Approximating subset k-connectivity problems

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A B S T R A C T

A subset $T \subseteq V$ of terminals is k-connected to a root s in a directed/undirected graph J if J has k internally-disjoint vs-paths for every $v \in T$; T is k-connected in J if for every $s \in T$ the set $T \setminus \{s\}$ is k-connected to s in J. We consider the Subset k-Connectivity Augmentation problem: given a graph $G = (V, E)$ with edge/node-costs, a node subset $T \subseteq V$, and a subgraph $J = (V, E_J)$ of G such that T is $(k-1)$-connected in J, find a minimum-cost augmenting edge-set $F \subseteq E \setminus E_J$ such that T is k-connected in $J \cup F$. The problem admits trivial ratio $O(|T|^2)$. We consider the case $|T| > k$ and prove that for directed/undirected graphs and edge/node-costs, a ρ-approximation algorithm for Rooted Subset k-Connectivity Augmentation implies the following approximation ratios for Subset k-Connectivity Augmentation:

(i) $b(\rho + k) + (\frac{|T|}{|T|-k})^2 O(\log \frac{|T|}{|T|-k})$, where $b = 1$ for undirected graphs and $b = 2$ for directed graphs.

(ii) $\rho \cdot O(\frac{|T|}{|T|-k} \log k)$.

The best known values of ρ on undirected graphs are $\min\{|T|, O(k)\}$ for edge-costs and $\min\{|T|, O(k \log |T|)\}$ for node-costs; for directed graphs $\rho = |T|$ for both versions. Our results imply that unless $k = |T| - o(|T|)$, Subset k-Connectivity Augmentation admits the same ratios as the best known ones for the rooted version. This improves the ratios in Nutov (2009) [19] and Laekhanukit (2011) [15].

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1. Introduction

In the Survivable Network problem we are given a graph $G = (V, E)$ with edge/node-costs and connectivity requirements $\{r(u, v): u, v \in T \subseteq V\}$ on a node subset T of terminals. The goal is to find a minimum-cost subgraph of G that contains $r(u, v)$ internally-disjoint uv-paths for all $u, v \in T$. In the node-costs version of Survivable Network, we seek a min-cost subset $V' \subseteq V \setminus T$ such that the graph induced by $T \cup V'$ in G satisfies the connectivity requirements.

In the Rooted Subset k-Connectivity problem there is $s \in T$ such that $r(s, t) = k$ for all $t \in T \setminus \{s\}$ and $r(u, v) = 0$ otherwise. In the Subset k-Connectivity problem $r(u, v) = k$ for all $u, v \in T$ and $r(u, v) = 0$ otherwise. In the *augmentation versions*, G contains a subgraph J of cost zero with $r(u, v) - 1$ internally disjoint paths for all $u, v \in T$. A subset $T \subseteq V$ of terminals is k-connected to a root s in a directed/undirected graph J if J has k internally-disjoint vs-paths for every $v \in T$; T is k-connected in J if for every $s \in T$ the set $T \setminus \{s\}$ is k-connected to s in J. Formally, the versions of Survivable Network we consider are as follows.

> **Rooted Subset k-Connectivity Augmentation**
> *Instance:* A graph $G = (V, E)$ with edge/node-costs, a set $T \subseteq V$ of terminals, a root $s \in T$, and a subgraph $J = (V, E_J)$ of G such that $T \setminus \{s\}$ is $(k-1)$-connected to s in J.
> *Objective:* Find a minimum-cost edge-set $F \subseteq E \setminus E_J$ such that $T \setminus \{s\}$ is k-connected to s in $J \cup F$.

> **Subset k-Connectivity Augmentation**
> *Instance:* A graph $G = (V, E)$ with edge/node-costs, a set $T \subseteq V$ of terminals, and a subgraph $J = (V, E_J)$ of G such that T is $(k-1)$-connected in J.
> *Objective:* Find a minimum-cost edge-set $F \subseteq E \setminus E_J$ such that T is k-connected in $J \cup F$.

Obtaining a polylogarithmic approximation ratio for these problems is unlikely, see [10,16,2]. It is known and easy to see that for both edge-costs and node-costs, if Subset k-Connectivity Augmentation admits approximation ratio $\rho(k)$ such that $\rho(k)$ is a monotone increasing function, then Subset k-Connectivity admits ratio $k \cdot \rho(k)$. Moreover, for edge costs, if in addition the approximation ratio $\rho(k)$ is w.r.t. a standard setpair/biset LP-relaxation to the problem, then Subset k-Connectivity admits ratio $H(k) \cdot \rho(k)$, where $H(k)$ denotes the kth harmonic number; see [19,15] for details. For edge-costs, a standard setpair LP-relaxation for Survivable Network (due to Frank and Jordán [6]) is as follows:

$$\min \left\{ \sum_{e \in E} c_e x_e : \sum_{e \in E(X, X^*)} x_e \geqslant r(X, X^*),\ X, X^* \subseteq V,\ X \cap X^* = \emptyset,\ 0 \leqslant x_e \leqslant 1 \right\}$$

where $r(X, X^*) = \max\{r(u, v):\ u \in X,\ v \in X^*\} - |V \setminus (X \cup X^*)|$ and $E(X, X^*)$ is the set of edges in E from X to X^*.

Subset k-Connectivity admits trivial ratios $O(|T|^2)$ for both edge-costs and node-costs, by computing for every $u, v \in T$ an optimal edge-set of k internally-disjoint uv-paths (this is essentially a Min-Cost k-Flow problem, that can be solved in polynomial time), and taking the union of the computed edge-sets. For metric edge-costs the problem admits an $O(1)$ ratio [3]. The edge-costs version of Subset k-Connectivity Augmentation such that every edge with positive cost has its both endnodes in T admits ratio $O(\min\{\frac{|T|}{|T|-k} \log \frac{|T|}{|T|-k}, \log |T|\})$ [20].

The case $T = V$ of Rooted Subset k-Connectivity problem is the k-Outconnected Subgraph problem; this problem admits a polynomial time algorithm for directed graphs [7], which implies ratio 2 for undirected graphs. The case $T = V$ of Subset k-Connectivity problem is the k-Connected Subgraph problem. This problem is NP-hard, and the best known ratio for it is $O(\log k \log \frac{n}{n-k})$ for both directed and undirected graphs [20]; for the augmentation version of increasing the connectivity by one the ratio in [20] is $O(\log \frac{n}{n-k})$.

For $|T| \geqslant k + 1$ Subset k-Connectivity can be decomposed into k instances of Rooted Subset k-Connectivity, cf. [12] for the case $T = V$ (this is the k-Connected Subgraph problem), where it is also shown that the number of k-Outconnectivity Augmentation instances can be reduced to $O(\frac{|T|}{|T|-k} \log k)$, which is $O(\log k)$ unless $k = |T| - o(|T|)$.

Recently, Laekhanukit [15] made an important observation that the method of [12] can be extended for the case of arbitrary $T \subseteq V$. Specifically, he proved that if $|T| \geqslant 2k$, then $O(\log k)$ instances of Rooted Subset k-Connectivity Augmentation will suffice. Thus for $|T| \geqslant 2k$, the $O(k)$-approximation algorithm of [19] for Rooted Subset k-Connectivity Augmentation leads to the ratio $O(k \log k)$ for Rooted Subset k-Connectivity. By exploiting additional properties of the algorithm of [19], he reduced the ratio to $O(k)$ in the case $|T| \geqslant k^2$.

Using a different approach, we will show by a much simpler proof, that for both directed and undirected graphs and edge-costs and node-costs, Subset k-Connectivity Augmentation can be reduced to solving *one* instance (or two instances, in the case of directed graphs) of Rooted Subset k-Connectivity Augmentation and $O(k) + (\frac{|T|}{|T|-k})^2 O(\log \frac{|T|}{|T|-k})$ instances of the Min-Cost k-Flow problem. This leads to a much simpler algorithm, improves the result of Laekhanukit [15] for $|T| < k^2$, and applies also for node-costs and directed graphs. In addition, we give a more natural and much simpler extension of the algorithm of [12] for $T = V$, that also enables the same bound $O(\frac{|T|}{|T|-k} \log k)$ as in [12] for arbitrary T with $|T| \geqslant k + 1$, and in addition applies also for directed graphs, for node-costs, and for an arbitrary type of edge-costs, e.g., metric costs, or uniform costs, or $0, 1$-costs. When we say "$0, 1$-edge-costs" we mean that the input graph G is complete, and the goal is to add to the subgraph J of G formed by the zero-cost edges a minimum size edge-set F (any edge is allowed) such that $J \cup F$ satisfies the connectivity requirements. Formally, our result is the following.

Theorem 1.1. *For both directed and undirected graphs, and edge-costs and node-costs the following holds. If Rooted Subset k-Connectivity Augmentation admits approximation ratio $\rho = \rho(k, |T|)$, then for $|T| \geqslant k + 1$ Subset k-Connectivity Augmentation admits the following approximation ratios:*

(i) $b(\rho + k) + (\frac{|T|}{|T|-k})^2 O(\log \frac{|T|}{|T|-k})$, *where $b = 1$ for undirected graphs and $b = 2$ for directed graphs.*

(ii) $\rho \cdot O(\frac{|T|}{|T|-k} \log \min\{k, |T| - k\})$, *and this is so also for $0, 1$-edge-costs.*

Furthermore, for edge-costs, if the approximation ratio ρ is w.r.t. the setpair LP-relaxation for the problem, then so are the ratios in (i) *and* (ii).

For $|T| > k$, the best known values of ρ on undirected graphs are $O(k)$ for edge-costs and $\min\{O(k\log|T|), |T|\}$ for node-costs [19]; for directed graphs $\rho = |T|$ for both versions. For 0, 1-edge-costs $\rho = O(\log k)$ for undirected graphs [21] and $\rho = O(\log|T|)$ for directed graphs [18]. For edge-costs, these ratios are w.r.t. a standard LP-relaxation. Thus Theorem 1.1 implies the following.

Corollary 1.2. *For $|T| \geqslant k+1$, Subset k-Connectivity Augmentation admits the following approximation ratios.*

- *For undirected graphs, the ratios are $O(k) + (\frac{|T|}{|T|-k})^2 O(\log\frac{|T|}{|T|-k})$ for edge-costs, $O(k\log|T|) + (\frac{|T|}{|T|-k})^2 O(\log\frac{|T|}{|T|-k})$ for node-costs, and $\frac{|T|}{|T|-k} \cdot O(\log^2 k)$ for 0, 1-edge-costs.*
- *For directed graphs, the ratio is $2(|T|+k) + (\frac{|T|}{|T|-k})^2 O(\log\frac{|T|}{|T|-k})$ for both edge-costs and node-costs, and $\frac{|T|}{|T|-k} \cdot O(\log|T|\log k)$ for 0, 1-edge-costs.*

For Subset k-Connectivity, the ratios are larger by a factor of $O(\log k)$ for edge-costs, and by a factor of k for node-costs.

Note that except the case of 0, 1-edge-costs, Corollary 1.2 is deduced from part (i) of Theorem 1.1. However, part (ii) of Theorem 1.1 might become relevant if Rooted Subset k-Connectivity Augmentation admits ratio better than $O(k)$. In addition, part (ii) applies for *any type* of edge-costs, e.g. metric or 0, 1-edge-costs.

We conclude this section by mentioning some additional related work. For metric costs k-Connected Subgraph admits ratios $2 + \frac{k-1}{n}$ for undirected graphs and $2 + \frac{k}{n}$ for directed graphs [11]. For 0, 1-edge-costs the problem admits a polynomial time algorithm for directed graphs [6], which implies ratio 2 for undirected graphs. It is an open question whether on undirected graphs k-Connected Subgraph with 0, 1-edge-costs admits a polynomial time algorithm, but the augmentation version of the problem is solvable in polynomial time [22]. The currently best known non-trivial ratios for the Survivable Network problem on undirected graphs are: $O(k^3\log|T|)$ for arbitrary edge-costs by Chuzhoy and Khanna [4], $O(\log k)$ for metric costs due to Cheriyan and Vetta [3], $O(k) \cdot \min\{\log^2 k, \log|T|\}$ for 0, 1-edge-costs [21,14], and $O(k^4\log^2|T|)$ for node-costs [19].

2. Proof of Theorem 1.1

We start by proving the following essentially known statement.

Proposition 2.1. *For both directed and undirected graphs, and edge-costs and node-costs the following holds. Suppose that Rooted Subset k-Connectivity Augmentation admits an approximation ratio ρ. If for an instance of Subset k-Connectivity Augmentation we are given a set of q edges (when any edge is allowed) and p stars (may be directed to or from the root, in the case of directed graphs) on T whose addition to J makes T k-connected, then we can compute a $(\rho p + q)$-approximate solution F to this instance in polynomial time. Furthermore, for edge-costs, if the ρ-approximation is w.r.t. a standard setpair LP-relaxation, then $c(F) \leqslant (\rho p + q)\tau^*$, where τ^* is an optimal setpair LP-relaxation value for Subset k-Connectivity Augmentation.*

Proof. For every edge uv among the q edges, compute a min-cost edge-set $F_{uv} \subseteq E \setminus E_J$ such that $J \cup F_{uv}$ contains k internally-disjoint uv-paths. This can be done in polynomial time for both edge and node costs, using a Min-Cost k-Flow algorithm. For edge-costs, it is known that $c(F_{uv}) \leqslant \tau^*$. Then replace uv by F_{uv}, and note that T remains k-connected. Similarly, for every star S with center s and leaf-set T', compute a ρ-approximate augmenting edge-set $F_S \subseteq E \setminus E_J$ such that $J \cup F_S$ contains k internally-disjoint sv-paths (or vs-paths, in the case of directed graphs and S being directed towards the root) for every $v \in T'$. Then replace S by F_S, and note that T remains k-connected. For edge-costs, it is known that if the ρ-approximation for the rooted version is w.r.t. the standard setpair LP-relaxation, then $c(F_S) \leqslant (\rho p + q)\tau^*$. The statement follows. $\square$

Motivated by Proposition 2.1, we consider the following question:

Given a $(k-1)$-connected subset T in a graph J, how many edges and/or stars on T one needs to add to J such that T will become k-connected?

We emphasize that we are interested in obtaining *absolute bounds* on the number of edges in the question, expressed in certain parameters of the graph; namely we consider the *extremal graph theory* question and not only the *algorithmic problem*. Indeed, the algorithmic problem of adding the minimum number of edges on T such that T will become k-connected can be shown to admit a polynomial-time algorithm for directed graphs using the result of Frank and Jordán [6]; this also implies a 2-approximation algorithm for undirected graphs. However, in terms of the parameters $|T|$, k, the result in [6] implies

only the trivial bound $O(|T|^2)$ on the number of edges one needs to add to J such that T will become k-connected. On the other hand, no exact or approximation algorithm is known for computing a minimum size collection of stars as above.

Our bounds will be expressed in terms $|T|$, k, and the number of inclusion-minimal "deficient" sets of the graph J; we will also give a polynomial time algorithm to compute a set of edges/stars within these bounds. We need some definitions to state our results.

Definition 2.1. An ordered pair $\hat{X} = (X, X^+)$ of subsets of a groundset V is called a *biset* if $X \subseteq X^+$; X is the *inner part* and X^+ is the *outer part* of $\hat{X}$, $\Gamma(\hat{X}) = X^+ \setminus X$ is the boundary of $\hat{X}$ (so $X^+ = X \cup \Gamma(\hat{X})$), and $X^* = V \setminus X^+$ is the *complementary set* of $\hat{X}$.

Given an instance of directed/undirected Subset k-Connectivity Augmentation we may assume that T is an independent set in J, namely, that no edge in J has both endnodes in T. Otherwise, we obtain an equivalent instance by subdividing every edge $uv \in J$ with $u, v \in T$ by a new node, cf. [19]. Let us say that a directed/undirected *edge covers a biset* $\hat{X}$ if it goes from X to X^*, and that $\hat{X}$ *divides* T if $X \cap T, X^* \cap T \neq \emptyset$.

Definition 2.2. Given a $(k-1)$-connected independent set T in a (directed or undirected) graph $J = (V, E_J)$ let us say that a biset $\hat{X}$ on V is tight in J, if $\hat{X}$ divides T, no edge of J covers $\hat{X}$, and $|\Gamma(\hat{X})| = k - 1$.

By Menger's Theorem, an independent set $T \subseteq V$ is ℓ-connected in J if, and only if, $|\Gamma(\hat{X})| \geqslant \ell$ for any biset $\hat{X}$ on V that divides T such that no edge of J covers $\hat{X}$. Hence F is a feasible solution to Subset k-Connectivity Augmentation if, and only if, F covers the biset-family $\mathcal{F}$ of tight bisets; see [6,13,21]. Thus denoting $\ell = k - 1$, our question can be reformulated as follows.

Given an ℓ-connected independent set T in a directed/undirected graph J, how many edges and/or stars on T are needed to cover the family $\mathcal{F}$ of tight bisets?

Definition 2.3. The intersection and the union of two bisets $\hat{X}, \hat{Y}$ is defined by $\hat{X} \cap \hat{Y} = (X \cap Y, X^+ \cap Y^+)$ and $\hat{X} \cup \hat{Y} = (X \cup Y, X^+ \cup Y^+)$. Two bisets $\hat{X}, \hat{Y}$ *intersect* if $X \cap Y \neq \emptyset$; if in addition $X^* \cap Y^* \neq \emptyset$ then $\hat{X}, \hat{Y}$ *cross*. We say that a biset-family $\mathcal{F}$ on T is:

- *crossing* if $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$ for any $\hat{X}, \hat{Y} \in \mathcal{F}$ that cross.
- ℓ-*regular* if $|\Gamma(\hat{X})| \leqslant \ell$ for every $\hat{X} \in \mathcal{F}$, and if $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y} \in \mathcal{F}$ for any intersecting $\hat{X}, \hat{Y} \in \mathcal{F}$ with $|X \cup Y| \leqslant |T| - \ell - 1$.

Note that $|X \cup Y| \leqslant |T| - \ell - 1$ if, and only if, $|T \setminus (X \cup Y)| \geqslant \ell + 1$. The following statement is essentially known (see [8, Lemma 1.2] for the case $T = V$, and [21] for the first part with arbitrary T); we provide a proof-sketch for completeness of exposition.

Lemma 2.2. *Let T be an ℓ-connected independent set in a directed/undirected graph J, and let $\hat{X}, \hat{Y}$ be tight bisets with $X \cap Y \cap T \neq \emptyset$. If $X^* \cap Y^* \cap T \neq \emptyset$ then $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y}$ are both tight. If $|T \setminus (X \cup Y)| \geqslant \ell$ then $\hat{X} \cap \hat{Y}$ is tight, and if a strict inequality holds (namely, if $|X \cup Y| \leqslant |T| - \ell - 1$) then also $\hat{X} \cup \hat{Y}$ is tight.*

Proof. It is easy to verify that the following holds for any two bisets $\hat{X}, \hat{Y}$ on V and $T \subseteq V$.

(i) If an edge covers one of $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y}$ then it covers one of $\hat{X}, \hat{Y}$ (see [20]). Hence if $\hat{X}, \hat{Y}$ are tight then no edge of J covers $\hat{X} \cap \hat{Y}$ or $\hat{X} \cup \hat{Y}$.

(ii) Suppose that $X \cap Y \cap T \neq \emptyset$. If $\hat{X}$ divides T then $\hat{X} \cap \hat{Y}$ divides T, and if $X^* \cap Y^* \cap T \neq \emptyset$ then each of $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y}$ divides T.

(iii) $|\Gamma(\hat{X})| + |\Gamma(\hat{Y})| = |\Gamma(\hat{X} \cap \hat{Y})| + |\Gamma(\hat{X} \cup \hat{Y})|$, namely, the function $|\Gamma(\cdot)|$ is modular. In particular, if both $\hat{X}, \hat{Y}$ are tight then we have $2\ell = |\Gamma(\hat{X})| + |\Gamma(\hat{Y})| = |\Gamma(\hat{X} \cap \hat{Y})| + |\Gamma(\hat{X} \cup \hat{Y})|$.

Suppose that $X^* \cap Y^* \cap T \neq \emptyset$. Then each of $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y}$ divides T. Hence $|\Gamma(\hat{X} \cap \hat{Y})| \geqslant \ell$ and $|\Gamma(\hat{X} \cup \hat{Y})| \geqslant \ell$, by Menger's Theorem. Thus the modularity of $|\Gamma(\cdot)|$ implies $|\Gamma(\hat{X} \cap \hat{Y})| = |\Gamma(\hat{X} \cup \hat{Y})| = \ell$. This proves that $\hat{X} \cap \hat{Y}, \hat{X} \cup \hat{Y}$ are both tight.

Now suppose that $X^* \cap Y^* \cap T = \emptyset$. Then $T \setminus (X \cup Y) \subseteq \Gamma(X \cup Y)$, so $|\Gamma(X \cup Y)| \geqslant |T \setminus (X \cup Y)|$. Also, $|\Gamma(\hat{X} \cap \hat{Y})| \geqslant \ell$ by (i) and (ii) above, and since T is ℓ-connected in J. Consequently, by the modularity of $|\Gamma(\cdot)|$ we have $\ell \leqslant |\Gamma(\hat{X} \cap \hat{Y})| \leqslant 2\ell - |T \setminus (X \cup Y)|$. Thus if $|T \setminus (X \cup Y)| \geqslant \ell$ then we must have $|T \setminus (X \cup Y)| = \ell$ and $|\Gamma(\hat{X} \cap \hat{Y})| = \ell$, which implies that $\hat{X} \cap \hat{Y}$ is tight. $\square$

Corollary 2.3. *Let T be an ℓ-connected independent set in a directed/undirected graph $J = (V, E_J)$. Then the biset-family on T*

$$\mathcal{F} = \left\{ (X \cap T, X^+ \cap T) \colon (X, X^+) \text{ is a tight biset in } J \right\}$$

and the reverse family $\bar{\mathcal{F}} = \{(T \setminus X^+, T \setminus X): \hat{X} \in \mathcal{F}\}$ of $\mathcal{F}$ are both crossing and ℓ-regular. Furthermore, if J is undirected then $\mathcal{F}$ is symmetric, namely, $\mathcal{F} = \bar{\mathcal{F}}$.

Given two bisets $\hat{X}$, $\hat{Y}$ we write $\hat{X} \subseteq \hat{Y}$ and say that $\hat{Y}$ contains $\hat{X}$ if $X \subseteq Y$ or if $X = Y$ and $X^+ \subseteq Y^+$; $\hat{X} \subset \hat{Y}$ and $\hat{Y}$ properly contains $\hat{X}$ if $X \subset Y$ or if $X = Y$ and $X^+ \subset Y^+$.

Definition 2.4. Let $\mathcal{F}$ be a biset family on T. A biset $\hat{C} \in \mathcal{F}$ is a *core* of $\mathcal{F}$ if $\hat{C}$ contains no biset in $\mathcal{F} \setminus \{\hat{C}\}$; namely, a core is an inclusion-minimal biset in $\mathcal{F}$. Let $\mathcal{C}(\mathcal{F})$ be the family of cores of $\mathcal{F}$. Let $\nu(\mathcal{F})$ denote the maximum number of bisets (or cores, which is equivalent) in $\mathcal{F}$ which inner parts are pairwise disjoint. For an integer ℓ let $\mathcal{F}^\ell = \{\hat{X} \in \mathcal{F}: |X| \leqslant (|T| - \ell)/2\}$.

Given a biset-family $\mathcal{F}$ and an edge-set I on T, the residual biset-family $\mathcal{F}_I$ of $\mathcal{F}$ consists of the members of $\mathcal{F}$ uncovered by I. We will assume that for any I, the cores of $\mathcal{F}_I$ and of $\bar{\mathcal{F}}_I$ can be computed in polynomial time. For $\mathcal{F}$ being as in Corollary 2.3 this can be implemented in polynomial time using the Ford–Fulkerson Max-Flow Min-Cut algorithm, cf. [21]. Note also that $\hat{C}$ is a core of $\mathcal{F}^\ell$ if, and only if, $\hat{C}$ is a core of $\mathcal{F}$ with $|C| \leqslant (|T| - \ell)/2$. Hence our assumption implies that also the cores of $\mathcal{F}_I^\ell$ can be computed in polynomial time. It is known and easy to see (cf. [20]), that if $\mathcal{F}$ is crossing and/or ℓ-regular, so is $\mathcal{F}_I$, for any edge-set I.

Lemma 2.4. *Let $\mathcal{F}$ be an ℓ-regular biset-family on T and let $\hat{X}, \hat{Y} \in \mathcal{F}^\ell$ intersect. Then $\hat{X} \cap \hat{Y} \in \mathcal{F}^\ell$ and $\hat{X} \cup \hat{Y} \in \mathcal{F}$. In particular, the inner parts of the min-cores of $\mathcal{F}^\ell$ are pairwise disjoint, $\nu(\mathcal{F}^\ell) = |\mathcal{C}(\mathcal{F}^\ell)|$, and $\nu(\mathcal{F}_I^\ell)$ can be computed in polynomial time for any edge set I on T.*

Proof. Since $|X|, |Y| \leqslant \frac{|T| - \ell}{2}$, we have $|X \cup Y| = |X| + |Y| - |X \cap Y| \leqslant |T| - \ell - 1$. Thus $\hat{X} \cap \hat{Y}, \hat{X} \cap \hat{Y} \in \mathcal{F}$, by the ℓ-regularity of $\mathcal{F}$. Moreover, $\hat{X} \cap \hat{Y} \in \mathcal{F}^\ell$, since $|X \cap Y| \leqslant |X| \leqslant \frac{|T| - \ell}{2}$. $\square$

Let $H(k)$ denote the kth Harmonic number. We prove the following two theorems that imply Theorem 1.1.

Theorem 2.5. *Let $\mathcal{F}$ be a biset-family on T such that both $\mathcal{F}, \bar{\mathcal{F}}$ are crossing and ℓ-regular, where $|T| > \ell$. Then there exists a polynomial time algorithm that computes an edge-cover I of $\mathcal{F}$ of size $|I| \leqslant \nu(\mathcal{F}^\ell) + \nu(\bar{\mathcal{F}}^\ell) + \alpha(|T|, \ell)$, where $\alpha(|T|, \ell) = 6(\frac{|T|}{|T| - \ell})^2 \cdot (H(\frac{3|T|}{|T| - \ell}) + \frac{1}{2}) = (\frac{|T|}{|T| - \ell})^2 O(\log \frac{|T|}{|T| - \ell})$. Furthermore, if $\mathcal{F}$ is symmetric then $|I| \leqslant \nu(\mathcal{F}^\ell) + \alpha(|T|, \ell)$.*

Theorem 2.6. *Let $\mathcal{F}$ be a biset-family on T such that both $\mathcal{F}$ and $\bar{\mathcal{F}}$ are ℓ-regular. Then there exists a collection of $O(\frac{|T|}{|T| - \ell} \times \log \min\{\nu(\mathcal{F}^\ell), |T| - \ell\})$ stars on T which union covers $\mathcal{F}$, and such a collection can be computed in polynomial time. Furthermore, the total number of edges in the stars is at most $\nu(\mathcal{F}^\ell) + \nu(\bar{\mathcal{F}}^\ell) + (\frac{|T|}{|T| - \ell})^2 \cdot O(\log \frac{|T|}{|T| - \ell})$.*

Note that the second statement in Theorem 2.6 implies (up to constants) the bound in Theorem 2.5. However, the proof of Theorem 2.5 is much simpler than the proof of Theorem 2.6, and the proof of Theorem 2.5 is a part of the proof of the second statement in Theorem 2.6.

Let us show that Theorems 2.5 and 2.6 imply Theorem 1.1. For that, we show that by applying b times the ρ-approximation algorithm for Rooted Subset k-Connectivity Augmentation, we obtain an instance with $\nu(\mathcal{F}^\ell), \nu(\bar{\mathcal{F}}^\ell) \leqslant k$, where $\mathcal{F}$ is as in Corollary 2.3 and $\ell = k - 1$. This is achieved by the following procedure due to Khuller and Raghavachari [9] that originally considered the case $T = V$, see also [1,5,11]; the same procedure is also used by Laekhanukit in [15]. Choose an arbitrary subset $T' \subseteq T$ of k nodes, add a new node s (the root) both to G and to J, and add to J all edges between s and T' of cost zero each. Then, using the ρ-approximation algorithm for the Rooted Subset k-Connectivity Augmentation problem, compute an augmenting edge set F such that $J \cup F$ contains k internally disjoint vs-paths and sv-paths for every $v \in T'$. Now, add F to J and remove s from J. It is a routine to show that $c(F) \leqslant b\text{opt}$, and that for edge-costs $c(F) \leqslant b\tau^*$. It is also known that if $\hat{X}$ is a tight biset of the obtained graph J, then $X \cap T', X^* \cap T' \neq \emptyset$, cf. [1,15]. Combined with Lemma 2.4 we obtain that $\nu(\mathcal{F}^\ell), \nu(\bar{\mathcal{F}}^\ell) \leqslant |T'| \leqslant k$ for the obtained instance, as claimed.

3. Proof of Theorem 2.5

Definition 3.1. Given a biset-family $\mathcal{F}$ on T, let $\Delta(\mathcal{F})$ denote the maximum degree in the hypergraph $\mathcal{F}^{in} = \{X: \hat{X} \in \mathcal{F}\}$ of the inner parts of the bisets in $\mathcal{F}$. We say that $T' \subseteq T$ is a *transversal* (or a *hitting set*) of $\mathcal{F}$ if $T' \cap X \neq \emptyset$ for every $X \in \mathcal{F}^{in}$; a function $t : T \to [0, 1]$ is a *fractional transversal* of $\mathcal{F}$ if $\sum_{v \in X} t(v) \geqslant 1$ for every $X \in \mathcal{F}^{in}$.

Lemma 3.1. *Let $\mathcal{F}$ be a crossing biset-family. Then $\Delta(\mathcal{C}(\mathcal{F})) \leqslant \nu(\bar{\mathcal{F}})$.*

Proof. Since $\mathcal{F}$ is crossing, the members of $\mathcal{C}(\mathcal{F})$ are pairwise non-crossing. Thus if $\mathcal{H}$ is a subfamily of $\mathcal{C}(\mathcal{F})$ such that the intersection of the inner parts of the bisets in $\mathcal{H}$ is non-empty, then $\bar{\mathcal{H}}$ is a subfamily of $\bar{\mathcal{F}}$ such that the inner parts of the bisets in $\bar{\mathcal{H}}$ are pairwise disjoint, so $|\bar{\mathcal{H}}| \leqslant \nu(\bar{\mathcal{F}})$. $\square$

Lemma 3.2. *Let T' be a transversal of a biset-family $\mathcal{F}'$ on T and let I' be an edge-set on T obtained by picking for every $s \in T'$ an edge from s to every inclusion-minimal member of the set-family $\{X^*\colon \hat{X} \in \mathcal{F}', \ s \in X\}$. Then I' covers $\mathcal{F}'$. Moreover, if $\mathcal{F}'$ is crossing then $|I'| \leqslant |T'| \cdot \nu(\bar{\mathcal{F}}')$.*

Proof. The statement that I' covers $\mathcal{F}'$ is obvious. If $\mathcal{F}'$ is crossing, then for every $s \in T$ the inclusion-minimal members of $\{X^*\colon \hat{X} \in \mathcal{F}', \ s \in X\}$ are pairwise-disjoint, hence their number is at most $\nu(\bar{\mathcal{F}}')$. The statement follows. $\quad\square$

The following statement from [12], easily follows from Lemma 2.4.

Claim 3.3 *(See [12].). Let $\mathcal{F}$ be an ℓ-regular biset-family on T. For $\hat{C} \in \mathcal{C}(\mathcal{F}^\ell)$ let $\hat{U}_C$ denote the union of the bisets in $\mathcal{F}^\ell$ that contain $\hat{C}$ and contain no other member of $\mathcal{C}(\mathcal{F}^\ell)$. Then the sets U_C are pairwise disjoint, and if $|U_C| \leqslant |T| - \ell - 1$ then $\hat{U}_C \in \mathcal{F}$.*

Lemma 3.4. *The following holds for any ℓ-regular biset-family $\mathcal{F}$ on T.*

(i) $\nu(\mathcal{F}) \leqslant \nu(\mathcal{F}^\ell) + \frac{2|T|}{|T|-\ell}$.

(ii) *If $\nu(\mathcal{F}^\ell_{\{e\}}) = \nu(\mathcal{F}^\ell)$ holds for every edge e on T then $\nu(\mathcal{F}^\ell) \leqslant \frac{|T|}{|T|-\ell}$.*

(iii) *There exists a polynomial time algorithm that finds a transversal T' of $\mathcal{C}(\mathcal{F})$ of size at most $|T'| \leqslant \nu(\mathcal{F}^\ell) + \frac{2|T|}{|T|-\ell} \cdot H(\Delta(\mathcal{C}(\mathcal{F})))$.*

Proof. Part (i) is immediate.

We prove (ii). If $|U_C| \leqslant |T| - \ell - 1$ for some $\hat{C} \in \mathcal{C}(\mathcal{F}^\ell)$, then $\hat{U}_C \in \mathcal{F}$, by Claim 3.3. In this case $\nu(\mathcal{F}^\ell_{\{e\}}) \leqslant \nu(\mathcal{F}^\ell) - 1$ for any edge e from C to U_C^*. Hence $|U_C| \geqslant |T| - \ell$ must hold for every $\hat{C} \in \mathcal{C}(\mathcal{F}^\ell)$. By Claim 3.3, the sets U_C are pairwise disjoint. Part (ii) follows.

We prove (iii). Let T^ℓ be an inclusion-minimal transversal of $\mathcal{C}(\mathcal{F}^\ell)$. By Lemma 2.4, $|T^\ell| = \nu(\mathcal{F}^\ell)$. We will show how to find a transversal T'' of the family $\mathcal{C}'' = \mathcal{C}(\mathcal{F}) \setminus \mathcal{C}(\mathcal{F}^\ell)$ of size at most $|T''| \leqslant \frac{2|T|}{|T|-\ell} \cdot H(\Delta(\mathcal{C}(\mathcal{F})))$. Such T'' is computed by a standard greedy algorithm, that starts with $T'' = \emptyset$, and repeatedly adds an element $v \in T \setminus T''$ that intersects the maximum number of sets in the family $\{C\colon \hat{C} \in \mathcal{C}'', \ C \cap T'' = \emptyset\}$. It is known that such a greedy algorithm computes a transversal of size at most $H(\Delta(\mathcal{C}''))$ times the minimum value of a fractional transversal of $\mathcal{C}''$ (cf. [17]). The statement follows by observing that the function $t(v) = \frac{2}{|T|-\ell}$ for every $v \in T$ is a fractional transversal of $\mathcal{C}''$ of value $\frac{2|T|}{|T|-\ell}$, and that $\Delta(\mathcal{C}'') \leqslant \Delta(\mathcal{C}(\mathcal{F}))$. $\quad\square$

The algorithm for computing I as in Theorem 2.5 starts with $I = \emptyset$ and then continues as follows.

Phase 1. While there exists an edge e on T such that $\nu(\mathcal{F}^\ell_{I \cup \{e\}}) \leqslant \nu(\mathcal{F}^\ell_I) - 1$, or such that $\nu(\bar{\mathcal{F}}^\ell_{I \cup \{e\}}) \leqslant \nu(\bar{\mathcal{F}}^\ell_I) - 1$, add e to I.

Phase 2. Find a transversal T' of $\mathcal{C}(\mathcal{F}')$ as in Lemma 3.4(iii), where $\mathcal{F}' = \mathcal{F}_I$. Then find an edge-cover I' of $\mathcal{F}'$ as in Lemma 3.2 and add I' to I.

The edge-set I computed covers $\mathcal{F}$, by Lemma 3.2. The algorithm can be implemented in polynomial time, by Lemma 2.4. Clearly, the number of edges in I at the end of Phase 1 is at most $\nu(\mathcal{F}^\ell) + \nu(\bar{\mathcal{F}}^\ell)$, and is at most $\nu(\mathcal{F}^\ell)$ if $\mathcal{F}$ is symmetric. Now we bound the size of I'. Note that at the end of Phase 1 we have $\nu(\mathcal{F}^\ell_I), \nu(\bar{\mathcal{F}}^\ell_I) \leqslant \frac{|T|}{|T|-\ell}$, by Lemma 3.4(ii). Thus $\nu(\bar{\mathcal{F}}_I) \leqslant \frac{3|T|}{|T|-\ell}$ by Lemma 3.4(i), and hence $\Delta(\mathcal{C}(\mathcal{F}_I)) \leqslant \nu(\bar{\mathcal{F}}_I) \leqslant \frac{3|T|}{|T|-\ell}$ by Lemma 3.1. Consequently, $|T'| \leqslant \nu(\mathcal{F}^\ell_I) + \frac{2|T|}{|T|-\ell} \cdot H(\Delta(\mathcal{C}(\mathcal{F}_I))) \leqslant \frac{2|T|}{|T|-\ell} \cdot (H(\frac{3|T|}{|T|-\ell}) + \frac{1}{2})$. From this we get $|I'| \leqslant |T'| \cdot \nu(\bar{\mathcal{F}}_I) \leqslant 6(\frac{|T|}{|T|-\ell})^2 \cdot (H(\frac{3|T|}{|T|-\ell}) + \frac{1}{2}) = \alpha(|T|, \ell)$.

The proof of Theorem 2.5 is now complete.

4. Proof of Theorem 2.6

Let $\mathcal{S}_T$ denote collection of spanning stars on T, so $|\mathcal{S}_T| = |T|$ in the case of undirected graphs (except that $|\mathcal{S}_T| = 1$ if $|T| = 2$) and $|\mathcal{S}_T| = 2|T|$ in the case of directed graphs. We start by analyzing the performance of a natural Greedy Algorithm for covering $\mathcal{F}^\ell$, that starts with $I = \emptyset$ and while $\nu(\mathcal{F}^\ell_I) \geqslant 1$ adds to I a star $S \in \mathcal{S}_T$ for which $\nu(\mathcal{F}^\ell_{I \cup S})$ is minimal. The algorithm terminates, since any in-star in $\mathcal{S}_T$ with center in the inner part of some core of $\mathcal{F}^\ell_I$, reduces the number of cores by one. Furthermore, the algorithm can be implemented in polynomial time, by Lemma 2.4. The proof of the following statement is similar to the proof of the main result of [12].

Lemma 4.1. *Let $\mathcal{F}$ be an ℓ-regular biset-family on T and let $\mathcal{S}$ be the collection of stars computed by the Greedy Algorithm. Then*

$$|\mathcal{S}| = O\left(\frac{|T|}{|T|-\ell} \log \min\{\nu(\mathcal{F}^\ell), |T| - \ell\}\right).$$

Recall that for $\hat{C} \in \mathcal{C}(\mathcal{F}^\ell)$ we denote by $\hat{U}_C$ the union of the bisets in $\mathcal{F}^\ell$ that contain $\hat{C}$ and contain no other member of $\mathcal{C}(\mathcal{F}^\ell)$, and that by Claim 2.4 the set in $\{U_C: \hat{C} \in \mathcal{C}(\mathcal{F}^\ell)\}$ are pairwise disjoint.

Definition 4.1 *(See [12].).* We say that $s \in T$ *out-covers* $\hat{C} \in \mathcal{C}(\mathcal{F}^\ell)$ if $s \in U_C^*$.

Lemma 4.2. *Let $\mathcal{F}$ be an ℓ-regular biset-family on T and let $\nu = \nu(\mathcal{F}^\ell)$.*

(i) *There is $s \in T$ that out-covers at least $\nu(1 - \frac{\ell}{|T|}) - 1$ members of $\mathcal{C}(\mathcal{F}^\ell)$.*

(ii) *If s out-covers $\hat{C} \in \mathcal{C}(\mathcal{F}^\ell)$ then any edge from C to s covers all the bisets in $\mathcal{F}^\ell$ that contain $\hat{C}$ and contain no other member of $\mathcal{C}(\mathcal{F}^\ell)$.*

(iii) *Let s out-cover the members of $\mathcal{C} \subseteq \mathcal{C}(\mathcal{F}^\ell)$ and let S be a star with one edge from the inner part of each member of $\mathcal{C}$ to s. Then $\nu(\mathcal{F}^\ell) \leqslant \nu(\mathcal{F}_S^\ell) - |\mathcal{C}|/2$.*

Consequently, there exists a star S on T such that

$$\nu\big(\mathcal{F}_S^\ell\big) \leqslant \frac{1}{2}\left(1 + \frac{\ell}{|T|}\right) \cdot \nu + \frac{1}{2} = \alpha \cdot \nu + \beta. \tag{1}$$

Proof. We prove (i). Consider the hypergraph

$$\mathcal{H} = \big\{T \setminus \Gamma(\hat{U}_C): \hat{C} \in \mathcal{C}\big(\mathcal{F}^\ell\big)\big\}.$$

Let $\nu \in T$. Note that if $\nu \in T \setminus \Gamma(\hat{U}_C)$ then ν out-covers $\hat{C}$, unless $\nu \in U_C$. By Claim 3.3, the sets U_C are pairwise disjoint, hence ν belongs to at most one of them. Hence the number of members of $\mathcal{C}(\mathcal{F}^\ell)$ out-covered by ν is at least the degree of ν in $\mathcal{H}$ minus 1. Thus all we need to prove is that there is a node $s \in T$ whose degree in $\mathcal{H}$ is at least $\nu(1 - \frac{\ell}{|T|})$. For every $\hat{C} \in \mathcal{C}(\mathcal{F}^\ell)$ we have $|T \setminus \Gamma(\hat{U}_C)| \geqslant |T| - \ell$, by the ℓ-regularity of $\mathcal{F}$. Hence the bipartite incidence graph of $\mathcal{H}$ has at least $\nu(|T| - \ell)$ edges, and thus has a node $s \in T$ of degree at least $\nu(1 - \frac{\ell}{|T|})$, which equals the degree of s in $\mathcal{H}$. Part (i) follows.

Part (ii) follows from the simple observation that any biset in $\mathcal{F}^\ell$, that contains $\hat{C}$ and contains no other member of $\mathcal{C}(\mathcal{F}^\ell)$, is contained in $\hat{U}_C$.

We prove (iii). It is sufficient to show that every $\hat{C} \in \mathcal{C}(\mathcal{F}_S^\ell)$ contains some $\hat{C}' \in \mathcal{C}(\mathcal{F}^\ell) \setminus \mathcal{C}$ or contains at least two members in $\mathcal{C}$. Clearly, $\hat{C}$ contains some $\hat{C}' \in \mathcal{C}(\mathcal{F}^\ell)$. We claim that if $\hat{C}' \in \mathcal{C}$ then $\hat{C}$ must contain some $\hat{C}'' \in \mathcal{C}(\mathcal{F}^\ell)$ distinct from $\hat{C}'$. This is so since $\hat{C} \in \mathcal{C}(\mathcal{F}_S^\ell)$ and since S covers all bisets in $\mathcal{F}^\ell$ that contain $\hat{C}$ and contain no other member of $\mathcal{C}(\mathcal{F}^\ell)$, by part (ii). Part (iii) follows. □

Let us use parameters α, β, γ, δ and j set to

$$\alpha = \frac{1}{2}\left(1 + \frac{\ell}{|T|}\right), \qquad \beta = \frac{1}{2}, \qquad \gamma = 1 - \frac{\ell}{|T|} = 2(1 - \alpha), \qquad \delta = 1.$$

Note that $\alpha < 1$ and that $\frac{\beta}{1-\alpha} = \frac{|T|}{|T|-\ell}$. Let j be the minimum integer satisfying $\alpha^j(\nu - \frac{\beta}{1-\alpha}) \leqslant \frac{2}{1-\alpha}$, namely,

$$j = \left\lceil \frac{\ln \frac{1}{2}(\nu(1-\alpha) - \beta)}{\ln(1/\alpha)} \right\rceil \leqslant \left\lceil \frac{\ln \frac{1}{2}\nu(1-\alpha)}{\ln(1/\alpha)} \right\rceil. \tag{2}$$

We assume that $\nu \geqslant \frac{2+\beta}{1-\alpha}$ to have $j \geqslant 0$ (otherwise Lemma 4.1 follows).

Lemma 4.3. *Let $0 \leqslant \alpha < 1$, $\beta \geqslant 0$, $\nu_0 = \nu$, and for $i \geqslant 1$ let*

$$\nu_{i+1} \leqslant \alpha \nu_i + \beta, \qquad s_i = \gamma \nu_{i-1} - \delta.$$

Then $\nu_i \leqslant \alpha^i(\nu - \frac{\beta}{1-\alpha}) + \frac{\beta}{1-\alpha}$ and $\sum_{i=1}^j s_i \leqslant \frac{1-\alpha^j}{1-\alpha} \cdot \gamma(\nu - \frac{\beta}{1-\alpha}) + j(\frac{\gamma\beta}{1-\alpha} - \delta)$. Moreover, for j given by (2)

$$\nu_j \leqslant \frac{2+\beta}{1-\alpha} = \frac{5|T|}{|T|-\ell} \quad \text{and} \quad \sum_{i=1}^j s_i \leqslant 2\left(\nu - \frac{|T|}{|T|-\ell}\right).$$

Proof. Unraveling the recursive inequality $\nu_{i+1} \leqslant \alpha \nu_i + \beta$ in the lemma we get:

$$\nu_i \leqslant \alpha^i \nu + \beta\big(1 + \alpha + \cdots + \alpha^{i-1}\big) = \alpha^i \nu + \beta \frac{1-\alpha^i}{1-\alpha} = \alpha^i\left(\nu - \frac{\beta}{1-\alpha}\right) + \frac{\beta}{1-\alpha}.$$

This implies $s_i \leqslant \gamma(v - \frac{\beta}{1-\alpha})\alpha^{i-1} + \frac{\gamma\beta}{1-\alpha} - \delta$, and thus

$$\sum_{i=1}^{j} s_i \leqslant \gamma\left(v - \frac{\beta}{1-\alpha}\right)\sum_{i=1}^{j}\alpha^{i-1} + j\left(\frac{\gamma\beta}{1-\alpha} - \delta\right)$$
$$= \gamma\left(v - \frac{\beta}{1-\alpha}\right) \cdot \frac{1-\alpha^j}{1-\alpha} + j\left(\frac{\gamma\beta}{1-\alpha} - \delta\right).$$

For j given by (2) we have $v_j \leqslant \alpha^j(v - \frac{\beta}{1-\alpha}) + \frac{\beta}{1-\alpha} \leqslant \frac{2}{1-\alpha} + \frac{\beta}{1-\alpha} = \frac{2+\beta}{1-\alpha}$, and

$$\sum_{i=1}^{j} s_i \leqslant \frac{1-\alpha^j}{1-\alpha} \cdot \gamma\left(v - \frac{\beta}{1-\alpha}\right) + j\left(\frac{\gamma\beta}{1-\alpha} - \delta\right)$$
$$\leqslant 2\left(v - \frac{\beta}{1-\alpha}\right) = 2\left(v - \frac{|T|}{|T|-\ell}\right). \quad \square$$

We now finish the proof of Lemma 4.1. At each one of the first j iterations we out-cover at least $v(\mathcal{F}_I^\ell)(1 - \frac{\ell}{|T|}) - 1$ members of $\mathcal{C}(\mathcal{F}_I^\ell)$, by Lemma 4.2. In each one of the consequent iterations, we can reduce $v(\mathcal{F}_I^\ell)$ by at least one, if we choose the center of the star in C for some $\hat{C} \in \mathcal{C}(\mathcal{F}_I^\ell)$. Thus using Lemma 4.3, performing the necessary computations, and substituting the values of the parameters, we obtain that the number of stars in $\mathcal{S}$ is bounded by

$$j + v_j \leqslant \left\lceil \frac{\ln\frac{1}{2}v(1-\alpha)}{\ln(1/\alpha)} \right\rceil + \frac{5|T|}{|T|-\ell} = O\left(\frac{|T|}{|T|-\ell}\log\min\{v, |T|-\ell\}\right).$$

Now we discuss a variation of this algorithm that produces $\mathcal{S}$ with a small number of leaves. Here at each one of the first j iterations we out-cover *exactly* $v(1 - \frac{\ell}{|T|}) - 1$ cores. For that, we need to be able to compute the bisets $\hat{U}_C$, and such a procedure can be found in [15]. The number of edges in the stars at the end of this phase is at most $2(v - \frac{|T|}{|T|-\ell})$ and $v_j \leqslant \frac{5|T|}{|T|-\ell}$. In the case of non-symmetric $\mathcal{F}$ and/or directed edges, we apply the same algorithm on $\bar{\mathcal{F}}^\ell$. At this point, we apply Phase 2 of the algorithm from the previous section. Since the number of cores of each one of $\mathcal{F}_I^\ell, \bar{\mathcal{F}}_I^\ell$ is now $O(\frac{|T|}{|T|-\ell})$, the size of the transversal T' computed is bounded by $|T'| = O(\frac{|T|}{|T|-\ell} \cdot \log\frac{|T|}{|T|-\ell})$. The number of stars is at most $|T'|$, while the number of edges in the stars is at most $|T'| \cdot v(\bar{\mathcal{F}}_I) = (\frac{|T|}{|T|-\ell})^2 \cdot O(\log\frac{|T|}{|T|-\ell})$.

This concludes the proof of Theorem 2.6.

Acknowledgement

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