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The Necessary Conditions for Optimal Control in Hilbert Spaces

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INTRODUCTION

The problem we consider is a nonlinear distributed control problem in a Hilbert space and therefore includes many problems of interest in practical applications. Of course the main point of an application is to construct an optimal control. The main tool used to construct the optimal control is the Pontryagin maximum principle, a set of necessary conditions which an optimal control must satisfy. The objective of this paper is to derive the Pontryagin principle for our general class of control problems of Lagrange type. We will prove the derived principle using dynamic programming and the resulting Hamilton–Jacobi–Bellman equation in infinite dimensions.

The method of proof, first developed in [3], uses the fact that the value function is a viscosity subsolution of the Bellman equation. The theory of viscosity solutions of first-order equations has turned out to be fundamental and has resolved many outstanding problems. This theory was originated by Crandall and Lions [6] and has been recently extended by them to infinite dimensional problems [5].

Despite the myriad of results concerning viscosity solutions it is very interesting that our proof of the maximum principle uses only the definition

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of viscosity solution and the fact that the value function is a viscosity solution (actually we only use the fact that it is a subsolution). The tour de force uniqueness results in viscosity solutions are not needed here. Furthermore, the assumptions of Crandall and Lions [5] on the hamiltonian which are needed to guarantee existence, uniqueness, etc., and which are not all satisfied in this paper, are not needed here. On the other hand, since we are working in Hilbert spaces there is sufficient smoothness of the norm to work with smooth test functions.

The proof used here presents a new emphasis on the Bellman equation. The Bellman equation has been used in the past to formally derive a maximum principle for certain problems but the derivation has depended on the existence of a C^2 solution to the equation. Of course the solution is almost always not C^2. We do not use this assumption.

Finally, let us just remark that the results of this paper will hold for much more general problems than those considered here.

1. The Control Problem in Hilbert Space

In this paper we will derive the necessary conditions for an optimal control for the following problem in a Hilbert space. This problem was also considered in Barbu [1] by a completely different method.

$$dx(t)/dt + Ax(t) + Fx(t) = Bu(t) \qquad \text{if} \quad s < t \leqslant T \tag{1.1}$$

$$x(s) = y \in \mathscr{D}(A) \equiv \text{domain of } A \tag{1.2}$$

Subject to (1.1)–(1.2) find $u \in L^2(s, T; U)$ which minimizes the cost functional

$$P_{s,y}(u) \equiv g(x(T; s, y)) + \int_s^T h(x(r; s, y)) + f(u(r))\, dr. \tag{1.3}$$

The solution of (1.1) for the initial condition $(\tau, \xi) \in [0, T] \times \mathscr{D}(A)$ is denoted by $x(\cdot;; \tau, \xi)$. We assume that H and U are real Hilbert spaces with inner products $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ and norms $\|\cdot\|_H$ and $\|\cdot\|_U$, respectively. The following assumptions will hold throughout this paper:

(i) $A: \mathscr{D}(A) \subseteq H \to H$ is a closed, densely defined linear operator; $-A$ is the infinitesimal generator of an analytic C_0 semigroup e^{-At} and $\forall t > 0$, e^{-At} is compact.

(ii) $B: U \to H$ is a linear, continuous operator and B^* is its adjoint.

(iii) $F: H \to H$ is (nonlinear) Frechet differentiable with a locally Lipschitz Frechet derivative $F': H \to \mathscr{L}(H, H)$. There are constants c_1 and c_2 such that $(Ax + Fx, x) \geqslant -c_1 \|x\|_H^2 - c_2$ for every $x \in \mathscr{D}(A)$.

(iv) $g: H \to \mathbb{R}^1$ and $h: H \to \mathbb{R}^1$ are Frechet differentiable with locally Lipschitz derivatives ∇g and ∇h. There are α, $\alpha_0 \in H$ and real numbers β and β_0 such that

$$g(x) \geqslant (\alpha, x) + \beta \qquad \forall x \in H$$

$$h(x) \geqslant (\alpha_0, x) + \beta_0 \qquad \forall x \in H.$$

(v) $f: H \to (-\infty, +\infty]$ is convex, lower semicontinuous, and satisfies the condition $f(u) \geqslant \omega \|u\|_U^2 + \gamma$ $\forall u \in U$, for some $\omega > 0$, $\gamma \in \mathbb{R}^1$.

(vi) The conjugate function $f^*: U \to \mathbb{R}^1$, $f^*(p) = \sup\{\langle p, u\rangle - f(u); u \in U\}$ is Gateaux differentiable with locally Lipschitz Gateaux derivative ∇f^*.

The preceding assumptions guarantee that for each $u \in L^2(s, T; U)$ and $y \in \mathscr{D}(A)$ there exists a unique trajectory $x(\cdot; s, y)$ on $[s, T]$ which is absolutely continuous and $dx/dt \in L^2(s, T; H)$. Furthermore, the mapping $u \mapsto x$ from $L^2(s, T; U)$ to $C([s, T]; H)$ is continuous and compact. Consequently [1, 2], the problem (1.1)–(1.3) admits at least one optimal control $u^* \in L^2(s, T; U)$ and associated trajectory $x^* \in C([s, T]; \mathscr{D}(A))$, $x^* = x^*(\cdot; s, y)$. Note that $y \in \mathscr{D}(A) \Rightarrow x^*(t) \in \mathscr{D}(A)$ $\forall t \geqslant s$.

Define the *value function* $V: [s, T] \times \mathscr{D}(A) \to \mathbb{R}^1$ for the problem (1.1)–(1.3) by

$$V(\tau, \xi) = \inf\{P_{\tau,\xi}(u); u \in L^2(s, T; U)\}. \tag{1.4}$$

By continuity (see below), we can define V on all of $[s, T] \times H$.

The main results of Barbu [1] are the following

THEOREM 1. *The function $V(\tau, \xi)$ is continuous on $[s, T] \times \mathscr{D}(A)$, locally Lipschitz in ξ for every $\tau \in [s, T]$, and absolutely continuous in τ for every $\xi \in \mathscr{D}(A)$. The subdifferential $D_\xi^- V(\tau, \xi) \neq \emptyset$ for every $(\tau, \xi) \in [s, T] \times H$ and*

$$\partial V/\partial \tau - (A\xi + F\xi, \eta) + h(\xi) + \inf_{u \in U} \{\langle B^*\eta, u\rangle + f(u)\} = 0 \tag{1.5}$$

for a.e. $\tau \in [s, T)$, all $\xi \in \mathscr{D}(A)$, and some $\eta \in D_\xi^- V(\tau, \xi)$;

$$V(T, \xi) = g(\xi), \qquad \forall \xi \in H. \tag{1.6}$$

If g satisfies $\nabla g(p) \in \mathscr{D}(A^*)$ for all $p \in H$, then (1.4) holds for every $\eta \in D_\xi^- V(\tau, \xi)$.

Equation (1.5) is the Hamilton–Jacobi–Bellman equation associated with the optimal control problem (1.1)–(1.4)

THEOREM 2. *Every optimal control u^* for the problem (1.1)–(1.3) is given by the feedback law*

$$u^*(t) \in \nabla f^*(-B^* D_x^- V(t, x^*(t))), \qquad \forall t \in [s, T]$$

where x^* is the optimal trajectory corresponding to u^*.

The purpose of the present paper is to prove, using the theory of viscosity solutions in Hilbert space, the following maximum principle result.

THEOREM 3. *Let $u^* \in L^2(s, T; U)$ be an optimal control and $x^*(\cdot; s, y)$ the corresponding optimal trajectory for the problem (1.1)–(1.3) for the initial conditions $(s, y) \in [0, T] \times \mathscr{D}(A)$. Then for a.e. $t \in [s, T]$ we have*

$$\begin{aligned}&\inf\langle B^* D_x \Gamma(t, x^*(t; s, y)), u\rangle + f(u); u \in U\}\\&\quad = \langle B^* D_x \Gamma(t, x^*(t; s, y)), u^*(t)\rangle + f(u^*(t)),\end{aligned}$$

where

$$\begin{aligned}D_x \Gamma(t, x^*(t)) = {}& \nabla g(x^*(T)) \cdot D_y x^*(T; s, y)\\&+ \int_t^T \nabla h(x^*(r)) \cdot D_y x^*(r; s, y)\, dr.\end{aligned}$$

We have here the definition

$$\Gamma(t, x^*(t)) = g(x^*(T)) + \int_t^T h(x^*(r)) + f(u^*(r))\, dr.$$

2. THE VALUE FUNCTION IS A VISCOSITY SOLUTION

The proof of Theorem 3 will be based on the Hamilton–Jacobi–Bellman equation for the value function and the elementary theory of viscosity solutions to first-order equations. The equation of interest here is

$$\begin{aligned}&\partial W/\partial t + \inf\{\langle B^* D_x W(t, x), u\rangle + f(u); u \in U\}\\&\quad - (Ax + Fx, D_x W(t, x)) + h(x) = 0 \qquad (2.1)\end{aligned}$$

if $s \leqslant t < T$, $x \in \mathscr{D}(A)$, with

$$W(T, x) = g(x) \qquad \text{for every} \quad x \in H. \qquad (2.2)$$

The following definition is an adaptation of that given in [5].

DEFINITION. A continuous function W: $[s, T] \times \mathscr{D}(A) \to \mathbb{R}^1$ is a *viscosity* solution of (2.1) if for any $\varphi \in C([s, T] \times H)$

(i) if $W-\varphi$ has a local max at $(t_0, x_0) \in [s, T] \times \mathscr{D}(A)$ and φ is differentiable at (t_0, x_0) then

$$\partial\varphi(t_0, x_0)/\partial t + \inf\{\langle B^* D_x \varphi(t_0, x_0), u\rangle + f(u); u \in U\}$$
$$- (Ax_0 + Fx_0, D_x \varphi(t_0, x_0)) + h(x_0) \geqslant 0$$

and

(ii) if $W-\varphi$ has a local min at $(t_0, x_0) \in [s, T] \times \mathscr{D}(A)$ and φ is differentiable at (t_0, x_0) then

$$\partial\varphi(t_0, x_0)/\partial t + \inf\{\langle B^* D_x \varphi(t_0, x_0), u\rangle + f(u); u \in U\}$$
$$- (Ax_0 + Fx_0, D_x \varphi(t_0, x_0)) + h(x_0) \leqslant 0.$$

Remark. We could take the test function φ to be continuously Frechet differentiable since the spaces are Hilbert spaces (see [6]). Note that we only *require* differentiability at the extreme points. Note also that φ is defined on all of H and not just $\mathscr{D}(A)$. This is necessary because $Ax + Fx \in H$, not $\mathscr{D}(A)$.

We begin by establishing that the value function defined in the previous section is a viscosity solution of (2.1).

THEOREM 4. *The value function V is a viscosity solution of* (2.1).

Theorem 4 was proved in Barbu [1]. The proof presented there, however, used the maximum principle stated in Theorem 3. Our interest here is to use the fact that V is a viscosity solution of the Bellman equation to prove the maximum principle. Therefore we prove Theorem 4 by a method based on the Bellman principle of optimality and which is standard in the connection between optimal control theory and viscosity solutions. We will use C^1 test functions φ to simplify the discussion but this is not necessary.

Proof. That V is at least continuous is established in Theorem 1. Therefore we need only establish the two remaining conditions in the definition of viscosity solution.

Suppose first that $V-\varphi$ has a local min at $(t_0, x_0) \in [s, T] \times \mathscr{D}(A)$ with $\varphi \in C^1([s, T] \times H)$. Let $u \in L^2([t_0, T]; U)$ be any control and let $x(\cdot; t_0, x_0)$ be the corresponding trajectory. Then, for small $\varepsilon > 0$ we have

$$V(t_0+\varepsilon, x(t_0+\varepsilon)) - V(t_0, x_0)$$
$$\geqslant \varphi(t_0+\varepsilon, x(t_0+\varepsilon)) - \varphi(t_0, x_0)$$

$$= \int_{t_0}^{t_0+\varepsilon} d\varphi(r, x(r))/dr\, dr$$

$$= \int_{t_0}^{t_0+\varepsilon} \varphi_t(r, x(r)) + \langle B^* D_x \varphi(r, x(r)), u(r) \rangle$$

$$- (Ax(r) + Fx(r), D_x \varphi(r, x(r)))\, dr. \tag{2.3}$$

Note that $x(r) \in \mathscr{D}(A)$ if $r \geqslant t_0$. Now suppose that

$$\partial\varphi(t_0, x_0)/\partial t + \inf\{\langle B^* D_x \varphi(t_0, x_0), u\rangle + f(u); u \in U\}$$
$$- (Ax_0 + Fx_0, D_x \varphi(t_0, x_0)) + h(x_0) \geqslant \lambda > 0.$$

Then for every $u \in U$,

$$\partial\varphi(t_0, x_0)/\partial t + \langle B^* D_x \varphi(t_0, x_0), u\rangle + f(u)$$
$$- (Ax_0 + Fx_0, D_x \varphi(t_0, x_0)) + h(x_0) \geqslant \lambda.$$

For sufficiently small ε this implies that if $t_0 \leqslant r \leqslant t_0 + \varepsilon$,

$$\partial\varphi(r, x(r))/\partial t + \langle B^* D_x \varphi(r, x(r)), u(r)\rangle + f(u(r))$$
$$- (Ax(r) + Fx(r), D_x \varphi(r, x(r))) + h(x(r)) \geqslant \lambda/2.$$

Using this in (2.3) we get

$$V(t_0 + \varepsilon, x(t_0 + \varepsilon)) - V(t_0, x_0)$$
$$\geqslant \int_{t_0}^{t_0+\varepsilon} \varphi_t(r, x(r)) + \langle B^* D_x \varphi(r, x(r)), u(r)\rangle$$
$$- (Ax(r) + Fx(r), D_x \varphi(r, x(r)))\, dr$$
$$\geqslant \varepsilon\lambda/2 - \int_{t_0}^{t_0+\varepsilon} h(x(r)) + f(u(r))\, dr.$$

Consequently,

$$V(t_0 + \varepsilon, x(t_0 + \varepsilon)) + \int_{t_0}^{t_0+\varepsilon} h(x(r)) + f(u(r))\, dr \geqslant \varepsilon\lambda/2 + V(t_0, x_0)$$

and this contradicts the optimality principle:

$$\inf\left\{V(t_0 + \varepsilon, x(t_0 + \varepsilon)) + \int_{t_0}^{t_0+\varepsilon} h(x(r)) + f(u(r))\, dr;\right.$$
$$\left. u \in L^2([t_0, t_0 + \varepsilon]; U)\right\} = V(t_0, x_0).$$

Now we suppose that $V-\varphi$ has a local max at $(t_0, x_0) \in [s, T] \times \mathscr{D}(A)$. We must show that

$$\partial\varphi(t_0, x_0)/\partial t + \inf\{\langle B^* D_x \varphi(t_0, x_0), u\rangle + f(u); u \in U\}$$
$$- (Ax_0 + Fx_0, D_x \varphi(t_0, x_0)) + h(x_0) \geqslant 0.$$

For sufficiently small $\varepsilon > 0$, we have for any control $u \in L^2([t_0, T]; U)$ and associated trajectory $x(\cdot; t_0, x_0)$,

$$\begin{aligned} &V(t_0+\varepsilon, x(t_0+\varepsilon)) - V(t_0, x_0) \\ &\quad \leqslant \varphi(t_0+\varepsilon, x(t_0+\varepsilon)) - \varphi(t_0, x_0) \\ &\quad = \int_{t_0}^{t_0+\varepsilon} d\varphi(r, x(r))/dr\, dr \\ &\quad = \int_{t_0}^{t_0+\varepsilon} \varphi_t(r, x(r)) + \langle B^* D_x \varphi(r, x(r)), u(r)\rangle \\ &\qquad - (Ax(r) + Fx(r), D_x \varphi(r, x(r)))\, dr. \end{aligned} \tag{2.4}$$

If

$$\partial\varphi(t_0, x_0)/\partial t + \inf\{\langle B^* D_x \varphi(t_0, x_0), u\rangle + f(u); u \in U\}$$
$$- (Ax_0 + Fx_0, D_x \varphi(t_0, x_0)) + h(x_0) \leqslant -\lambda < 0$$

then for $t_0 \leqslant r \leqslant t_0 + \varepsilon$ with sufficiently small ε, we have

$$\partial\varphi(r, x(r))/\partial t + \inf\{\langle B^* D_x \varphi(r, x(r)), u\rangle + f(u); u \in U\}$$
$$- (Ax(r) + Fx(r), D_x \varphi(r, x(r))) + h(x(r)) \leqslant -\lambda/2.$$

Now, there exists a control $\omega(r)$ so that

$$\inf\{\langle B^* D_x \varphi(r, x(r)), u\rangle + f(u); u \in U\} + \lambda/4$$
$$\geqslant \langle B^* D_x \varphi(r, x(r)), \omega(r)\rangle + f(\omega(r)),$$

where $x(\cdot)$ is the trajectory corresponding to $\omega(\cdot)$. Using this control and trajectory in the preceding we get

$$\partial\varphi(r, x(r))/\partial t + \langle B^* D_x \varphi(r, x(r)), \omega(r)\rangle + f(\omega(r))$$
$$- (Ax(r) + Fx(r), D_x \varphi(r, x(r))) + h(x(r)) \leqslant -\lambda/4.$$

Consequently, for small ε, using (2.4) we obtain

$$V(t_0+\varepsilon, x(t_0+\varepsilon)) - V(t_0, x_0) \leqslant -\lambda\varepsilon/4 - \int_{t_0}^{t_0+\varepsilon} h(x(r)) + f(\omega(r))\, dr$$

so that

$$V(t_0+\varepsilon, x(t_0+\varepsilon)) + \int_{t_0}^{t_0+\varepsilon} h(x(r)) + f(\omega(r))\, dr$$
$$\leqslant V(t_0, x_0) - \lambda\varepsilon/4 < V(t_0, x_0).$$

This is a contradiction of the principle of optimality, so the proof is complete.

3. Proof of the Maximum Principle

Define the function Γ: $[s, T] \times \mathscr{D}(A) \to \mathbb{R}^1$ by

$$\Gamma(\tau, \xi) = g(x(T; \tau, \xi)) + \int_\tau^T h(x(r; \tau, \xi)) + f(u^*(r))\, dr,$$

where

$$dx/dt + Ax(t) + Fx(t) = Bu^*(t) \qquad \text{if} \quad s \leqslant \tau < t \leqslant T,$$
$$x(\tau; \tau, \xi) = \xi \in \mathscr{D}(A).$$

Note that $\Gamma(T, \xi) = g(\xi)$.

Recall that u^* is the optimal control on $[s, T]$ and $x^*(\cdot : s, y)$ is the corresponding optimal trajectory ($x \neq x^*$, in general).

Lemma 5. *The function $\Gamma(\tau, \xi)$ is absolutely continuous in $\tau \in [s, T]$ and continuously (Frechet) differentiable in $\xi \in \mathscr{D}(A)$.*

Proof. The proof of the lemma depends on the properties of the function $\varphi(t) \equiv \partial x(t; \tau, \xi)/\partial \tau$ and the function $\psi(t) \equiv D_\xi x(t; \tau, \xi)$. Under our assumptions on the operators A, B, and F it is true that the function $\psi(t)$ is the unique absolutely continuous solution of the evolution system

$$d\psi/dt + A\psi + F'x \circ \psi = 0 \qquad \text{with} \quad \psi(\tau) = 1_{\mathscr{D}(A)}.$$

A reference for this fact is, for example, Smoller [6].

Similarly, $\varphi(t)$ is the unique absolutely continuous solution of the problem $d\varphi/dt + A\varphi + F'x \circ \varphi = 0$, with $\varphi(\tau) = A\xi + F\xi$.

Using the assumptions on A, F, g, f, and h the conclusion of the lemma now follows directly from the definition of Γ.

Remark. From the formula

$$D_\xi \Gamma(\tau, \xi) = \nabla g(x(T; \tau, \xi))\, D_\xi x(T; \tau, \xi) + \int_\tau^T \nabla h(x(r; \tau, \xi)) \cdot D_\xi x(r; \tau, \xi)\, dr$$

and the preceding *linear* equation for $\psi(t) = D_\xi x(t; \tau, \xi)$ we see that $D_\xi \Gamma$ may be extended using continuity to all of $H^* = H$ and not just $\mathscr{D}(A)^*$. From this point on we assume this extension.

COROLLARY 6. *For almost every* $s \leqslant \tau \leqslant t \leqslant T$, *the function* Γ *is differentiable at* $(t, x(t; \tau, \xi)) \in [\tau, T] \times \mathscr{D}(A)$ *and for a.e.* $t \in [\tau, T]$ *we have*

$$\Gamma_t(t, x(t; \tau, \xi)) - (D_x \Gamma(t, x(t; \tau, \xi)), Ax(t; \tau, \xi) + Fx(t; \tau, \xi)) + \langle B^* D_x \Gamma(t, x(t; \tau, \xi)), u^*(t) \rangle + h(x(t; \tau, \xi)) + f(u^*(t)) = 0$$

with

$$\Gamma(T, x(T; \tau, \xi)) = g(x(T; \tau, \xi)).$$

Proof. The fact that Γ is differentiable at $(t, x(t; \tau, \xi))$ for a.e. $t \in [\tau, T]$ follows from the fact that Γ_t exists for a.e. $t \in [\tau, T]$ and $D_x \Gamma$ exists everywhere and (see the preceding remark) is a continuous linear operator on H. Notice that this is a simple generalization of the classical result that a function is differentiable if it has continuous partial derivatives. That is, a function is differentiable at a point when the partials exist there and at least one of the partials is continuous.

The second assertion of the corollary follows from the chain rule and the fact that

$$d\Gamma(t, x(t; \tau, \xi))/dt = -h(x(t; \tau, \xi)) - f(u^*(t))$$

for a.e. $t \in [\tau, T]$. This completes the proof.

We are now prepared to complete the proof of Theorem 3.

Notice that from the definitions of Γ, V, and P we have $\Gamma(s, y) = P_{s,y}(u^*) = V(s, y)$ and $\Gamma(\tau, \xi) \geqslant V(\tau, \xi)$ if $\tau \in [s, T]$ and $\xi \in \mathscr{D}(A)$. Further, the principle of dynamic programming states that if a control is optimal on $[s, T]$, then it is also optimal on any subinterval of $[s, T]$. Consequently,

$$V(t, x^*(t; s, y)) = \Gamma(t, x^*(t; s, y)) \qquad \text{for} \quad s \leqslant t \leqslant T.$$

It follows that $V - \Gamma$ achieves a maximum of 0 along every point $(t, x^*(t; s, y))$ of the optimal trajectory. Since V is a viscosity solution of the Bellman equation and Γ is differentiable at almost every point of the optimal trajectory we get from the definition

$$\partial\Gamma(t, x^*(t))/\partial t + \inf\{\langle B^*D_x\Gamma(t, x^*(t)), u\rangle + f(u); u \in U\}$$
$$- (Ax^*(t) + Fx^*(t), D_x\Gamma(t, x^*(t)) + h(x^*(t)) \geqslant 0$$

for a.e. $t \in [s, T]$.

Notice that we have previously established that $D_x\Gamma \in H$ so that Γ is indeed an appropriate test function for V.

Next, Corollary 6 for Γ evaluated along the optimal trajectory $(t, x^*(t; s, y))$ tells us that

$$\partial\Gamma(t, x^*(t))/\partial t + \inf\{\langle B^*D_x\Gamma(t, x^*(t)), u\rangle + f(u); u \in U\}$$
$$- (Ax^*(t) + Fx^*(t), D_x\Gamma(t, x^*(t)) + h(x^*(t)) \leqslant 0$$

for a.e. $t \in [s, T]$. Combining these two inequalities, we must have

$$\partial\Gamma(t, x^*(t))/\partial t + \inf\{\langle B^*D_x\Gamma(t, x^*(t)), u\rangle + f(u); u \in U\}$$
$$- (Ax^*(t) + Fx^*(t), D_x\Gamma(t, x^*(t)) + h(x^*(t)) = 0$$

for a.e. $t \in [s, T]$. Again invoking Corollary 6 for the argument $(t, x^*(t; s, y))$ and comparing the equation there with the preceding equation we conclude that

$$\inf\{\langle B^*D_x\Gamma(t, x^*(t)), u\rangle + f(u); u \in U\}$$
$$= \langle B^*D_x\Gamma(t, x^*(t)), u^*(t)\rangle + f(u^*(t))$$

for a.e. $t \in [s, T]$. This concludes the proof of the maximum principle.

We conclude by defining the function $p(t) = D_x\Gamma(t, x^*(t))$.

THEOREM 7. *p is the solution of the adjoint problem*

$$p'(t) - A^*p(t) - F'(x^*(t))^* \cdot p(t) + \nabla h(x^*(t)) = 0 \qquad \text{if} \quad s \leqslant t \leqslant T,$$
$$p(T) = \nabla g(x^*(T)),$$

where $x^*(\cdot; s, y)$ is the optimal trajectory.

Proof. For simplicity we will take $g = 0$. Let $\gamma(\cdot) \in \mathscr{D}(A)$ and put $\psi(t) = D_y x^*(t; \tau, \xi) \cdot \gamma$, $\psi(\cdot) \in \mathscr{D}(A)$, where $d\psi/st + A\psi + F'(x^*) \cdot \psi = 0$, $\psi(\tau) = \gamma(\tau)$. Then, *letting* p be the solution of the adjoint problem, we have

$$(p'(t), \gamma(t)) - (p(t), A\gamma(t)) - (p(t), F'(x^*) \cdot \gamma(t)) + (\nabla h, \gamma(t)) = 0.$$

Next, using the equation for p and integration by parts we have

$$\begin{aligned}(D\Gamma(t, x^*(t)), \gamma(t)) &= \int_t^T (\nabla h(x^*(r; t, x^*(t))), \gamma(r))\, dr \\ &= \int_t^T -(p'(r), \psi(r)) + (p(r), A\psi(r)) + (p(r), F'(x^*)\cdot\psi(r))\, dr \\ &= (p(t), \gamma(t)) \qquad (\text{since } \psi' + A\psi + F'\cdot\psi = 0 \text{ and } \psi(t) = \gamma(t)).\end{aligned}$$

Since this is true for any $\gamma \in \mathscr{D}(A)$, and $\mathscr{D}(A)$ is dense in H, by continuity we conclude that $p = D\Gamma$ and the proof is complete.

(The proof of this theorem is based on a suggestion of L. D. Berkovitz, personal communication.)

EXAMPLE. We conclude this paper with a typical problem which falls in the class considered here.

Let $\Omega \subseteq R^n$ and let $Q \equiv [0, T] \times \Omega$. The Hilbert spaces under consideration here are $U = H = L^2(\Omega)$. Let $H_0^1(\Omega)$ be the usual Sobolev space of functions with one derivative which vanish on $\partial\Omega$ (in the generalized sense). Let $H^2(\Omega)$ be the space of functions with 2 derivatives in $L^2(\Omega)$. The dynamics are given by

$$\begin{gathered} y \in H^1(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \qquad u \in L^2(Q) \\ \partial y/\partial t - \Delta y + F(y) = u \qquad \text{on} \quad (0, T) \times \Omega, \\ y = 0 \qquad \text{on} \quad \Sigma = (0, T) \times \partial\Omega, \end{gathered}$$

where F is a continuously differentiable, bounded, real-valued function with bounded derivative. We are taking $B \equiv I$ and $A \equiv -\Delta$, φ and ψ in $\mathscr{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

We seek to minimize the payoff

$$P(u) = \int_0^T \int_\Omega h(y(t, x)) + f(u(t, x))\, dx\, dt.$$

Applying the necessary conditions derived in this paper to this problem we obtain that an optimal control u^* must satisfy the condition

$$\begin{aligned}\inf_{u \in L^2(\Omega)} \int_\Omega \{p(t, x)\, u(x) + f(u(x))\}\, dx \\ = \int_\Omega p(t, x)\, u^*(t, x) + f(u^*(t, x))\, dx\end{aligned}$$

for a.e. t in $[0, T]$ and $p=p(t, x) \in H^1(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ is the solution of the problem

$$\partial p/\partial t + \Delta p - p\, dF(y(t, x))/dy + h(y(t, x)) = 0$$
$$p(T, x) = 0, \qquad x \in \Omega.$$

In [4] the techniques of this paper are also applied to the control problems governed by differential-difference equations and to nonlinear, divergence form parabolic equations with control in the coefficients.

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