This paper \(^1\) is a sequel to the paper "Ultraproducts and elementary classes" [3]. In [3], a number of results were stated without proof in an appendix, and our purpose here is to give the proofs of two of the main results stated there. As the title indicates we shall study the connection between the ultraproduct construction and the notion of a saturated (that is, homogeneous universal) structure. \(\alpha\)-saturated structures (see § 1 below), where \(\alpha\) is a cardinal \(\omega_p\), are the natural analogues, for arbitrary theories, of Hausdorff’s \(\eta\)-sets for the theory of simple order. It is known that they have many desirable properties (see [7]); for instance any two elementarily equivalent \(\alpha\)-saturated structures of power \(\alpha\) are isomorphic. In § 2 we prove Theorem A. 4 of [3], which states that if \(D\) is an ultrafilter with the set-theoretical property \(D \in \mathcal{Q}(\alpha)\) (see § 1), then every ultraproduct modulo \(D\) is an \(\alpha\)-saturated structure. As a corollary, any two elementarily equivalent structures \(\mathcal{A}\), \(\mathcal{B}\) have some isomorphic ultrapowers \(\mathcal{A}/D \cong \mathcal{B}/D\). In § 3 we prove Theorem A. 9 of [3], which gives a structure \(\mathcal{A}(\alpha)\) such that, for each \(D\), \(\mathcal{A}(\alpha)/D\) is \(\alpha\)-saturated if and only if \(D \in \mathcal{Q}(\alpha)\). Finally, in § 4 we give counterexamples to some plausible conjectures involving ultraproducts, where either the construction of the example or the original conjecture is related to the results of this paper.

As much as possible, we shall not repeat here the notation, references, or historical remarks which can be found in [3]. We do, however, give enough reminders of the notation from [3] to permit a reader with a good background in model theory to follow this paper without referring to [3]. The proofs of the results in the appendix of [3] which are not given here may be found elsewhere: Theorems A.1 and A. 2 in [7], Lemma A. 3 in [4], and Theorems A.10–A.12 in [5].

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§ 1. Preliminaries

We shall assume all the notation introduced in [3]. In particular: \(\alpha\), \(\beta\), \(\gamma\) are arbitrary cardinals; \(\mu \in \omega^\beta\) is a similarity type; \(\mathcal{A} = \langle A, R_{\lambda}\rangle_{\lambda < \gamma}\).

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and $\mathfrak{B} = \langle B, S_\lambda \rangle_{\lambda \leq \text{q}}$ are arbitrary structures of type $\mu$ (the $R_\lambda, S_\lambda$ being characteristic functions of $\mu(\lambda)$-ary relations); $L(\mu)$ is a first order logic with identity and $\mu(\lambda)$-ary predicate symbols $P_\lambda, \lambda \leq \text{q}$; $D$ is an ultrafilter on a set $I$; $\prod_{i \in I} \mathfrak{U}_i/D$ is an ultraproduct and $\mathfrak{U}/D$ is an ultrapower. We recall that for any ordinal $\alpha$, the type $\mu_{\alpha}$ is defined so that $\mu \subseteq \mu_{\alpha} \subseteq \xi$ and $\mu \oplus \xi(\alpha + \xi) = 1$ for all $\xi < \xi$; moreover, if $a \in A^\alpha$ then $(\mathfrak{U}, a)$ is the structure $\langle A, R_\alpha \rangle_{\alpha \leq \text{q}}$ of type $\mu \oplus \xi$ such that, for all $\xi < \xi$, $R_{\alpha + \xi}$ is the characteristic function of the set $\{a_\xi\}$.

Let us denote by $F(\mu)$ the set of all formulas $\varphi(v_0)$ of $L(\mu)$ which have at most the one free variable $v_0$. We shall say that a set $\Gamma \subseteq F(\mu)$ is satisfiable in $\mathfrak{A}$ if there is an element $a_0 \in A$ such that every formula $\varphi(v_0) \in \Gamma$ holds in $\mathfrak{A}$ with $v_0$ interpreted as $a_0$. $\Gamma$ is said to be finitely satisfiable in $\mathfrak{A}$ if every finite subset of $\Gamma$ is satisfiable in $\mathfrak{A}$.

**Definition.** A structure $\mathfrak{A}$ is said to be $\alpha$-saturated if, for each $\xi < \alpha$ and $a \in A_\xi$, every set of formulas $\Gamma \subseteq F(\mu \oplus \xi)$ which is finitely satisfiable in $(\mathfrak{A}, a)$ is satisfiable in $(\mathfrak{A}, a)$.

In case $\alpha$ is the power of $\mathfrak{A}$, the above definition reduces to the notion of a saturated structure introduced by Vaught, for example in [10]. Saturated structures are studied in detail in [7]; although they are not given a name there, it is shown (in Theorem 3.4, p. 49) that a structure is saturated if and only if it is homogeneous and universal. Similarly, in [3] we used the term $\alpha$-homogeneous $\alpha$-universal structure instead of $\alpha$-saturated structure. The following result is proved in [7], p. 43 and p. 49.

**Lemma 1.1.** Any two elementarily equivalent $\alpha$-saturated structures of power $\alpha$ are isomorphic.

We shall also need the following two simple lemmas.

**Lemma 1.2.** If $A$ is finite, then $\mathfrak{A}$ is $\alpha$-saturated for all $\alpha$.

**Lemma 1.3.** If $\mathfrak{A}$ is $\alpha$-saturated, then $A$ is either finite or of power at least $\alpha$.

**Proof:** Suppose $\omega < \beta < \alpha$, where $\beta$ is the power of $A$, and let $a \in A^\beta$ be an enumeration of $A$. Let $\Gamma \subseteq F(\mu \oplus \beta)$ be the set $\{P_{\xi + \xi}(v_0) \mid \xi < \beta\}$. Then $\Gamma$ is finitely satisfiable, but not satisfiable, in $(\mathfrak{A}, a)$.

Let us recall some more notation from [3]. If $X$ is a set, then $\alpha(X)$ is the power of $X$, $S(X) = \{Y \mid Y \subseteq X\}$, $S_\alpha(X) = \{Y \in S(X) \mid \alpha(Y) < \alpha\}$, and $S_\alpha(X) = \{X - Y \mid Y \in S_\alpha(X)\}$. If $G$ and $H$ are functions with domain $X$ whose values are sets, we shall write $H \leq G$ if $H(x) \subseteq G(x)$ for all $x \in X$. We denote by $Q(\alpha)$ the class of all ultrafilters $D$ such that (i) $D$ is countably incomplete (that is, not countably complete), and (ii) for all cardinals $\beta < \alpha$ and every monotonic function $G$ on $S_\alpha(\beta)$ into $D$, there exists a multiplicative function $H$ on $S_\alpha(\beta)$ into $D$ such that $H \leq G$. Obviously, if $\beta < \alpha$ then $Q(\beta) \subset Q(\alpha)$.
An investigation of the class $Q(\alpha)$, but with a different terminology, can be found in [4] 2). We state below the main results which are proved in [4].

**Lemma 1.4.** If $D \in Q(\alpha^+)$, then every $x \in D$ has power at least $\alpha$.

**Lemma 1.5.** If $D$ is countably incomplete, then $D \in Q(\omega_1)$.

**Lemma 1.6.** Assume $\kappa(2^\omega) = \alpha^+$ and $\kappa(I) = \alpha$. Then there exists an ultrafilter $D \in Q(\alpha^+)$ on $I$. Indeed, any set $F$ such that:

(i) $F \subseteq S(I) - S_{\alpha}(I)$;

(ii) $\kappa(F) < \alpha$; and

(iii) $F$ is closed under finite intersection;

can be extended to an ultrafilter $D \in Q(\alpha^+)$.

**Lemma 1.7.** Let $\alpha > \omega$ and $\kappa(I) = \alpha$. Then there exists a countably incomplete ultrafilter $D \notin Q(\omega_2)$ on $I$. Indeed, any set $F$ satisfying (i)–(iii) of Lemma 1.6 can be extended to such a $D$.

We shall use, often without explicit mention, the fundamental result 3) of Los that: a sentence $\varphi$ holds in $\prod_{i \in I} \mathfrak{A}_i / D$ if and only if

$$\{ i \in I \mid \varphi \text{ holds in } \mathfrak{A}_i \} \in D.$$ 

§ 2. $\alpha$-saturated ultraproducts

We obtain a sufficient condition for an ultraproduct to be an $\alpha$-saturated structure.

**Theorem 2.1.** Let $\alpha > \omega$ and $D \in Q(\alpha)$. Then for any structures $\mathfrak{A}_i$, $i \in I$, of type $\mu$, the ultraproduct $\prod_{i \in I} \mathfrak{A}_i / D$ is $\alpha$-saturated.

**Proof:** Let $\mathfrak{A} = \prod_{i \in I} \mathfrak{A}_i / D$, $\xi < \alpha$, $a \in A^I$, $\mathfrak{B} = (\mathfrak{A}, a)$, and $\Gamma \subseteq F(\varnothing \oplus \xi)$. Suppose $\Gamma$ is finitely satisfiable in $\mathfrak{B}$. It suffices to show that $\Gamma$ is satisfiable in $\mathfrak{B}$. It is easily seen that $\mathfrak{B}$ can be expressed as an ultraproduct $\mathfrak{B} = \prod_{i \in I} \mathfrak{B}_i / D$.

By Lemma 1.5 we have $D \in Q(\omega_1)$, so we may assume that $\alpha > \omega$. Since $\varnothing$, $\xi < \alpha$, we have $\kappa(\Gamma) < \alpha$. Define the function $G$ on $S_{\alpha}(\Gamma)$ into $S(I)$ so that, for all $\Delta \in \mathfrak{B}_i(\Gamma)$,

$$G(\Delta) = \{ i \in I \mid \Gamma - \Delta \text{ is satisfiable in } \mathfrak{B}_i \}.$$ 

It is easily seen that $G$ is monotonic. For each $\Delta \in \mathfrak{B}_i(\Gamma)$, let

$$\psi_\Delta = (\exists v_0) \land_{\varphi \in \Gamma - \Delta} \varphi(v_0).$$ 

Since $\Gamma$ is finitely satisfiable in $\mathfrak{B}$, each $\psi_\Delta$ holds in $\mathfrak{B}$. Then by the

\[ \text{A countably incomplete ultrafilter } D \text{ belongs to } Q(\alpha) \text{ if and only if, in the notation of [4], we have } G(J) \geq \alpha \text{ where } J \text{ is the ideal which has } D \text{ as its dual.} \]

\[ \text{See [3], p. 483, for references; the result is numbered Theorem 1.1 there.} \]
theorem of Łoś, we have
\[ \{ i \in I \mid \psi_i \text{ holds in } B_i \} \in D \]
for each \( \Delta \), and therefore \( G(\Delta) \in D \) for each \( \Delta \).

Since \( D \) is countably incomplete, there is a decreasing chain \( X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n \supseteq \ldots \) such that \( X_0 = I \), each \( X_n \in D \), and \( \bigcap_{n < \omega} X_n = 0 \). Define \( G' \) so that, for each \( \Delta \in S_\omega(\Gamma) \),
\[ G'(\Delta) = G(\Delta) \cap X_n, \]
where \( n \) is the power of \( \Gamma - \Delta \).

Then \( G' \) is clearly a monotonic function on \( S_\omega(\Gamma) \) into \( D \), and \( G' \ll G \).

Since \( D \in Q(x) \) there exists a multiplicative function \( H \ll G' \) on \( S_\omega(\Gamma) \) into \( D \). For each \( i \in I \), let \( n(i) \) be the greatest \( n < \omega \) such that \( i \in X_n \), and let
\[ \Gamma(i) = \{ \varphi \in \Gamma \mid i \in H(\Gamma - \{ \varphi \}) \}. \]

The power of \( \Gamma(i) \) is at most \( n(i) \); for otherwise there would be a subset \( \Gamma_0 \subseteq \Gamma(i) \) of power \( n(i) + 1 \), and by the multiplicativity of \( H \) we would have
\[ i \in \bigcap_{\varphi \in \Gamma_0} H(\Gamma - \{ \varphi \}) = H(\Gamma - \Gamma_0) \subseteq G'(\Gamma - \Gamma_0) \subseteq X_{n(i) + 1}, \]
contradicting the fact that \( i \notin X_{n(i) + 1} \). Hence \( \Gamma(i) \in S_\omega(\Gamma) \) and \( i \in \Gamma(\Gamma - \Gamma(i)) \) for all \( i \in I \). Since \( H \ll G \), we have \( i \in G(\Gamma - \Gamma(i)) \), and therefore \( \Gamma(i) \) is satisfiable in \( B_i \).

Choose \( g \in \prod_{i \in I} B_i \) so that, for each \( i \in I \), \( g(i) \) satisfies \( \Gamma(i) \) in \( B_i \). Then for all \( \varphi \in \Gamma \) we have
\[ \{ i \in I \mid g(i) \text{ satisfies } \varphi \text{ in } B_i \} \supseteq \{ i \in I \mid \varphi \in \Gamma(i) \} = H(\Gamma - \{ \varphi \}) \in D. \]

Therefore, by the theorem of Łoś, the set \( \Gamma \) is satisfied by \( g \upharpoonright D \) in \( B \).

Our proof is complete.

At this point it is natural to include a remark on the existence of \( \alpha \)-saturated structures. Let \( \beta \) be a cardinal such that \( g < \beta^+ \). MORLEY and VAUGHT have proved in [7] that every infinite structure of power at most \( \kappa(2^\beta) \) has a \( \beta^+ \)-saturated elementary extension of power \( \kappa(2^\beta) \); (see Theorem 2.10 and the remark (4) following Theorem 2.8 in the paper [7]). In the case \( \beta = \omega \), the above result of Morley and Vaught follows from Theorem 2.1 using Lemma 1.5, although the proof of Morley and Vaught was by a different method. For \( \beta > \omega \), we can obtain the result of Morley and Vaught from Theorem 2.1 only if we assume the generalized continuum hypothesis (GCH) and use Lemma 1.6.

Theorem 2.1 has a number of other consequences, some of which are stated in the appendix of [3]. We shall state just two consequences of the theorem in the following corollaries.

Corollary 2.2. Suppose \( \mu \) is denumerable and \( B_i, i \in I \) are structures of type \( \mu \). Then for every countably incomplete ultrafilter \( D \) on \( I \), the ultraproduct \( \prod_{i \in I} B_i \upharpoonright D \) is \( \omega_1 \)-saturated.
Proof: By Lemma 1.5 and Theorem 2.1.

Corollary 2.3. Assume the GCH. \( \mathfrak{A} \) and \( \mathfrak{B} \) are elementarily equivalent (in symbols \( \mathfrak{A}^* = \mathfrak{B}^* \)) if and only if there exists a set \( I \) and an ultrafilter \( D \) on \( I \) such that \( \mathfrak{A}/D \cong \mathfrak{B}/D \). Moreover, if \( \kappa(\alpha) < \alpha \) and \( \kappa(A) \), \( \kappa(B) < \alpha^+ \), then we may take \( I \) to be of power \( \alpha \).

Proof: By the theorem of Łoś we have \( \mathfrak{A}^* = (\mathfrak{A}/D)^* \) and \( \mathfrak{B}^* = (\mathfrak{B}/D)^* \), and hence \( \mathfrak{A}/D \cong \mathfrak{B}/D \) implies \( \mathfrak{A}^* = \mathfrak{B}^* \).

Suppose \( \mathfrak{A}^* = \mathfrak{B}^* \) and let \( \kappa(\alpha) < \alpha \) and \( \kappa(A) \), \( \kappa(B) < \alpha^+ \). By Lemma 1.6 and the GCH, there exists an ultrafilter \( D \in Q(\alpha^+) \) on \( \alpha \) (unless \( \alpha \) is finite, in which case we take a principal ultrafilter for \( D \)). Then

\[
(\mathfrak{A}/D)^* = \mathfrak{A}^* = (\mathfrak{B}/D)^*.
\]

If \( \alpha \) is finite, then \( \mathfrak{A}, \mathfrak{B}, \mathfrak{A}/D, \) and \( \mathfrak{B}/D \) are all isomorphic. If \( \alpha \) is infinite, then \( \alpha \) is infinite. Hence by Theorem 2.1, \( \mathfrak{A}/D \) and \( \mathfrak{B}/D \) are \( \alpha^+ \)-saturated. Using the GCH, we see that \( A^*/D \) and \( B^*/D \) each have power at most \( \alpha^+ \). But by Lemma 1.3, \( A^*/D \) and \( B^*/D \), being infinite, have power at least \( \alpha^+ \). Therefore by Lemma 1.1, \( \mathfrak{A}/D \cong \mathfrak{B}/D \).

Notice that the GCH was not used at all in the proof of Theorem 2.1, but was used twice in the proof of Corollary 2.3. Most of the work needed to arrive at Corollary 2.3 is done in the proof of Lemma 1.6, which can be found in [4].

§ 3. THE CONVERSE

In this section we prove that the condition \( D \in Q(\alpha) \) is not only sufficient, but also necessary, for every ultraproduct modulo \( D \) to be \( \alpha \)-saturated. For each infinite cardinal \( \alpha \), let us define \( \mathfrak{A}(\alpha) = (S(\alpha), R(\alpha)) \), where \( R(\alpha)(x, y) = 1 \) if and only if \( x \subseteq y \). Thus \( \mathfrak{A}(\alpha) \) is a structure of type \( \langle 2 \rangle \) and of power \( \alpha(2^\alpha) \).

Theorem 3.1. Suppose \( \omega < \alpha < \beta^+ \). Then \( D \in Q(\alpha) \) if and only if the ultrapower \( \mathfrak{A}(\beta)^D/D \) is \( \alpha \)-saturated.

Proof: If \( D \in Q(\alpha) \), then \( \mathfrak{A}(\beta)^D/D \) is \( \alpha \)-saturated by Theorem 2.1.

Before proceeding to the converse, it is convenient to introduce the special notation \( q_\xi \) for the formula

\[
(\mathfrak{A}v_1) [P_{1+\xi}(v_1) \land P_0(v_0, v_1)] \land (\mathfrak{A}v_2) [P_0(v_2, v_0) \land \neg v_2 = v_0].
\]

Thus \( q_\xi \in F(\langle 2 \rangle \oplus \gamma) \) for all \( \gamma > \xi \). If \( a \in S(\beta)^\gamma \) then \( q_\xi(v_0) \) states, for the model \( (\mathfrak{A}(\alpha), a) \), that "\( v_0 \) is a non-empty subset of \( a_\xi \)."

Now let \( \mathfrak{A}(\beta)^D/D = \mathfrak{A} = (\mathfrak{A}, R) \) and suppose \( \mathfrak{A} \) is \( \alpha \)-saturated. We first prove that \( D \) is countably incomplete. Since \( \beta \) is infinite, we may choose a decreasing chain \( a_0 \supseteq a_1 \supseteq \ldots \supseteq a_n \supseteq \ldots \) of sets \( a_n \in S(\beta) \) such that each \( a_n \) is non-empty but \( \bigcap_{n<\omega} a_n = 0 \). Then the set \( \Gamma = \{ q_n \mid n < \omega \} \) of formulas \( q_n \) defined above is finitely satisfiable in the structure \( (\mathfrak{A}(\beta), a) \) of type \( \langle 2 \rangle \oplus \omega \). For an appropriate \( a' \in A^\omega \) we have \( (\mathfrak{A}(\beta), a')^D/D = (\mathfrak{A}, a') \). Then
\( \Gamma \) is finitely satisfiable in \((\mathfrak{A}, \alpha')\); since \( \omega < \alpha \) and \( \mathfrak{A} \) is \( \alpha \)-saturated, \( \Gamma \) is satisfiable in \((\mathfrak{A}, \alpha')\), say by the element \( f/D \). For each \( n < \omega \), we have

\[ x_n = \{ i \in I \mid f(i) \text{ satisfies } q_n(v_0) \text{ in } (\mathfrak{A}(\beta, \alpha)) \} \in D. \]

But

\[ x_n = \{ i \in I \mid 0 \neq f(i) \subseteq a_n \}, \]

so \( \bigcap_{n < \omega} x_n = \emptyset \). Hence \( D \) is countably incomplete.

Now let \( \gamma < \alpha \); since \( \alpha < \beta^+ \) we have \( \gamma < \beta \). Let \( G \) be a monotonic function on \( \bar{S}_w(\gamma) \) into \( D \). We must find a multiplicative function \( H < G \) on \( \bar{S}_w(\gamma) \) into \( D \). If \( \gamma \) is finite, then \( 0 \in \bar{S}_w(\gamma) \) and we may take \( H(t) = G(0) \in D \) for all \( t \in \bar{S}_w(\gamma) \). Assume \( \gamma \) is infinite. Then \( \kappa(\bar{S}_w(\gamma)) = \gamma \), and hence there is a one-one function \( h \) on \( \bar{S}_w(\gamma) \) onto \( \gamma \). For each \( i \in I \) and \( s \in \bar{S}_w(\gamma) \), we define

\[ f_s(i) = \{ h(t) \mid t \in \bar{S}_w(\gamma), t \subseteq s, \text{ and } i \in G(t) \}. \]

Then since \( G \) is monotonic we have

\[ f_s(i) = 0 \text{ if and only if } i \notin G(s). \]

Moreover, for all \( s, t, \) and \( i \) we have

\[ f_s \cap i(i) = f_s(i) \cap f_t(i) \]

and

\[ \{ j \in I \mid h(s) \in f_s(j) \} = G(s) \in D. \]

For each \( \xi < \gamma \) let \( \bar{\xi} = \gamma - \{ \xi \} \), and define \( b \in A' \) by

\[ b_\xi = f_\xi/D \text{ for all } \xi < \gamma. \]

Let \( \Gamma \subseteq F(\mu \oplus \gamma) \) be the set \( \Gamma = \{ q_\xi(v_0) \mid \xi < \gamma \} \), where the formulas \( q_\xi(v_0) \) are as defined above. It follows from (3) and (4) that \( \Gamma \) is finitely satisfiable in \((\mathfrak{A}, b)\); since \( \mathfrak{A} \) is \( \alpha \)-saturated and \( \gamma < \alpha \), \( \Gamma \) is satisfiable in \((\mathfrak{A}, b)\), say by the element \( g/D \). We define the function \( H \) on \( \bar{S}_w(\gamma) \) into \( S(I) \) by:

\[ H(s) = \{ i \in I \mid 0 \neq g(i) \subseteq f_s(i) \}. \]

It follows from (3) that \( H \) is multiplicative. By (2) we have \( H < G \). Finally, for each \( \xi < \gamma \) we have

\[ H(\bar{\xi}) = \{ i \in I \mid 0 \neq g(i) \subseteq f_\xi(i) \} \in D, \]

because \( g/D \) satisfies \( q_\xi(v_0) \) in \((\mathfrak{A}, b)\). Hence \( H(s) \in D \) for all \( s \in \bar{S}_w(\gamma) \), and our proof is complete.

Consider the class \( M(\alpha) \) of all structures \( \mathfrak{A} \) such that, whenever \( D \) is countably incomplete, we have \( D \in Q(\alpha) \) if and only if \( \mathfrak{A}/D \) is \( \alpha \)-saturated. Corollary 2.3 shows that \( M(\omega_1) \) is the class of all structures, and Theorem 3.1 shows that \( \mathfrak{A}(\beta) \in M(\alpha) \) whenever \( \omega < \alpha < \beta^+ \). Furthermore, by examining the proof of Theorem 3.1 it can be seen that certain other structures belong to \( M(\alpha) \). For example, let \( \omega < \alpha < \beta^+ \). Then any extension \( \mathfrak{B} = \langle B, S \rangle \)
of $\mathfrak{U}(\beta)$, such that $S$ is transitive and any two elements $a, b \in S(\beta)$ have the greatest lower bound $a \cap b$ with respect to $S$, belongs to $M(\alpha)$. In particular, any elementary extension of $\mathfrak{U}(\beta)$, and any ultrapower of $\mathfrak{U}(\beta)$, belong to $M(\alpha)$. Finally, one can find a member of $M(\alpha)$ of power slightly less than $\omega(2^\beta)$, as follows. Let $h$ be a one-one function on $\beta$ onto $S_\omega(\beta)$, and for each $\eta < \beta$ let $x(\eta) = \{ \xi < \beta \mid h(\eta) \subseteq h(\xi) \}$. Let $\mathfrak{U} = \langle B, S \rangle$ be the substructure of $\mathfrak{U}(\beta)$ such that $B$ is the set of all sets of the form $x(\eta) \cap \bigcap_{\zeta < \eta} (\beta - x(\xi))$, where $\gamma, \eta < \beta$ and each $\xi < \beta$. Then $\mathfrak{U}$ belongs to $M(\alpha)$, and the power of $B$ is $\bigcup_{\eta < \beta} \omega(\beta^n)$.

For $\alpha > \omega_1$ there are structures $\mathfrak{U} \notin M(\alpha)$. For example, if $\alpha > \omega_1$, no finite structure belongs to $M(\alpha)$, and by Lemma 1.7 no structure of type 0 belongs to $M(\alpha)$. There are several open problems in connection with $M(\alpha)$, and we state a few of them below.

1. Does $\alpha < \beta$ imply $M(\alpha) \supseteq M(\beta)$?
2. Evaluate $\bigcap \{ \alpha(A) \mid \mathfrak{U} \in M(\alpha) \}$.
3. Does $\bigcap_{\alpha} M(\alpha) = 0$?
4. Does $\mathfrak{U} \in M(\alpha)$ imply that every elementary extension of $\mathfrak{U}$ is in $M(\alpha)$?
5. Does $M(\alpha)$ contain any structures $\mathfrak{U} = \langle A, R \rangle$ where $R$ is a simple ordering?
6. Find a syntactical characterization of $M(\alpha)$, or of the elementary class generated by $M(\alpha)$.

§ 4. Examples

In this section we give a series of counterexamples. The first two examples are applications of the results of § 2 and § 3 above to answer questions raised elsewhere. The remaining examples show that Corollary 2.4 cannot be improved in certain directions. The following gives a counterexample to Theorem 2 in KOCHEN [6].

Example 4.1. (GCH). Let $\mathfrak{U} = \mathfrak{U}(\omega_1)$. Suppose $\alpha > \omega_1, \kappa(\alpha) = \alpha$, and $F$ is any set which satisfies conditions (i)-(iii) of Lemma 1.6. Then $F$ can be extended to two ultrafilters $D, E$ on $\omega_1$ such that the ultrapowers $\mathfrak{U}^\omega|D$ and $\mathfrak{U}^\omega|E$ are not isomorphic.

Proof. By Lemmas 1.6 and 1.7, we may choose $D, E \supseteq F$ such that $D \in Q(\alpha^+) \subseteq Q(\omega_2)$ and $E \notin Q(\omega_2)$, but $E$ is countably incomplete. Then by Theorem 3.1, $\mathfrak{U}^\omega|D$ is $\omega_2$-saturated but $\mathfrak{U}^\omega|E$ is not.

The next result is due to C. C. CHANG and answers a question raised in [2], p. 207.

Example 4.2. (GCH). Let $\mathfrak{U} = \mathfrak{U}(\omega_1)$. Then there exist ultrafilters $D, E$ on $\omega_1$ such that $(\mathfrak{U}^\omega|D)^{\omega_1}|E$ and $(\mathfrak{U}^\omega|E)^{\omega_1}|D$ are not isomorphic. Indeed, whenever $\kappa(I) = \alpha > \omega_1, \kappa$ any set $F$ satisfying conditions (i)-(iii) of
Lemma 1.6 can be extended to two ultrafilters $D, E$ on $I$ such that $(\mathfrak{U}/D)/E$ and $(\mathfrak{U}/E)/D$ are not isomorphic.

Proof: Choose $D, E \supseteq F$ such that $D \in Q(\omega_2)$ and $E \notin Q(\omega_2)$, but $E$ is countably incomplete. Then by Theorem 2.1, the ultrapower $(\mathfrak{U}/E)/D$ is $\omega_2$-saturated. However by the remarks following Theorem 3.1, we have $\mathfrak{U}/D \in M(\omega_2)$, and hence the ultrapower $(\mathfrak{U}/D)/E$ is not $\omega_2$-saturated.

In the two preceding examples, we may take any structure $\mathfrak{B}$ all of whose ultrapowers belong to $M(\omega_2)$ for $\mathfrak{A}$, instead of taking $\mathfrak{A}(\omega_1)$. Hence by the remarks following Theorem 3.1, $\mathfrak{A}$ may be taken with power $\omega_1$ or any larger power.

Example 4.3. Let $\kappa(I)=\alpha \geq \omega$ and $\beta=(\kappa(2^\alpha))^+$. Then there exist structures $\mathfrak{A}, \mathfrak{B}$ of type $\langle 1 \rangle$ such that

(i) $\kappa(A)=\kappa(B)=\beta$;
(ii) $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent;
(iii) there are no ultrafilters $D, E$ on $I$ such that $\mathfrak{U}/D \cong \mathfrak{B}/E$.

Proof: Let $X, Y \subseteq \beta$ be sets such that $\kappa(X)=\kappa(2^\alpha)$ and $\kappa(Y)=\kappa(\beta-Y)=\beta$. Let $\mathfrak{A}=\langle \beta, R \rangle$ and $\mathfrak{B}=\langle \beta, S \rangle$, where $R, S$ are the characteristic functions of $X, Y$. Then $\mathfrak{A}^*=\mathfrak{B}^*$. For any ultrafilter $D$ on $I$ we have $\kappa(X/D)=\kappa(2^\alpha)$, but $\kappa(Y/D)\geq \beta$. Thus (i)-(iii) hold.

Example 4.4. Let $\kappa(I)=\alpha \geq \omega$. Then there exist structures $\mathfrak{A}, \mathfrak{B}$, with $\alpha=\omega^+$, such that $\kappa(A)=\kappa(B)=\alpha^+$ and conditions (ii), (iii) of Example 4.3 hold.

Proof: Let $A=S_\omega(\alpha^+)$; thus $\kappa(A)=\alpha^+$. For each $\zeta<\alpha^+$, let $X_\zeta=\{a \in A \mid \zeta \in a\}$ and let $R_\zeta$ be the characteristic function of $X_\zeta$. Let $\mathfrak{A}=\langle A, R_\zeta, \zeta<\alpha^+ \rangle$, and consider the set of formulas $\Gamma=\{P_\zeta(\nu_0) \mid \zeta<\alpha^+\}$. We first show that

(1) $\Gamma$ is not satisfiable in any ultrapower $\mathfrak{U}/D$.

To verify (1), suppose that $g \in A^I$. For each $i \in I$, there are only finitely many $\zeta<\alpha^+$ such that $g(i) \in X_\zeta$. Therefore the set

$Y=\{\zeta<\alpha^+ \mid g(i) \in X_\zeta \text{ for some } i \in I\}$

has power $\alpha$, and we may choose $\xi \in \alpha^+-Y$. The formula $P_\xi(\nu_0)$ is not satisfied in $\mathfrak{U}/D$ by $g/D$, and hence (1) is verified.

On the other hand, $\Gamma$ is clearly finitely satisfiable in $\mathfrak{A}$. By the compactness theorem, there is a structure $\mathfrak{B}$ such that $\kappa(B)=\alpha^+, \mathfrak{B}^*=\mathfrak{A}^*$, and $\Gamma$ is satisfiable in $\mathfrak{B}$. Then $\Gamma$ is satisfiable in every ultrapower $\mathfrak{B}/E$. Hence by (1) we can never have $\mathfrak{U}/D \cong \mathfrak{B}/E$.

Example 4.4 and Corollary 2.4 still leave open the following question: can 4.4 be improved by taking $\mathfrak{A}, \mathfrak{B}$ so that $\kappa(A)=\kappa(B)=\alpha$? The answer is not known even for $\alpha=\omega$. 

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The following result answers a question of A. Robinson.

Example 4.5. There is a structure $\mathcal{U}$, with $\varphi = \kappa(2^\omega)$ and $\kappa(A) = \omega$, and there exist two non-principal ultrafilters $D, E$ on $\omega$, such that $\mathcal{U}/D$ and $\mathcal{U}/E$ are not isomorphic.

Proof: Let $\beta = \kappa(2^\omega)$, let $X_\zeta$, $\zeta < \beta$ be an enumeration of $S(\omega)$, and let $\mathcal{V} = (\omega, R_{\zeta} < \beta)$ be such that each $R_{\zeta}$ is the characteristic function of $X_\zeta$. Let $D$ be an arbitrary non-principal ultrafilter on $\omega$. Then $\kappa(A^\omega/D) < \beta$. Let $Z$ be the set of all non-principal ultrafilters $D'$ on $\omega$ such that the set of formulas $\Gamma_{D'} = \{P_{\zeta}(v_0) | X_{\zeta} \in D'\}$ is satisfiable in $\mathcal{U}^\omega/D$. Then $\kappa(Z) < \beta$, because if $D' \neq D'$, then $\Gamma_{D'}$ and $\Gamma_{D''}$ cannot both be satisfied by the same element in $\mathcal{U}^\omega/D$. However, by a theorem of Popišil [8] and Tarski [9], there are $\kappa(2^{2^\omega})$ different non-principal ultrafilters $E$ on $\omega$. Hence we may choose a non-principal ultrafilter $E \notin Z$ on $\omega$. It is easily seen that $\Gamma_E$ is satisfiable in $\mathcal{U}^\omega/E$, in fact by the element $g/E$ where $g$ is the identity function on $\omega$. Therefore $\mathcal{U}^\omega/D$ is not isomorphic to $\mathcal{U}^\omega/E$.

By an easy modification of the above proof, using the result proved in [4] that (assuming the $GCH$) there are $\kappa(2^{2^\omega})$ different ultrafilters $D \in Q(\kappa^+)$ on $\alpha$, we obtain the following result.

Example 4.6. ($GCH$). Let $\kappa(I) = \kappa > \omega$. Then there is a structure $\mathcal{U}$, with $\kappa(A) = \kappa$ and $\varphi = \kappa^+$, and there exist two ultrafilters $D, E \in Q(\kappa^+)$ on $I$, such that $\mathcal{U}/D$ and $\mathcal{U}/E$ are not isomorphic.

REFERENCES