PARTITIONS OF DIGRAPHS INTO PATHS OR CIRCUITS

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The problem of partitioning the arcs of a digraph into elementary paths has been considered first by B. Alspach and N.J. Pullman in [2]. We consider the slightly different problem of partitioning the arcs of a digraph into elementary paths or circuits. A general conjecture is given which is solved in particular cases (with in fact slightly stronger results).

Definition and notations

Definitions and notations are classical (see [3]). A digraph will be denoted $G = (V, E)$, where $V$ is the set of vertices and $E$ the set of arcs. A demi-cocycle denoted $\omega^+(A)$ will be the subset of $E$ whose arcs go from $A \subset V$ to $V - A$. Let us define $\lambda$ as $\sup_{A \subset V} |\omega^+(A)|$.

Conjecture

Our main conjecture is

Conjecture 1. We can partition the arcs of a digraph into $\lambda$ or fewer elementary paths or circuits.

From now on we shall omit the word elementary.

The problem considered is similar to a problem first considered by Alspach and Pullman in [2], namely, to partition the arcs of a digraph into paths and one of their conjectures (see [2]) has been solved by O'Brien [4], who showed that for $|V| = n \geq 4$ the arcs of a digraph can be partitioned into $\lfloor \frac{3}{4}n^2 \rfloor$ or fewer paths. Our conjecture is of course closely related to this problem since $\lambda$ is clearly bounded above by $\lfloor \frac{3}{4}n^2 \rfloor$.

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We prove our conjecture for pseudo-symmetric digraphs (i.e., digraphs with \( d^+(v) = d^-(v) \) for each vertex \( v \)), acircuitous digraphs and bipartite digraphs.

As a consequence we obtain the bound 2\( \lambda \) for any digraph.

**Theorem 1.** It is always possible to partition the arcs of a pseudo-symmetric digraph into \( \lambda \) or fewer circuits.

**Proof.** In fact, we prove a slightly stronger result, namely, that there is a circuit which meets all the demi-cocycles of maximal size; then the result follows by an easy induction on \( \lambda \).

For this purpose let us consider a path of maximal length \( \mu = (v_1 \cdots v_k) \) and the terminal endpoint \( v_k \). Then \( \Gamma^+_G(v_k) \) is contained in \( \{v_1, v_2, \ldots, v_k\} \). Let \( v_i \) be the first vertex on \( \mu \) belonging to \( \Gamma^+_G(v_k) \). Let \( C \) be the circuit \( \{v_i, v_{i+1}, \ldots, v_k, v_i\} \). \( C \) contains all the vertices of the set \( \Gamma^+_G(v_k) \). This circuit meets every demi-cocycle of maximal size. Indeed, if \( \omega^+(A) \) is such a cocycle, then \( \omega^+(V-A) \) also has cardinality \( \lambda \) since the digraph is pseudo-symmetric. Then, it is sufficient to prove that \( C \) cannot be contained in \( A \). But, in this case, a simple counting argument shows (as \( d^+(v_k) = d^-(v_k) \) and \( \Gamma^+_G(v_k) \subseteq C \)) that we would have \( |\omega^+(V-A \cup v_k)| > |\omega^+(V-A)| = \lambda \), a contradiction. This achieves the proof of Theorem 1. \( \square \)

**Theorem 2.** It is always possible to partition the arcs of an acircuitous digraph into \( \lambda \) paths.

This is a simple corollary of a theorem by Alspach and Pullman [2] which states that the exact number of paths of a minimal partition is exactly \( \sum_{v \in V} \max(d^+(v) - d^-(v), 0) \) and it is sufficient to consider the set \( A \) of vertices such that \( d^+(v) - d^-(v) > 0 \). Then \( \sum_{v \in V} \max(d^+(v) - d^-(v), 0) \leq |\omega^+(A)| \).

**Remark 1.** One could also prove that each path of maximal length meets every demi-cocycle of size \( \lambda \) with arguments similar to those used in the demonstration of Theorem 1.

**Remark 2.** As the arcs of a digraph can always be partitioned into a pseudo-symmetric digraph and an acircuitous digraph, we easily deduce the bound 2\( \lambda \) mentioned in the Introduction.

**Theorem 3.** It is always possible to partition the arcs of a bipartite digraph into \( \lambda \) paths or circuits of length at most two and this bound is the best possible.

**Proof.** Of course, this bound is the best possible since such a path or circuit can meet a demi-cocycle of size \( \lambda \) at most once.
As in Theorem 1, we show there is a path or circuit of length at most two which meets every demi-cocycle of size $\lambda$ and the result follows.

For this purpose let us consider the graph $H$ in which a vertex represents an arc of $G$ and in which two vertices are linked if and only if the represented arcs of $G$ are consecutive in $G$. Then, $G$ being bipartite, $H$ is bipartite. Indeed, we can obtain a bicoloration of $H$ by giving to an arc of $G$ the colour of its terminal endpoint ($G$ being supposed bicoloured).

We can also remark that there is a bijective mapping between the demi-cocycles of size $\lambda$ in $G$ and the stable sets of maximal size in $H$. (If we consider a stable set of maximal size in $H$, it is sufficient to consider the demi-cocycle of $G$ associated to the set of the initial endpoints of the represented arcs in $G$.) Then there exists in $H$ (as it is bipartite and hence a perfect graph (see [3, Chapter 16])) a complete graph, namely a vertex or an edge which meets every stable set of maximal size. This vertex or this edge induces in $G$ a path or a circuit of length at most two which meets every demi-cocycle of size $\lambda$. This completes the proof of Theorem 3.

In addition to our conjecture, we add the following

**Conjecture 2.** It is always possible to partition the arcs of a pseudo-symmetric digraph with at most $\alpha n$ circuits ($n = |V|$), $\alpha$, a constant independent of $G$.

**Problem 3.** Is it true that if the digraph $G$ is $k$ chromatic it is always possible to partition the arcs into at most $\alpha n$ paths or directed cycles of length at most $k$. (This is true for $k = 2$ in view of Theorem 3)?

**Acknowledgments**

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**References**


$^1$ G. Etienne (private communication) gave us examples which show that $\alpha \geq \frac{4}{3}$. Simply consider a collection of $K^*_n$ having one vertex in common.

N.B. Our main conjecture has been solved since this paper has been submitted. See the paper by M. Maamoun, "Decompositions of digraphs into paths and cycles", J. Combin. Theory Ser. B 38 (1985) 97–101.