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Note

Graphs and digraphs with given girth and connectivity

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Abstract

In this paper, we show that for any given two positive integers g and k with $g \geq 3$, there exists a graph (digraph) G with girth g and connectivity k. Applying this result, we give a negative answer to the problem proposed by M. Junger, G. Reinelt and W.R Pulleyblank (1985).

1. Introduction

We consider both graphs and digraphs. Loops and multiple edges are not allowed. Let G be a graph (or digraph) with vertex set $V(G)$ and edge set $E(G)(D(G))$. The *order* of G is the number of vertices of G. The *girth* of a graph (digraph) G is the length of the smallest (directed) cycle in G. A graph (digraph) G is said to be *k-vertex connected* if for every pair of two vertices u and v there are at least k vertex-disjoint (directed) paths from u to v, the smallest such k is called the *vertex connectivity*, denoted by $\kappa(G)$. A graph (digraph) G is said to be *k-edge connected* if for every pair of two vertices u and v there are at least k edge disjoint (directed) paths from u to v, the smallest such k is called the *edge connectivity*, denoted by $\lambda(G)$. For the definitions and terminologies not mentioned here we refer to [2].

The existence and construction of graphs with given parameters are extensively studied. We mention two examples here. There exists a (k, g)-cage [7] which is the minimal graph with girth g and regularity k; there also exists a regular graph with given girth pair [3]. In this note, we propose the following problem.

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For any given positive integers g and κ with $g \geqslant 3$, does there exist a graph (digraph) G with girth g and vertex (edge) connectivity $\kappa(\lambda)$?

In Section 2, we give an existence theorem of graphs with arbitrary girth and connectivity. As a byproduct, we disprove a problem proposed by Junger et al. [4]. In Section 3, we construct circulant digraphs of arbitrary girth and connectivity.

2. The existence theorem for undirected graphs

To prove the existence theorem, we need two lemmas.

Lemma 2.1. (Mader [5]). *If G is a graph with minimum degree at least $4d$, then G contains a d-vertex (edge) connected subgraph.*

Lemma 2.2. (Tutte [7]). *Let $k \geqslant 3, g \geqslant 3$ and*

$$n=(k-1)\left[\frac{(k-1)^{g-1}-1}{k-2}+\frac{(k-1)^{g-2}-1}{k-2}+1\right].$$

Then there exists a graph G such that G is k-regular with girth g and order n.

Now we can give our existence theorem as follows.

Theorem 2.3. *For any given positive integer g and κ with $g \geqslant 3$, there is a graph G with girth g and vertex (edge) connectivity $\kappa(\lambda)$.*

Proof. Let $k=4\kappa$. By Lemma 2.2, there is a k-regular graph H with girth $2g$ and order n. Therefore, H contains an κ vertex (edge) connected subgraph H_1 by Lemma 2.1. It is obvious that the girth of H_1 is at least $2g$. Let H_2 be the graph obtained by adding one edge to H_1 such that the girth of H_2 is g, and let C be the resulting cycle of length g. The connectivity of H_2 is at least κ. Deleting one edge will reduce the vertex (or edge) connectivity by at most one. By deleting edges of H_2 not in the cycle C if necessary, we will finally obtain a graph G with girth g and vertex (or edge) connectivity κ. □

For a positive integer s, an *s-partition* of a graph G was defined by Junger et al. in [4]. It is a partition of $E(G)$ into $E_1, E_2, \ldots, E_k$ such that $|E_i|=s$ for $1 \leqslant i \leqslant k-1$ and $1 \leqslant |E_k| \leqslant s$, and each E_i $(1 \leqslant i \leqslant k)$ induces a connected subgraph of G. They proved a very interesting result. That is, if G is 4-edge connected, then there exists an s-partition for all s. Based on this result they proposed the following problem (problem 4 in [4]).

What edge connectivity can ensure the existence of s-partition of graphs such that each part is 2-edge connected?

We give this problem a negative answer.

Theorem 2.4. *For given integer s ($s \geqslant 3$) and any integer λ, there exists a graph G with edge connectivity λ, G has no s-partition such that each part is 2-edge connected.*

Proof. By Theorem 2.3, there is a graph G with girth $s+1$ and edge connectivity λ. For any s-partition of G, each part of the partition does not contain a cycle, since the smallest cycle of G is of length $s+1$. Therefore, each part of the partition is not 2-edge connected. □

Remark. The referee noticed that the conjecture of Junger et al. can be disproved by applying the result of [6].

3. Circulant digraphs with arbitrary girth and connectivity

The *circulant digraph* $D(n, S)$, where S is a subset of $\{1, 2, \ldots, n-1\}$, has vertices $0, 1, 2, \ldots, n-1$ and directed edges from i to every $i+s$ ($s \in S$) for each vertex i.

Lemma 3.1 (Ayoub and Frisch [1]). *The circulant digraph $D(n, S)$ has vertex (edge) connectivity $|S|$, if $S=\{1, 2, \ldots, s-1, s\}$.*

Lemma 3.2. *The circulant digraph $D(n, S)$ has girth g, if $S=\{1, 2, \ldots, s\}$ and $n=gs$.*

Proof. Any directed cycle of length k has vertices $a_0, a_0+a_1, a_0+a_1+a_2, \ldots, a_0+a_1+a_2+\cdots+a_k \equiv a_0 \pmod n$, where $a_i \in S$ and $a_1+a_2+\cdots+a_k \equiv 0 \pmod n$. Let $b=\max\{a_1, \ldots, a_k\}$. Then $n \leqslant a_1+\cdots+a_k \leqslant bk \leqslant sk$. Therefore, $k \geqslant n/s$. This means that any directed cycle must have length at least n/s. Moreover, $0, s, 2s, \ldots, (g-1)s$ is a directed cycle of length $g=n/s$. Hence the girth of $D(n, S)$ is $g=n/s$. □

Theorem 3.3. *For any two positive integers κ and g, we can always construct a circulant digraph with vertex (edge) connectivity κ and girth g.*

Proof. Let $S=\{1, 2, \ldots, \kappa\}$ and $n=g\kappa$. Then the circulant digraph $D(n, S)$ has the vertex (edge) connectivity κ by Lemma 3.1 and the girth g by Lemma 3.2. □

This theorem answered our problem of digraph case.

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