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Horispherical Subalgebras of Real Lie Algebras

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INTRODUCTION

Let $\mathfrak{L}$ be a finite-dimensional real Lie algebra and let x be an element of $\mathfrak{L}$. The horispherical subalgebra of $\mathfrak{L}$ relative to x is defined as

$$\mathfrak{Z} = \{z \text{ in } \mathfrak{L} \mid \lim_{t\to\infty} \exp(t \operatorname{ad}(x))(z) = 0\}.$$

This definition was first given by Maruyama [13], when he showed that if $\mathfrak{L}$ is semisimple and if G is a Lie group with Lie algebra $\mathfrak{L}$, then H defined by

$$H = \{h \text{ in } G \mid \lim_{t\to\infty} \exp(tx)\, h \exp(-tx) = e\},$$

e the identity of G, is closed and connected and has Lie algebra $\mathfrak{Z}$. H is called the horispherical subgroup relative to the one-parameter subgroup $P(t) = \exp(tx)$. Maruyama [14] pursued the subject of horispherical subalgebras of real semisimple Lie algebras and showed that there is a one-to-one correspondence between the classes of conjugate horispherical subalgebras and the set of faces of a Weyl chamber of $(\mathfrak{L}, \mathfrak{K})$, where $\mathfrak{K}$ is a maximal compactly embedded subalgebra of the Lie algebra $\mathfrak{L}$.

Gel'fand and Graev [4], and Gel'fand and Pyateckiĭ-Šapiro [6], had previously used horispherical subgroups in their study of group representations. In a related note Gel'fand and Pyateckiĭ-Šapiro [7] showed that there are as many classes of conjugate horispherical subgroups in SL(n, R) as there are representations of n in the form $n = k_1 + k_2 + \cdots + k_s$ where each k_i is positive.

The limit condition which defines horispherical subgroups appears in several instances in which neither horispherical subgroups or horispheres (orbits of horispherical subgroups) are mentioned. Auslander [1, p. 14] gives a proof which effectively shows that if G is a group with $G = P(t) \cdot H$, where H is horispherical relative to $P(t)$, and if U is a unitary representation of G in a

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separable Hilbert space W, then the pair $(G, P(t))$ exhibits the Mautner phenomenon; i e., $U_{P(t)}\psi = \psi$ for all t implies $U_g\psi = \psi$ for all g in G

Horne [9, p 47] gives a construction of a semigroup on a half-space in which horispherical subgroups play an important role Previously Horne [8] had shown how to construct the half-space $G \cup G/P(t)$, where G is a Lie group on a Euclidean space that is the semidirect product of its derived group G' and a one parameter subgroup $P(t)$. The multiplication defined in [9] is continuous if and only if G' is the horispherical subgroup of G relative to $P(t)$.

In this paper, we consider horispherical subalgebras of real finite dimensional Lie algebras. We begin Section 1 by replacing the above definition of horispherical subalgebras by an equivalent definition which has a more algebraic flavor. We then show that a horispherical subalgebra $\mathfrak{Z}$ of a Lie algebra $\mathfrak{L}$ is strongly nilpotent in the sense that ad(z) is nilpotent on $\mathfrak{L}$ for all z in $\mathfrak{Z}$

The work of Auslander and Brezin [2] on almost-algebraic Lie algebras is used heavily in Section 2, where it is shown that a Lie algebra and its small almost-algebraic hull $A(\mathfrak{L})$ have exactly the same set of horispherical subalgebras This fact is used to obtain a statement on conjugate horispherical subalgebras. Finally, it is shown that if $\mathfrak{Z}$ is horispherical in $\mathfrak{L}$ then $\mathfrak{Z} = \mathfrak{Z}_1 \ \mathfrak{Z}_2$, where $\mathfrak{Z}_1$ is a horispherical subalgebra of a Levi factor of $\mathfrak{L}$ and $\mathfrak{Z}_2$ is a horispherical subalgebra of a solvable subalgebra of $\mathfrak{L}$ which contains the radical of $\mathfrak{L}$ and is of codimension 1 with the radical of $\mathfrak{L}$.

In Section 3 questions concerning conjugate horispherical subalgebras of real solvable Lie algebras are considered. Bounds for the number of classes of conjugate horispherical subalgebras are found. Examples show these bounds to be the best possible in the general case considered

1. Equivalent Definitions and Basic Facts

Let $\mathfrak{L}$ be a finite-dimensional real Lie algebra and let x be an element of $\mathfrak{L}$. Let $\mathfrak{L}^c$ be the complexification of $\mathfrak{L}$ and let $\mathfrak{B}$ be the subalgebra of $\mathfrak{L}^c$ generated by $1 \otimes x$. Let $R(\alpha(x))$ represent the real part of the eigenvalue $\alpha(x)$, where α is a root.

Definition 1 1. The *horispherical subalgebra* $\mathfrak{Z}$ of $\mathfrak{L}$ relative to x is defined as the intersection of $\mathfrak{L}$ with the direct sum of those root spaces $\mathfrak{L}_\alpha$ of $\mathfrak{B}$ in $\mathfrak{L}^c$ for which $R(\alpha(x)) < 0$.

In [14, Proposition 1.1], Maruyama proved the equivalence of Definition 1.1 with the definition of horispherical subalgebra previously given in the introduction of this paper. In so doing, Maruyama also proved that if $\mathfrak{Z}$ is horispherical in $\mathfrak{L}$ relative to x and if $x = s + n$ with ad(x) = ad(s) + ad(n) the Jordan decomposition of ad(x), where ad(s) is the semisimple part, then $\mathfrak{Z}$ is horispherical in $\mathfrak{L}$ relative to s In [14], Maruyama was specifically concerned with real semisimple Lie algebras, but it is easy to see that his Proposition 1.1 is more general. In all that follows, we will use Definition 1 1 for horispherical subalgebras.

THEOREM 1.1. *If $\mathfrak{Z}$ is horispherical in $\mathfrak{L}$ relative to x, then* ad(z) *is nilpotent on $\mathfrak{L}$ for all z in $\mathfrak{Z}$.*

Proof First we notice that ad(x) is nonsingular on $\mathfrak{Z}$. But any Lie algebra with a nonsingular derivation is nilpotent [10], so $\mathfrak{Z}$ is nilpotent

Since $\mathfrak{Z}$ is nilpotent, $\mathfrak{L}^c$ is the direct sum of the root spaces of $\mathfrak{Z}^c$. Suppose α is a nonzero root. Let $\mathfrak{B}$ be the sum of all root spaces corresponding to nonzero roots β, where $\beta = k\alpha$ for some integer $k > 0$. $\mathfrak{B}$ is a solvable subalgebra of $\mathfrak{L}^c$ since either $\mathfrak{L}_\gamma \mathfrak{L}_\sigma \subset \mathfrak{L}_{\gamma+\sigma}$ if $\gamma + \sigma$ is a root, or $\mathfrak{L}_\gamma \mathfrak{L}_\sigma = \{0\}$ Since x belongs to the root space corresponding to the zero root, it is further clear that $\mathfrak{S} \oplus \mathfrak{B}$ is a solvable subalgebra of $\mathfrak{L}^c$, where $\mathfrak{S}$ is the subalgebra of $\mathfrak{L}^c$ generated by x and $\mathfrak{Z}^c$. But then ad$(x)(\mathfrak{Z}) = \mathfrak{Z}$ implies that $\mathfrak{Z}$ is contained in the nil radical of $\mathfrak{S} \oplus \mathfrak{B}$. Hence all roots of $\mathfrak{Z}^c$ in $\mathfrak{L}^c$ are zero.

Dixmier and Lister [3], have given an example of a nilpotent Lie algebra for which every derivation is nilpotent. Clearly such an algebra cannot be embedded in any Lie algebra so as to be a horispherical subalgebra. It would be interesting to have a characterization of those nilpotent Lie algebras which are horispherical in some Lie algebra.

2. HORISPHERICAL SUBALGEBRAS AND ALMOST-ALGEBRAIC LIE ALGEBRAS

In order that we might pursue questions of conjugacy and decomposition of horispherical subalgebras, we need certain information about almost-algebraic Lie algebras. Such Lie algebras have been carefully studied [2, 12]. In particular, [2] contains a complete account of all the information we will need on this subject. So that we might more compactly state our results, a brief discussion of some aspects of almost-algebraic Lie algebras will be given However, particular details will be carefully referenced as needed.

Let $\mathfrak{A}$ be a Lie algebra with nilradical $\mathfrak{N}$ $\mathfrak{A}$ is called *almost-algebraic* if $\mathfrak{N}$ is complemented in $\mathfrak{A}$ by an $\mathfrak{A}$-reductive subalgebra $\mathfrak{C}$ $\mathfrak{C}$ is called a *Malcev factor* for $\mathfrak{A}$, and the semidirect product decomposition $\mathfrak{A} = \mathfrak{C} \cdot \mathfrak{N}$ is called a *Malcev decomposition* of $\mathfrak{A}$. Let $\mathfrak{L}$ be a subalgebra of the almost-algebraic Lie algebra $\mathfrak{A}$. $\mathfrak{A}$ is called a *small almost-algebraic hull* for $\mathfrak{L}$ if $\mathfrak{L}$ is an ideal in $\mathfrak{A}$ and no almost-algebraic proper subalgebra of $\mathfrak{A}$ contains $\mathfrak{L}$. Every Lie algebra admits one and, up to isomorphism, only one small almost-algebraic hull [2, p. 299]. Hereafter, we will denote by $A(\mathfrak{L})$, the small almost-algebraic hull of $\mathfrak{L}$.

THEOREM 2.1 *Let $\mathfrak{L}$ be a Lie algebra and let $A(\mathfrak{L})$ be its small almost-algebraic* *hull The set of horispherical subalgebras of $A(\mathfrak{L})$ is exactly the set of horispherical subalgebras of $\mathfrak{L}$.*

Proof If $\mathfrak{Z}$ is horispherical in $\mathfrak{L}$ relative to x, then since $\mathfrak{L}$ is an ideal in $A(\mathfrak{L})$, $\mathfrak{Z}$ is clearly horispherical in $A(\mathfrak{L})$ relative to x.

Let $\mathfrak{Z}$ be horispherical in $A(\mathfrak{L})$ relative to x. Let $x = h + n$ be a decomposition for x for which $\mathrm{ad}(x) = \mathrm{ad}(h) + \mathrm{ad}(n)$ is the Jordan decomposition of $\mathrm{ad}(x)$, with $\mathrm{ad}(h)$ semisimple [2, p. 301]. We recall that $\mathfrak{Z}$ is horispherical in $A(\mathfrak{L})$ relative to h Let $\mathfrak{C}$ be a maximal $A(\mathfrak{L})$-reductive subalgebra of $A(\mathfrak{L})$ containing h and let $\mathfrak{C} = \mathfrak{S} \oplus \mathfrak{T}$ be a decomposition of $\mathfrak{C}$ with $\mathfrak{S}$ a Levi factor and $\mathfrak{T}$ the center of $\mathfrak{C}$. $\mathfrak{S}$ is a Levi factor of $\mathfrak{L}$ [2, p. 303], and $\mathrm{ad}(t)$ is semisimple for each t in $\mathfrak{T}$ [11, p 47]. There are elements s in $\mathfrak{S}$ and t in $\mathfrak{T}$ such that $h = s + t$. Now $\mathrm{ad}(h)$ and $\mathrm{ad}(t)$ are semisimple and commute so that $\mathrm{ad}(s)$ is semisimple also

Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{C}$ generated by s and $\mathfrak{T}$. Let $\mathfrak{R}$ be the radical of $\mathfrak{L}$ and let $\mathfrak{R}_0$ and $\mathfrak{R}_1$ be the Fitting Null and Fitting one components, respectively, of $\mathfrak{R}$ with respect to $\mathrm{ad}(\mathfrak{B})$ [11, p. 39] Since $\mathfrak{R}$ and $\mathfrak{N}$ generate $\mathfrak{T}$ $\mathfrak{N}$ [2, p. 305], there is an x in $\mathfrak{R}$ and an n in $\mathfrak{N}$ such that $t = x + n$. Let $x = x_0 + x_1$ with x_0 in $\mathfrak{R}_0$ and x_1 in $\mathfrak{R}_1$. Since $\mathfrak{R}$ is solvable it is clear that $\mathfrak{R}_1$ is a subset of $\mathfrak{N}$. Now $\mathrm{ad}(s)$ both nilpotent and semisimple on $\mathfrak{R}_0$ implies that $\mathrm{ad}(s)$ is zero on $\mathfrak{R}_0$. Hence

$$[s, x_1 + n] = [s, x_0 + x_1 + n] = [s, t] = 0.$$

Similarly, $[t, x_1 + n] = 0$ so that

$$s + x_0 = (s + t) - (x_1 + n)$$

is a decomposition of $s + x_0$ with $\mathrm{ad}(s + t)$ the semisimple part of $\mathrm{ad}(s + x_0)$. Thus $\mathrm{ad}(h) = \mathrm{ad}(s + t)$ is the semisimple part of $\mathrm{ad}(s + x_0)$ and $\mathfrak{Z}$ is horispherical relative to $s + x_0$ Hence $\mathfrak{Z}$ is horispherical in $\mathfrak{L}$ since both s and x_0 belong to $\mathfrak{L}$.

THEOREM 2 2. *Let $\mathfrak{L}$ be a Lie algebra and let $A(\mathfrak{L})$ be its small almost-algebraic hull. Let $(\mathfrak{S} \oplus \mathfrak{T})$ $\mathfrak{N}$ be a Malcev decomposition of $A(\mathfrak{L})$ with $\mathfrak{S}$ a Levi factor. If $\mathfrak{Z}$ is horispherical in $\mathfrak{L}$ then $\mathfrak{Z}$ is conjugate under an inner automorphism of $\mathfrak{L}$ to a horispherical subalgebra $\mathfrak{Z}^*$ relative to $h + t$, where h is an element of $\mathfrak{S}$ with* $\mathrm{ad}(h)$ *semisimple and t is in $\mathfrak{T}$.*

Proof. Let $\mathfrak{Z}$ be horispherical in $\mathfrak{L}$ relative to x and let $x = s + n$ be a decomposition for x, where $\mathrm{ad}(x) = \mathrm{ad}(s) + \mathrm{ad}(n)$ is the Jordan decomposition for $\mathrm{ad}(x)$ with $\mathrm{ad}(s)$ semisimple Thus $\mathfrak{Z}$ is horispherical in $A(\mathfrak{L})$ relative to s. Let $\mathfrak{B}$ be a maximal $A(\mathfrak{L})$-reductive subalgebra of $A(\mathfrak{L})$ containing s. For y in $\mathfrak{N}$, let $E(y)$ represent the inner automorphism generated by y; i.e , $E(y) = \exp(\mathrm{ad}(y))$ Then there exists an element n^* in $\mathfrak{N} \cap [A(\mathfrak{L}), A(\mathfrak{L})]$ such that

$E(n^*)(\mathfrak{B}) = (\mathfrak{S} \oplus \mathfrak{T})$ [2, p. 306; 15] Since $[A(\mathfrak{L}), A(\mathfrak{L})] = [\mathfrak{L}, \mathfrak{L}]$, n^* belongs to $\mathfrak{L}$ so that $E(n^*)$ is an inner automorphism of $\mathfrak{L}$. There is an h in $\mathfrak{S}$ and a t in $\mathfrak{T}$ such that $E(n^*)(s) = h + t$. Since $\mathrm{ad}(E(n^*)(s))$ and $\mathrm{ad}(t)$ are semisimple and commute, $\mathrm{ad}(h)$ is semisimple. Clearly, $\mathfrak{Z}^* = E(n^*)(\mathfrak{Z})$ is horispherical in $\mathfrak{L}$ relative to $h + t$

In the following corollary we see that if $\mathfrak{Z}$ is horispherical in $\mathfrak{L}$ then $\mathfrak{Z} = \mathfrak{Z}_1 \cdot \mathfrak{Z}_2$ where $\mathfrak{Z}_1$ is a horispherical subalgebra of a Levi factor of $\mathfrak{L}$ and $\mathfrak{Z}_2$ is a horispherical subalgebra of a solvable subalgebra of $\mathfrak{L}$ containing the radical.

COROLLARY 2.1. *Let $\mathfrak{L}, \mathfrak{S}, \mathfrak{T}, \mathfrak{N}, \mathfrak{Z}, \mathfrak{Z}^*, h,$ and t remain as in the previous theorem and let $\mathfrak{R}$ be the radical of $\mathfrak{L}$. Then $\mathfrak{Z}^* = \mathfrak{Z}_1 \cdot \mathfrak{Z}_2$, where $\mathfrak{Z}_1$ is horispherical in $\mathfrak{S}$ relative to h and $\mathfrak{Z}_2$ is horispherical in the subalgebra generated by h and $\mathfrak{R}$ relative to $h + x$ for some x in $\mathfrak{R}$ such that $[h, x] = 0$ and the semisimple part of* $\mathrm{ad}(h + x)$ *is* $\mathrm{ad}(h + t)$

Proof. Let z be in $\mathfrak{Z}^*$. Then $z = z_1 + z_2$, where z_1 is in $\mathfrak{S} \oplus \mathfrak{T}$ and z_2 is in $\mathfrak{N}$ Since $\mathrm{ad}(h + t)(\mathfrak{Z}^*) = \mathfrak{Z}^*$ and $\mathfrak{S}$ is an ideal in $\mathfrak{S} \oplus \mathfrak{T}$ it follows that z_1 belongs to $\mathfrak{S}$ and $\mathfrak{Z}$ Let $\mathfrak{Z}_1 = \mathfrak{Z}^* \cap \mathfrak{S}$ and $\mathfrak{Z}_2 = \mathfrak{Z}^* \cap \mathfrak{N}$. Thus $\mathfrak{Z}^* = \mathfrak{Z}_1 \cdot \mathfrak{Z}_2$. Clearly $\mathfrak{Z}_1$ is horispherical in $\mathfrak{S}$ relative to h. Let $\mathfrak{R} = \mathfrak{R}_0 + \mathfrak{R}_1$ be the Fitting decomposition of $\mathfrak{R}$ relative to $\mathrm{ad}(\mathfrak{B})$ as described in the proof of Theorem 2.1 and let $t = x_0 + x_1 + n$ be the representation of t given by this decomposition and by the fact that $\mathfrak{R}$ and $\mathfrak{N}$ generate $\mathfrak{T} \cdot \mathfrak{N}$. Clearly, $\mathfrak{Z}_2$ is horispherical relative to $h + x_0$ in the subalgebra generated by h and $\mathfrak{R}$.

3. HORISPHERICAL SUBALGEBRAS OF SOLVABLE LIE ALGEBRAS

Let $\mathfrak{L}$ be a sovable Lie algebra, $A(\mathfrak{L})$ its small almost-algebraic hull, and $\mathfrak{T} \cdot \mathfrak{N}$ a Malcev decomposition for $A(\mathfrak{L})$ with $\mathfrak{N}$ the nilradical. Consider the root space decomposition of $\mathfrak{N}^c$ under the actions of $\mathfrak{T}$. Let $\mathfrak{N}_\alpha$ represent the root space in $\mathfrak{N}^c$ corresponding to the root α

LEMMA 3.1. *Let n be in $\mathfrak{N}$ and let z be any nonzero element of $\mathfrak{N}_\alpha$ Then the projection of $E(n)(z)$ onto $\mathfrak{N}_\alpha$ is not* 0.

Proof. Let p represent the projection of $\mathfrak{N}$ onto $\mathfrak{N}_\alpha$. Suppose $p(E(n)(z)) = 0$. Recall that

$$E(n)(z) = z + \mathrm{ad}(n)(z) + (\mathrm{ad}(n))^2(z)/2! + \quad + (\mathrm{ad}(n))^m(z)/m!$$

for some positive integer m. Thus

$$z = p(z - E(n)(z)) \tag{1}$$

and

$$z = p(p(z - E(n)(z)) - E(n)(p(z - E(n)(z)))). \tag{2}$$

Let $n = \sum x_\beta$ be the representation of n given by the root space decomposition. Now it is clear that $z - E(n)(z)$ consists of a sum of products each of which is of length at least 2. Since z is in $\mathfrak{N}_\alpha$, it is clear that $z - E(n)(z)$ is a sum of products each of which is of length at least 2 and belongs to some root space. Hence in (1), z is represented as a sum of products each of which is of length at least 2 Further, in (2), z is represented as a sum of products each of which is of length at least 3 Clearly, by continuing this substitution process, we could obtain a representation for z in which each term is a product of length at least k for $k \geqslant N$, where N is any positive integer But $\mathfrak{N}$ is nilpotent Thus z must be 0.

THEOREM 3.1. *Let $\mathfrak{L}$ be a solvable Lie algebra and let $\mathfrak{T} \cdot \mathfrak{N}$ be a Malcev decomposition of $A(\mathfrak{L})$, the small almost-algebraic hull of $\mathfrak{L}$. Suppose that $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ are horispherical in $\mathfrak{L}$ relative to t_1 and t_2, respectively, with both t_1 and t_2 in $\mathfrak{T}$. Then $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ are inner conjugates if and only if $\mathfrak{Z}_1 = \mathfrak{Z}_2$.*

Proof. Clearly $\mathfrak{Z}_1 = \mathfrak{Z}_2$ implies $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ are inner conjugates. Suppose n is an element of $\mathfrak{L}$ in $\mathfrak{N}$ such that $E(n)(\mathfrak{Z}_1) = \mathfrak{Z}_2$. By the above lemma we see that $R(\alpha(t_1)) < 0$ implies $R(\alpha(t_2)) < 0$ Thus $\mathfrak{Z}_1 \subset \mathfrak{Z}_2$. Hence $\mathfrak{Z}_1 = \mathfrak{Z}_2$.

THEOREM 3 2 *Let $\mathfrak{L}$ be a solvable Lie algebra and let $\mathfrak{T}$ $\mathfrak{N}$ be a Malcev decomposition for $A(\mathfrak{L})$, the small almost-algebraic hull of $\mathfrak{L}$. If k is the number of roots α for $\mathfrak{T}$ acting on $\mathfrak{N}^c$ for which $R(\alpha(t)) \neq 0$ for some t in $\mathfrak{T}$, then there are at most 2^k classes of conjugate horispherical subalgebras of $\mathfrak{L}$*

Proof. Let Δ be the set of roots α for $\mathfrak{T}$ acting on $\mathfrak{N}^c$ for which $R(\alpha(t)) \neq 0$ for some t in $\mathfrak{T}$ Let $P(\alpha)$ be the null space of $R \circ \alpha$ and let P be the union of all such null spaces with α in Δ. Consider the components of $\mathfrak{T} \backslash P$. There are at most 2^k such components, with equality in the event that no α in Δ is a multiple of another β in Δ If t_1 and t_2 belong to the same component then $R(\alpha(t_1)) < 0$ is equivalent to $R(\alpha(t_2)) < 0$ for all α in Δ. Thus any two elements of the same component define the same horispherical subalgebra The previous theorem shows that any two elements from different components would define non-conjugate horispherical subalgebras

The above theorem says that if $\mathfrak{L}$ is a solvable Lie algebra with nilradical $\mathfrak{N}$ and $\dim \mathfrak{N} = m$, and if k is the number of classes of conjugate horispherical subalgebras, then $1 \leqslant k \leqslant 2^m$. It is easy to see that both upper and lower bounds are sharp The upper triangular matrices of trace zero provide a class of examples in which the upper bound is achieved. The Lie algebra with basis x, y, and z and multiplication given by

$$[x, y] = 0, \qquad [x, z] = y, \qquad \text{and} \qquad [y, z] = -x$$

is a solvable, nonnilpotent Lie algebra with only the trivial horispherical subalgebra.

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