Limits of tangents and minimality of complex links

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Received 10 October 2001; accepted 27 February 2002

Abstract

We show that the complex link of a large class of space germs \((X,x_0)\) is characterized by its “simplicity”, among the Milnor fibres of functions with isolated singularity on \(X\). This amounts to the minimality of the Milnor number, whenever this number is defined. Such a phenomenon has been first pointed out in case \((X,x_0)\) is an isolated hypersurface singularity, by Teissier (Cycles évanescentes, sections planes et conditions de Whitney, in: Singularités à Cargèse 1972, Astérisque, Nos. 7 et 8, Soc. Math. France, Paris, 1973, pp. 285–362).

MSC: 32S30; 14J17; 32C42

Keywords: Complex link; Limits of tangents; Milnor fibre; Polar curve

1. Introduction and main results

Let \((X,x_0)\) denote the germ at some point \(x_0\) of a reduced complex analytic space \(X\), embedded into \(\mathbb{C}^N\), for some \(N\). The complex link \(\text{lk}^\mathbb{C}(X,x_0)\) of \(X\) at \(x_0\) is the Milnor fibre of the restriction to \(X\) of a general function germ \(l:(\mathbb{C}^N,x_0) \to (\mathbb{C},0)\), where \(x_0\) is viewed as a point stratum in some Whitney stratification of \(X\). The attribute “general” attached to a function \(l\) means that \(l\) is a local parameter (equivalently, \(l \in m \setminus m^2\), where \(m\) denotes the maximal ideal of the local algebra of holomorphic functions) and that the tangent hyperplane \(T_{x_0}l^{-1}(0)\) is not a limit of tangent hyperplanes to \(X\) (see Section 2.3 for the precise definition).

The complex link measures in some sense the singularity of \(X\) at \(x_0\). It is the central object of study in the complex Morse theory of singular complex spaces, developed by Goresky and MacPherson [5]. It is independent (up to isotopy) from the choice of general \(l\), as shown in [5, Chapter II, Section 2.3].
On the other hand, the complex link has the “simplest” topology among the Milnor fibres of functions with isolated singularity on \((X,x_0)\), see Corollary 2.6. One may ask if this minimality property characterizes \(\text{lk}^C(X,x_0)\). More precisely:

**Question.** Let \(F : \mathbb{C}^N \to \mathbb{C}\) be a holomorphic function such that the restriction \(F|_X\) has stratified isolated singularity at \(x_0\). If its Milnor fibre \(M(F|_X,x_0)\) has the same homology groups as the complex link \(\text{lk}^C(X,x_0)\), does it follow that \(F\) is a general function?

This seems to be a long-standing open question, at least for linear functions. Teissier proved in his 1973 well known paper [20, Section 1.6], the following: If \((X,x_0)\) is an isolated hypersurface singularity (abbreviated, ihs) then, for a linear function \(l : \mathbb{C}^N \to \mathbb{C}\) with isolated singularity on \(X\), the Milnor number of \(M(l|_X,x_0)\) is minimal if and only if the hyperplane \(l^{-1}(l(x_0))\) is general.

In the particular case of an ihs \((X,x_0)\), Teissier’s result answers positively to the above problem, since the minimality of the Milnor number is equivalent to the minimality of homology groups, a property which is satisfied by \(\text{lk}^C(X,x_0)\), cf. Section 2.3.

In this paper we prove a more general result. Let us fix some notations. Consider the coarsest Whitney stratification \(\mathcal{W} = \{W_i\}_{i \geq 0}\) of \(X\) at \(x_0\), such that \(W_0 = \{x_0\}\) is a stratum (see Section 2.1). We assume that strata are connected and we denote by \(W_1, \ldots, W_q\) the strata having \(x_0\) in their closure, other than \(W_0\). Let us recall the definition of the complex link \(\text{lk}^C(X,W_i)\) of a stratum \(W_i\). Let \(k_i = \dim_C W_i\) and let \(N_i \subset \mathbb{C}^N\) be a manifold of codimension \(k_i\), transversal to \(W_i\) at some point \(x_i \in W_i\). Then \(\text{lk}^C(X,W_i) := \text{lk}^C(X \cap N_i, x_i)\). It is shown in [5] that this does not depend on choices of \(N_i\) and \(x_i\). Notice that the complex link of the highest dimensional stratum (which is the nonsingular part of \(X\)) is empty. We say that \(\text{lk}^C(X,W_i)\) is acyclic if it has the homology of a point. We tacitly work with homology over \(\mathbb{Z}\), and specify when we use \(\mathbb{Q}\)-coefficients.

**Theorem 1.1.** Let \(F : (\mathbb{C}^N,x_0) \to (\mathbb{C},0)\) be a local parameter (i.e. \(F \in \mathfrak{m} \setminus \mathfrak{m}^2\)) such that the restriction \(F|_X\) to \(X\) has at most isolated singularity at \(x_0\). Assume that the complex link \(\text{lk}^C(X,W_i)\) is not acyclic, for any \(i \neq 0\). Then we have the equivalences:

(a) The Milnor fibre \(M(F|_X,x_0)\) is homotopy equivalent to the complex link \(\text{lk}^C(X,x_0)\).

(b) The Milnor fibre \(M(F|_X,x_0)\) has the same homology groups as \(\text{lk}^C(X,x_0)\).

(c) \(F\) is general with respect to \(X\) and \(\mathcal{W}\) at \(x_0\).

If the nonacyclicity condition holds over \(\mathbb{Q}\), then the above are also equivalent to:

(d) The sum of the Betti numbers of \(M(F|_X,x_0)\) does not exceed the sum of the Betti numbers of \(\text{lk}^C(X,x_0)\).

The nonacyclicity assumption for the complex links is satisfied, for instance, by any space \(X\) with isolated singularity at \(x_0\) (by our remark above) and by any complete intersection with at most one-dimensional singularities (Proposition 4.1).

Theorem 1.1 may also be viewed as part of the study of the limits of tangent hyperplanes. This study has been founded by Whitney [26] and developed by several authors; we refer the reader to the survey article [16] by Lê and Teissier and to Zak’s monograph [27]. Special attention was directed to the case \(X\) is a surface, where the blow-up techniques allow deeper insight, see e.g. [15,19,2].
Back to our Question: When $F \in m^2$, the problem seems to be even more delicate. The only known case (and classical) is that of a nonsingular $X$, when $M(F|_X,x_0) \not\cong \text{lk}^C(X,x_0)$, due to the fact that the Milnor number of $F$ is positive, whereas the complex link is obviously contractible. For singular $X$ we prove the following:

**Theorem 1.2.** Assume that $X$ is irreducible and that its rectified $\mathbb{Q}$-homological depth is greater or equal to its complex dimension. If $F \in m^2$ is a holomorphic function such that the restriction $F|_X$ to $X$ has at most isolated singularity at $x_0$, then the $\mathbb{Q}$-homology of the Milnor fibre $M(F|_X,x_0)$ is different from the one of the complex link $\text{lk}^C(X,x_0)$.

In case of $\mathbb{Q}$-coefficients, the condition $\text{rHd} X \geq \dim_C X$ implies that the locally constant sheaf $\mathbb{Q}_X$ is perverse (see [7]) and so the reduced $\mathbb{Q}$-homology of $M(F|_X,x_0)$ is concentrated in dimension $\dim_C X - 1$. From Theorems 1.1 and 1.2, it follows that a nongeneral function $F$ with isolated singularity on $X$ has Milnor number strictly greater than that of the complex link. For spaces $X$ with *Milnor property* [13] (i.e. the reduced homology of the complex links is concentrated in the top dimension), being "homotopy equivalent to the complex link $\text{lk}^C(X,x_0)$" just means having the same Milnor number, thus having "minimal Milnor number". Teissier’s result for his cited above, as well as more recent Teissier type results for normal surfaces in [19,2], belong to this class.

2. Preliminaries

2.1. Isolated singularities

Allover in this paper, when speaking about the dimension of complex analytic spaces, we mean the complex dimension. We have assumed that our space $X$ is endowed with an analytic Whitney stratification ("analytic" means that each stratum is a complex analytic manifold) having connected strata and having a one-point stratum $W_0 = \{x_0\}$. To avoid "unnecessary strata", we shall work with the *coarsest Whitney stratification* $\mathcal{W}$ (in the sense of Teissier’s [21]), unless otherwise specified. For instance, if $X$ is a space with isolated singularity at $x_0$, then the coarsest Whitney stratification $\mathcal{W}$ has as strata $W_0$ and the connected components of $X \setminus x_0$.

There are finitely many Whitney strata having $x_0$ in their closure. Let us denote these strata, different from $W_0$, by $W_1, \ldots, W_q$.

**Definition 2.1** (Lê [12]). The *singular locus* of a holomorphic function germ $f : (X,x_0) \rightarrow \mathbb{C}$, with respect to the stratification $\mathcal{W}$, is the following space germ at $x_0$:

$$\text{Sing}_{\mathcal{W}} f := \bigcup_{i=0}^{q} \text{Sing}(f|_{W_i}),$$

where $\text{Sing}(f|_{W_i})$ is the set of points where the restriction of $f$ to the manifold $W_i$ is not a submersion. We say that $f$ has an *isolated singularity* if $\text{Sing}_{\mathcal{W}} f = \{x_0\}$.

One may remark that $\text{Sing}_{\mathcal{W}} f$ is a closed analytic subset of $X$, due to the Whitney (a) property of the stratification. Since the stratum $W_0$ is a single point, the definition implies that $x_0 \in \text{Sing}_{\mathcal{W}} f$. 


For another definition of isolated singularity, in terms of limits of hyperplanes, we send to Remark 2.4.

**Definition 2.2.** Let \( f : (X, x_0) \to (\mathbb{C}, 0) \) be a holomorphic germ and let \( B_\varepsilon \subset \mathbb{C}^N \) be an open ball of radius \( \varepsilon > 0 \) centred at \( x_0 \). The Milnor fibre of \( f \) at \( x_0 \), denoted by \( M(f, x_0) \), is \( f^{-1}(\eta) \cap B_\varepsilon \), for \( \varepsilon \) small enough and \( 0 < |\eta| \ll \varepsilon \). This is independent of \( \varepsilon \) and \( \eta \), as shown by Milnor (cf. [18] in case of nonsingular \( X \)) and by Lê (cf. [10] for general \( X \)).

Let us denote by \( \mathcal{W}^i \) the union of strata of dimension \( \leq i \). After [7], one says that \( \text{rhd} X \geq m \) if for any \( i \) and any point \( x \in \mathcal{W}^i \setminus \mathcal{W}^{i-1} \), the homology groups of \( (U_x, U_x \setminus \mathcal{W}^i) \) are trivial up to the order \( m - 1 - i \), where \( \{U_x\} \) is some fundamental system of neighbourhoods of \( x \). It is shown in loc.cit. that this does not depend on the chosen Whitney stratification. A similar definition holds in homotopy instead of homology, giving rise to the rectified homotopical depth, denoted \( \text{rhd} X \). The condition \( \text{rhd} X \geq \dim X \) holds, for instance, when \((X, x_0)\) is a complete intersection (see [14] and also [11,4]). Note that, if \((X, x_0)\) is nonsingular, then \( \text{rhd} X = 2n > \dim X \).

### 2.2. Limits of hyperplanes

We define the space of limits of hyperplanes associated to a stratified complex analytic space \( X \) embedded into \( \mathbb{C}^N \).

Let \( \mathbb{P}(T^* C^N) \) denote the projectivization of the cotangent bundle of \( \mathbb{C}^N \) and let us identify it with \( \mathbb{C}^N \times \tilde{\mathbb{P}}^{N-1} \), where \( \tilde{\mathbb{P}}^{N-1} \) denotes the dual of \( \mathbb{P}^{N-1} \).

**Definition 2.3.** We call conormal of \( X \) with respect to the stratification \( \mathcal{W} \) the space \( T^*_{X,\mathcal{W}} := \bigcup_{i=1}^N T^*_i \), where

\[
T^*_i := \text{closure} \left\{ (x, \omega) \in \mathbb{P}(T^* C^N) \mid x \in W_i, \omega(T_x W_i) = 0 \right\} \subset \tilde{W}_i \times \tilde{\mathbb{P}}^{N-1}.
\]

One says that a function \( l : \mathbb{C}^N \to \mathbb{C} \) is *general* with respect to \( X \) and \( \mathcal{W} \), at \( x_0 \), if \((x_0, d l_{x_0}) \notin (T^*_X)_{x_0}\), where \((T^*_X)_{x_0} := \pi^{-1}(x_0)\) and \( \pi : T^*_X \to X \) denotes the projection on the first factor.

The space \( T^*_X \) is a closed analytic subspace of \( X \times \mathbb{P}^{N-1} \) (see e.g. [16]). It is of dimension \( N - 1 \) and consists of the duals of the limits of hyperplanes in \( \mathbb{C}^N \) which are tangent to the regular part of \( X \) or tangent to some other Whitney stratum of \( X \). Let \( p : T^*_X \to \mathbb{P}^{N-1} \) denote the second projection.

**Remark 2.4.** The singular locus of a function \( f : (X, x_0) \to \mathbb{C} \) (cf. Definition 2.1) can be alternatively defined, as follows. Let \( F : \mathbb{C}^N \to \mathbb{C} \) be an extension of \( f \) to a neighbourhood of \( x_0 \) in \( \mathbb{C}^N \). Then, for some \( x \neq x_0 \), \( x \in \text{Sing}_{\mathcal{W}} f \) if and only if \((x, dF_x) \in T^*_X \). In particular, \( x_0 \) is an isolated singularity of \( f \) if and only if \((x, dF_x) \notin T^*_X \), for all \( x \neq x_0 \).

### 2.3. Minimality of homotopy type

It has been proved in [23] that the complex link \( \text{lk}^C(X, x_0) \) is contained, via a natural embedding, in the Milnor fibre \( M(f, x_0) \) of any function germ \( f : (X, x_0) \to \mathbb{C} \) with isolated singularity at \( x_0 \). Moreover, the Milnor fibre \( M(f, x_0) \) has the following bouquet structure, up to homotopy type:
Theorem 2.5 (Tibăr [23]).

\[ M(f, x_0) \overset{\text{ht}}{=} \text{lk}^C(X, x_0) \vee \bigvee_{i=1}^q \left( S^{k_i}(\text{lk}^C(X, W_i)) \right), \]

where \( k_i = \dim_C W_i \) and \( S^{k_i} \) means the \( k_i \)-times repeated suspension.

The number of objects in the last bouquet \( \bigvee \) depends on polar invariants; for precise details, we send to [23]. By convention, the suspension over the empty set is the 0-sphere \( S^0 \), i.e. two points. From this bouquet structure, one can immediately derive the following:

Corollary 2.6. The complex link \( \text{lk}^C(X, x_0) \) is minimal among the Milnor fibres of functions \( f \) with isolated singularity, either with respect to homology or with respect to homotopy type. The homology of the complex link \( H_*(\text{lk}^C(X, x_0), \mathbb{Z}) \) is included as direct summand into \( H_*(M(f, x_0), \mathbb{Z}) \).

3. Proof of the main results

3.1. Proof of Theorem 1.1

(a) \( \Rightarrow \) (c). If the function \( F : \mathbb{C}^N \to \mathbb{C} \) is general with respect to \( X \) at \( x_0 \), then the Milnor fibre \( M(F|_{X}, x_0) \) is homotopy equivalent to the complex link \( \text{lk}^C(X, x_0) \). This is a well-known general fact, proved by Goresky and MacPherson in their monograph [5, part II, Section 2.3].

Since (a) \( \Rightarrow \) (b) is clear, the only thing left is:

(b) \( \Rightarrow \) (c). By a local analytic change of coordinates \( \varphi : (\mathbb{C}^N, x_0) \to (\mathbb{C}^N, x_0) \), we get that \( F \circ \varphi^{-1} \) is a germ of a linear function and we consider its restriction to \( X':=\varphi(X) \). We compare its Milnor fibre to the complex link of \( X' \). There is a homotopy equivalence (induced by \( \varphi \)) of Milnor fibres: \( M(F|_{X'}, x_0) \overset{\text{ht}}{=} M(F \circ \varphi^{-1}|_{X'}, x_0) \). Since the tangent map \( T_{x_0} \varphi \) yields a one-to-one correspondence between noncharacteristic covectors, the complex links of \( X \) and \( X' \) are homotopy equivalent. We have thus proved that, by eventually changing coordinates and working on \( X' \) instead of \( X \), one may assume without loss of generality that \( F \) is a linear function.

The case \( \dim X = 1 \) being obvious (multiplicity argument: \( \deg_{x_0} X < \text{mult}_{x_0} (X, H) \) is equivalent to \( H \) containing a line in the tangent cone of \( X \)), we concentrate on the case: \( \dim X \geq 2 \) and all the connected components of \( X\setminus\{x_0\} \) are of dimension \( \geq 2 \).

Now, since \( F \) is linear, the 1-form \( dF_X \) is an element of \( \mathbb{P}^{N-1} \) and does not depend on \( x \). Therefore, we shall use the notation \( dF \). We show that, if \( (x_0, dF) \in (T^*_x \mathcal{N}, x_0) \) then \( H_*(\text{lk}^C(X, x_0), \mathbb{Z}) \) is strictly included into \( H_*(M(F|_{X}, x_0), \mathbb{Z}) \) (compare to Corollary 2.6). The proof is divided into three steps.

Step 1: We identify \( x_0 \) with the origin of \( \mathbb{C}^N \). Let \( l_{\text{gen}} : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0) \) be a linear function, general with respect to \( X \) and \( \mathcal{N} \). Consider the relative conormal \( T^*_{l_{\text{gen}}} := \bigcup_{j=1}^q T^*_{l_{\text{gen}}|_{W_j}} \), where

\[ T^*_{l_{\text{gen}}|_{W_j}} := \text{closure}\{x \in W_j, dI \in \mathbb{P}^{N-1} | dl(T_{x}(l_{\text{gen}}|_{W_j}(l_{\text{gen}}(x)))) = 0\} \subset W_j \times \mathbb{P}^{N-1}, \]

where \( l : \mathbb{C}^N \to \mathbb{C} \) denotes a linear function (modulo a multiplicative constant).
We have that $T_{l_{\text{gen}}}^*$ is an analytic subspace of $X \times \mathbb{P}^{N-1}$, of dimension $N$. Let us denote by $p_r$ the projection to $\mathbb{P}^{N-1}$ and by $\pi_r$ the projection to $X$. It follows, by dimension reasons, that the analytic subspace $p_r^{-1}(dF)$ is of dimension $\geq 1$, provided that it is nonempty. This is indeed nonempty since $T_{X,W}^* \subset T_{l_{\text{gen}}}^*$ and since, by our assumption, $(x_0,dF) \in T_{X,W}^*$.

Since $\pi_r$ is proper, the set $\pi_r(p_r^{-1}(dF))$ is analytic. Moreover, we claim that there exists a Zariski-open subset $\Omega \subset \mathbb{P}^{N-1}$ such that, for any $l_{\text{gen}}$ belonging to it, we have

$$\pi_r(p_r^{-1}(dF)) \cap F^{-1}(0) = \{x_0\}. \tag{1}$$

Then condition (1) implies that $\dim \pi_r(p_r^{-1}(dF)) \leq 1$. Remark in addition that, by its definition, $\pi_r$ realizes an analytic isomorphism between $p_r^{-1}(dF)$ and $\pi_r(p_r^{-1}(dF))$. Since $\dim p_r^{-1}(dF) \geq 1$, we conclude that $\dim \pi_r(p_r^{-1}(dF)) = 1$.

To justify (1), choose a Thom $(a_F)$-stratification of $X$ at $x_0$, which is finer than the Whitney stratification $\mathcal{W}$. (This exists by [8], see also [6, Théorème 1.2.1].) Take $l_{\text{gen}}$ such that it is general with respect to all strata of this $(a_F)$-stratification, except at $x_0$. Then $\pi_r(p_r^{-1}(dF))$ cannot meet those strata outside $x_0$.

**Step 2:** From now on we work with $l_{\text{gen}} \in \Omega$. By Step 1, the following analytic subset of $X$, as germ at $x_0$:

$$\Gamma(l_{\text{gen}},F;X) := \pi_r(p_r^{-1}(dF))$$

is a nonempty curve. One calls it the *polar curve* of $F$ with respect to $l_{\text{gen}}$. It can have several irreducible components, in different strata. For instance, it follows from the definition that all strata $W_i$ of dimension 1 are contained in the polar curve $\Gamma(l_{\text{gen}},F;X)$. Polar curves and polar varieties proved to be useful tools for studying topological and algebro-geometric aspects of singularities (work of Lê, Teissier and other authors).

Consider the set $p^{-1}(dF)$. It has dimension 0, since having dimension $\geq 1$ would mean that the function $F$ has nonisolated singularity at $x_0$, which is excluded by the hypotheses of the theorem. This implies that, in some neighbourhood of $(x_0,dF)$ in $T_{X,W}^*$, the projection $p$ is a finite analytic map.

Let then $L$ denote the line in $\mathbb{P}^{N-1}$ spanned by $dF$ and $dl_{\text{gen}}$. It follows that $p^{-1}(L)$ is a curve, as germ at $x_0$. We claim the following equality of germs of reduced curves:

$$\pi(p^{-1}(L)) = \pi_r(p_r^{-1}(dF)). \tag{2}$$

An important point in the proof is that both sets are germs of curves at $x_0$.

“$\supset$”. Let $x \in \pi_r(p_r^{-1}(dF))$, $x \neq x_0$ close enough to $x_0$. Since $dF$ annihilates the hyperplane $T_x(l_{\text{gen}}(x)) \cap T_xW_i$ of $T_xW_i$, it follows that there is a linear combination of the two independent co-vectors $dF$ and $dl_{\text{gen}}$ which annihilates $T_xW_i$.

“$\subset$”. If $x \in \pi(p^{-1}(L)) \cap W_i$ then there is a linear combination of $dF$ and $dl_{\text{gen}}$ which annihilates $T_xW_i$. This implies that $dF$ annihilates $T_{x,l_{\text{gen}}}(l_{\text{gen}}(x)) \cap T_xW_i$. Our claim is proved.

The equality (2) gives two interpretations of the polar curve $\Gamma(l_{\text{gen}},F;X)$. Since $(x_0,dF)$ belongs to the curve $p^{-1}(L)$, every irreducible component of $p^{-1}(L)$ contains some sequence of points $(x_n,\omega_n) \in T_{X,W}^*$ tending to $(x_0,dF)$, where $x_n \neq x_0$. This implies that $\omega_n(T_xW_i) = 0$, if $x_n$ belongs to the Whitney stratum $W_i$. Since the stratification $\mathcal{W}$ is analytic, each component of $\Gamma(l_{\text{gen}},F;X)$ is contained into a single stratum of $\mathcal{W}$, provided we take out the point $x_0$. It follows that $\omega_n$ annihilates
the tangent space\(^1\) to the curve \(\Gamma(l_{\text{gen}}, F; X)\), at \(x_n\). Then the tangent map \(T_{x_n}(l_{\text{gen}}, F)\) sends the hyperplane \(H_n\), the kernel of \(\omega_n\), to a line in \(\mathbb{C}^2\). This line is the tangent space at \((l_{\text{gen}}(x_n), F(x_n))\) to the discriminant \(\Delta:=\text{the image of } \Gamma(l_{\text{gen}}, F; X) \text{ by the map } (l_{\text{gen}}, F)\). The analytic curve \(\Delta \subset \mathbb{C}^2\) is viewed as a germ at the origin.

Summing up, we have proved that \(T_{x_n}(l_{\text{gen}}, F)(H_n)\) is tangent to \(\Delta\). By our assumption, the limit hyperplane \(H:=\lim_{n \to \infty} H_n\) is the kernel of \(dF\). We conclude that the image of \(H\) by the tangent map \(T_{x_0}(l_{\text{gen}}, F)\) is in the tangent cone of \(\Delta\) at the origin. More precisely, we have proved that the image of \(H\) coincides with the tangent cone of each component of \(\Delta\).

**Step 3.** We have defined the polar curve \(\Gamma(l_{\text{gen}}, F; X)\) and its image \(\Delta\) by the application \((l_{\text{gen}}, F) : (\mathbb{C}^N, 0) \to (\mathbb{C}^2, 0)\). By Step 2, the coordinate line \(\{F = 0\} \subset \mathbb{C}^2\) is tangent at 0 to every component \(\Delta_j\) of \(\Delta\) (Fig. 1).

Then the intersection multiplicity \(\text{mult}_0(\Delta_j, \{F = 0\})\) is greater than \(\text{mult}_0(\Delta_j, \{l_{\text{gen}} = 0\})\). It follows that, for \(\eta \neq 0\) and \(\varepsilon \neq 0\) small enough, we have the following inequality for the numbers of points of intersection:

\[
\#(\Delta_j \cap \{F = \eta\}) > \#(\Delta_j \cap \{l_{\text{gen}} = \varepsilon\}). \tag{3}
\]

This inequality has an important implication on the bouquet structure (cf. Theorem 2.5, see its proof in [23]), for our function \(F\):

\[
M(F|_X, x_0) \cong \text{lk}^C(X, x_0) \vee \bigvee_{i=1}^q \left(\bigvee_{k=1} S^k(\text{lk}^C(X, W_i))\right), \tag{4}
\]

namely: the number of objects in each bouquet \(\bigvee S^k(\text{lk}^C(X, W_i))\) is \(\geq 1\), since equal to the difference of the numbers in (3) (cf. [23, Section 4.1]). At this point, we use the assumption about the complex links \(\text{lk}^C(X, W_i)\) to conclude that the homology of the bouquet \(\bigvee_{i=1}^q \left(\bigvee_{k=1} S^k(\text{lk}^C(X, W_i))\right)\) is nontrivial. Hence, \(H_\ast(M(F|_X, x_0), \mathbb{Z})\) is different from the homology of the complex link \(\text{lk}^C(X, x_0)\). This ends the proof of (b) \(\Rightarrow\) (c).

For the last equivalent statement, in case of \(\mathbb{Q}\)-coefficients, we still need to justify:

(d) \(\Rightarrow\) (c). The above conclusion now reads: \(H_\ast(M(F|_X, x_0), \mathbb{Q}) \neq H_\ast(\text{lk}^C(X, x_0), \mathbb{Q})\). Moreover, the relation (4) shows that the Betti numbers of \(M(F|_X, x_0)\) are greater or equal to the corresponding Betti numbers of \(\text{lk}^C(X, x_0)\). But, since at least one complex link \(\text{lk}^C(X, W_i)\) has nontrivial homology over \(\mathbb{Q}\), then at least some Betti number of \(M(F|_X, x_0)\) is strictly greater. This completes the proof of our Theorem.

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1 Remark that \(x_n\) is a regular point of \(\Gamma(l_{\text{gen}}, F; X)\).
In case of functions in $m^2$, we cannot prove in general the nonemptiness of the polar curve, which was one of the important points in the above proof. The proof of Theorem 1.2 needs therefore another argument.

3.2. Proof of Theorem 1.2

We may define the relative conormal $T^*_F$ and the polar locus $\Gamma(F, l_{\text{gen}}; X)$, similarly as at Step 1. Note however the difference: the roles of $F$ and $l_{\text{gen}}$ are switched, i.e. $\Gamma(F, l_{\text{gen}}; X) = \pi_*(p^{-1}(dF_{l_{\text{gen}}}))$. It is well known (see e.g. [10]) that, for general enough $l_{\text{gen}}$, the polar locus $\Gamma(F, l_{\text{gen}}; X)$ has dimension $\leq 1$, and no curve component is contained in $F^{-1}(0)$ or in $l_{\text{gen}}^{-1}(0)$; this follows by considerations similar to those in Step 1. Note that $\Gamma(F, l_{\text{gen}}; X)$ is either a curve or is void, by the same dimension reasons as in Step 1.

If $\Gamma(F, l_{\text{gen}}; X)$ is a curve, then Lê’s result [11, Proposition 2.1] tells that $\Delta = \Delta(F, l_{\text{gen}})$ has the axis $\{F = 0\}$ as reduced tangent cone at the origin. Therefore Step 3 can be applied and, if the complex links $lk^C(X, W_i)$, for $i \neq 0$, are not acyclic over $\mathbb{Q}$, then it leads to the conclusion $H_*(M(F|_X, x_0), \mathbb{Q}) \neq H_*(lk^C(X, x_0), \mathbb{Q})$.

Let then assume that either $\Gamma(F, l_{\text{gen}}; X) = \emptyset$, or $\dim \Gamma(F, l_{\text{gen}}; X) = 1$ and all complex links $lk^C(X, W_i)$, for $i \neq 0$, are acyclic over $\mathbb{Q}$. In both situations, we claim that we have the equalities of homologies with $\mathbb{Q}$-coefficients

$$H_*(lk^C(X, x_0)) = H_*(lk^C(X \cap \{F = 0\}, x_0)) = H_*(M(F|_X \cap \{l_{\text{gen}} = 0\}, x_0)) = H_*(M(F|_X, x_0)).$$

The second equality comes from the corresponding homotopy equivalence and is true for $\mathbb{Z}$-coefficients, whether or not $\Gamma(F, l_{\text{gen}}; X)$ is empty. This is due to the fact that, except of the origin, the values on the axes of $\mathbb{C}^2 \setminus (0, 0)$ are regular values for the restriction $(F, l_{\text{gen}})|_X : (X, x_0) \to (\mathbb{C}^2, 0)$. In case $\Gamma(F, l_{\text{gen}}; X) = \emptyset$, the first and third equalities are obvious since the discriminant of $(F, l_{\text{gen}})|_X$ consists of the origin only. Let us see what happens when $\dim \Gamma(F, l_{\text{gen}}; X) = 1$. By Lê’s attaching result [9], see also [23], the space $M(F|_X, x_0)$, respectively, $lk^C(X, x_0)$, can be obtained from $lk^C(X \cap \{F = 0\}, x_0)$ by attaching thimbles over the Milnor fibres of the singularities of the restriction $l_{\text{gen}}|_{X \cap \{F = 1\}}$, respectively, of the restriction $F|_X \cap \{l_{\text{gen}} = 0\}$ (follow Fig. 1). These Milnor fibres are suspensions over the complex links $lk^C(X, W_i)$ of the corresponding strata $W_i$, as shown in [23]. Now, if those complex links are acyclic, then the above-mentioned attaching does not change the homology. Consequently, we get the first and third equalities in (5).

Next, we use the condition $\text{rHd} X \geq \dim X$ of our theorem. After [7], this implies that $lk^C(X, x_0)$, respectively, $lk^C(X \cap \{F = 0\}, x_0)$, has the homology of a bouquet of spheres of dimension $\dim X - 1$, respectively, of dimension $\dim X - 2$. Due to the first equality in (5), this can happen in the same time only if both complex links have the homology of a point. Then also $M(F|_X, x_0)$ has the homology of a point, by (5). This gives a contradiction, since the Milnor fibre of $F|_X$ must have a nontrivial monodromy. Indeed, A’Campo’s result [1], see also [22, Corollary 2.3], says that, if $F$ is not a local parameter, then the Lefschetz number of the monodromy of the Milnor fibre $M(F|_X, x_0)$ is zero. \qed
As a consequence of both theorems, we may prove:

**Corollary 3.1.** If \( X \) is of pure dimension, \( r\text{Hd}_\mathbb{Q} X \geq \dim X \) and the complex link \( \text{lk}^\mathbb{C}(X,W_i) \) is not \( \mathbb{Q} \)-acyclic, \( \forall i \neq 0 \), then, for any holomorphic germ \( F:(\mathbb{C}^n,x_0) \to \mathbb{C} \) with isolated singularity on \( X \), we have the equivalence:

(a) \( \chi(M(F|_X,x_0)) = \chi(\text{lk}^\mathbb{C}(X,x_0)) \).

(b) \( F \) is a local parameter, general with respect to \( X \) and \( W \).

**Proof.** If \( F \) is not a local parameter at \((\mathbb{C}^n,x_0)\) or is not general with respect to \( X \) and \( W \), then Theorems 1.2 and 1.1 say that the Milnor fibre \( M(F|_X,x_0) \) is different from the complex link \( \text{lk}^\mathbb{C}(X,x_0) \). Since the space \( X \) satisfies \( r\text{Hd}_\mathbb{Q} X \geq \dim X \), it follows that the reduced homology over \( \mathbb{Q} \) of \( M(F|_X,x_0) \) and that of \( \text{lk}^\mathbb{C}(X,x_0) \) are concentrated in the top dimension, i.e. \( \dim X - 1 \). Therefore, the difference between \( M(F|_X,x_0) \) and \( \text{lk}^\mathbb{C}(X,x_0) \) is detectable by the Euler characteristic. \( \square \)

4. Further remarks

4.1. Nonacyclicity of complex links

As we have seen, spaces with isolated singularity have the property asked by Theorem 1.1. It would be interesting to find algebraic criteria implying this property in case of other spaces. An example of a space \( X \) which does not satisfy the nonacyclicity condition in the theorem is the following: take a \( \mu \)-constant family of hypersurface germs at \( 0 \in \mathbb{C}^n \), which is not \( \mu^* \)-constant (cf. [3]), and let \( Y \) denote the total space. Let \( X: = Y \times D_\epsilon \), for a small disk \( D_\epsilon \). Then the complex link of the stratum \( \{0\} \times D_\epsilon^* \) is homeomorphic to the complex link \( \text{lk}^\mathbb{C}(Y,0) \), hence contractible. In case of one-dimensional singular locus, we have the following nonacylicity result:

**Proposition 4.1.** If \( X \) is a complete intersection at \( x_0 \), having a nonempty singular locus of dimension \( \leq 1 \) then the complex link \( \text{lk}^\mathbb{C}(X,W_i) \) is not contractible, \( \forall i \neq 0 \).

**Proof.** We first reduce the problem to the case of an isolated complete intersection singularity (icis). If \( W_i \) is a curve component of the singular locus \( \text{Sing} X \) of \( X \), then take a point \( x \in W_i \), \( x \neq x_0 \) and take a hyperplane \( H \subset \mathbb{C}^N \) transversal to \( W_i \) through \( x \). Then \( X^i := X \setminus H \) is an icis at \( x \).

We have to show that \( \text{lk}^\mathbb{C}(X^i,x) \) is not contractible. Since \( X^i \) is an icis, it is well known (see [17, p. 68]) that one may choose holomorphic function germs \( f_j: (\mathbb{C}^n,0) \to (\mathbb{C},0) \) such that: the ideal \( (f_1,\ldots,f_k) \) defines \( X^i \) as a complete intersection in \((\mathbb{C}^n,0)\), the ideal \( (f_2,\ldots,f_k) \) defines an icis \( X^i_1 \) and \( f_1 \in m_{X^i,0}^2 \cdot m_{X^i_1,0} \). We identify \( x \) to \( 0 \in \mathbb{C}^n \).

We take a linear function \( l_{\text{gen}}: (\mathbb{C}^n,0) \to (\mathbb{C},0) \), general relative to \( X^i_1 \). The slice \( Y_1:=X^i_1 \cap l_{\text{gen}}^{-1}(0) \) is itself an icis at \( 0 \) and the restriction \( f_{1|Y_1} \) is again in the square of the maximal ideal. It follows that \( M(f_{1|Y_1},x) \) cannot be contractible, by the same argument of Lefschetz number of the monodromy (cf. [1,22]) as used in the proof of Theorem 1.2. Moreover, we have the homotopy equivalence \( M(f_{1|Y_1},x) \approx \text{lk}^\mathbb{C}(X^i_1 \cap \{f_1 = 0\},0) \), as shown in the proof of Theorem 1.2 (namely the second
equality in (5) and the comment after it, for the functions $f_1$ and $l_{\text{gen}}$ on the space $X'_1$). Since $X'_1 \cap \{f_1 = 0\} = X'$, it follows that $\text{lk}_C(X', 0)$ cannot be contractible. □

4.2. Open spaces

Instead of a closed analytic space $X$, one might consider $X = Y \setminus V$, where $V \subset Y$ are germs of reduced analytic spaces at $x_0$, embedded into $\mathbb{C}^N$.

One takes the coarsest Whitney stratification $\mathcal{W}$ of $Y$ having $V$ as union of strata. Inspite of the fact that $x_0 \not\in X$, one may still define Milnor fibres at $x_0$ and in particular the complex link $\text{lk}_C(X, x_0)$. Milnor fibres of germs on $X$ have been considered by Hamm and Lê [7]. We may refer to [24, 25] for the definitions and other results involving these objects. Briefly speaking, from the Milnor fibres defined over the closed analytic space $Y$, we take out the set $V$. The new Milnor fibres are well defined, since $V$ is a union of strata.

Then our Question in Section 1 still makes sense. Since the answer involves new technical ingredients, we postpone the discussion to some future publication.

Acknowledgements

The author thanks the anonymous referee for valuable remarks which helped to improve the paper.

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