_2),(\ell_1,\ell_2)}] = \begin{cases} q^{\ell_2}, & \text{if } \ell_1 > \ell_2; \\ q^{\ell-1}(q+1), & \text{if } \ell_1 = \ell_2 = \ell. \end{cases} \tag{7.1}$$

As it happens for the groups G_{1^n}, geometric induction might be reducible or irreducible. As it turns out, the characters which induce irreducibly are

$$\hat{C}_{(\ell_1),(\ell_2)} := \{\theta = (\theta_i, \theta_{ii}) \mid \theta(u, u^{-1}) = 1,\ \forall u \in 1 + \mathfrak{p}^{\ell_1 - 1}\} \subset \hat{G}_{(\ell_1)} \times \hat{G}_{(\ell_2)}. \tag{7.2}$$

For $\ell_1 > \ell_2$ this means that θ_i does not factor through $\mathfrak{o}_{\ell_1-1}^\times$ (as it is non-trivial on $1 + \mathfrak{p}_{\ell_1}^{\ell_1-1}$), and if $\ell_1 = \ell_2 = \ell$ it means that $\theta_{i|_{1+\mathfrak{p}^{\ell-1}}} \neq \theta_{ii|_{1+\mathfrak{p}^{\ell-1}}}$. Let $p_{\ell,m} : \mathfrak{o}_\ell \to \mathfrak{o}_m$ denote the reduction map. The following claim connects the infinitesimal and geometric inductions.

Claim 7.5. *For $1 \leqslant m < \ell_2$ the following equalities hold*

$$\mathbf{i}_{(\ell_1),(\ell_2)}^{(\ell_1,\ell_2)} \circ (\mathrm{id} \boxtimes p_{\ell_2,m}^*) = \mathbf{i}_{(\ell_1,m),\hookrightarrow}^{(\ell_1,\ell_2)} \circ \mathbf{i}_{(\ell_1),(m)}^{(\ell_1,m)},$$
$$\mathbf{i}_{(\ell_2),(\ell_1)}^{(\ell_1,\ell_2)} \circ (\mathrm{id} \boxtimes p_{\ell_2,m}^*) = \mathbf{i}_{(\ell_1,m),\twoheadrightarrow}^{(\ell_1,\ell_2)} \circ \mathbf{i}_{(m),(\ell_1)}^{(\ell_1,m)}.$$

Proof. We argue for the first equality, the second follows along the same lines. Unraveling definitions (e.g. (2.1), (3.3) and (3.5)) we have that

$$\mathbf{i}_{(\ell_1),(\ell_2)}^{(\ell_1,\ell_2)} \circ (\mathrm{id} \boxtimes p_{\ell_2,m}^*) = \mathrm{Ind}_{P_2}^{G} \circ \mathrm{Inf}_{P_2/U_2}^{P_2}, \quad \text{and}$$
$$\mathbf{i}_{(\ell_1,m),\hookrightarrow}^{(\ell_1,\ell_2)} \circ \mathbf{i}_{(\ell_1),(m)}^{(\ell_1,m)} = \mathrm{Ind}_{P_1}^{G} \circ \mathrm{Inf}_{P_1/U_1}^{P_1} \circ \mathrm{Ind}_{P_2/U_1}^{P_1/U_1} \circ \mathrm{Inf}_{P_2/U_2}^{P_2/U_1},$$

where $P_2 = P_{(\ell_1),(\ell_1,\ell_2)} \subset P_1 = P_{(\ell_1,m),(\ell_1,\ell_2)} \subset G = G_{(\ell_1,\ell_2)}$, and

$$U_1 = \begin{bmatrix} 1 & \mathfrak{p}_{\ell_1}^{\ell_1-\ell_2+m} \\ & 1+\mathfrak{p}_{\ell_2}^m \end{bmatrix} \subset U_2 = \begin{bmatrix} 1 & \mathfrak{p}_{\ell_1}^{\ell_1-\ell_2} \\ & 1+\mathfrak{p}_{\ell_2}^m \end{bmatrix} \subset P_2.$$

Since U_j is normal in P_j, we can use Lemma 7.1 and get the desired equality. □

7.3.1. Irreducible geometrically induced representations

The following proposition is analogous to Proposition 7.3.

Proposition 7.6. *Let $\lambda = (\ell_1, \ell_2) \in \Lambda_2$ and let ρ be an irreducible representation of G_λ.*

(1) *If $\ell_1 > \ell_2 > 1$ and $\rho_{|K_\lambda} \geqslant \left(\begin{smallmatrix} \hat{u} & \hat{0} \\ \hat{0} & \hat{z} \end{smallmatrix}\right)$ with $\hat{u} \in \mathfrak{o}_1^\times$, then* $\dim(\rho) = q^{\ell_2}$.

(2) *If $\ell_1 = \ell_2 = \ell > 1$ and $\rho_{|K_{\ell^2}} \geqslant \left(\begin{smallmatrix} \hat{u} & \hat{0} \\ \hat{0} & \hat{z} \end{smallmatrix}\right)$ with $\hat{u} \neq \hat{z}$, then* $\dim(\rho) = q^{\ell-1}(q+1)$.

Proof. (1) We argue by induction on ℓ_2. Let ρ be an irreducible representation of G_λ which lies above the orbit of $\left(\begin{smallmatrix} \hat{u} & \hat{0} \\ \hat{0} & \hat{z} \end{smallmatrix}\right) \in \hat{K}_{(\ell_1,\ell_2)}$. By twisting with a one-dimensional character we may assume that $\hat{z} = 0$. Using the action (5.1) we verify that this orbit contains the q^2 characters $\{\left(\begin{smallmatrix} \hat{u} & c\hat{u} \\ b\hat{u} & bc\hat{u} \end{smallmatrix}\right) \mid b, c \in \mathfrak{o}_1\}$, out of which q survive the application of the functor $\mathbf{r}_{(\ell_1,\ell_2-1),\hookrightarrow}^{(\ell_1,\ell_2)}$, namely those having $c=0$. It follows that if $\xi=\mathbf{r}^{(\ell_1,\ell_2)}_{(\ell_1,\ell_2-1),\hookrightarrow}(\rho)$ then $\dim(\xi)=\dim(\rho)/q$. Using the explicit form (3.3) we observe that $\langle \xi_{|V_1},\hat{0}_{V_1}\rangle=0$, which means that $\xi_{|K_{(\ell_1,\ell_2-1)}}\geqslant\left(\begin{smallmatrix}\hat{u} & \hat{0}\\ \hat{0} & \hat{z}'\end{smallmatrix}\right)$ with the same $\hat{u}\neq 0$ and some $\hat{z}'\in\mathfrak{o}_1$. By the induction hypothesis it follows that ξ is irreducible and that $\dim(\xi)=q^{\ell_2-1}$, hence the assertion follows. The basis of the induction is $\ell_2=2$ in which case the role of K is taken over by H of Section 4 and ξ lies above the orbits q-dimensional representations of the Heisenberg group.

(2) Assume $\rho\in\hat{G}_{\ell^2}$ such that ρ lies above the orbit $[\left(\begin{smallmatrix}\hat{u} & \hat{0}\\ \hat{0} & \hat{z}\end{smallmatrix}\right)]$ with $\hat{u}\neq\hat{z}$. By twisting with a one-dimensional character we may assume that $\hat{z}=0$ and that $\hat{u}\neq 0$. This orbit contains the (q^2-1) characters

$$\begin{pmatrix}\frac{ad}{ad-bc}\hat{u} & \frac{ac}{ad-bc}\hat{u}\\ \frac{bd}{ad-bc}\hat{u} & \frac{-bc}{ad-bc}\hat{u}\end{pmatrix},\qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{GL}_2(\mathfrak{o}_1)$$

(using the action 5.2), out of which $(q-1)$ survive the application of the functor $\mathbf{r}^{(\ell,\ell)}_{(\ell,\ell-1),\hookrightarrow}$, namely those having $c=0$. It follows that if $\xi=\mathbf{r}^{(\ell,\ell)}_{(\ell,\ell-1),\hookrightarrow}(\rho)$ then $\dim(\xi)=\dim(\rho)/(q+1)$. Observing that $\xi_{|K_{(\ell,\ell-1)}}\geqslant\left(\begin{smallmatrix}\hat{u} & \hat{0}\\ \hat{0} & \hat{z}\end{smallmatrix}\right)$ with $\hat{u}\neq 0$ we can use part (1) and the result follows. □

Theorem 7.7. *For any $\theta\in\hat{C}_{(\ell_1),(\ell_2)}$ the geometric inductions ξ_θ and $\xi_\theta^\vee$ are equivalent and irreducible. If $\ell_1>\ell_2$ then θ determines ξ_θ completely. If $\ell_1=\ell_2$ then $\xi_\theta\simeq\xi_{\theta'}$ if and only if $\theta=\theta'$ or $\theta^{\mathrm{op}}=\theta'$.*

Proof. Since $\theta_{|V_1V_2}\leqslant\xi_{\theta|V_1V_2}$ and $\theta_{|V_1V_2}\leqslant\xi^\vee_{\theta|V_1V_2}$, the representations ξ_θ and $\xi_\theta^\vee$ contain the orbits of Proposition 7.6 and by dimension counting must be irreducible. To prove the equivalence of ξ_θ and $\xi_\theta^\vee$ we shall use the identity

$$\xi_\theta=\mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_1),(\ell_2)}(\theta_i,\theta_{ii})=\det{}^*\theta_2\otimes\mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_1),(\ell_2)}\big(\theta_{ii}^{-1}\theta_i,1\big)$$

(and analogously for $\xi_\theta^\vee$), where we abused notation and wrote θ_{ii}^{-1} instead of $p^*_{\ell_1,\ell_2}(\theta_{ii}^{-1})$. Note that by (7.2), the character $\theta_{ii}^{-1}\theta_i$ is non-trivial on $1+\mathfrak{p}_{\ell_1}^{\ell_1-1}$, hence we are reduced to the case $\theta_{ii}=1$. In such case we also have

$$\mathbf{r}^{(\ell_1,\ell_2)}_{(\ell_1,1),\hookrightarrow}(\xi_\theta)=\mathbf{r}^{(\ell_1,\ell_2)}_{(\ell_1,1),\twoheadrightarrow}(\xi_\theta)=\mathbf{r}^{(\ell_1,\ell_2)}_{(\ell_1,1),\hookrightarrow}\big(\xi_\theta^\vee\big)=\mathbf{r}^{(\ell_1,\ell_2)}_{(\ell_1,1),\twoheadrightarrow}\big(\xi_\theta^\vee\big),$$

since all these representations are q-dimensional irreducible representations of $G_{(\ell_1,1)}$ with the same central character. In particular, using Claim 7.5 we get

$$\big\langle\xi_\theta,\xi_\theta^\vee\big\rangle=\big\langle\mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_1,1),\hookrightarrow}\mathbf{i}^{(\ell_1,1)}_{(\ell_1),(1)}(\theta),\xi_\theta^\vee\big\rangle=\big\langle\mathbf{i}^{(\ell_1,1)}_{(\ell_1),(1)}(\theta),\mathbf{r}^{(\ell_1,\ell_2)}_{(\ell_1,1),\hookrightarrow}\big(\xi_\theta^\vee\big)\big\rangle=1.$$

Finally, for $\theta_1,\theta_2\in\hat{C}_{(\ell_1),(\ell_2)}$ let $\left(\begin{smallmatrix}\hat{u}_j & \hat{0}\\ \hat{0} & \hat{z}_j\end{smallmatrix}\right)$ denote the representative of $\xi_{\theta_j|K_\lambda}$ $(j=1,2)$. We should prove that

$$\xi_{\theta_1}\simeq\xi_{\theta_2}\quad\Longleftrightarrow\quad\begin{cases}\theta_1=\theta_2, & \text{if }\lambda\text{ is non-rectangular;}\\ \theta_1=\theta_2\text{ or }\theta_1=\theta_2^{\mathrm{op}}, & \text{if }\lambda\text{ is rectangular.}\end{cases}$$

We argue by induction on ℓ_2. If λ is non-rectangular, then $\langle \xi_{\theta_1}, \xi_{\theta_2} \rangle = 0$ unless $(\hat{u}_1, \hat{z}_1) = (\hat{u}_2, \hat{z}_2)$. If the latter holds then we can write $\xi_{\theta_j} = \det^* \chi_{\hat{z}} \otimes \xi_{\theta'_j}$ with $\theta'_j = (\chi_{\hat{z}}^{-1}, \chi_{\hat{z}}^{-1}) \cdot \theta_j = (\mathrm{id} \boxtimes p^*_{\ell_2, \ell_2 - 1})(\theta''_j)$ for some $\theta''_j \in \hat{C}_{(\ell_1),(\ell_2-1)}$. Using Claim 7.5 we now have $\xi_{\theta'_j} = \mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_1,\ell_2-1),\hookrightarrow}(\xi_{\theta''_j})$, and by the induction hypothesis $\langle \xi_{\theta''_1}, \xi_{\theta''_2} \rangle = \langle \theta''_1, \theta''_2 \rangle$, which in turn implies $\theta_1 = \theta_2$. If λ is rectangular then $\langle \xi_{\theta_1}, \xi_{\theta_2} \rangle = 0$ unless either $(\hat{u}_1, \hat{z}_1) = (\hat{u}_2, \hat{z}_2)$ or $(\hat{u}_1, \hat{z}_1) = (\hat{z}_2, \hat{u}_2)$. If the former holds we proceed as in the non-rectangular case. If the latter holds, replacing θ_2 with θ_2^{op} brings us back to the first case. The replacement does not change the representation since the representations ξ_θ and $\xi^\vee_{\theta^{\mathrm{op}}}$ are equal, being intertwined by the Weyl element. The basis for the induction is $\ell_2 = 1$ which holds by the remarks above. $\square$

7.3.2. Reducible geometrically induced representations

The last ingredient of the induced representations is geometric induction which is reducible. Let ρ be one of the $2q^{\ell_1 - 2}(q-1)\mathcal{D}^{q-1} \subset \hat{G}_{(\ell_1,1)}$ irreducible representations which lie above the orbits $B_\pm$ of Section 4. Let

$$\xi_\rho = \begin{cases} \mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_1,1),\hookrightarrow}(\rho), & \text{if } \rho \text{ lies above } \mathrm{B}_+; \\ \mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_1,1),\twoheadrightarrow}(\rho), & \text{if } \rho \text{ lies above } \mathrm{B}_-. \end{cases}$$

Proposition 7.8. *The representations ξ_ρ are irreducible of dimension ι_λ, distinct, and contained in geometrically induced representations.*

Proof. The same arguments given in Theorem 7.4 apply to ξ_ρ and imply that $\dim \xi_\rho = \iota_\lambda$ and that it is irreducible. Applying the functor $\mathbf{r}^{(\ell_1,1)}_{(\ell_1),(1)}$ to ρ, if ρ lies above B_+, or the functor $\mathbf{r}^{(\ell_1,1)}_{(1),(\ell_1)}$ if ρ lies above B_-, gives a non-zero representation of $G_{(\ell_1)} \times G_{(1)}$ which proves that ρ is contained in a geometrically induced representation, say

$$\rho \leqslant \begin{cases} \mathbf{i}^{(\ell_1,1)}_{(\ell_1),(1)}(\theta), & \text{if } \rho \text{ lies above } \mathrm{B}_+; \\ \mathbf{i}^{(\ell_1,1)}_{(1),(\ell_1)}(\theta), & \text{if } \rho \text{ lies above } \mathrm{B}_-, \end{cases} \qquad \theta \in \hat{G}_{(\ell_1)} \times \hat{G}_{(\ell_2)} \setminus \hat{C}_{(\ell_1),(\ell_2)}.$$

This means that

$$\xi_\rho \leqslant \begin{cases} \mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_1,1),\hookrightarrow} \circ \mathbf{i}^{(\ell_1,1)}_{(\ell_1),(1)}(\theta), & \text{if } \rho \text{ lies above } \mathrm{B}_+; \\ \mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_1,1),\twoheadrightarrow} \circ \mathbf{i}^{(\ell_1,1)}_{(1),(\ell_1)}(\theta), & \text{if } \rho \text{ lies above } \mathrm{B}_-, \end{cases}$$

and the proposition now follows from Claim 7.5. $\square$

8. Classification of irreducible representations

We are now in a position to integrate all the results obtained so far and give a complete classification of the irreducible representations of G_λ ($\lambda \in \Lambda_2$).

Table 3
Primitive representations of G_λ

Type	Dimension	Number
Cuspidal	$q^{\ell_2-1}(q-1)$	$q^{\ell_1+\ell_2-3}(q-1)^2$
$\mathbf{i}^{\lambda}_{\mu,\hookrightarrow}(\hat{C}_\mu),\ \mu\in\{(\ell_1,m)\}_{m=1}^{\ell_2-1}$	$q^{\ell_2-1}(q-1)$	$q^{\ell_1+m-3}(q-1)^2$
$\mathbf{i}^{\lambda}_{\mu,\twoheadrightarrow}(\hat{C}_\mu),\ \mu\in\{(\ell_1,m)\}_{m=1}^{\ell_2-1}$	$q^{\ell_2-1}(q-1)$	$q^{\ell_1+m-3}(q-1)^2$
$\mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_1),(\ell_2)}(\hat{C}_{(\ell_1),(\ell_2)})$	q^{ℓ_2}	$q^{\ell_1+\ell_2-3}(q-1)^3$
$\xi_\rho<\mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_1),(\ell_2)}(\hat{G}_{(\ell_1)}\times\hat{G}_{(\ell_2)}\smallsetminus\hat{C}_{(\ell_1),(\ell_2)})$	$q^{\ell_2-1}(q-1)$	$q^{\ell_1-2}(q-1)$
$\xi_\rho<\mathbf{i}^{(\ell_1,\ell_2)}_{(\ell_2),(\ell_1)}(\hat{G}_{(\ell_1)}\times\hat{G}_{(\ell_2)}\smallsetminus\hat{C}_{(\ell_1),(\ell_2)})$	$q^{\ell_2-1}(q-1)$	$q^{\ell_1-2}(q-1)$

8.1. Non-rectangular case

Let $\lambda=(\ell_1,\ell_2)$ with $\ell_1>\ell_2>1$. The primitive irreducibles representations of G_λ which were obtained in Sections 6 and 7 are listed in Table 3.

The following theorem ties up all the loose ends we have left concerning the groups G_λ for λ non-rectangular and completes the classification of the irreducible representations of G_λ.

Theorem 8.1. *The representations in Table* 3 *exhaust all the primitive irreducible representations of G_λ ($\lambda\in\Lambda_2$ non-rectangular). In particular, Theorem* 1.4 *part* (2) *and Theorem* 1.2 *for λ non-rectangular hold.*

Proof. The families to which the representations are divided are disjoint, and the result follows by checking that we have accumulated enough representations:

$$|G_\lambda|-q|G_{\flat\lambda}|=\sum \text{Number}\cdot(\text{Dimension})^2=q^{\ell_1+3\ell_2-5}\big(q^3-1\big)(q-1)^2,$$

where the sum runs over all the representations in Table 3. □

8.2. Rectangular case

In the case of the group G_{ℓ^2}, the classification is different in two conceptual points and of course in the numerics. First, the images of infinitesimal functors $\mathbf{i}^{\ell^2}_{\mu,\hookrightarrow}$ and $\mathbf{i}^{\ell^2}_{\mu,\twoheadrightarrow}$ coincide. Second, the infinitesimal induction from cuspidal representations of G_μ ($\mu\in\mathcal{I}_{\ell^2}$) should be further twisted in order to cover all the orbits, as only the orbit with $\hat{u}=0$ in (iii) of Table 2 survives after an application of infinitesimal restriction functors.

Theorem 8.2. *The representations in Table* 4 *exhaust all the primitive irreducible representations of G_{ℓ^2}. In particular, Theorem* 1.4 *part* (3) *and Theorem* 1.2 *for λ rectangular hold.*

Proof. The families to which the representations are divided are disjoint, and the result follows by checking that we have accumulated enough representations:

$$|G_{\ell^2}|-q|G_{\flat\ell^2}|=\sum \text{Number}\cdot(\text{Dimension})^2=q^{4\ell-7}\big(q^3-1\big)\big(q^2-q\big)\big(q^2-1\big),$$

where the sum runs over all the representations in Table 4. □

Table 4
Primitive representations of $G_{\ell 2}$

Type	Dimension	Number
Cuspidal	$q^{\ell-1}(q-1)$	$\frac{1}{2}(q-1)(q^2-1)q^{2\ell-3}$
$q \cdot \mathbf{i}_{\mu}^{\ell^2}(\hat{C}_{\mu}),\ \mu \in \{(\ell, m)\}_{m=1}^{\ell-1}$	$q^{\ell-2}(q^2-1)$	$q^{\ell+m-2}(q-1)^2$
$\mathbf{i}_{(\ell),(\ell)}^{(\ell,\ell)}(\hat{C}_{(\ell),(\ell)})$	$q^{\ell-1}(q+1)$	$\frac{1}{2}q^{2\ell-3}(q-1)^3$
$\rho < \mathbf{i}_{(\ell),(\ell)}^{(\ell,\ell)}(\hat{G}_{(\ell)} \times \hat{G}_{(\ell)} \smallsetminus \hat{C}_{(\ell),(\ell)})$	$q^{\ell-2}(q^2-1)$	$q^{\ell-1}(q-1)$

To make a link with the literature we remark that the representations of type (ii) were constructed in [22] where they are called *la série déployée*. The cuspidal representations were constructed there as well and are called *la série non ramifiée*. The infinitesimally induced representations turn out to be the missing part in the description of the irreducible representations of G_{ℓ^2}, and was the prime motivation behind the present study.

It is perhaps worthwhile mentioning that though a complete and unified description of the rectangular case seems to be missing in the literature, for this particular case all the necessary ingredients are handy and one could complete the classification, see a recent preprint by Stasinski [31].

Another remark which is in order regards even characteristic. In [23,24] a separate discussion is devoted to the even characteristic case, and even that only for $\mathrm{GL}_2(\mathbb{Z}_2)$. It is important to note that the difference lies in the method and not in the objects under study. This has been just shown to hold in general: for any G_λ ($\lambda \in \Lambda_2$), and over any $\mathfrak{o}$, the cardinality of the residue field $\mathfrak{o}/\mathfrak{p}$ appears only as a parameter and not in any essential way.

9. Conjugacy classes

Recall that for any finite $\mathfrak{o}$-module M, an element $f \in \mathrm{End}_{\mathfrak{o}}(M)$ is called cyclic if there exist $m \in M$ such that $m, f \cdot m, f^2 \cdot m, \ldots, f^j \cdot m$ span M over $\mathfrak{o}$ for some $j \in \mathbb{N}$.

Definition 9.1. An element $f \in \mathrm{End}_{\mathfrak{o}}(M)$ is called *almost cyclic* if it can be written as

$$f = aI + \pi^i h$$

with $a \in \mathfrak{o}$, i a non-negative integer and $h \in \mathrm{End}_{\mathfrak{o}}(M)$ a cyclic element with $\pi^i h \neq 0$.

The classification of conjugacy classes of G_λ ($\lambda \in \Lambda_2$) is greatly simplified by the following dichotomy:

Lemma 9.2. *Any element in $E_\lambda = \mathrm{End}_{\mathfrak{o}}(M_\lambda)$ ($\lambda \in \Lambda_2$) is either scalar or almost cyclic.*

Proof. Let $f \in E_\lambda$ be non-scalar. Then $f = aI + \pi^i h$ for some non-negative integer i, $a \in \mathfrak{o}$, and $h \in E_\lambda$ such that $\pi^i h \neq 0$ and $\bar{h} \in \mathrm{End}_{\mathfrak{o}}(M_\lambda/\pi M_\lambda)$ is non-scalar. This implies that $\bar{h}$ is cyclic, since $M_\lambda/\pi M_\lambda \simeq \mathfrak{o}_1^2$, and any element in $\mathrm{End}_{\mathfrak{o}}(\mathfrak{o}_1^2)$ is either scalar or cyclic. By Nakayama's lemma h must be cyclic since it is a lift of a cyclic element. $\square$

We shall now build the conjugacy classes of G_λ inductively. The inverse image $p^{-1}(C)$ of a conjugacy class $C \subset G_{\flat\lambda}$ with respect to the reduction map $p: G_\lambda \to G_{\flat\lambda}$ is a disjoint union of conjugacy classes in G_λ. An almost cyclic class has precisely q^2 classes above it while the classes in G_λ which lie above a scalar class $aI \in G_{\flat\lambda}$ are parameterized by the G_λ-orbits in $p^{-1}(aI) = aI \cdot K_\lambda \subset G_\lambda$ under conjugation. This is given by

$$\begin{aligned} &\#\text{Orbits of } G_{\ell^2} \text{ on } K_{\ell^2}: \quad q^2+q, \quad \ell > 1, \\ &\#\text{Orbits of } G_\lambda \text{ on } K_\lambda: \quad q^2+q+1, \quad \ell_1 > \ell_2 > 1 \end{aligned}$$

where the first equality is simply the number of similarity classes in $K_{\ell^2} \simeq M_2(\mathfrak{o}_1)$: q scalar and q^2 cyclic. The second equality follows by recalling the action (1) in the special case $K_\lambda = K_\lambda^{1,0} \simeq M_2(\mathfrak{o}_1)$. Explicitly the orbits are represented in this case by

$$\left\{ \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, u \in \mathfrak{o}_1 \right\}, \qquad \left\{ \begin{pmatrix} 0 & v \\ 1 & z \end{pmatrix} \,\middle|\, v, z \in \mathfrak{o}_1 \right\}, \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

With this in hand we can prove the following.

Theorem 9.3. *For any $\lambda \in \Lambda_2$ the number of conjugacy classes in G_λ is*

$$\left|\mathrm{Conj}(G_\lambda)\right| = \begin{cases} q^{2\ell} - q^{\ell-1}, & \text{if } \ell_1 = \ell_2 = \ell; \\ q^{\ell_1+\ell_2-2}(q^2-q+2) - q^{\ell_1-2}(q+1), & \text{if } \ell_1 > \ell_2. \end{cases} \tag{9.1}$$

Proof. We argue by induction on the type. The basis for the induction is given by $G_{(\ell,1)}$ ($\ell \geqslant 1$):

$$\begin{aligned} \mathrm{Conj}(G_{(1,1)}) &\simeq \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \,\middle|\, u \in \mathfrak{o}_1^\times \right\} \sqcup \left\{ \begin{pmatrix} 0 & v \\ 1 & z \end{pmatrix} \,\middle|\, v \in \mathfrak{o}_1,\ z \in \mathfrak{o}_1^\times \right\}, \\ \mathrm{Conj}(G_{(\ell,1)}) &\simeq \left\{ \begin{pmatrix} u & 0 \\ 0 & z \end{pmatrix} \,\middle|\, u, z \in \mathfrak{o}_\ell^\times \right\} \sqcup \left\{ \begin{pmatrix} u & \pi^{\ell-1} \\ 0 & u \end{pmatrix} \,\middle|\, u \in \mathfrak{o}_{\ell-1}^\times \right\} \\ &\quad \sqcup \left\{ \begin{pmatrix} u & w \\ 1 & u \end{pmatrix} \,\middle|\, u \in \mathfrak{o}_{\ell-1}^\times,\ w \in \mathfrak{o}_1 \right\}, \quad \text{if } \ell > 1. \end{aligned}$$

In particular, $|\mathrm{Conj}(G_{(1,1)})| = q^2 - 1$ and $|\mathrm{Conj}(G_{(\ell,1)})| = q^3 - q^2 + q - 1$ for $\ell > 1$, which establishes the basis for the induction. Assuming (9.1) for λ we shall prove it for $\sharp\lambda = (\ell_1 + 1, \ell_2 + 1)$. Indeed, $\mathrm{Conj}(G_\lambda)$ contains $|\mathfrak{o}_\ell^\times|$ scalar elements and the rest are almost cyclic. Lifting them to $G_{\sharp\lambda}$ gives

$$\begin{aligned} \left|\mathrm{Conj}(G_{\sharp\lambda})\right| &= \left(q^2+q\right)\left|\mathfrak{o}_\ell^\times\right| + q^2\left(\left|\mathrm{Conj}(G_\lambda)\right| - \left|\mathfrak{o}_\ell^\times\right|\right) \\ &= \left(q^2+q\right)q^{\ell-1}(q-1) + q^2\left[\left(q^{2\ell} - q^{\ell-1}\right) - q^{\ell-1}(q-1)\right] \\ &= q^{2\ell+2} - q^\ell \end{aligned}$$

if $\lambda = \ell^2$, and

$$\begin{aligned} \left|\mathrm{Conj}(G_{\sharp\lambda})\right| &= \left(q^2+q+1\right)\left|\mathfrak{o}_{\ell_1}^\times\right| + q^2\left(\left|\mathrm{Conj}(G_\lambda)\right| - \left|\mathfrak{o}_{\ell_1}^\times\right|\right) \\ &= q^{\ell_1+\ell_2}\left(q^2-q+2\right) - q^{\ell_1-1}(q+1) \end{aligned}$$

if $\ell_1 > \ell_2 > 1$. $\square$

The result of course matches the polynomials defined by the representations (see Theorem 1.4), that is $R_\lambda(1) = |\mathrm{Conj}(G_\lambda)|$.

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