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Theoretical Computer Science 316 (2004) 113–136

Theoretical Computer Science

www.elsevier.com/locate/tcs

On the relationship between compact regularity and Gentzen's cut rule

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Abstract

The patch topology on a stably compact space, generalizing the Lawson topology on a domain, is a coreflection of stably compact spaces in compact regular spaces. This paper investigates compact regularity and the patch coreflection in multilingual sequent calculus (MLS), which can be regarded as a category of predicative representations of stably compact spaces. An object of MLS is a certain sort of generalization of the positive fragment of Gentzen's sequent calculus. We show that an object of MLS represents a compact regular space if and only if every sequent arises as an instance of Gentzen's cut rule with complete freedom to choose the placement of the cut formula.

The relationship between compact regularity and Gentzen's cut rule is further explicated by the patch coreflection in MLS. The construction is a universal solution (up to a certain equivalence of tokens) to the problem of adding opposites to a logic, i.e., tokens that obey Gentzen's rules for negation. In the spectral case, this is equivalent to adding Boolean complements. The paper closes by considering the full subcategory of MLS consisting of objects with opposites. By taking contrapositives of sequents, we obtain an anti-involution on morphisms making this category equivalent to the Freyd/Scedrov allegory of compact regular spaces and closed binary relations. Moreover, the category of "maps" of this allegory is predicatively equivalent to the image of the patch functor.

Keywords: Sequent calculus; Predicativity; Compact regular locales; Stone duality

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0304-3975/$ - see front matter

doi:10.1016/j.tcs.2004.01.027

1. Introduction

A diagram similar to the following is at the heart of this paper:

$$\begin{array}{ccc} \mathsf{KRegSp} & \underset{\mathsf{Patch}}{\overset{\subseteq}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} & \mathsf{SCSp}^{\mathsf{p}} \\ \Omega \downarrow \; \equiv \; \uparrow \mathsf{pt} & & \Omega \downarrow \; \equiv \; \uparrow \mathsf{pt} \\ \mathsf{KRegLoc} & \underset{\mathsf{Patch}}{\overset{\subseteq}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} & \mathsf{SCLoc}^{\mathsf{p}} \end{array} \tag{1}$$

where KRegSp and $\mathsf{KRegLoc}$ are the categories of compact regular spaces and compact regular locales, respectively, and $\mathsf{SCSp}^{\mathsf{p}}$ and $\mathsf{SCLoc}^{\mathsf{p}}$ are the categories of stably compact spaces with perfect maps and stably compact locales with perfect locale morphisms, respectively. Details of the diagram are discussed below, but consider it informally for now.

The top of the diagram represents a coreflection of stably compact spaces with perfect maps in compact regular spaces with continuous maps. This coreflection appears first to have been stated explicitly by Escardó [4], although it is implicit in work as early as Nachbin [14], investigating order in relation to topology. Stably compact spaces are useful for connecting classical mathematics and domain theory because they include compact regular spaces and the Scott topologies of domains. Indeed, the topology $\mathsf{Patch}(D)$ for a domain D is none other that its Lawson topology. So the top of the diagram generalizes the relation between the Scott and Lawson topologies on domains to stably compact spaces.

The functors Ω and pt appearing on the sides of the diagram are the familiar adjoints between Top and Loc that arise from Stone duality, cut down to equivalences for the categories here. The proof that $\mathsf{KRegLoc}$ and $\mathsf{SCLoc}^{\mathsf{p}}$ consist of spatial locales (and so Ω and pt cut down to equivalences in these cases) is non-constructive. On the other hand, Escardó [4,5] offers an intuitionistically valid localic version of the patch coreflection, as indicated at the bottom of the diagram. The effect is to isolate the non-constructive aspects of the spatial construction in the equivalence functor pt. Thus, the lower coreflection is valid in any topos, whereas the upper coreflection is not.

Essentially the same story can be told for other spatial constructions versus their localic counter-parts: one isolates the intuitionistically valid aspects of the construction in the locales, and transfers the results to spaces via Ω and pt. Thus the move from spaces to locales is a move in the direction of greater constructive generality. While localic proofs may be intuitionistically valid, however, they are bound to involve impredicative principles for the simple reason that a locale is really a certain kind of complete lattice. Quantification over the power set of formal opens of the locale is generally unavoidable. So one needs the sub-object classifier of a topos simply to speak about locales in general.

In important special cases, however, there are alternatives. The two best known, and oldest, examples are Stone spaces and spectral spaces [15], where one does not need the entire (complete) lattice of opens. Instead, one can work on bases (of compact opens) that form Boolean and distributive lattices, respectively. Thus Stone's original theorems skip over locale theory to reach categories that can be formalized in weaker-type theories than toposes. For example following Meinke and Tucker [13], the type theory $\lambda P\bar{\omega}$ [1] or Martin Löf's *intuitionistic-type theory* (*ITT*) [12] with at least two universes is strong enough to formalize universal algebra, and hence distributive and Boolean lattices. Stone gives us a way to isolate aspects of constructions on spectral spaces and Stone spaces that are valid in all models of such predicative type theories.

In this paper we reconsider the patch construction predicatively, augmenting the previous diagram with another row:

$$\begin{array}{ccc}
\mathsf{KRegSp} & \underset{\text{Patch}}{\overset{\subseteq}{\rightleftarrows}}\ (\perp) & \mathsf{SCSp}^{\mathsf{p}} \\
\Omega \downarrow \ \equiv \ \uparrow \mathsf{pt} & & \Omega \downarrow \ \equiv \ \uparrow \mathsf{pt} \\
\mathsf{KRegLoc} & \underset{\text{Patch}}{\overset{\subseteq}{\rightleftarrows}}\ (\perp) & \mathsf{SCLoc}^{\mathsf{p}} \\
\mathsf{ang} \downarrow \ \equiv \ \uparrow \mathsf{idl} & & \mathsf{lang} \downarrow \ \equiv \ \uparrow \mathsf{idl} \\
\mathsf{MLS}^{\mathsf{u}}_{\neg} & \underset{\text{Patch}}{\overset{\subseteq}{\rightleftarrows}}\ (\perp) & \mathsf{MLS}^{\mathsf{u}}
\end{array} \tag{2}$$

where $\mathsf{MLS}^{\mathsf{u}}_{\neg}$ and $\mathsf{MLS}^{\mathsf{u}}$ are categories equivalent to $\mathsf{KRegLoc}$ and $\mathsf{SCLoc}^{\mathsf{p}}$. The proofs of these lower equivalences are intuitionistically valid. The proof that the bottom construction is a coreflection, on the other hand, is valid in any type theory strong enough to formalize the construction of free algebras of finite signature, free semi-lattices and inductively defined relations thereon. For a more recent practical example of how such a formalization might proceed, Capretta [2] investigates universal algebra within the system Coq, although that type system is stronger than necessary for the present work.

MLS, which stands for *multilingual sequent calculus*, was first introduced in [10]. It is based on Gentzen's sequent calculus [7], and yields finitary formal representations of stably compact locales (or spaces, non-constructively) that are especially amenable to predicative constructions. Specifically, we characterize the objects of MLS that represent compact regular locales and give a patch construction in MLS. In brief, it turns out that regularity is intimately connected to the behavior of Gentzen's cut rule:

$$\frac{\Gamma_0 \to \Delta_0, \alpha \qquad \alpha, \Gamma_1 \to \Delta_1}{\Gamma_0, \Gamma_1 \to \Delta_0, \Delta_1}\ (\mathsf{Cut})$$

and the patch construction amounts to freely adding opposites (tokens that behave according to Gentzen's negation rules) to a given logic.

One key idea distinguishes MLS from simple (algebraic) propositional logic. The tokens (formulas) that appear on the left and right sides of a sequent need not be drawn from the same language. This allows us to consider MLS as a category in which the consequence relations are morphisms and composition of morphisms is characterized by a special case of Gentzen's cut rule. Also, in place of identity axioms $\phi \vdash \phi$, we stipulate that objects (that is, identity morphisms) satisfy certain weaker conditions that otherwise would follow from the identity axioms. These conditions ensure that the resulting morphisms act as identities for the composition rule, and that the logical connectives are proof-theoretically well-behaved. In [10], MLS is proved to be equivalent as an order enriched category to the category SCSp^* of stably compact spaces and binary relations $R \subseteq X_\kappa \times Y$ that are compact saturated subsets of the given product topology. Here X_κ is the co-compact topology on X, i.e., the topology generated by compact saturated sets. In his Ph.D. thesis, Kegelman [11] shows further that SCSp^* is equivalent to the category of stably compact pre-locales, i.e., the opposite of arithmetic lattices and Scott continuous meet semi-lattice homomorphisms.

Because MLS is defined as a generalization of the propositional sequent calculus, one expects proof-theoretic ideas to play an important role in our investigations. Specifically, consider the Cut rule again. Gentzen's classical sequent calculus (his System **K**) can be regarded as a binary relation $\rightarrow_{\mathbf{K}}$ between finite sets of formulas. By definition, $\rightarrow_{\mathbf{K}}$ is closed under the Cut rule. It also enjoys the following metatheorem.

Theorem 1.1. *If $\Gamma_0, \Gamma_1 \rightarrow_{\mathbf{K}} \Delta_0, \Delta_1$ holds, then there exists a formula α so that $\Gamma_0 \rightarrow_{\mathbf{K}} \Delta_0, \alpha$ and $\alpha, \Gamma_1 \rightarrow_{\mathbf{K}} \Delta_1$.*

Thus, a derivable sequent can be decomposed into an instance of Cut by any possible splittings of the left- and right-hand sides. Let us call a binary relation R between finite subsets a *sequent relation*, and say that R enjoys *parallel cut decomposition* if it satisfies the condition satisfied by $\rightarrow_{\mathbf{K}}$ in Theorem 1.1. Whereas the standard proof of Theorem 1.1 depends on negation, a weaker form of the theorem holds in the positive fragment of Gentzen's system. Let $\rightarrow_{\mathbf{P}}$ denote the sequent relation for the positive fragment of System **K**. That is, formulas are negation-free and the proof rules for negation are omitted.

Theorem 1.2. *If $\Gamma_0, \Gamma_1 \rightarrow_{\mathbf{P}} \Delta_0, \Delta_1$ holds with either $\Gamma_1 = \emptyset$ or $\Delta_0 = \emptyset$, then there exists a formula α so that $\Gamma_0 \rightarrow_{\mathbf{P}} \Delta_0, \alpha$ and $\alpha, \Gamma_1 \rightarrow_{\mathbf{P}} \Delta_1$.*

The difference between the two theorems is our freedom to split the right- and left-hand sides of a sequent. In the classical setting, we can decompose a sequent into an instance of Cut by splitting the two sides independently: $\Gamma_0, \Gamma_1 \rightarrow_{\mathbf{K}} \Delta_0, \Delta_1$. In the positive setting, we can only decompose one or the other side of the sequent at one time: either $\Gamma_0, \Gamma_1 \rightarrow_{\mathbf{P}} \Delta_1$ or $\Gamma_0 \rightarrow_{\mathbf{P}} \Delta_0, \Delta_1$. In general, say that a sequent relation R enjoys *sequential cut decomposition* if it satisfies the condition satisfied by $\rightarrow_{\mathbf{P}}$ in Theorem 1.2.

To be clear, when we say that Theorem 1.1 "depends on negation" we mean that it is a consequence of Theorem 1.2 plus Gentzen's rules for negation:

$$\frac{\Gamma \rightarrow_{\mathbf{K}} \Delta, \phi}{\neg\phi, \Gamma \rightarrow_{\mathbf{K}} \Delta} \quad \frac{\phi, \Gamma \rightarrow_{\mathbf{K}} \Delta}{\Gamma \rightarrow_{\mathbf{K}} \Delta, \neg\phi}$$

Note that these two rules plus the identity axioms $\phi \rightarrow_{\mathbf{K}} \phi$ imply that (up to equivalence of propositions) $\neg\phi$ is the Boolean complement of ϕ. And yet, it is the presence of Gentzen's rules for negation and not Boolean complements, per se, that yield Theorem 1.1 as a consequence of Theorem 1.2.

The objects of MLS are defined as certain sequent relations that are closed under Cut and enjoy sequential cut decomposition. By the equivalence of [10], these correspond to stably compact locales. The objects of MLS that enjoy parallel cut decomposition are exactly the ones that correspond to compact regular locales by this equivalence. Thus, the important topological property of compact regularity translates to a proof-theoretic property in MLS. Moreover, the patch construction in MLS is a universal solution to the problem of adding Gentzen's rules for negation to an existing object.

Independent of the author, Coquand and Zhang [3] develop a patch construction based on sequents similar to the one given here. Several things, however, distinguish the present work from Coquand and Zhang's. First, the authors develop representations only of the spaces and not the morphisms, so the patch construction as a coreflection in a category of sequents is not explicitly available to them. As a result, their construction can be given predicatively, but its proof of correctness rests on an impredicative equivalence of categories. Also, the present paper considers compact regular spaces in the larger ambient category SCSp^* as reflected in MLS. Another recent work along lines similar to Coquand and Zhang is Vickers' unpublished manuscript [16]. One way to read the Vickers paper is as a careful development of the provenance of a category equivalent to MLS as the Karoubi envelope of a category Ent of sets and (reflexive) entailment relations. The result can be regarded as an extension of Coquand and Zhang's object representations to morphisms, or better yet, a thorough explanation of how these representations arise from standard category theoretic concerns. The paper does not consider the patch construction, but it seems possible that by putting the two works together, the patch construction as a coreflection similar to the one given in this paper would be possible. Nevertheless, in contrast to the present paper, neither [3] nor [16] assume any logical structure on the tokens that comprise sequents, whereas the present paper's characterization of regularity via parallel cut decomposition depends crucially on the logic.

The following section introduces the spatial and localic ideas to be employed in the sequel. Because of the facts summarized in diagram (1), we concentrate on locales. Section 3 introduces the category MLS and its subcategory $\mathsf{MLS}^{\mathsf{u}}$, and sketches the proofs that (a) MLS is equivalent to the category $\mathsf{SCPreLoc}$ of stably compact pre-locales and (b) the equivalence cuts down to an equivalence between $\mathsf{MLS}^{\mathsf{u}}$ and $\mathsf{SCLoc}^{\mathsf{p}}$. In Section 4, useful proof-theoretic techniques are introduced for constructing morphisms of $\mathsf{MLS}^{\mathsf{u}}$. Section 5 characterizes the largest full subcategory of $\mathsf{MLS}^{\mathsf{u}}$ that is equivalent to $\mathsf{KRegLoc}$, and Section 6 establishes the patch construction in $\mathsf{MLS}^{\mathsf{u}}$. This also shows that the objects equipped with opposites, i.e., tokens that relate to other tokens via Gentzen's rules for negation, determine a full subcategory $\mathsf{MLS}^{\mathsf{u}}_{\neg}$ of $\mathsf{MLS}^{\mathsf{u}}$ that is also equivalent to the category $\mathsf{KRegLoc}$. In Section 7, we consider the objects of $\mathsf{MLS}^{\mathsf{u}}_{\neg}$ in the larger ambient category of MLS, showing that their full subcategory $\mathsf{MLS}_{\neg}$ forms an allegory in the sense of Freyd and Scedrov [6]. The anti-involution of this allegory is defined by yet another proof-theoretic concept: the law of contraposition. The proof that this anti-involution satisfies the necessary modularity condition for an allegory, however, seems to require the axiom of choice. Nevertheless, the fact that contraposition is an anti-involution is predicatively valid. The section closes by showing that the category of "maps" in $\mathsf{MLS}_\neg$ is precisely $\mathsf{MLS}^{\mathsf{u}}_\neg$.

2. Stably compact locales and pre-locales

A *frame* is a complete lattice in which finite meets distribute over arbitrary joins. A *frame homomorphism* preserves finite meets and all joins; a *pre-frame homomorphism* preserves finite meets and directed joins. The category (of *pre-locales*) PreLoc is the opposite of the category of frames and pre-frame homomorphisms; the category (of *locales*) Loc is the subcategory of PreLoc determined by frame homomorphisms.

We follow the convention of many writers in the field by viewing a locale as a formal topology, denoting a locale by X and its corresponding frame by $\Omega(X)$. Elements of the lattice $\Omega(X)$ are called "opens" of X and are denoted by U, V and so on; morphisms in Loc are called *continuous maps* and are denoted by f, g and so on. If $f: X \to Y$ is a locale or pre-locale morphism, we denote by $f_*: \Omega(Y) \to \Omega(X)$ the corresponding frame or pre-frame homomorphism.

The following definitions are from [9]. Say that a locale X is *locally compact* if its frame $\Omega(X)$ is a continuous lattice. That is, the map $\bigvee : \mathsf{idl}(\Omega(X)) \to \Omega(X)$ sending order ideals in $\Omega(X)$ to their joins has a lower adjoint: $\Downarrow U \subseteq I$ if and only if $U \leqslant \bigvee I$. This adjoint determines a relation $\ll$ (way-below) on opens by $V \ll U$ if and only if $V \in \Downarrow U$. A locally compact locale is *stably compact* if $\Downarrow$ preserves finite meets. In terms of $\ll$, this means that (a) $U \ll 1$ holds for all U (in particular, $1 \ll 1$) and (b) $U \ll V$ and $U \ll W$ together imply $U \ll V \wedge W$. Condition (a) is compactness, the condition (b) is described by saying that $\ll$ is multiplicative. Let SCLoc and $\mathsf{SCPreLoc}$ denote the full subcategories of Loc and of PreLoc determined by stably compact locales.

Because a frame homomorphism $f_*: \Omega(Y) \to \Omega(X)$ preserves all joins, viewed as a map between posets it has an upper adjoint $f^*: \Omega(X) \to \Omega(Y)$ given by $V \mapsto \bigvee \{U \mid f_*(U) \leqslant V\}$. Say that continuous map $f: X \to Y$ between locally compact locales is *perfect* if f_* preserves $\ll$. This will hold if and only if f^* is Scott continuous. As an upper adjoint, f^* preserves meets, so if f is perfect then f^* is a pre-frame homomorphism. Conversely, if $h: \Omega(Y) \to \Omega(X)$ preserves meets and has an upper adjoint, then it preserves all joins and hence is a frame homomorphism. Thus, the perfect maps between locally compact locales are (opposites of) lower adjoint pre-frame homomorphisms. Let $\mathsf{SCLoc}^{\mathsf{p}}$ denote the category of stably compact locales and perfect maps.

Say that U *is well inside* V (written $U \lessdot V$) iff there exists W so that $W \wedge U = 0$ and $V \vee W = 1$. A locale is *regular* iff every open U is the join of opens well-inside U. Notice that $1 \lessdot 1$ always holds, and $\lessdot$ is multiplicative. Also, the opens well inside any V form an ideal. In a compact locale, $U \lessdot V$ implies $U \ll V$. In a regular locale, $U \ll V$ implies $U \lessdot V$. Furthermore, every frame homomorphism f_* preserves finite joins and meets, and so preserves $\lessdot$. Thus the category of compact regular locales, denoted by $\mathsf{KRegLoc}$, is a full subcategory of $\mathsf{SCLoc}^{\mathsf{p}}$.

A sober topological space is *stably compact* if its frame of opens is stably compact as a frame. Likewise, a sober space is (*compact*) *regular* if its frame of opens is (compact) regular. For a stably compact space X, let X_κ denote the co-compact topology for X. That is, the opens of X_κ are complements of compact saturated sets of X. The perfect continuous maps between stably compact spaces are especially simple to describe: a continuous function f from X to Y is perfect (i.e., f^{-1} preserves $\ll$ on opens) if and only if f is also continuous from X_κ to Y_κ. In a compact regular space, every subset is saturated and compact sets are the same as closed sets. So if X is compact regular, then $X = X_\kappa$. Conversely, for a stably compact space X, if $X = X_\kappa$, then X is regular. The reader may consult [8] for the basic facts regarding stably compact spaces and their relationship to lattice theory and domain theory.

3. MLS

In this section, we introduce the category MLS and sketch the proof that it is equivalent to the category of stably compact pre-locales. MLS can be motivated purely by proof-theoretic concerns, but can also be motivated by considering how one might deal with $\ll$ predicatively. Evidently, to check $U \ll V$ one must universally quantify over ideals. In the case of stably compact locales, the following localic corollary of the Hoffman–Mislove Theorem helps. Let $K(X)$ denote the poset of Scott open filters on $\Omega(X)$. For $U \in \Omega(X)$ and $F \in K(X)$, define a relation $U \sqsubset F$ if and only if $U \leqslant V$ for all $V \in F$.

Theorem 3.1. *In a stably compact locale X, $U \ll V$ if and only if there exists $F \in K(X)$ so that* (i) $U \sqsubset F$ *and* (ii) $V \in F$. *Moreover, $K(X)$ forms the frame of a stably compact locale in which $F \ll G$ holds if and only if there exists an open $U \in \Omega(X)$, so that* (i) $U \sqsubset F$ *and* (ii) $U \in G$.

Thus, the universal quantification over ideals needed to determine $U \ll V$ can be replaced by existential quantification over Scott open filters. This suggests that we may obtain a predicative description of a stably compact locale X by axiomatizing $\ll$ as it relates to $\sqsubset$. Specifically, if $U \sqsubset F$ and $V \in F$, then $U \ll V$. Thus we postulate a set of tokens (surrogates for propositions $U \sqsubset F$) that are closed under formal finite meets and joins, and axiomatize a *sequent relation* $\Vdash$ to capture the conditions $U \sqsubset F$ and $V \in F$. To simplify notation, we follow the standard convention by not distinguishing between a token ϕ and the singleton $\{\phi\}$, and by writing Γ, Γ' to mean the union $\Gamma \cup \Gamma'$ when Γ and Γ' are finite sets of tokens.

Define a *token algebra* to be an algebra for the signature $\langle \wedge, \vee, \top, \bot \rangle$, where $\wedge$ and $\vee$ are binary operators and $\top$ and $\bot$ are constants. In this paper, every token algebra we construct will be a free token algebra, denoted by $T(A)$ where A is the set of generators. On the other hand, we do not need to assume freeness for token algebras that are given. A *consequence relation* from token algebra L to token algebra M is a binary relation on finite subsets of L and M satisfying the familiar positive logical rules of Gentzen's calculus:

$$\frac{}{\bot \vdash}\,(L\bot) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot}\,(R\bot)$$

$$\frac{\Gamma \vdash \Delta}{\top, \Gamma \vdash \Delta}\,(L\top) \qquad \frac{}{\vdash \top}\,(R\top)$$

$$\frac{\phi, \psi, \Gamma \vdash \Delta}{\phi \wedge \psi, \Gamma \vdash \Delta}\,(L\wedge) \qquad \frac{\Gamma \vdash \Delta, \phi \quad \Gamma \vdash \Delta, \psi}{\Gamma \vdash \Delta, \phi \wedge \psi}\,(R\wedge)$$

$$\frac{\phi, \Gamma \vdash \Delta \quad \psi, \Gamma \vdash \Delta}{\phi \vee \psi, \Gamma \vdash \Delta}\,(L\vee) \qquad \frac{\Gamma \vdash \Delta, \phi, \psi}{\Gamma \vdash \Delta, \phi \vee \psi}\,(R\vee)$$

and the structural rule:

$$\frac{\Gamma \vdash \Delta}{\Gamma', \Gamma \vdash \Delta, \Delta'}\,(W)$$

Because we take $\vdash$ as a relation between finite sets, contraction and exchange are implicit. Also, note that because L and M differ, it makes no sense to require $\phi \vdash \phi$.

For two consequence relations $\vdash : L \to M$ and $\vdash' : M \to N$ define the composition $\vdash;\vdash' : L \to N$ by

$$\frac{\Gamma \vdash \phi \qquad \phi \vdash' \Delta}{\Gamma \vdash;\vdash' \Delta}\,(\mathsf{Cut}')$$

We refer to the construction of $\vdash;\vdash'$ as *cut composition*.

Lemma 3.2. *The cut composition of two consequence relations is a consequence relation. Moreover, cut composition is associative.*

A *stable calculus* (*referred to as a continuum sequent* calculus in [10]) is a token algebra L equipped with a consequence relation $\Vdash_L$ from L to L so that the following hold:

(1) $\Vdash_L$ is closed under Cut:

$$\frac{\Gamma_0 \Vdash_L \Delta_0, \alpha \qquad \alpha, \Gamma_1 \Vdash_L \Delta_1}{\Gamma_0, \Gamma_0 \Vdash_L \Delta_0, \Delta_1}\,(\mathsf{Cut})$$

(2) $\Vdash_L$ enjoys sequential cut decomposition: If $\Gamma_0, \Gamma_1 \Vdash_L \Delta_0, \Delta_1$ where either $\Gamma_1 = \emptyset$ or $\Delta_0 = \emptyset$, then there exists a token α so that $\Gamma_0 \Vdash_L \Delta_0, \alpha$ and $\alpha, \Gamma_1 \Vdash_L \Delta_1$.

(3) the logical rules for $\wedge$ and $\vee$ also hold in reverse, e.g., if $\phi \wedge \psi, \Gamma \Vdash_L \Delta$ holds, then $\phi, \psi, \Gamma \Vdash_L \Delta$ holds.

Notice that $\Vdash_L;\Vdash_L \subseteq \Vdash_L$ is a special case of (1) and $\Vdash_L \subseteq \Vdash_L;\Vdash_L$ is a special case of (2). If we were to assume the identity axioms $\phi \Vdash_L \phi$, then (1) would imply both

(2) and (3). So conditions (2) and (3) can be seen as weakened forms of the identity axioms ensuring that $\Vdash_L$ is idempotent with respect to cut composition.

Say that a consequence relation $\vdash$ from L to M is *compatible* with the stable calculi $(L, \Vdash_L)$ and $(M, \Vdash_M)$ provided that $\Vdash_L;\vdash;\Vdash_M \ = \ \vdash$. Let MLS denote the category of stable calculi and compatible consequence relations, writing $\vdash : L \to_c M$ to indicate compatibility.

Theorem 3.3 (Jung et al. [10], Kegelmann [11]). *The category* MLS *is equivalent as an order-enriched category to the category of stably compact pre-locales, where the order on hom-sets in* MLS *is given by* $\subseteq$ *and the order in stably compact pre-locales is given by point-wise order on the corresponding pre-frame homomorphisms. These are equivalent to the category* SCSp^* *of stably compact spaces and binary relations* $R \subseteq X_\kappa \times Y$ *that are compact saturated in the given product topology.*

A detailed proof of the equivalence with the spatial category can be found in [10]. The localic proof can be found in [11]. A sketch of the proof will suffice here.

Consider a stably compact (pre-)locale X. Define the token algebra $\mathsf{lang}(X)$ to be freely generated from pairs $(U, F) \in \Omega(X) \times K(X)$ so that $U \sqsubset F$. To make the notation clearer, we write $[U \sqsubset F]$ for such a pair. Now suppose that $f : X \to Y$ is a pre-locale morphism. The corresponding pre-frame homomorphism f_* is Scott continuous and preserves finite meets, so f_*^{-1} preserves Scott open filters. Thus define $\vdash_f : \mathsf{lang}(X) \to_c \mathsf{lang}(Y)$ as the least consequence relation so that for all pairs $[U_1 \sqsubset F_1], \ldots, [U_m \sqsubset F_m]$ in $\mathsf{lang}(X)$ and $[V_1 \sqsubset G_1], \ldots, [V_n \sqsubset G_n]$ in $\mathsf{lang}(Y)$, the following implication holds:

$$\frac{f_*(V_1 \vee \cdots \vee V_n) \in F_1 \sqcup \cdots \sqcup F_m}{[U_1 \sqsubset F_1], \ldots, [U_m \sqsubset F_m] \vdash_f [V_1 \sqsubset G_1], \ldots, [V_n \sqsubset G_n]} \ (\vdash_f)$$

where $F \sqcup G$ is the join of filters. Now we may check that for an identity locale map $\mathsf{id} : X \to X$, the consequence relation $\vdash_{\mathsf{id}}$ is a stable calculus, and the construction $f \mapsto \vdash_f$ is functorial. To see that it preserves and reflects order, note that the pairs $[U \sqsubset F]$ constitute a sublattice $L(X)$ of $\Omega(X) \times K(X)^{\mathsf{op}}$, so the tokens of $\mathsf{lang}(X)$ can be interpreted in $L(X)$ via the structure map $\eta_X : \mathsf{lang}(X) \to L(X)$. With this, the definition of $\vdash_f$ implies that

$$\frac{\phi_1, \ldots, \phi_m \vdash_f \psi_1, \ldots, \psi_n}{\eta_X(\phi_1 \wedge \cdots \wedge \phi_m) \vdash_f \eta_X(\psi_1 \vee \cdots \vee \psi_n)}$$

where the double bar indicates bi-implication.

Given a compatible consequence relation $\vdash : L \to_c M$ and set $X \subseteq M$, define

$$[\vdash]X = \{\psi \in L \mid \exists \Delta \subseteq X. \psi \vdash \Delta\}.$$

The function $[\vdash](-)$ is obviously Scott continuous from $\mathfrak{P}(M)$ to $\mathfrak{P}(L)$. Moreover, $[\vdash]([\vdash']X) = [\vdash;\vdash']X$ holds for $\vdash : L \to_c M$ and $\vdash' : M \to_c N$. In particular, $[\Vdash_L](-)$ is an idempotent, Scott continuous function on $\mathfrak{P}(L)$ for any stable calculus L. Define

$\mathsf{idl}(L)$ as the image of $[\Vdash_L](-)$, ordered by $\subseteq$. We refer to $I \in \mathsf{idl}(L)$ as a (*round*) *ideal*. Because of compatibility, for any $\vdash : L \to_c M$, $[\vdash](-)$ restricts to a map from $\mathsf{idl}(M)$ to $\mathsf{idl}(L)$.

The poset $\mathsf{idl}(L)$ is continuous as a retract of the continuous lattice $\mathfrak{P}(L)$. Finite meets in $\mathsf{idl}(L)$ are intersections and directed joins are unions, and $[\vdash](-)$ preserves them. Finally, arbitrary joins are given by $[\Vdash_L]\bigcup_i I_i$. Putting these observations together we see that $\mathsf{idl}(L)$ is a stably compact pre-frame, and $[\vdash](-)$ is a pre-frame homomorphism. This provides a contravariant functor into stably compact pre-frames, hence a covariant functor into SCPreLoc. Moreover, $\vdash \subseteq \vdash'$ holds if and only if $[\vdash](I) \subseteq [\vdash'](I)$ holds for all ideals, so the functor also preserves and reflects order.

The reader may consult [10,11] for the verifications that these data define an equivalence between the categories MLS and SCPreLoc. The important point for this paper is that stable calculi are essentially proof-theoretic counter-parts to stably compact (pre-) locales. The compatible consequence relations that correspond to continuous maps can also be characterized syntactically, yielding a subcategory of MLS that is equivalent to the category SCLoc of stably compact locales.

Lemma 3.4 (Jung et al. [10]). *For compatible consequence relation $\vdash : L \to_c M$, the following are equivalent*:
(1) *The map $I \mapsto [\vdash]I$ from $\mathsf{idl}(M)$ to $\mathsf{idl}(L)$ is a frame homomorphism*;
(2) *pre-composition with $\vdash$, as in $\vdash;(-)$, preserves finite joins of compatible consequence relations*;
(3) *whenever $\Gamma \vdash \delta_1,\ldots,\delta_m$ holds, there exists $\lambda_1,\ldots,\lambda_m$ so that* (i) $\Gamma \Vdash_L \lambda_1,\ldots,\lambda_n$ *and* (ii) *for each i, $\lambda_i \vdash \delta_i$.*

The co-compact topology of a stably compact space also has a simple description in MLS. For the purposes of this paper, we need the construction in MLS, but do not explicitly need a proof that it correctly represents the co-compact topology. So we omit the proof.

For a token algebra L, let L_κ denote the token algebra obtained by taking a copy of the tokens in L, but swapping the interpretations of $\perp$ and $\top$, and similarly $\wedge$ and $\vee$. Writing ϕ° for token $\phi \in L$, we have $\top^\circ$ is $\perp$ of L_κ, $(\phi \wedge \psi)^\circ$ is $\phi^\circ \vee \psi^\circ$ of L_κ, etc. For compatible $\vdash : L \to_c M$, let $\vdash_\kappa$ be the consequence relation from M_κ to L_κ defined by

$$\frac{\Gamma \vdash \Delta}{\Delta^\circ \vdash_\kappa \Gamma^\circ}\,(\kappa).$$

It is easy to check that $(-)_\kappa$ is a contravariant functor that exhibits MLS as a self-dual category.

Recall that the category $\mathsf{SCLoc}^{\mathsf{p}}$ (stably compact locales and perfect maps) is equivalent to the category of stably compact locales and adjoint pairs of pre-locale morphisms. By the equivalence of Theorem 3.3, a pair of compatible consequence relations $\vdash^\dagger : L \to_c M$ and $\vdash_\dagger : M \to_c L$ that correspond to such an adjoint pair are also adjoint, in the sense that $\vdash^\dagger;\vdash_\dagger \subseteq \Vdash_L$ and $\Vdash_M \subseteq \vdash_\dagger;\vdash^\dagger$.

Define the category $\mathsf{MLS^u}$ as the category consisting of upper adjoint compatible consequence relations. If $\vdash^{\dagger}$ is a morphism of $\mathsf{MLS^u}$ we will write $\vdash_{\dagger}$ for its lower adjoint. Because the equivalence of Theorem 3.3 preserves order on hom-sets, it cuts down to an equivalence between $\mathsf{MLS^u}$ and stably compact pre-locales with adjoint pairs of pre-locale morphisms. Therefore,

Theorem 3.5. *The category* $\mathsf{MLS^u}$ *is equivalent to the category* $\mathsf{SCLoc^p}$.

4. Pluperfect maps

In this section, we investigate our chief tool for defining adjoint pairs of consequence relations, and show that products in $\mathsf{MLS^u}$ exist. The product functor extends to a functor in MLS, but is not the categorical product there. So we denote it by $\otimes$ to avoid confusion.

Consider a function $h: M \to L$ between the token algebras of two continuous sequent calculi. Say that h *cooperates with* $\Vdash_L$ (or *is cooperative* for short) if the following relations hold:

$$\frac{}{h(\bot) \Vdash_L}\,(Lh\bot) \qquad \frac{\Gamma \Vdash_L \Delta}{\overline{\Gamma \Vdash_L \Delta, h(\bot)}}\,(Rh\bot)$$

$$\frac{\Gamma \Vdash_L \Delta}{\overline{h(\top), \Gamma \Vdash_L \Delta}}\,(Lh\top) \qquad \frac{}{\Vdash_L h(\top)}\,(Rh\top)$$

$$\frac{h(\phi), h(\psi), \Gamma \Vdash_L \Delta}{\overline{h(\phi \wedge \psi), \Gamma \Vdash_L \Delta}}\,(Lh\wedge) \qquad \frac{\Gamma \Vdash_L \Delta, h(\phi) \quad \Gamma \Vdash_L \Delta, h(\psi)}{\overline{\Gamma \Vdash_L \Delta, h(\phi \wedge \psi)}}\,(Rh\wedge)$$

$$\frac{h(\phi), \Gamma \Vdash_L \Delta \quad h(\psi), \Gamma \Vdash_L \Delta}{\overline{h(\phi \vee \psi), \Gamma \Vdash_L \Delta}}\,(Lh\vee) \qquad \frac{\Gamma \Vdash_L \Delta, h(\phi), h(\psi)}{\overline{\Gamma \Vdash_L \Delta, h(\phi \vee \psi)}}\,(Rh\vee)$$

The idea is that h preserves the proof-theoretic interpretation of the logical connectives, even if it does not preserve the algebraic structure "on the nose". The composition of two cooperative maps is cooperative, and any homomorphism of token algebras is cooperative.

Also consider the following properties of maps $h: M \to L$:

- [smooth] Whenever $\Gamma \Vdash_L h(\phi)$, there exists $\phi' \in M$ such that $\phi' \Vdash_M \phi$ and $\Gamma \Vdash_L h(\phi')$. Likewise, whenever $h(\phi) \Vdash_L \Gamma$ there exists $\phi' \in M$ such that $\phi \Vdash_M \phi'$ and $h(\phi') \Vdash_L \Gamma$.
- [$\Vdash$-preserving] $\Delta \Vdash_M \Delta'$ implies $h(\Delta) \Vdash_L h(\Delta')$ (where $h(\Delta)$ is short for $h(\psi_1), \ldots, h(\psi_n)$ whenever $\Delta = \psi_1, \ldots, \psi_n$).
- [$\Vdash$-reflecting] $h(\Delta) \Vdash_L h(\Delta')$ implies $\Delta \Vdash_M \Delta'$.
- [dense] $\Gamma \Vdash_L \Gamma'$ implies that there exists $\phi \in M$ with $\Gamma \Vdash_L h(\phi) \Vdash_L \Gamma'$.

Lemma 4.1. *Suppose $h: M \to L$ is a map between the underlying token algebras of stable calculi. Define relations $\vdash^h \subseteq \mathfrak{P}_{\mathsf{fin}}(L) \times \mathfrak{P}_{\mathsf{fin}}(M)$ and $\vdash_h \subseteq \mathfrak{P}_{\mathsf{fin}}(M) \times \mathfrak{P}_{\mathsf{fin}}(L)$ by* *the proof rules*

$$\frac{\Gamma \Vdash_L h(\Delta)}{\Gamma \vdash^h \Delta}\,(Rh) \qquad \frac{h(\Theta) \Vdash_L \Lambda}{\Theta \vdash_h \Lambda}\,(Lh)$$

(1) *If h is smooth and cooperative then $\vdash^h$ and $\vdash_h$ are compatible consequence relations.*
(2) *If h is a smooth and $\Vdash$-preserving then h is also cooperative and $\vdash^h$ is the upper adjoint to $\vdash_h$. That is, $(\vdash^h;\vdash_h) \subseteq \Vdash_L$ and $\Vdash_M \subseteq (\vdash_h;\vdash^h)$.*
(3) *If h is smooth, cooperative and $\Vdash$-reflecting then $(\vdash_h;\vdash^h) \subseteq \Vdash_M$.*
(4) *If h is smooth, cooperative and dense then $\Vdash_L \subseteq (\vdash^h;\vdash_h)$.*

We refer to a map from M to L that is smooth and $\Vdash$-preserving as a *pluperfect map* because it gives rise to an adjoint pair, and hence to a perfect map from the locale $\mathsf{idl}(L)$ to the locale $\mathsf{idl}(M)$.

Lemma 4.2. *The composition of pluperfect maps is pluperfect. Moreover, if $g : N \to M$ and $h : M \to L$ are both pluperfect, then $\vdash^h;\vdash^g = \vdash^{hg}$ and $\vdash_g;\vdash_h = \vdash_{hg}$. Also, for the identity pluperfect map $\mathsf{id} : L \to L$, $\vdash^{\mathsf{id}} = \Vdash_L$.*

So we can define a category $\mathsf{MLS}^{\mathsf{plu}}$ consisting of stable calculi and pluperfect maps. The assignment $h \mapsto \vdash^h$ is a contravariant functor (remember that we write composition in MLS left to right) from $\mathsf{MLS}^{\mathsf{plu}}$ to $\mathsf{MLS}^{\mathsf{u}}$. One can easily construct examples to show that this functor is neither faithful nor full, but the following gives a very simple relation between pluperfect maps that is equivalent to the order on hom-sets in $\mathsf{MLS}^{\mathsf{u}}$.

Lemma 4.3. *For pluperfect maps $g : M \to L$ and $h : M \to L$, $\vdash^g \subseteq \vdash^h$ if and only if for all $\Gamma, \Delta \subseteq M$ the following implication holds*:

$$\frac{\Gamma \Vdash_M \Delta}{g(\Gamma) \Vdash_L h(\Delta)}\,(g \leqslant h).$$

Proof. Suppose the implication $(g \leqslant h)$ holds. Consider $\Theta \vdash^g \Delta$. Then $\Theta \Vdash_L g(\Delta)$. By smoothness, we can find a token $\phi \in M$ so that $\Theta \Vdash_L g(\phi)$ and $\phi \Vdash_M \Delta$. Thus

$$\cfrac{\cfrac{\Theta \Vdash_L g(\phi) \qquad \cfrac{\phi \Vdash_M \Delta}{g(\phi) \Vdash_L h(\Delta)}\,(g \leqslant h)}{\Theta \Vdash_L h(\Delta)}\,(\mathsf{Cut})}{\Theta \vdash^h \Delta}\,(Rh)$$

Suppose $\vdash^g \subseteq \vdash^h$. Then $\Vdash_M \subseteq (\vdash_g;\vdash^g) \subseteq (\vdash_g;\vdash^h)$. So if $\Gamma \Vdash_M \Delta$, then there exists a token in L for which $\Gamma \vdash_g \phi \vdash^h \Delta$. That is, $g(\Gamma) \Vdash_L \phi \Vdash_L h(\Delta)$. $\square$

The functor $h \mapsto \vdash^h$ from $(\mathsf{MLS}^{\mathsf{plu}})^{\mathsf{op}}$ to $\mathsf{MLS}^{\mathsf{u}}$ has a full and faithful right adjoint, but the construction of this adjoint requires quantification over all filters and ideals of stable calculi. For our purposes, we will only need a local form of the adjunction that can be carried out predicatively.

Lemma 4.4. *Suppose* $\vdash^{\dagger_1} : L \to_c M_1, \ldots, \vdash^{\dagger_m} : L \to_c M_m$ *are upper adjoint consequence relations with corresponding lower adjoints* $\vdash_{\dagger_i} : M_i \to_c L$. *Then there is a continuous sequent calculus* $(\hat{L}, \Vdash_{\hat{L}})$ *and pluperfect maps* $\lambda : L \to \hat{L}$ *and* $\mu_i : M_i \to \hat{L}$ *so that* (i) λ *is also dense and* $\Vdash$*-reflecting* (*hence* $\vdash^{\lambda}$ *is an isomorphism*), (ii) $\vdash^{\dagger_i} = \vdash_{\lambda} ; \vdash^{\mu_i}$ *and* (iii) $\vdash_{\dagger_i} = \vdash_{\mu_i} ; \vdash^{\lambda}$.

Proof. To construct $\hat{L}$ predicatively, we use the data L, M_i, $\vdash^{\dagger_i}$ and the corresponding lower adjoints $\vdash_{\dagger_i}$ directly. The proof for $m > 1$ is essentially the same as the case of a single consequence relation, so we prove only the unary case here.

Let $\vdash^{\dagger} : L \to_c M$ have lower adjoint $\vdash_{\dagger} : M \to_c L$. Define the token algebra for $\hat{L}$ to be $T(L \uplus M)$. To be explicit, we write $\lambda\phi$ and $\mu\psi$ for generators with $\phi \in L$ and $\psi \in M$.

Take $\Vdash_{\hat{L}}$ to be the least consequence relation on $\hat{L}$ satisfying the following rule:

$$\frac{\Gamma \vdash_{\dagger} \alpha \qquad \alpha, \Theta \Vdash_L \Delta, \beta \qquad \beta \vdash^{\dagger} \Lambda}{\mu(\Gamma), \lambda(\Theta) \Vdash_{\hat{L}} \lambda(\Delta), \mu(\Lambda)}$$

We must check that $\Vdash_{\hat{L}}$ is a continuous sequent calculus, but this amounts to checking that the consequences of this rule are closed under weakening and Cut and that the maps $\phi \mapsto \lambda\phi$ and $\psi \mapsto \mu\psi$ are smooth. Also note that

(1) $\Theta \Vdash_L \Delta$ if and only if $\lambda(\Theta) \Vdash_{\hat{L}} \lambda(\Delta)$,
(2) $\Gamma \Vdash_M \Lambda$ implies $\mu(\Gamma) \Vdash_{\hat{L}} \mu(\Lambda)$,
(3) $\Theta \vdash^{\dagger} \Lambda$ if and only if $\lambda(\Theta) \Vdash_{\hat{L}} \mu(\Lambda)$, and
(4) $\Theta \vdash_{\dagger} \Delta$ if and only if $\mu(\Theta) \Vdash_{\hat{L}} \lambda(\Delta)$.

Thus λ and μ are pluperfect, and λ is $\Vdash$-reflecting. Furthermore, by induction on the ranks of tokens in $\Xi \Vdash_{\hat{L}} \Psi$, there exist $\xi, \psi \in L$ so that $\Xi \Vdash_{\hat{L}} \lambda(\xi)$, $\xi \Vdash_L \psi$ and $\lambda(\psi) \Vdash_{\hat{L}} \Psi$. That is, λ is also dense.

Finally, the fact that $\vdash^{\dagger} = \vdash_{\lambda} ; \vdash^{\mu}$ follows directly from the above characterization of $\vdash^{\dagger}$. □

This result can be generalized with a predicative proof to arbitrary finite diagrams, but for simplicity we have only stated it in the special case that is useful in this paper.

To illustrate the utility of pluperfect maps we construct binary products in $\mathsf{MLS}^{\mathsf{u}}$ together with the projections. For token algebras L_0 and L_1 define $L_0 \otimes L_1$ to be $T(L_0 \uplus L_1)$. We write generators as $0{:}\,\phi$ and $1{:}\,\psi$ for $\phi \in L_0$ and $\psi \in L_1$. For two compatible consequence relations (not necessarily upper adjoints) $\vdash_0 : L_0 \to_c M_0$ and $\vdash_1 : L_1 \to_c M_1$, define a consequence relation $\vdash_0 \otimes \vdash_1$ to be the least consequence relation from $L_0 \otimes L_1$ to $M_0 \otimes M_1$ satisfying

$$\frac{\Gamma \vdash_0 \Delta}{0{:}\,\Gamma \vdash_0 \otimes \vdash_1 0{:}\,\Delta}\,(0{:}\,) \qquad \frac{\Gamma \vdash_1 \Delta}{1{:}\,\Gamma \vdash_0 \otimes \vdash_1 1{:}\,\Delta}\,(1{:}\,)$$

This defines a binary endo-functor $\otimes$ on MLS that clearly is order-preserving. Hence $\otimes$ cuts down to a functor on $\mathsf{MLS}^{\mathsf{u}}$.

The rules defining $\vdash_0 \otimes \vdash_1$ show immediately that the maps 0: and 1: defined by $\phi \mapsto 0{:}\,\phi$ and $\psi \mapsto 1{:}\,\psi$ are pluperfect and $\Vdash$-reflecting. Thus, $\vdash^{0:}$ and $\vdash^{1:}$ are typed correctly to be the projections. To show that these are indeed the projections, consider a pair of upper adjoints $\vdash^{\dagger_0}\ : M \to_c L_0$ and $\vdash^{\dagger_1} : M \to_c L_1$. By Lemma **??**, we can assume that $\vdash^{\dagger_0}$ and $\vdash^{\dagger_1}$ are given by pluperfect maps, h_0 and h_1. Define $\langle h_0, h_1\rangle$ to be the homomorphism of token algebras from $L_0 \otimes L_1$ to M given by $0{:}\,\phi \mapsto h_0(\phi)$ and $1{:}\,\psi \mapsto h_1(\psi)$. So $\langle h_0, h_1\rangle$ is pluperfect, and $\langle h_0, h_1\rangle \circ 0 :\ = h_0$ and $\langle h_0, h_1\rangle \circ 1 :\ = h_1$. Thus, $\vdash^{\langle h_0, h_1\rangle}$ is the desired mediating morphism from M to $L_0 \otimes L_1$. To show uniqueness, one can employ Lemma 4.3 to show that if $\vdash^{g};\vdash^{0:}\ =\ \vdash^{h_0}$ and $\vdash^{g};\vdash^{1:}\ =\ \vdash^{h_1}$, then $\vdash^{g}\ =\ \vdash^{\langle h_0, h_1\rangle}$.

5. Compact regular calculi

In this section, we characterize the objects L of MLS for which $\mathsf{idl}(L)$ is a compact regular locale. It turns out that compact regularity is equivalent to parallel cut decomposition.

To characterize regularity of $\mathsf{idl}(L)$, consider the relations $\ll$ and $\lessdot$ in $\mathsf{idl}(L)$ for a stable calculus L.

Lemma 5.1. *For stable calculus L and ideals $I, J \in \mathsf{idl}(L)$,*
(1) *$I \ll J$ if and only if there exists $\psi \in J$ so that for all $\phi \in I$, $\phi \Vdash_L \psi$;*
(2) *$I \wedge J = 0$ if and only if for all $\phi \in I$ and all $\psi \in J$, $\phi, \psi \Vdash_L$;*
(3) *$I \vee J = 1$ if and only if there exists $\phi \in I$ and $\psi \in J$ for which $\Vdash_L \phi, \psi$;*
(4) *$I \lessdot J$ if and only if there exists $\alpha \in L$ so that* (a) *for all $\phi \in I$, $\alpha, \phi \Vdash_L$ and* (b) *there exists $\psi \in J$ for which $\Vdash_L \psi, \alpha$.*

Proof. Every ideal J is the directed union of ideals of the form $[\Vdash_L]\psi$ for $\psi \in J$. Thus if $I \ll J$, then $I \subseteq [\Vdash_L]\psi$ for some $\psi \in J$. Conversely, suppose that $\psi \in J$ is such that $\phi \Vdash_L \psi$ holds for all $\phi \in I$. Let $\{K_i\}_i$ be a directed set of ideals such that $J \subseteq \bigcup_i K_i$. Then $\phi \in K_i$ for some i. Hence $I \subseteq [\Vdash_L]\psi \subseteq K_i$.

The meet of ideals is their intersection. So the second item follows from the fact that $\phi \in I$ and $\psi \in J$ implies $\phi \wedge \psi \in I \cap J$.

The join of ideals is computed as $[\Vdash_L](I \cup J)$. So $I \vee J = 1$ implies that $\top \Vdash_L \phi, \psi$ for some $\phi \in I$ and $\psi \in J$. Conversely, if $\Vdash_L \phi, \psi$ then $\theta \Vdash_L \phi, \psi$ holds for all θ.

Suppose $K \wedge I = 0$ and $J \vee K = 1$. Then for all $\alpha \in K$ and all $\phi \in I$, $\alpha, \phi \Vdash_L$, and for some $\alpha \in K$ and some $\psi \in J$, $\Vdash_L \psi, \alpha$. Hence for some $\alpha \in K$, (a) for all $\phi \in L$, $\alpha, \phi \Vdash_L$ and (b) for some $\psi \in J$, $\Vdash_L \psi, \alpha$. Conversely, suppose that α is such that (a) holds, and suppose $\psi \in J$ is such that $\Vdash_L \psi, \alpha$ holds. Using sequential cut decomposition, find α' so that $\Vdash_L \psi, \alpha'$ and $\alpha' \Vdash_L \alpha$. Define $K_0 = \{\beta \in L \mid \forall \phi \in I. \beta, \phi \Vdash_L\}$, and $K = [\Vdash_L]K_0$. Certainly, K is an ideal. Consider $\gamma \in K$ and $\phi \in I$. Then by a cuts with members of K_0, we have $\gamma, \phi \Vdash_L$. Because $\alpha \in K_0$, $\alpha' \in K$. Hence $J \vee K = 1$. $\square$

Recall that we say that a sequent relation R enjoys parallel cut decomposition if and only if whenever $\Gamma_0, \Gamma_1\ R\ \Delta_0, \Delta_1$, there exists α so that $\Gamma_0\ R\ \Delta_0, \alpha$ and $\alpha, \Gamma_1\ R\ \Delta_1$.

Lemma 5.2. *For a stable calculus L, the following are equivalent*:
(1) $\mathsf{idl}(L)$ *is regular*.
(2) $\Theta \Vdash_L \Delta$ *holds if and only if there exists* α *so that* $\alpha, \Theta \Vdash_L$ *and* $\Vdash_L \Delta, \alpha$;
(3) $\Vdash_L$ *enjoys parallel cut decomposition*.

Proof. Suppose that $\mathsf{idl}(L)$ is regular, so that $I \lll J$ implies $I \leqslant J$. Consider $\Theta \Vdash_L \Delta$. By sequential cut decomposition choose β so that $\Theta \Vdash_L \beta \Vdash_L \Delta$. In particular, then $([\Vdash_L]\beta) \lll ([\Vdash_L]\Delta)$. So there exists a token α such that (a) $\phi \Vdash_L \beta$ implies $\alpha, \phi \Vdash_L$ and (b) there exists ψ such that $\phi \Vdash_L \Delta$ and $\Vdash_L \phi, \alpha$. By (a) and a use of Cut, $\alpha, \Theta \Vdash_L$. By (b) and a use of Cut, $\Vdash_L \Delta, \alpha$.

Suppose (2) holds and $I \lll J$. That is, there exists $\psi \in J$ so that $\phi \in I$ implies $\phi \Vdash_L \psi$. Because J is a round ideal, there is also second token $\psi' \in J$ so that $\psi \Vdash_L \psi'$. By (2), find α for which $\alpha, \psi \Vdash_L$ and $\Vdash_L \psi', \alpha$. Thus for all $\phi \in I$, $\alpha, \phi \Vdash_L$.

Clearly (2) is a special case of (3). So it remains to show that (2) implies (3). Suppose $\Gamma, \Theta \Vdash_L \Delta, \Lambda$. Then find tokens ϕ and ψ such that $\Gamma \Vdash_L \phi$, $\phi, \Theta \Vdash_L \Delta, \psi$ and $\psi \Vdash_L \Lambda$. By (2), find a token α so that $\alpha, \phi, \Theta \Vdash_L$ and $\Vdash_L \Delta, \psi, \alpha$. By weakenings, $\Gamma \Vdash_L \Delta, \phi$ and $\Gamma \Vdash_L \Delta, \psi, \alpha$. Similarly, $\alpha, \phi, \Theta \Vdash_L \Lambda$ and $\psi, \Theta \Vdash_L \Lambda$. So by a series of applications of the logical rules, $(\phi \wedge \alpha) \vee \psi$ is the desired token. $\square$

Say that a stable calculus is *regular* if it enjoys parallel cut decomposition, and let $\mathsf{MLS}^{\mathsf{u}}_{\mathsf{r}}$ denote the full subcategory of $\mathsf{MLS}^{\mathsf{u}}$ consisting of regular calculi. Then,

Theorem 5.3. $\mathsf{MLS}^{\mathsf{u}}_{\mathsf{r}}$ *is the largest full subcategory of* $\mathsf{MLS}^{\mathsf{u}}$ *for which the equivalence* $\mathsf{MLS}^{\mathsf{u}} \equiv \mathsf{SCLoc}^{\mathsf{p}}$ *cuts down to an equivalence with* $\mathsf{KRegLoc}$.

6. Opposites in MLS

In this section, we consider objects of MLS that are equipped with "opposites" (to distinguish from "negations" as complements) and construct a coreflection of $\mathsf{MLS}^{\mathsf{u}}$ in the full subcategory determined by these objects. Earlier versions of the paper used the term "negation", but as an anomymous reviewer and others have pointed out, the reader has a reasonable expectation to interpret negation as a complement. While opposites do not behave semantically as complements, they do behave proof-theoretically exactly in accord with Gentzen's rules for negation in systems **K** and **J**. Thus, although "negation" is historically justifiable, the word "opposite" helps to avoid confusion. In my view, the confusion arises from the fact that in the case of spectral frames, opposites and Boolean complements conflate.

In a stable calculus L, say that two tokens $\phi, \phi' \in L$ are *opposite* (written $\phi \sharp \phi'$) if and only if for every Γ and Δ, both of the following hold:

$$\frac{\Gamma \Vdash \Delta, \phi}{\phi', \Gamma \Vdash \Delta}\,[\mathrm{L}\,\sharp] \qquad \frac{\phi, \Gamma \Vdash \Delta}{\Gamma \Vdash \Delta, \phi'}\,[\mathrm{R}\,\sharp]$$

A *separated calculus* is a stable calculus in which every token has an opposite. A *canonically separated calculus* is a stable calculus L equipped with a function

$\neg : L \to L$ such that $\neg\phi \sharp \phi$ for all tokens $\phi \in L$. Neither the law of the excluded middle ($\Vdash_L \phi, \neg\phi$) nor the law of non-contradiction ($\neg\phi, \phi \Vdash_L$) necessarily holds in a canonically separated calculus, in spite of the fact that the classical rules for negation obtain. This emphasizes the fact that opposites are not the same as Boolean complements. Let $\mathsf{MLS}_\neg$ denote the full subcategory of MLS determined by canonically separated calculi. Likewise, let $\mathsf{MLS}^{\mathsf{u}}_\neg$ denote the corresponding full subcategory of $\mathsf{MLS}^{\mathsf{u}}$. In the presence of choice, obviously $\mathsf{MLS}_\neg$ is equivalent to the subcategory consisting of separated calculi. Most of the results of this section do not make use of canonicity, as the statements and proofs make clear. Nevertheless, a key lemma in proving the universality of our patch construction employs a pluperfect map that is defined in terms of canonical opposites. In principle, this could be avoided by developing a relational analogue of pluperfect map.

To avoid some possible confusion, we note that opposites behave proof-theoretically like involutions. In a stable calculus L, say that two tokens $\phi, \phi' \in L$ are *equivalent* (written $\phi \equiv \phi'$) if and only if they have the same behavior proof-theoretically. That is, for every Γ and Δ, both of the following hold:

$$\frac{\phi, \Gamma \Vdash \Delta}{\phi', \Gamma \Vdash \Delta}\,[\mathrm{L}\equiv] \quad \frac{\Gamma \Vdash \Delta, \phi}{\Gamma \Vdash \Delta, \phi'}\,[\mathrm{R}\equiv]$$

Then

Lemma 6.1. *Let $\phi, \phi', \psi, \psi \in L$ be tokens of a stable calculus.*
- *If $\phi \equiv \phi' \sharp \psi$, then $\phi \sharp \psi$.*
- *If $\phi \sharp \psi \sharp \phi'$, then $\phi \equiv \phi'$.*

Proof. Obvious. □

The reader is invited to check that $\equiv$ is indeed a congruence on the algebra L, and the quotient $L/\equiv$ is a distributive lattice. If L is separated, then $L/\equiv$ is a de Morgan lattice, i.e., a distributive lattice with involution with respect to which $\wedge$ and $\vee$ satisfy the de Morgan laws.

Lemma 6.2. *Let $h : M \to L$ be a pluperfect map. Then h preserves $\equiv$. Moreover, $[h]$ is a homomorphism.*

Proof. Suppose $\phi \equiv \psi$ and $h(\phi), \Gamma \Vdash_L \Delta$. Because h is smooth, there exists $\theta \in M$ so that $\phi \Vdash_M \theta$ and $h(\theta), \Gamma \Vdash_L \Delta$. By equivalence, $\psi \Vdash_M \theta$. Because h is $\Vdash$-preserving, $h(\psi) \Vdash_L h(\theta)$. So an application of cut yields $h(\psi), \Gamma \Vdash_L \Delta$. The proof for $h(\phi)$ appearing on the right is identical.

The fact that $[h]$ is a homomorphism follows from cooperativity of h. □

With these technicalities, we make the relation between opposites and regularity clear.

Lemma 6.3. *A separated sequent calculus L is regular.*

Proof. Suppose $\Gamma \Vdash_L \Delta$. By sequential cut decomposition, $\Gamma \Vdash_L \alpha \Vdash_L \Delta$ holds for some α. Let β be an opposite of α. Then $\beta, \Gamma \Vdash_L$ and $\Vdash_L \Delta, \beta$. □

Next, we turn attention to the patch construction. The spatial intuition is that $\mathsf{Patch}(X)$ is (homeomorphic to) the "skewed" diagonal in $X \times X_\kappa$. That is, it is embedded in $X \times X_\kappa$ via the (perfect) continuous map $x \mapsto (x,x)$. So we construct the patch of a stable calculus L as a sub-object of $L \otimes L_\kappa$. To make the notation clearer, however, we define $\mathsf{Patch}(L)$ on an isomorphic copy of the token algebra $L \otimes L_\kappa$, Thus take the token algebra $\mathsf{Patch}(L)$ to be freely generated by $+\phi$ and $-\phi^\circ$ for $\phi \in L$. Let $i : L \otimes L_\kappa \to \mathsf{Patch}(L)$ be the obvious isomorphism of token algebras.

Now define $\Vdash_{\mathsf{Patch}(L)}$ to be the least consequence relation on $\mathsf{Patch}(L)$ satisfying:

$$\frac{\Theta, \Gamma \Vdash_L \Delta, \Lambda}{-\Lambda^\circ, +\Gamma \Vdash_{\mathsf{Patch}(L)} +\Delta, -\Theta^\circ}$$

Lemma 6.4. *The consequence relation* $\Vdash_{\mathsf{Patch}(L)}$ *is a canonically separated calculus. Moreover, the homomorphism i from* $L \otimes L_\kappa$ *to* $\mathsf{Patch}(L)$ *is pluperfect and dense.*

Proof. Restricted to generators, $\Vdash_{\mathsf{Patch}(L)}$ is obviously closed under weakening and Cut. Moreover, it has interpolants because $\Vdash_L$ has them. So $\Vdash_{\mathsf{Patch}(L)}$ is a stable calculus. Define $\neg\phi$ in the obvious way: $\neg(+\phi) = -\phi^\circ$, $\neg(-\phi^\circ) = +\phi$, $\neg(\phi \wedge \psi) = \neg\phi \vee \neg\psi$, and so on. Induction on the ranks of tokens shows that this defines canonical opposites for $\mathsf{Patch}(L)$.

If $1 : \Lambda^\circ, 0 : \Gamma \Vdash_{L \otimes L_\kappa} 0 : \Delta, 1 : \Theta^\circ$, then either $\Lambda^\circ \Vdash_{L_\kappa} \Theta^\circ$ or $\Gamma \Vdash_L \Delta$. In either case, i is $\Vdash$-preserving when restricted to generators. It is smooth when restricted to generators because the maps $\phi \mapsto 0 : \phi$ and $\phi^\circ \mapsto 1 : \phi^\circ$ are smooth. Induction on ranks of tokens shows that it is fully $\Vdash$-preserving and smooth. Any onto map is dense. □

The reader may think of $\mathsf{Patch}(L)$ as embedded in the product $L \otimes L_\kappa$ by the (regular) monomorphism $\vdash^i$. Define the map $+ : L \to \mathsf{Patch}(L)$ by $\phi \mapsto +\phi$ and similarly $- : L_\kappa \to \mathsf{Patch}(L)$. Clearly, $+ = i \circ 0{:}$ and $- = i \circ 1{:}$, so both maps are pluperfect. Both $+$ and $-$ are also $\Vdash$-reflecting, so $\vdash_+ ; \vdash^+ = \Vdash_L$ and $\vdash_- ; \vdash^- = \Vdash_{L\kappa}$. In particular, both $\vdash^+$ and $\vdash^-$ are epimorphisms in MLS. Moreover,

Lemma 6.5. *The consequence relations* $\vdash^+$ *and* $\vdash^-$ *are monomorphisms in the category* $\mathsf{MLS^u}$.

Proof. Suppose that $\vdash^g : M \to_c \mathsf{Patch}(L)$ and $\vdash^h : M \to_c \mathsf{Patch}(L)$ are upper adjoints such that $\vdash^g ; \vdash^+ = \vdash^h ; \vdash^+$. Without loss of generality, assume these are given by pluperfect maps h and g. Then by Lemma 4.3, $\Gamma \Vdash_L \Delta$ implies both $g(+(\Gamma)) \Vdash_M h(+(\Delta))$ and $h(+(\Gamma)) \Vdash_M g(+(\Delta))$.

To prove that $\vdash^g = \vdash^h$, we must show that

$$\frac{\Theta \Vdash_{\mathsf{Patch}(L)} \Lambda}{g(\Theta) \Vdash_M h(\Delta)}$$

and likewise for g and h swapped. It suffices to proof this for the basis of an induction on ranks of tokens because g and h are cooperative.

Consider $-\Delta_1^\circ, +\Gamma_0 \Vdash_{\mathsf{Patch}(L)} + \Delta_0, -\Gamma_1^\circ$. Then $\Gamma_1, \Gamma_0 \Vdash_L \Delta_0, \Delta_1$. By interpolating, find tokens ϕ, ϕ', ψ and ψ', so that

(1) $\Gamma_1 \Vdash_L \phi$,
(2) $\phi \Vdash_L \phi'$,
(3) $\phi', \Gamma_0 \Vdash_L \Delta_0, \psi'$,
(4) $\psi' \Vdash_L \psi$ and
(5) $\psi \Vdash_L \Delta_1$.

Thus,

$$\dfrac{\dfrac{\Gamma_1 \Vdash_L \phi}{-\phi^\circ \Vdash_{\mathsf{Patch}(L)} -\Gamma_1^\circ}}{g(-\phi^\circ) \Vdash_M g(-\Gamma_1^\circ)}$$

Applying similar derivations to items (2), (4) and (5), we get (2′) $\Vdash_M g(-\phi^\circ), g(+\phi')$, (4′) $h(-\psi^\circ), h(+\psi') \Vdash_{\mathsf{Patch}(L)}$ and (5′) $h(-\Delta_1^\circ) \Vdash_{\mathsf{Patch}(L)} h(-\psi^\circ)$. Combining by Cut yields the two sequents $\Vdash_M g(-\Gamma_1^\circ), g(+\phi')$ and $h(-\Delta_1^\circ), h(+\psi')$. Finally, because $\vdash^g; \vdash^+ = \vdash^h; \vdash^+$, we can apply Lemma 4.3 to item (3), obtaining $g(+\phi'), g(+\Gamma_0) \Vdash_M h(+\Delta_0), h(+\psi')$. One more application of Cut yields $g(-\Delta_1^\circ, +\Gamma_0) \Vdash_M h(+\Delta_0, -\Gamma_1^\circ)$. So $\vdash^g \subseteq \vdash^h$. The symmetric argument shows the other containment. The proof that $\vdash^-$ is a monomorphism is identical. □

The next lemma shows that if L is regular, then the canonically separated $\mathsf{Patch}(L)$ is isomorphic to L in the category $\mathsf{MLS}^{\mathsf{u}}$.

Lemma 6.6. *If L is regular, then $+ : L \to \mathsf{Patch}(L)$ is dense.*

Proof. Suppose L is regular and $-\Delta_0, +\Gamma_0 \Vdash_{\mathsf{Patch}(L)} + \Delta_1, -\Gamma_1$ holds. Then $\Gamma_1, \Gamma_0 \Vdash_L \Delta_1, \Delta_0$ also holds. By regularity, there is a token α so that

$$\Gamma_0 \Vdash_L \Delta_0, \alpha \qquad \text{and} \qquad \alpha, \Gamma_1 \Vdash_L \Delta_1.$$

Hence,

$$-\Delta_0^\circ, +\Gamma_0 \Vdash_{\mathsf{Patch}(L)} +\alpha \qquad \text{and} \qquad +\alpha \Vdash_{\mathsf{Patch}(L)} +\Delta_1, -\Gamma_1^\circ.$$

The result now follows by induction on ranks of tokens. □

Remember that we intend to think of $\mathsf{Patch}(L)$ as isomorphic to the "skewed" diagonal of $L \otimes L_\kappa$. Thus, for an upper adjoint $\vdash^\dagger : M \to_c L$ where M is canonically separated, we actually wish to think of it as determining a "curve" in $L \otimes L_\kappa$, and then to show that the image of this curve lies entirely on the diagonal. So, we need to find a morphism from M to $L \otimes L_\kappa$. But this amounts to finding a morphism $\vdash^{\hat{h}}$ from M to L_κ and taking the mediating morphism $\vdash^{\langle h, \hat{h} \rangle}$.

Lemma 6.7. *Suppose $h : L \to M$ is a pluperfect map and M is canonically separated. Define $\hat{h} : L_\kappa \to M$ by $\phi^\circ \mapsto \neg h(\phi)$. Then $\hat{h}$ is also pluperfect.*

Proof. $\hat{h}$ is $\Vdash$-preserving:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma^\circ \Vdash_{L_\kappa} \Delta^\circ}{\Delta \Vdash_L \Gamma}}{h(\Delta) \Vdash_M h(\Gamma)}}{\neg h(\Gamma) \Vdash_M \neg h(\Delta)}}{\hat{h}(\Gamma^\circ) \Vdash_M \hat{h}(\Delta^\circ)}$$

If $\hat{h}(\phi^\circ), \Theta \Vdash_M \Lambda$, then $\Theta \Vdash_M \Lambda, h(\phi)$. By the smoothness of h, there exists ψ so that $\psi \Vdash_L \phi$ and $\Theta \Vdash_M \Lambda, h(\psi)$. Hence $\hat{h}(\psi^\circ), \Theta \Vdash_M \Lambda$ and $\phi^\circ \Vdash_{L_\kappa} \psi^\circ$. The symmetric argument obtains for $\hat{h}(\phi^\circ)$ appearing on the right. So $\hat{h}$ is also smooth. □

Suppose that $\vdash^{\langle h,\hat{h}\rangle}$ factors through $\vdash^i$. That is, suppose there is a pluperfect map $g : \mathsf{Patch}(L) \to M$ satisfying $\vdash^{\langle h,\hat{h}\rangle} = \vdash^g ; \vdash^i = \vdash^{g \circ i}$. Then

$$\begin{aligned} \vdash^g ; \vdash^+ &= \vdash^{g \circ +} \\ &= \vdash^{g \circ i \circ 0:} \\ &= \vdash^{\langle h,\hat{h}\rangle} ; \vdash^{0:} \\ &= \vdash^h \end{aligned}$$

And $\vdash^g$ is unique as an upper adjoint with this property because $\vdash^+$ is a monomorphism. The obvious choice for g is the homomorphism $\langle h, \hat{h}\rangle \circ i^{-1}$. Note, however, that i^{-1} by itself is not $\Vdash$-preserving, so we can not simply conclude that g is pluperfect as a composite of pluperfect maps.

Lemma 6.8. *Suppose $h : L \to M$ is a pluperfect map with M a canonically separated calculus. Then the homomorphism $g = \langle h, \hat{h}\rangle \circ i^{-1}$ from $\mathsf{Patch}(L)$ to M is also pluperfect.*

Proof. The map g is determined by its value on generators: $+\phi \mapsto h(\phi)$ and $-\phi^\circ \mapsto \neg h(\phi)$. To show that g is pluperfect, consider a sequent of $\mathsf{Patch}(L)$ consisting of generators only.

$$\cfrac{\cfrac{\cfrac{-\Delta_1^\circ, +\Gamma_0 \Vdash_{\mathsf{Patch}(L)} +\Delta_0, -\Gamma_1^\circ}{\Gamma_1, \Gamma_0 \Vdash_L \Delta_0, \Delta_1}}{h(\Gamma_1), h(\Gamma_0) \Vdash_M h(\Delta_0), h(\Delta_1)}}{\neg h(\Delta_1), h(\Gamma_0) \Vdash_M h(\Delta_0), \neg(\Gamma_1)}$$

The last sequent in this derivation is $g(-\Delta_1^\circ), g(+\Gamma_0) \Vdash_M g(+\Delta_0), g(-\Gamma_1^\circ)$, so g, restricted to generators, is $\Vdash$-preserving. For smoothness, again we can consider generators first. Suppose $g(+\phi) \Vdash_M \Delta$. Then $h(\phi) \Vdash_M \Delta$. So smoothness of h yields a ψ so that $\phi \Vdash_L \psi$ and $h(\psi) \Vdash_M \Delta$. Thus $+\psi$ is the desired token. The other cases of generators are proved similarly. The general case of $\Vdash$-preservation and smoothness follows by induction, using the fact that g is a homomorphism and is thus cooperative. □

Now we may summarize with the main result of this section.

Theorem 6.9. *The construction* $\mathsf{Patch}(L)$ *constitutes a coreflection of the category* $\mathsf{MLS^u}$ *in* $\mathsf{MLS^u_\neg}$ *with co-unit* $\vdash^+ : \mathsf{Patch}(L) \to L$.

Proof. Given upper adjoint $\vdash^\dagger : M \to_c L$ with M a canonically separated calculus, assume without loss of generality that $\vdash^\dagger = \vdash^h$ for some pluperfect h. Define $g : \mathsf{Patch}(L) \to M$ as $\langle h, \hat{h} \rangle \circ i^{-1}$. Then $g \circ + = h$ so $\vdash^g ; \vdash^+ = \vdash^h$. Uniqueness is immediate because $\vdash^+$ is monomorphic in $\mathsf{MLS^u}$. □

Corollary 6.10. *The full subcategories* $\mathsf{MLS^u_\neg}$ *and* $\mathsf{MLS^u_r}$ *of* $\mathsf{MLS^u}$ *are equivalent.*

Proof. If L is regular, then $\vdash^+$ is an isomorphism. So the patch coreflection cuts down to an equivalence. □

As briefly noted at the beginning of this section, the existence of canonical opposites is not in general the same as the existence of Boolean complements. On the other hand, if L is reflexive ($\phi \Vdash_L \phi$ holds for each token) and separated, then the lattice $L/\equiv$ is Boolean. As Escardó [4,5] notes, the patch construction on spectral frames is a universal solution to the problem of adding Boolean complements (of the compact opens). The results of this section suggest that it is better to think in terms of adding opposites.

The foregoing construction via pluperfect maps almost shows that $\mathsf{Patch}(L)$ is the free canonically separated object over L in the category $\mathsf{MLS^{plu}}$. What is missing is only the uniqueness of the mediating morphism $\langle h, \hat{h} \rangle \circ i^{-1}$ for a given pluperfect map $h : L \to M$ into a canonically separated object M. The objective of finding a unique pluperfect map is hopelessly confounded, however, by our decisions (a) to base all constructions on free token algebras (in the spirit of traditional logic) and (b) to give a liberal definition of pluperfect maps that permits non-homomorphisms. Had we constructed $\mathsf{Patch}(L)$ as the co-product algebra $L + L_\kappa$ and insisted that pluperfect maps must be homomorphisms, then $\langle h, \hat{h} \rangle$ would actually be the mediating morphism for the co-product, and uniqueness would follow from the fact that i is an isomorphism of algebras. In light of our decisions, this can be remedied by accepting uniqueness of pluperfect maps up to $\equiv$. That is, say that two parallel pluperfect maps $g : L \to M$ and $h : L \to M$ are *equivalent* (written $g \equiv h$) provided that $g(\phi) \equiv h(\phi)$ for all $\phi \in L$.

Theorem 6.11. *Suppose* $h : L \to M$ *is a pluperfect map into a canonically separated calculus* M, *and* $j : \mathsf{Patch}(L) \to M$ *is a pluperfect map such that* $j \circ + \equiv h$. *Then* $j \equiv \langle h, \hat{h} \rangle \circ i$.

Proof. Let $g = \langle h, \hat{h} \rangle \circ i$. By the hypothesis, $j(+\phi) \equiv g(+\phi)$ for all $\phi \in L$. For generators of the form $-\phi^\circ$, consider a sequent $j(-\phi^\circ), \Gamma \Vdash_M \Delta$. By smoothness of j, there is a token $\psi \in L$ so that $j(-\psi^\circ), \Gamma \Vdash_M \Delta$ and $\psi \Vdash_L \phi$. Hence $\Vdash_{\mathsf{Patch}(L)} -\psi^\circ, +\phi$. Because j is $\Vdash$-preserving, $\Vdash_M j(-\psi^\circ), j(+\phi)$. The hypothesis means that $\Vdash_M j(-\psi^\circ), h(\phi)$. So $\neg h(\phi) \Vdash_M g(-\psi^\circ)$. An application of Cut yields $\neg h(\phi), \Gamma \Vdash_M \Delta$, but $\neg(h) = g(-\phi^\circ)$. Conversely, $g(-\psi^c irc), \Gamma \Vdash_M \Delta$ implies $\Gamma \Vdash_M \Delta, h(\phi)$. Smoothness of h yields a token $\psi \in L$ so that $\Gamma \Vdash_M \Delta, h(\psi)$ and $\psi \Vdash_L \phi$. So $+\psi, -\phi^\circ \Vdash_{\mathsf{Patch}(L)}$, and hence $j(+\psi), j(-\phi^\circ) \Vdash_M$. Again the hypothesis means that $h(\psi), j(-\phi^\circ) \Vdash_M$, so an application of Cut yields $j(-\phi^\circ), \Gamma \Vdash_M \Delta$. The equivalence for generators appearing on the right follows *mutatis mutandis*. Finally, the result follows by induction on tokens in $\mathsf{Patch}(L)$. □

Therefore up to $\equiv$, $\mathsf{Patch}(L)$ is indeed the free canonically separated object over L in the category $\mathsf{MLS}^{\mathsf{plu}}$.

7. $\mathsf{MLS}_\neg$ as an allegory

The patch construction as described above lives in the category of upper adjoint consequence relations because these correspond to perfect maps between stably compact locales. Nevertheless, it is also worth considering the full subcategory of MLS determined by regular calculi. Because of the equivalence of Corollary 6.10, we consider $\mathsf{MLS}_\neg$, the full sub-subcategory of MLS in which objects are canonically separated. We show that $\mathsf{MLS}_\neg$ exhibits the structure of an allegory [6]. In particular, $\mathsf{MLS}_\neg$ is a semi-lattice enriched category equipped with an anti-involution functor $(-)^\circ$ that leaves objects fixed and semi-lattice structure fixed, and satisfies a certain modularity condition with respect to composition. The proof that $(-)^\circ$ satisfies the modularity condition, however, appears to require choice, while the rest of the proof remains predicative.

As in any allegory, one defines $\mathsf{Map}(\mathsf{MLS}_\neg)$ to be the sub-category of morphisms $R: X \longrightarrow Y$ so that $\mathsf{id}_X \leqslant R; R^\circ$ and $R^\circ; R \leqslant \mathsf{id}_Y$. Indeed, the definition is reasonable whether or not one accepts the non-constructive proof of modularity. In our setting, it follows (predicatively again) that $\mathsf{Map}(\mathsf{MLS}_\neg)$ is a subcategory of $\mathsf{MLS}^{\mathsf{u}}_\neg$. We show that all adjoint pairs in $\mathsf{MLS}_\neg$ take the form $(\vdash, \vdash^\circ)$, so the two categories are actually the same. Furthermore, $\mathsf{Map}(\mathsf{MLS}_\neg)$ is exactly the sub-category of $\mathsf{MLS}_\neg$ consisting of compatible consequence relations that satisfy Lemma 3.4. As these morphisms correspond to locale morphisms, $\mathsf{MLS}^{\mathsf{u}}_\neg$ is equivalent to the category of compact regular locales.

Recall that in a compact regular space X, $\mathsf{Patch}(X) = X$. So $X_\kappa = X$. The full subcategory KRegSp of SCSp^* consisting of compact regular spaces is (non-constructively) equivalent to $\mathsf{MLS}_\neg$. Moreover, it is easily seen to be an allegory, essentially because the morphisms are compact saturated subsets of the product $X \times Y = X_\kappa \times Y$. As the spaces here are compact regular, all subsets are saturated and compact sets are the same as closed sets. Inversion of a closed relation $R \subseteq X \times Y$ gives us the necessary anti-involution. Intersection of closed relations gives the necessary meet. In short, the forgetful functor from KRegSp to Rel creates the allegory structure. So (using choice),

Theorem 7.1. $\mathsf{MLS}_\neg$ *is an allegory*.

We already know how the order on hom-sets of KRegSp translates to $\mathsf{MLS}_\neg$: $\vdash \subseteq \vdash'$ if and only if $R_\vdash \supseteq R_{\vdash'}$. In order to understand $\mathsf{MLS}_\neg$ as an allegory, it remains to calculate the anti-involution.

Lemma 7.2. *For a canonically separated calculus L, the map* $\neg^\circ : L \to L_\kappa$ *given by* $\phi \mapsto (\neg\phi)^\circ$ *is a smooth, dense,* $\Vdash$*-preserving and reflecting map from L to* L_κ.

Proof. All of the listed properties follow easily from the fact that $\Gamma \Vdash_L \Delta$ holds if and only if $\Delta^\circ \Vdash_{L_\kappa} \Gamma^\circ$ if and only if $\neg\Delta \Vdash_L \neg\Gamma$. □

So the canonically separated calculi are isomorphic to their own duals via the given map. However, simply being isomorphic to one's own dual does not suffice for regularity. For example, consider the stable calculus $\Vdash_{[0,1]}$ defined on the free token algebra generated by $\mathbb{Q} \cap (0,1)$ to be the least consequence relation containing $p \Vdash_{[0,1]} q$ whenever $p > q$. The locale determined by this calculus is isomorphic to the locale of upper open subsets of $[0,1]$, so is not regular. It is, however, isomorphic to its own dual by the pluperfect token algebra homomorphism: $p \mapsto (1-p)^\circ$.

The isomorphism $\vdash_{\neg^\circ} : L \to L_\kappa$ yields a way to invert any compatible consequence relation $\vdash : L \to_c M$ between canonically separated calculi by the construction $\vdash \;\mapsto\; \vdash_{\neg^\circ}; \vdash_\kappa; \vdash^{\neg^\circ}$. Because $\vdash_{\neg^\circ}$ and $\vdash^{\neg^\circ}$ are inverses of one another, this construction is a contravariant endofunctor that is fixed on objects. It clearly is an order isomorphism on hom-sets. We can state more directly how this functor works.

Lemma 7.3. *For* $\vdash : L \to_c M$ *between canonically separated calculi, define* $\vdash_\circ$ *by*

$$\frac{\neg\Gamma \vdash \neg\Delta}{\Delta \vdash^\circ \Gamma}\,(\mathsf{contra})$$

Then $\vdash^\circ$ *is a compatible consequence relation from M to L and* $\vdash_\circ \;=\; \vdash_{\neg^\circ}; \vdash_\kappa; \vdash^{\neg^\circ}$.

Proof. This follows immediately from the definitions of $\neg^\circ$ and $\vdash_\kappa$. □

The next technical lemmas show that $(-)^\circ$ is the anti-involution for the allegory $\mathsf{MLS}_\neg$. The proof appears to require the law of the excluded middle. Say that $P \subseteq L$ is a *prime ideal* provided that for every Γ, $\Gamma \cap P \neq \emptyset$ if and only if there exists $\Delta \subseteq P$ so that $\Gamma \Vdash_L \Delta$. The name is justified by noticing that P is indeed an ideal (restrict to singletons Γ) and is a prime in the lattice $\mathsf{idl}(L)$. That is, the space $\mathsf{pt}(\mathsf{idl}(L))$ consists of prime ideals.

Lemma 7.4. *For any canonically separated calculus L, prime ideal P and token* $\phi \in L$, *either* $\phi \notin P$ *or* $\neg\phi \notin P$.

Proof. Suppose that P is prime and both ϕ and $\neg\phi$ belong to P. Then for some $\psi \in P$, we must have $\phi \Vdash_L \psi$ and $\neg\phi \Vdash_L \psi$. But by taking negations, $\Vdash_L \psi, \neg\phi$ and $\Vdash_L \psi, \phi$ both hold. This is a contradiction. □

Lemma 7.5. *For a compatible consequence relation* $\vdash : L \to_c M$ *in* $\mathsf{MLS}_\neg$, *let* P *and* Q *be prime ideals of* L *and* M, *respectively. Then* $P \supseteq [\vdash]Q$ *if and only if* $Q \supseteq [\vdash^\circ]P$.

Proof. Because $\vdash = (\vdash^\circ)^\circ$, it suffices to prove one direction.

For prime ideals P and Q, $P \supseteq [\vdash]Q$ equivalent to the implication: $\Gamma \vdash \Delta$ and $\Delta \subseteq Q$ implies $P \cap \Gamma \neq \emptyset$.

Let $P \supseteq [\vdash]Q$. Suppose $\gamma_1, \ldots, \gamma_n \vdash^\circ \Delta$ for some $\Delta \subseteq P$. Our goal is to show that one of the γ_i must belong to Q. Taking the contrapositive, $\neg\Delta \vdash \neg\gamma_1, \ldots, \neg\gamma_n$ By sequential cut decomposition, there are tokens $\theta_i \in L$ so that $\neg\Delta \vdash \theta_1, \ldots, \theta_n$ and $\theta_i \Vdash_M \neg\gamma_i$ for each i. Thus $\theta_i, \gamma_i \Vdash_M$ also holds for each i.

Toward a contradiction, suppose that $\gamma_i \notin Q$ for each i. Then $\theta_i \in Q$ for each i. Thus, $P \cap \neg\Delta \neq \emptyset$. This contradicts Lemma 7.4. □

These results give us a complete description of $\mathsf{MLS}_\neg$ as an allegory. So we continue by considering the category $\mathsf{Map}(\mathsf{MLS}_\neg)$. The definition of $\mathsf{Map}(-)$ does not actually depend on $\mathsf{MLS}_\neg$ being an allegory, but only on it having the right "signature." Let $\mathscr{A}$ be an order enriched category with order isomorphism $I : \mathscr{A} \Rightarrow \mathscr{A}^{\mathsf{op}}$ so that I^2 is the identity and I keeps objects fixed. In particular, any allegory meets these conditions, as does $\mathsf{MLS}_\neg$. In fact, our exposition shows that $\mathsf{MLS}_\neg$ meets these conditions predicatively whether it is provably an allegory or not. Following [6], for such a category $\mathscr{A}$, define $\mathsf{Map}(\mathscr{A})$ to be the subcategory consisting of morphisms $R : X \to Y$ so that $R \circ I(R) \leqslant \mathsf{id}_X$ and $I(R) \circ I \leqslant \mathsf{id}_Y$. Clearly, in the case of $\mathsf{MLS}_\neg$, if we take the ordering on hom-sets to be $\supseteq$, then $\mathsf{Map}(\mathsf{MLS}_\neg)$ consists of those morphisms $\vdash$ for which $\vdash^\circ$ happens to be the lower adjoint. The obvious question arises as to whether these are all of the upper adjoints.

Lemma 7.6. *For* $\vdash : L \to_c M$ *in* $\mathsf{MLS}_\neg$, *the following are equivalent*:
(1) $\vdash$ *has a lower adjoint with respect to* $\subseteq$;
(2) $\vdash^\circ$ *is the lower adjoint of* $\vdash$;
(3) $\vdash$ *is a functional morphism*, *i.e.*, *satisfies the equivalent conditions of Lemma* 3.4.

Proof. Obviously (2) implies (1) and (1) implies (3). Consider a functional compatible consequence relation $\vdash : L \to_c M$. Suppose that $\Gamma \vdash ; \vdash^\circ \Delta$. Then for some token ϕ, $\Gamma \vdash \phi$ and $\neg\Delta \vdash \neg\phi$. Interpolating, there is also a token ψ so that $\neg\Delta \vdash \psi$ and $\phi, \psi \Vdash_M$. By weakening and an application of Cut', $\neg\Delta, \Lambda \vdash$ holds. Because $\vdash$ is functional, $\neg\Delta, \Lambda \Vdash_L$ holds as well. Thus $\Gamma \Vdash_L \Delta$. Suppose that $\gamma_1, \ldots, \gamma_m \Vdash_M \delta_1, \ldots, \delta_n$. Then $\Vdash_M \delta_1, \ldots, \delta_n, \neg\gamma_1, \ldots, neg\gamma_m$. As $\vdash$ is functional and $\vdash \Delta, \neg\Gamma$, there exists $\theta_1, \ldots, \theta_m$ and $\lambda_1, \ldots, \lambda_n$ so that (a) $\Vdash_L \theta_1, \ldots, \theta_m, \lambda_1, \ldots, \lambda_n$, (b) $\theta_i \vdash \delta_i$ for each i and (c) $\lambda_j \vdash \neg\gamma_j$ for each j. Thus for another token ψ, we have $\neg\lambda_1, \ldots, \neg\lambda_n \Vdash_L \psi$ and $\psi \Vdash_L \theta_1, \ldots, \theta_m$. Combining these with (b) and (c), and taking contrapositives for (c) yields $\Gamma \vdash^\circ ; \vdash \Delta$. □

Theorem 7.7. *The categories* $\mathsf{Map}(\mathsf{MLS}_\neg)$ *and* $\mathsf{MLS}^{\mathsf{u}}_\neg$ *are equivalent*.

8. Conclusion

We have established a surprising connection between stable compactness, compact regularity and the proof-theoretic notion of cut decomposition. These results suggest that categories based in the sequent calculus such as MLS are useful tools for investigating topological concepts in a predicative setting. In particular, the proof that the patch construction in $\mathsf{MLS}^{\mathsf{u}}$ is indeed a coreflection is carried out entirely in the proof-theoretic setting of MLS. Thus the coreflection holds in any predicative type theory strong enough to formalize universal algebra. Similarly, the proofs that contraposition determines an anti-involution in $\mathsf{MLS}_{\neg}$ and that the subcategory of maps defined in $\mathsf{MLS}_{\neg}$ coincides with $\mathsf{MLS}^{\mathsf{u}}_{\neg}$ are predicatively valid. In contrast it is not clear whether $\mathsf{MLS}_{\neg}$ can be constructively shown to form an allegory.

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