Desingularization of Schubert varieties and orbit closures of prehomogeneous vector spaces of commutative parabolic type

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ABSTRACT

Let $G$ be a simple algebraic group, and $P$ its parabolic subgroup whose unipotent radical $U$ is commutative. We construct desingularizations of the closures of the $P$-orbits in $G/P$, which in the same time give desingularizations of the orbit closures of the prehomogeneous vector space $(P/U, \text{adjoint}, U)$. Our construction is based on a formal argument concerning Coxeter systems and BN-pairs. As an application, we calculate the Frobenius traces of the intersection cohomology complexes of these orbit closures. Then using it, we study certain character sums associated to the above prehomogeneous vector spaces.

1. MAIN RESULTS

In [4], we have developed Intersection Cohomology Method in the theory of prehomogeneous vector spaces, taking up as an example the space of square matrices. In particular, we have shown that this method is effective to study certain character sums associated to the prehomogeneous vector space. In the present paper, we take up prehomogeneous vector spaces of commutative parabolic type, and show that the same method works in these cases as well. As we have seen in [4], the crux of the argument is the study of intersection cohomology complexes. For the example taken up in [4], a small resolution of every orbit closure exists. This fact makes it very easy to study the intersection cohomology complexes. But for the examples studied here, the small resolutions are not available any more. Instead, we study them via an explicit desingularization of every orbit closure.
1.1. Parabolic subgroups with commutative unipotent radical. Let $G$ be a simple algebraic group, $B$ its Borel subgroup, $T$ a maximal torus contained in $B$, $W$ the Weyl group, and $\{s_i\}_{i \in S}$ ($S = \{1, 2, \ldots, l\}$) the reflections corresponding to the simple roots $\{\alpha_i\}_{i \in S}$. We assume that these are defined over a fixed field $k$, and we identify these with the sets of $\bar{k}$-rational points, where $\bar{k}$ is a fixed algebraic closure of $k$. We follow [2] for the numbering of the simple roots. For $I \subset S$, let $W_I$ be the Weyl subgroup generated by $\{s_i\}_{i \in I}$, $w_I$ the longest element of $W_I$, and $P_I := BW_IB$ the parabolic subgroup corresponding to $I$. We assume that the unipotent radical $U$ of $P_I$ is commutative. Such $P_I$'s are intimately related with the Hermitian symmetric spaces (more precisely, every irreducible compact Hermitian symmetric space can be obtained as $G/P_I$ with such $P_I$ and with $k = \bar{\mathbb{C}}$), and also related with the Jordan algebras. They are classified as follows.

Table 1.

<table>
<thead>
<tr>
<th>$P_I$</th>
<th>$I'$ $\leq I' \leq I''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(A_{l'}, l')$</td>
<td>$l' \geq 3$</td>
</tr>
<tr>
<td>$(B_l, 1)$</td>
<td>$l' \geq 2$</td>
</tr>
<tr>
<td>$(C_l, l)$</td>
<td>$l' \geq 4$</td>
</tr>
<tr>
<td>$(D_l, 1)$</td>
<td></td>
</tr>
<tr>
<td>$(D_l, l)$</td>
<td>$l' \geq 4$</td>
</tr>
<tr>
<td>$(E_6, 1)$</td>
<td></td>
</tr>
<tr>
<td>$(E_7, 7)$</td>
<td></td>
</tr>
</tbody>
</table>

(Here $P_I$ for the case $(X, i_0)$ is the maximal parabolic subgroup corresponding to the white node labeled by $i_0$. Since $(A_{l'+l''-1}, l') \simeq (A_{l'+l''-1}, l'')$, $(D_l, l) \simeq (D_l, l - 1)$ and $(E_6, 1) \simeq (E_6, 6)$, we omit the types $(A_{l'+l''-1}, l'')$, $(D_l, l - 1)$ and $(E_6, 6)$ from our consideration.)

1.2. $P_I$-Orbits in $G/P_I$. Let $\mathcal{J}(I)$ ($\subset 2^S$) be the family consisting of $\phi$ and subsets $J \subset S$ satisfying the following two conditions.

1. $(W_J, J)$ is an irreducible Coxeter system, and
2. $w_J(\alpha_{i_0}) = -\alpha_{i_0}$, where $i_0$ is the unique element of $S \setminus I$.

Then $\{w_J \mid J \in \mathcal{J}(I)\}$ is a complete system of $P_I$-orbit representatives in $G/P_I$ (Cf. [9, Theorem 1.2. (b)]. Note that the orbit representatives $\{w_I\}$, given there are different from ours in general.) We list $J \in \mathcal{J}(I)$ in Table 2.

In the second column, $\{a, \cdots, b\}$ means $\{x \in \mathbb{Z} \mid a \leq x \leq b\}$. According to
Table 2.

<table>
<thead>
<tr>
<th>$I$</th>
<th>$J$</th>
<th>$\dim P_I w_J P_I / P_I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(A_{l'+l''-1, l''})$</td>
<td>$J(0) := \phi$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$J(i) := {l' - i + 1, \ldots, l' + i - 1}$</td>
<td>$i(l' + l'' - i)$</td>
</tr>
<tr>
<td>$(B_l, 1)$</td>
<td>$J(0) := \phi$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$J(1) := {1}$</td>
<td>$2l - 2$</td>
</tr>
<tr>
<td></td>
<td>$J(2) := S$</td>
<td>$2l - 1$</td>
</tr>
<tr>
<td>$(C_l, 1)$</td>
<td>$J(0) := \phi$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$J(i) := {l - i + 1, \ldots, l}$</td>
<td>$il - \binom{l}{2}$</td>
</tr>
<tr>
<td>$(D_l, 1)$</td>
<td>$J(0) := \phi$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$J(1) := {1}$</td>
<td>$2l - 3$</td>
</tr>
<tr>
<td></td>
<td>$J(2) := S$</td>
<td>$2l - 2$</td>
</tr>
<tr>
<td>$(D_l, l)$</td>
<td>$J(0) := \phi$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$J(i) := {l - 2i + 1, \ldots, l}$</td>
<td>$2i(l - 2i) + \binom{l}{2}$</td>
</tr>
<tr>
<td>$(E_6, 1)$</td>
<td>$J(0) := \phi$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$J(1) := {1}$</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>$J(2) := S \setminus {6}$</td>
<td>16</td>
</tr>
<tr>
<td>$(E_7, 7)$</td>
<td>$J(0) := \phi$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$J(1) := {7}$</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>$J(2) := S \setminus {1}$</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>$J(3) := S$</td>
<td>27</td>
</tr>
</tbody>
</table>

Using these subsets $I, J, L \subset S$, we can now state our main result.
1.3. Theorem.

(1) \( Bw_J w_J \cap w_I B = P_L P_I \).

(2) \( Bw_I w_L \cap w_I w_J \cap w_I \bar{B} = P_I w_J \bar{P}_I \).

(3) The morphism
\[
\pi_J : P_I \times_{P_L P_I} (P_L P_I / P_I) \to P_I w_J \bar{P}_I / P_I,
\]
\((x, yP_I) \mapsto xyP_I,
\]
gives a desingularization of the right member.

(4) The \( P_I \)-orbits of both members of (3) are in one-to-one correspondence via \( \pi_J \).

More precisely, the complete set of the \( P_I \)-representatives for the left (resp. right) member is given by \( \{ (1, w_J, P_I) \}_{J_1} \) (resp. \( \{ w_J, P_I \}_{J_1} \)), where \( J_1 \) runs over the set \( \{ J_1 \in J(I) \mid J_1 \subset J \} \).

(In general, if a group \( H \) acts on \( X \) and \( Y \) from right and left, respectively, then \( X \times_H Y \) denotes the quotient of \( X \times Y \) with respect to the \( H \)-action defined by \( h \cdot (x, y) = (xh^{-1}, hy) \) for \( h \in H, \ x \in X \) and \( y \in Y \).) The above desingularization is motivated by \([10, p.453, II.4-91]\) where the same desingularization is constructed for two cases \( (C_i, 1) \) and \( (D_i, 1) \).

1.4. Prehomogeneous vector spaces. Let \( P'_I \) be the parabolic subgroup which contains \( T \) and such that \( G_I := P_I \cap P'_I \) is a common Levi subgroup of \( P_I \) and \( P'_I \). Let \( U' \) be the unipotent radical of \( P'_I \). Then

(1) the natural morphism \( U' \to G/P_I \) is an open imbedding,

(2) it induces a bijection \( \{ \text{ad}(G_I)\text{-orbits in } U' \} \to \{ P_I\text{-orbits in } G/P_I \} \), and

(3) \( (\text{ad}(G_I), U') \) is an irreducible reduced prehomogeneous vector space.

Cf. \([11]\).

(As for (2) and (3), we should exclude the cases \( (B_i, 1) \) and \( (C_i, 1) \) when the characteristic of the base field is 2.) Hence (1.3) gives desingularizations of the orbit closures of the prehomogeneous vector space \( (\text{ad}(G_I), U') \) as well. For the sake of convenience, we describe explicitly the inverse images of \( P_I w_J P_I / P_I \) in \( U' \), i.e., the orbit closures of \( (\text{ad}(G_I), U') \).

\[
(\text{ad}(G_I), U') = \{ x \in M_l(\tilde{k}) \mid \text{rank}(x) \leq i \}
\]
\[
(\text{ad}(G_I), U') = \{ x \in M_l(\tilde{k}) \mid x = x, \text{rank}(x) \leq i \}
\]
\[
(\text{ad}(G_I), U') = \{ x \in M_l(\tilde{k}) \mid x = x, \text{rank}(x) \leq 2i \}
\]
In the case \( (E_6, 1) \) (resp. \( (E_7, 7) \)), we may identify \( (\text{ad}(L), U') \) with the half-spin representation of \( \text{Spin}_{10}(\tilde{k}) \times \tilde{k}^\times \) (resp. the natural representation of \( E_6 \times \tilde{k}^\times \) on the Jordan algebra of dimension 27), up to the effective kernel.

Using \( \pi_J \in P_I w_J P_I \subset P_I w_J \bar{P}_I \), we can see the explicit form of the orbits from \([6, \text{Prop. 2 in p. 1011}]\) (resp. \([7, \text{61}]\)).

1.5. Intersection cohomology complexes. In §3, using (1.3) and assuming \( k \) is a finite field \( F_q \), we give formulae for the Frobenius traces
\[
(-1)^{\dim O_J} \lambda_J(x) := \sum_i (-1)^i \text{tr}(\text{Frob}_q^i, H^i(\mathcal{L}_J[\dim O_J])(x)), \quad x \in (G/P_I)(F_q)
\]
of the intersection cohomology complexes $L_J[\dim O_J]$ of the closures of the $P_J$-orbits $O_J$ in $G/P_I$. Here $\text{Frob}_q$ denotes the geometric Frobenius over $\mathbb{F}_q$. See (3.5) for the explicit form of $\lambda_J$.

1.6. Fourier transform. Applying the methods of [4], we determine in §4 the Fourier transform of $\lambda_J|_{U'}$ except for the two cases $(B_t, 1)$ and $(C_t, 1)$, where $U'$ is the prehomogeneous vector space associated to $P_I$, cf. (1.4). I want to discuss these excluded cases elsewhere.

2. PROOF OF (1.3)

2.1. Review. For $I, J \subset S$, let $(W_I \setminus W/W_J)_I$ (resp. $(W_I \setminus W/W_J)_J$) be the set of the shortest (resp. the longest) representatives of $W_I \setminus W/W_J$. Put $(W_I \setminus W)^J := (W_I \setminus W/W_J)^J$ etc. Let $l(w)$ denote the length of $w \in W$. Then

1. $w_I^J = e$,
2. for any $i \in I$, $w_I(\alpha_i) = -\alpha_i'$ with some $i' \in I$,
3. for $w \in W_I$, $l(w w_I) = l(w_I) - l(w)$,
4. for $w' \in (W/W_I)_I$ and $w \in W_I$, $l(w' w) = l(w') + l(w)$,
5. for $w' \in (W/W_I)_J$ and $w \in W_I$, $l(w' w) = l(w') - l(w)$,
6. $w \in (W/W_I)_J$ if and only if $w(\alpha_i) > 0$ for all $i \in I$,
7. $w \in (W/W_I)_I$ if and only if $w w_I \in (W/W_I)_I$,
8. $\dim BwB = \dim B + l(w)$,
9. $(W/W_I)_I \cap (W/W_I)_J = (W/W_I)_I$.

See [2]. We use these facts constantly, sometimes tacitly. For a root $\alpha = \sum_{i \in S}^S c_i \alpha_i$, its support is defined to be $\text{supp}(\alpha) := \{i \in S \mid c_i \neq 0\}$.

2.2. Lemma. (1) For each $i \in I \cap J$, $w_J(\alpha_i) = -\alpha_i'$ with some $i' \in I \cap J$.
(2) $w_J w_I \cap J = w_I \cap J w_J$.
(3) For each $i \in I \cap L$, $w_I(\alpha_i) = \varepsilon \alpha_i$, and $w_J \cap L(\alpha_i) = \varepsilon \alpha_i$ with some $\varepsilon = \pm 1$ and $i', i'' \in I \cap L$. (In both places, $\varepsilon$ is the same.)
(4) $w_J w_I \cap L = w_I \cap L w_J$.

Proof. We can show the implications (1.2, (2)) $\Rightarrow$ (1) $\Rightarrow$ (2) and (1.2, (4)) $\Rightarrow$ (3) $\Rightarrow$ (4). $\square$

2.3. Lemma. (1) $J \subset L$.
Now assume that $(\partial J \setminus \{i_0\} =) I \setminus L \neq \emptyset$.
(2) $(\{i_0\} =) S \setminus I \subset J \setminus L$ and $I \setminus L = \partial J$.
(3) For $i_1 \in I \setminus L$, let $\alpha$ be the highest root in the root system generated by $\{\alpha_k \mid k = t_1 \text{ or } k \in J\}$, and $n_{i_1}$ the coefficient of $\alpha_{i_1}$ in $\alpha$. Then $n_{i_1} = 1$ or 2, and

$$w_J(\alpha_{i_1}) = \begin{cases} \hat{\alpha} & \text{if } n_{i_1} = 1, \\ \hat{\alpha} - \alpha_{i_1} & \text{if } n_{i_1} = 2. \end{cases}$$

In particular, $\text{supp}(w_J(\alpha_{i_1})) \supset S \setminus I$ (i.e., $\exists i_0$).
Proof. (1) follows from (1.2, (4)). (2) follows from (1.2, (2)). A case study shows the part of (3) concerning the value of $n_i$. By (1), the coefficient of $\alpha_i$ in $w_f(\alpha_i)$ is 1. For each $j \in J$, $w_f(\alpha_j) = -\alpha_j$ with some $j' \in J$, and hence $w_f(\alpha_i) + \alpha_j = w_f(\alpha_i - \alpha_j)$ is not a root. These two facts yield the remaining part of (3). □

2.4. Lemma. $w_f w_J \cap w_J \in (W_L \setminus W/W_I)_I$.

Proof. It suffices to show that $w_f w_J \cap w_J \in (W/W_I)_I$. We use (2.1, (6)). For $i \in I \cap J$, $w_f w_J \cap w_J(\alpha_i)$ is a simple root by (2.1, (2)). For $i \in I \setminus J$, $w_f w_J \cap w_J(\alpha_i)$ is a positive root whose support contains $i \notin J$. Hence $w_f w_J \cap w_J(\alpha_i) > 0$. □

2.5. Lemma. $w_I w_J \cap w_I \cap w_J \in (W_I \setminus W/W_I)_I$.

Proof. It suffices to prove the following two assertions.

(1) For every $i \in I$, $w_I w_J \cap w_I w_J(\alpha_i) > 0$.
(2) For every $i \in I$, $w_I w_J \cap w_I w_J(\alpha_i) > 0$.

2.5.1. Proof of (1).
2.5.1.1. Assume that $i \in I \cap J$. Then $w_I w_J \cap w_J(\alpha_i) = \alpha_j$ with some $j \in J \subset L$. If $j \in I \cap L$, then $w_I w_J \cap w_J(\alpha_j) = \alpha_j + \sum_{j \in J} d_j \alpha_j$ with some $d_j \in Z$ and $\alpha > 0$. If $w_J(\alpha_j) < 0$, then supp($w_J(\alpha_j)$) $\subset L \cap I$ and hence $w_I w_J \cap w_J(\alpha_j) > 0$. Therefore it suffices to show that

(3) $w_I(\alpha) > 0$, and
(4) $w_I w_J \cap w_J(\alpha) < 0$

do not hold in the same time. Assume the contrary. By (4), supp($\alpha$) $\subset I$, i.e., $\alpha = \alpha_i + \sum_{j \in J \cap L} d_j \alpha_j$. By (3), supp($\alpha$) $\not\subset L$. By (2.3, (1)), this means that $i \notin L$, i.e., $i \in I \setminus L$. By (2.3, (3)), supp($w_J(\alpha_i)$) $\subset S \setminus I$. Hence

$w_I w_J \cap w_J(\alpha) = w_I w_J \cap w_J(\alpha_i)$ by (2.2, (2))

$> 0$ since $w_J(\alpha_i) > 0$ and supp($w_J(\alpha_i)$) $\not\subset I$.

This contradicts (4).

2.5.2. Proof of (2).
2.5.2.1. Assume that $i \in L \cap I$. By (2.1, (2)), $w_I w_J \cap w_J(\alpha_i) = \pm \alpha_j$ with some $j \in L \cap J$. Hence by (2.2, (3)), $w_I w_J \cap w_J(\alpha_i) = \varepsilon \alpha_j$ with some $\varepsilon = \pm 1$ and $j \in I \cap L$. If $j \in I \cap J$, then $\varepsilon = +1$ by (2.3, (1)), and hence $w_I w_J \cap w_J(\alpha_j)$ is a simple root by (2.1, (2)). If $j \in I \cap L \setminus J$, then $\varepsilon = -1$ and $w_I w_J \cap w_J(\alpha_j) > 0$.

2.5.2.2. Assume that $i \in I \setminus L$. Then $w_I w_J \cap w_J(\alpha_i) = \alpha_i + \sum_{k \in L \setminus J} e_k \alpha_k$ with some $e_k \in Z$, whose support contains $i$ and hence is not contained in $J$ by (2.3, (1)). Hence $w_I w_J \cap w_J \cap w_J(\alpha_i) > 0$. If supp($w_J \cap w_J \cap w_J(\alpha_i)$) $\subset I$, then by (2.2, (2)) and (2.2, (4)), supp($w_J(\alpha_i)$) $\subset I$, which contradicts (2.3, (3)). Hence supp($w_J \cap w_J \cap w_J(\alpha_i)$) $\not\subset I$ by (2.2, (2)) and (2.2, (4)), and we get (2). □
2.6. Proof of (1.3).

2.6.1. Proof of (1.3, (1)). By (2.4) and (2.1, (9)), the left member of (1.3, (1)) is $P_L w_I w_J L_1 w_I P_I$, which is equal to $P_L P_I$ by (2.3, (1)). Since $P_L P_I / P_I \simeq P_L / P_L \cap I$ is a projective variety, $P_L P_I (\subset G)$ is closed, and we get the result.

2.6.2. (1.3, (2)) follows from (2.5) and (2.1, (9)).

2.6.3. Proof of (1.3, (3)). It is easy to see that both members are projective varieties and the left one is non-singular. In $P_I \times_{P_I \cap I} P_L P_I$, we have

$$B(w_I w_J L_1, w_J w_J L_1 w_I P_I) = (B w_I w_J L_1, w_J P_I) = (B w_I w_J L_1, P_L \cap I w_J P_I)$$

(The second equality follows from the following three facts.

(1) If supp($\alpha$) $\subset L \cap I$, then supp$(w_J \alpha) \subset I$ by (2.2, (3)).

(2) If $\alpha > 0$ and $i_0 \notin$ supp$(\alpha) \subset L \cap I (= L \setminus \{i_0\})$, then supp$(\alpha) \cap \partial J \neq \phi$ by (1.2, (4)), and hence we $w_J (\alpha) > 0$.

(3) If $\alpha > 0$ and $i_0 \in$ supp$(\alpha)$, then $w_I w_J L_1 (\alpha_{i_0}) > 0$.

In fact, from these facts, we can see that all the root subspaces of Lie($P_L \cap I$) are absorbed by $B$ or $P_I$.)

Hence the $B$-orbit of

$$(w_I w_J L_1, w_J w_J L_1 w_I P_I) \in P_I \times_{P_I \cap I} (P_L P_I / P_I)$$

is open by (1.3, (1)) and (2.1, (9)), and its image in $P_I P_J P_I / P_I$ is also open by (1.3, (2)). Hence it suffices to show the isotropy subgroups of $B$ at both points are the same. As is easily seen, both isotropy subgroups contain the maximal torus $T$, and hence are the semidirect products of $T$ and some unipotent subgroups. In particular, both are connected. Hence it suffices to prove the coincidence of the dimension, i.e., it suffices to show that

$$\dim P_I \times_{P_I \cap I} (P_L P_I / P_I) = \dim P_I w_J P_I / P_I.$$  

The left member of (4) is equal to

$$l(w_I) - l(w_J L_1) + l(w_J w_J L_1 w_I) - l(w_I),$$

by (1.3, (1)). Cf. (2.1, (8) and (9)) and (2.4). The right member is equal to

$$l(w_I w_J L_1 w_J L_1 w_I) - l(w_I),$$

by (1.3, (2)). Cf. (2.1, (8) and (9)) and (2.5). By (2.1, (4)), (2.1, (7)) and (2.5), (6) is equal to

$$l(w_I) + l(w_I w_J w_J L_1 w_I) - l(w_I).$$

By the definition of $L$, by (2.4), and by (2.1, (5)), (7) is equal to (5). Thus we get (1.3, (3)).

2.6.4. Proof of (1.3, (4)). The $P_I$-orbits of the left member of (1.3, (3)) are naturally in one-to-one correspondence with the $P_L \cap I$-orbits of $P_L P_I / P_I =
$P_L/P_L \cap I$. Since $P_L/P_L \cap I$ is also “an irreducible compact Hermitian symmetric space”, we can see that the orbit representatives are given by $\{w_J \mid J_1 \subset J, J_1 \in \mathcal{J}(I)\}$. (Note that $P_L/P_L \cap I = P_J/P_J \cap I$. Cf. the picture in (1.2).) Since the same set gives the $P_I$-orbit representatives of the right member of (1.3, (3)), we get (1.3, (4)). □

3. INTERSECTION COHOMOLOGY COMPLEXES

In this section, we take a finite field $\mathbb{F}_q$ as the base field, and determine the Frobenius trace of the intersection cohomology complex of every $P_I$-orbit closure in $G/P_I$. We keep the notation of §1 and §2.

3.1. Lemma.

(1) $w_J w_{J \cap I} = w_{J \cap I} w_J \in (W_I \setminus W/W_I)_T$.
(2) $(w_{J \cap I} w_J)(w_{J \cap I} w_J)^{-1} \cap I = I \cap L$.
(3) $(w_{J \cap I} w_J)W_I(w_{J \cap I} w_J)^{-1} \cap W_I = W_I \cap L$.

(4) Every element $w \in W_I w_J W_J$ can be uniquely expressed as $w = w'(w_{J \cap I} w_J)w''$ with $w' \in (W_I/W_I \cap L)_T$ and $w'' \in W_I$.

(5) If $w$ is expressed as in (4), then $l(w) = l(w') + l(w_{J \cap I} w_J) + l(w'')$.

Proof. (1) By (2.4) and (2.1, (7)), $w_J w_{J \cap I} \in (W_I/W_I)_T$. By (2.1, (1)) and (2.2, (2)), $w_J w_{J \cap I}$ is an involution. Hence we get the result.

(2) For $i \in I \setminus L$, $\text{supp}(w_{J \cap I} w_J(\alpha_i)) \supset S \setminus I$ by (2.3, (3)), and hence $(w_{J \cap I} w_J)s_i(w_{J \cap I} w_J)^{-1} \notin I$. Therefore $(w_{J \cap I} w_J)(w_{J \cap I} w_J)^{-1} \cap I = I \cap L$ by (2.2, (3)).

(3) follows from (2) and [12, lemma 2].

(4) Consider the isotropy subgroup of $W_I$ at $w_J w_{J \cap I} W_I \in W_I w_J W_I/W_I$, and use (3).

(5) If $w'' = e$, then (5) follows from (1). Hence it suffices to show that $w' w_J w_{J \cap I}(\alpha_i) > 0$ for all $i \in I$ and all $w' \in (W_I/W_I \cap L)_T$. First, if $i \in I \cap J$, then $w_J w_{J \cap I}(\alpha_i) = \alpha_i$ with some $i' \in I \cap J$ by (2.2, (1)). Since $J \cap I \subset L \cap I$ by (2.3, (1)), $w'(\alpha_i) > 0$. Second, if $i \in I \setminus L$, then $w_J(\alpha_i) > 0$ and its support contains $S \setminus I$ by (2.3, (3)). Hence $w' w_J w_J(\alpha_i) > 0$. Last, assume that $i \in I \cap L \setminus J$. By (1.2, (4)), no $(j, k) \in J \times (L \setminus J)$ is an edge of the Dynkin diagram of $(W, S)$. Hence $w' w_J w_J(\alpha_i) = w'(\alpha_i) > 0$. □

3.2. Poincaré polynomials. For a subset $X$ of $W$, put $f(t, X) := \sum_{x \in X} t^{l(x)}$. In the remainder of this section, we assume that $G$, $B$, $T$ and $P_I$ in §1 are defined over a finite field $\mathbb{F}_q$. Let $F$ be the automorphism of $W$ induced by the Frobenius endomorphism of $G$. Then $F$ is given by the symmetry of the diagram in Table 1; $F$ is trivial except for the two cases $(A_{2l-1}, l)$ and $(B_l, 1)$, and in these two cases $F$ is of order $\leq 2$. If the order is 2, we refer to these cases as $(2A_{2l-1}, l)$ and $(^2D_l, 1)$. For an algebraic variety $X$ over $\mathbb{F}_q$, let $|X|$ denote the cardinality of $X(\mathbb{F}_q)$. Thus $|BwB|/|B| = q^{l(w)}$ for $w \in W^F$, where the superscript $F$ means the set of $F$-fixed points. Hence

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We call \( f(t, W_t^F) \) the Poincaré polynomial of \( W_t^F \). Since \( f_S = |G|/|B| \), we can find the explicit form of \( f_S \) from that of \( |G| \), which is well-known.

### 3.3. Direct image

\( M_J := \mathcal{R}(\pi_J)_* \overline{O} \). (See (1.3, (3)) for \( \pi_J \).) Put \( O_{J_1} := P_I w_{J_1} P_I / P_I \) for \( J_1 \in \mathcal{J}(I) \), and \( e_{J_1} := w_{J_1} P_I \in O_{J_1} \). In this paragraph, we determine the values of the Frobenius trace

\[
(1) \quad \mu_J(x) = \mu_J(q, x) := \text{tr}(\text{Frob}_q, (M_J)_x), \quad \text{for } x \in \overline{O_J}(\mathbb{F}_q),
\]

where \( \text{Frob}_q \) denotes the geometric Frobenius over \( \mathbb{F}_q \), and \( \text{tr}(-, -) = \sum_i (-1)^i \text{tr}(-, H^i(-)) \). Since \( \mu_J \) is constant on each \( P_I(\mathbb{F}_q) \)-orbit, it suffices to determine the values of \( \mu_J(e_{J_1}) \) for all \( J_1 \in \mathcal{J}(I) \). (In fact it is easy to see that every cohomology sheaf of \( M_J \) is locally constant on each \( P_I \)-orbit \( O_{J_1} \), and constant on the dense \( B \)-orbit in \( O_{J_1} \). Hence it is constant on \( O_{J_1} \).) Put \( L_1 := S \setminus \partial J_1 \). (Cf. (1.2).) Then (3.1) yields

\[
(2) \quad |P_I w_{J_1} P_I| / |B| = f(q, (W_I / W_{I \cap L_1})^F) q^{l(w_{J_1}, w_{J_1 \cap I})} f(q, W_t^F).
\]

From (3.2, (1)) and (2), we get

\[
(3) \quad |P_I w_{J_1} P_I / P_I| = f_I f_{L_1 / L_1} q^{l(w_{J_1}, w_{J_1 \cap I})}.
\]

Here, replace \( S \) (resp. \( I \)) with \( L \) (resp. \( L \cap I \)), and put \( L_2 := L \setminus \partial J_1 = S \setminus (\partial J \cup \partial J_1) \). Then we get, in place of (3),

\[
(4) \quad |P_{L \cap I} w_{J_1} P_{L \cap I} / P_{L \cap I}| = f_{L \cap I} f_{L_1 / L_1} q^{l(w_{J_1}, w_{J_1 \cap L_1})}.
\]

Since the natural isomorphism \( P_L / P_{L \cap I} \rightarrow P_L P_I / P_I \) induces an isomorphism \( P_{L \cap I} w_{J_1} P_{L \cap I} / P_{L \cap I} \rightarrow P_{L \cap I} w_{J_1} P_I / P_I \), (4) yields

\[
(5) \quad |\pi^{-1}_J(P_I w_{J_1} P_I / P_I)| = |P_I \times_{P_I P_I} P_{L \cap I} w_{J_1} P_I / P_I| = f_{1/2} f_{L_1 / L_1} q^{l(w_{J_1}, w_{J_1 \cap I})}.
\]

(The first equality follows from (1.3, (4)).) Besides we have used \( L \cap I \cap L_2 = L_2 \cap I \) and \( J_1 \cap L \cap I = J_1 \cap I \). By (3) and (5), we get

\[
(6) \quad \mu_J(q, e_{J_1}) = |\pi^{-1}_J(e_{J_1})| = \frac{f_{I \setminus I}}{f_{L_1 / L_1}} \frac{f(q, S \setminus (\partial J_1 \cup \{i_0\}))}{f(q, S \setminus (\partial J \cup \partial J_1 \cup \{i_0\}))}.
\]

(See (1.2) for \( \partial J \). Recall that \( I = S \setminus \{i_0\} \). In particular, we can see that \( \mu_J(q, e_{J_1}) \) is a polynomial in \( q \). For an actual calculation, it is convenient to modify this polynomial as

\[
(7) \quad \mu_J, J_1(t) := \begin{cases} \text{dim} O_{J_1} \mu_J(t^2, e_{J_1}) & \text{if } J_1 \subset J \text{ and } J_1 \in \mathcal{J}(I), \\ 0, & \text{otherwise}, \end{cases}
\]

\[
\mu_J(t) := (\mu_J, J_1(t))_{J_1} \in \mathbb{C}[t, t^{-1}]^{\mathcal{J}(I)}.
\]

### 3.4. Intersection cohomology complex

\( L_J[\text{dim } O_J] \) on \( \overline{O_J} \). Let \( \mathcal{L}_J[\text{dim } O_J] \) be the simple perverse sheaf on \( \overline{O_J} = P_I w_{J_1} P_I / P_I \) such that \( \mathcal{L}_J |_{O_J} = \overline{O_J} \). (We follow [1]
for the dimension shift.) In this paragraph, we determine the values of the Frobenius traces

\[ \lambda_j(x) = \lambda_j(q, x) := \text{tr}(\text{Frob}_q, (\mathcal{L}_j)_x) \quad \text{for } x \in \overline{O}_j(\mathbb{F}_q). \]

As in (3.3), it suffices to calculate the values at \( e_{J_1} \) for \( \{ J_1 \in \mathcal{J}(I) \mid J_1 \subset J \} \).

First let us explain how to calculate \( \lambda_j(e_{J_1}) \). We can see that \( \lambda_j(e_{J_1}) = 1 \) if \( J_1 = \phi \), and \( = 0 \) otherwise. So, assume that \( \lambda_j(K \not
sim J) \) are already calculated. If the vanishing

\[ H^i(\mathcal{M}_j[\dim O_j]|_{o_{J_1}}) = 0 \quad \text{for } i \geq -\dim O_j \]

of cohomology sheaves occurs for all \( J_1 \not
sim J \), then \( \mathcal{M}_j[\dim O_j] = \mathcal{L}_j[\dim O_j] \).

(2) For some \( J_1 \not
sim J \), take the largest one among such \( J_1 \). If \( H^{-\dim O_j + i}(\mathcal{M}_j[\dim O_j]|_{o_{J_1}}) \neq 0 \) with some \( i \geq 0 \), then \( \mathcal{M}_j[\dim O_j] \) contains \( \mathcal{L}_j[\dim O_j + i] \) and \( \mathcal{L}_j[\dim O_j - i] \) as direct summands with the same multiplicity. (Note that \( \mathcal{M}_j[\dim O_j] \) is self-dual with respect to the Verdier duality. Note also that \( \mathcal{M}_j[\dim O_j] \) is a direct sum of simple perverse sheaves with shifts. Cf. [1].) Subtract all such summands, and replace \( \mathcal{M}_j[\dim O_j] \) by the remainder. Since the remainder is also self-dual, we can repeat the same argument with a smaller \( J_1 \) as far as (2) is not satisfied for some \( J_1 \). Thus we get the desired description of \( \mathcal{M}_j[\dim O_j] \) as a direct sum of \( \mathcal{L}_j \)'s with shifts, after a finite number of repetitions.

Reformulating the above procedure, we obtain the following characterization (3) – (5) of \( \lambda_j \), by which we can calculate the values of \( \lambda_j \). (The result of the calculation is listed in the next paragraph.)

(3) \( \lambda_j(q, e_{J_1}) \) is a polynomial in \( q \). Put

\[ \lambda_{J_1}(t) := \begin{cases} 
 t^{-\dim O_j} \lambda_j(t^2, e_{J_1}), & \text{if } J_1 \subset J \text{ and } J_1 \in \mathcal{J}(I), \\
 0, & \text{otherwise.} 
\end{cases} \]

(4) \( \mu_j(t) = \lambda_j(t) + \sum_{k \not
sim J} c_{J,K}(t) \lambda_j(t) \) with some \( c_{J,K}(t) \in \mathbb{C}[t, t^{-1}] \) such that \( c_{J,K}(t) = c_{J,K}(t^{-1}) \).

(5) \( \lambda_{J_1}(t) = \sum_{i< -\dim O_{J_1}} \lambda_{J_1,i} t^i \) with some \( \lambda_{J_1,i} \in \mathbb{C} \).

3.5. Tables of \( \lambda_{J(i)}(e_{J(i')}) \). Here we use the following notation:

\[ [a]_q = \frac{(q^a - 1)/(q - 1)}{[1]_q [2]_q \cdots [a]_q}, \quad \left[ \begin{array}{c} a \\ \hline b \end{array} \right]_q = \frac{[a]_q!}{[b]_q! [a - b]_q!}. \]

See Table 2 for \( J(i) \).

3.5.1. \( (\mathcal{A}_{l' + i' - 1}, l') \) with \( 1 \leq l' \leq l'' \).

\[ \lambda_{J(i)}(e_{J(i')}) = \left[ \begin{array}{c} l' - i' \\ \hline l' - i \end{array} \right]_q \quad (0 \leq i' \leq i \leq l') \]
3.5.2. \((A_{2l-1}, l)\).

\[ \lambda_{J(i)}(e_{J(i')}) = \left[ \begin{array}{c} l - i' \\ l - i \end{array} \right] \quad (0 \leq i' \leq i \leq l) \]

3.5.3. \((B_l, 1)\) with \(l \geq 3\).

\[ \lambda_{J(i)}(e_{J(i')}) - 1 \quad (0 \leq i' \leq i \leq 2) \]

3.5.4. \((C_l, l)\) with \(l \geq 2\).

\[ \lambda_{J(i)}(e_{J(i')}) = \begin{cases} \left[ \begin{array}{c} \frac{l - i'}{2} \\ \frac{l - i}{2} \end{array} \right] q^2 & \text{if } i \equiv i'(2), \\ \left[ \begin{array}{c} \frac{l - i'}{2} \\ \frac{l - i}{2} \end{array} \right] q^2 & \text{if } i \not\equiv i'(2). \end{cases} \]

(Here we always assume that \(0 \leq i' \leq i \leq l\), and \([x]\) denotes the largest integer not exceeding \(x\).)

3.5.5. \((D_l, 1)\) and \((2D_l, 1)\) with \(l \geq 4\).

\[ \lambda_{J(i)}(e_{J(i')}) = \begin{cases} 1 & \text{if } i \geq i' \text{ and } (i, i') \neq (1, 0), \\ 1 + \varepsilon q^{l-2} & \text{if } (i, i') = (1, 0), \end{cases} \]

where \(\varepsilon = +1\) (resp. \(\varepsilon = -1\)) in the \((D_l, 1)\)-case (resp. \((2D_l, 1)\)-case).

3.5.6. \((D_l, l)\) with \(l \geq 4\).

\[ \lambda_{J(i)}(e_{J(i')}) = \left[ \begin{array}{c} \frac{l - i'}{2} \\ \frac{l - i}{2} \end{array} \right] q^2, \quad (0 \leq 2i' \leq 2i \leq l). \]

3.5.7. \((E_6, 1)\).

\[ \lambda_{J(i)}(e_{J(i')}) = \begin{cases} 1 & \text{if } i \geq i' \text{ and } (i, i') \neq (1, 0), \\ 1 + q^3 & \text{if } (i, i') = (1, 0). \end{cases} \]

3.5.8. \((E_7, 7)\).

\[ \lambda_{J(i)}(e_{J(i')}) = \begin{cases} 1 + q^4 + q^8 & \text{if } (i, i') = (1, 0) \text{ or } (i, i') = (2, 0), \\ 1 + q^4 & \text{if } (i, i') = (2, 1), \\ 1 & \text{if } i \geq i' \text{ and } (i, i') \neq (1, 0), (2, 0), (2, 1). \end{cases} \]

3.6. Indication of the actual calculation. Except for the cases \((A_{l'+l'-1}, l')\), \((2A_{2l-1}, l)\), \((C_l, l)\) and \((D_l, l)\), we can work out the procedure explained in (3.4) without difficulty. In the \((A_{l'+l'-1}, l')\)-case, we can construct a small resolution of each \(\overline{O}_{J(i)}\) in the same way as in \([4, (3.1)]\), and we get the result. In order to study the cases \((2A_{2l-1}, l)\), \((C_l, l)\) and \((D_l, l)\), we need to know \(c_{J(i), J(i')}\) (cf. (3.4, (4))). Put \(t = \sqrt{q}\), and
\begin{align*}
\{2\}_{t}^{[a,b]} & := \prod_{k \in [a,b]} (t^k + t^{-k}), \quad \{2\}_{t}^{[a,b], \text{odd}} := \prod_{k \in [a,b], k = \text{odd}} (t^k + t^{-k}), \\
\{a\}_{t} := (t^a - t^{-a})/(t - t^{-1}), \quad \{a\}_{t}! := \{1\}_{t}\{2\}_{t} \cdots \{a\}_{t}, \quad \{a\}_{t}^{[b]} := \{b\}_{t}!\{a - b\}_{t}!
\end{align*}

In the \((2A_{2l-1}, l)\)-case, \(c_{I,(i), J(I')} = a_{l-i, l-i'}\) with
\[a_{l,j} = \{2\}_{t}^{[2l-j+1, j], \text{odd}} \left\{ \begin{array}{l}
\frac{1}{2} \{2\}_{t}^{[l-j, l-j+1]} \{j - i\}_{t} \\
\{2\}_{t}^{[j-l, j-l+1]} \{j - i\}_{t}
\end{array} \right\} \]
In the \((C_{l}, l)\)-case, \(c_{I,(i), J(I')} = a_{l-i, l-i'}\) with
\[a_{l,j} = \left\{ \begin{array}{ll}
\{2\}_{t}^{[2l-j+1, j], \text{odd}} & \text{if } j \text{ is odd,} \\
\frac{1}{2} \{2\}_{t}^{[l-j, l-j+1]} & \text{if } j \text{ is even.}
\end{array} \right\}
\]
In the \((D_{l}, l)\)-case, \(c_{I,(i), J(I')} = a_{l-2i, l-2i'}\) with
\[a_{2l, 2j} = \{2\}_{t}^{[2l-j+1, j]} \left\{ \begin{array}{l}
\frac{1}{2} \{2\}_{t}^{[j-l, j-l+1]} \\
\{2\}_{t}^{[j-l, j-l+1]}
\end{array} \right\} + \{2\}_{t}^{[j-l, j-l+1]} \left\{ \begin{array}{l}
\frac{1}{2} \{2\}_{t}^{[j-l, j-l+1]} \\
\{2\}_{t}^{[j-l, j-l+1]}
\end{array} \right\}.
\]

3.7. Remark. It is also possible to deduce (3.5.1) from [8, (10.1)], and, (3.5.4) and (3.5.6) from [3, (15.4)].

4. Fourier transform of \(\lambda_{J} \mid U'^{(F_{q})}\):

In this section, we keep the notation and the assumption of §3, and we calculate the Fourier transform of \(\lambda_{J} \mid U'^{(F_{q})}\) using the method given in [4]. Here we exclude two cases \((B_{l}, 1)\) and \((C_{l}, l)\) from our consideration because they need a different technique. I want to discuss these excluded cases elsewhere.

4.1. Fourier transformation. Let \(U'\) be as in (1.4), which we shall naturally regard as a vector space defined over \(F_{q}\), \(U'^{\vee}\) its dual, and \(\langle, \rangle : U'^{\vee} \times U' \to A_{1}\) the natural pairing. Fix a non-trivial additive character \(\psi : F_{q} \to \mathbb{C}^{\times}\). For a function \(\varphi : U'((F_{q})) \to \mathbb{C}\), define its Fourier transform \(\mathcal{F}(\varphi) : U'^{\vee}((F_{q})) \to \mathbb{C}\) by \(\mathcal{F}(\varphi)(v') := \sum_{v \in U'^{\vee}((F_{q}))} \varphi(v) \psi(\langle v', v \rangle)\).

4.2. Functions \(\lambda_{J}^{\prime} \text{ and } \lambda_{J}^{\prime \vee}\). Since we are excluding the cases \((B_{l}, 1)\) and \((C_{l}, l)\), we can identify the image of \(\text{ad} : G_{I} \to GL(U')\) with that of \(G_{I} \to GL(U'^{\vee})\) even in the case of small characteristic. Hence suitably identifying \(U'\) with \(U'^{\vee}\), each \(G_{I}\)-orbit \(O_{J}^{\prime} := O_{J} \cap U'\) of \(U'\) can be identified with those of \(U'^{\vee}\), which we shall denote by \(O_{J}^{\prime \vee}\). Via such identification, we regard \(\lambda_{J}^{\prime} = \lambda_{J} \mid U'^{(F_{q})}\) as a function on \(U'^{\vee}((F_{q}))\), which we shall denote by \(\lambda_{J}^{\prime \vee}\). Cf. (1.4, (1)) and (1.4, (2)). Let \(J := J(I) = \{J(i)\}_{i}\) be as in (1.2).
4.3. **Lemma.** Let $X$ be a variety over $\mathbb{F}_q$ and $\mathcal{L} \in D^b_c(X, \mathcal{O}_X)$ a simple perverse sheaf whose support has a dense subvariety $Z$ such that $\mathcal{L}|_Z = \mathcal{O}_Z[\dim Z]$. Then
\[
\sum_{x \in \chi(\mathbb{F}_q)} \sum_i (-1)^i \text{tr}(\text{Frob}_q, H^i(\mathcal{L}[-\dim Z]),) \sim (q^r)^{\dim \text{supp } \mathcal{L}},
\]
i.e., the ratio of both members tends to 1 when $r \to +\infty$. (Here $\text{supp } \mathcal{L}$ is the union of the supports of the cohomology sheaves $H^i(\mathcal{L})$.)

The proof is immediate from the definition of pure perverse sheaf.

4.4. **Theorem.** Excluding the cases $(B_1, 1)$ and $(C_1, I)$, but over an arbitrary finite base field, we have

\[
\mathcal{F}(\lambda_{J(i)^{\vee}}) = c^i \lambda_{J(m-i)^{\vee}}
\]
for $0 \leq i \leq m := |J(I)| - 1$, where
\[
\begin{align*}
\lambda &= q^l'' \text{ for } (A_1, \mathbb{F}_q, I') \text{ with } l' \leq l'' ,
\lambda &= (-1)^{l-1} q^l \text{ for } (2A_{2l-1}, I) \text{ with } l \geq 1 ,
\lambda &= q^{l-1} \text{ for } (D_l, I) ,
\lambda &= -q^l \text{ for } (2D_l, I) ,
\lambda &= q^{l-1} \text{ for } (D_l, I) \text{ with even } l ,
\lambda &= q^l \text{ for } (D_l, I) \text{ with odd } l ,
\lambda &= q^{I} \text{ for } (E_6, I) ,
\lambda &= q^9 \text{ for } (E_7, 7).
\end{align*}
\]

**Proof.** By the same argument as in [4, 3.4–3.6], we get
\[
\mathcal{F}(\lambda_{J(i)^{\vee}}) = c_i \lambda_{J(m-i)^{\vee}} \quad \text{with } c_i = \frac{\langle \lambda_{J(i)^{\vee}}, 1 \rangle}{\lambda_{J(m-i)^{\vee}}(0)},
\]
where $\langle \varphi_1, \varphi_2 \rangle = \sum_{v \in U'(\mathbb{F}_q)} \varphi_1(v)\varphi_2(v)$. If
\[
\lambda_{J(m-i)^{\vee}}(0) = \alpha q^N + (\text{lower terms of } q),
\]
where $\alpha$ is a non-zero constant `independent of $q$`, then $c_i$ can be more easily calculated by the formula
\[
c_i = \frac{q^{\dim O_{J(i)}}}{\alpha q^N}
\]
because of (4.3). (In order that the word `independent of $q$` has a sense, it suffices to replace $q$ with $q^{Mq^r}$ with a sufficiently divisible positive integer $M$ and to consider the expression (2) for all $r \in \mathbb{Z}_{\geq 0}$ in the same time.) Now the calculation is immediate. (See Table 2 for the dimension of $O_{J} = P_I w_{J} P_I / P_I$. See (3.5) for the value of $\lambda_{J(m-i)^{\vee}}(0) = \lambda_{J(m-i)}(e_{J(i)}).$) \(\square\)

4.5. **Corollary.** Let $\varphi_{J(i)}$ be the characteristic function of $O_{J(i)}(\mathbb{F}_q)$, and $\varphi_{J(i)^{\vee}}$ the function on $U'^{\vee}(\mathbb{F}_q)$ which is identified with $\varphi_{J(i)}$ as in (4.2). Put
\[
A := (\lambda_{J(i)}(e_{J(i)}^\vee))_{0 \leq i, j \leq m},
\]
where \( m = |J(I)| - 1 \) and \( e_{j(i)}^\vee \) is an element of \( O_{J(i)}(F_q) \). Then we have
\[
\begin{pmatrix}
\mathcal{F}(\varphi_{J(0)}) \\
\mathcal{F}(\varphi_{J(1)}) \\
\vdots \\
\mathcal{F}(\varphi_{J(m)})
\end{pmatrix}
= A^{-1}
\begin{pmatrix}
ce_m & \cdots & c \\
c_m & \cdots & c \\
\vdots & \ddots & \vdots \\
c_m & \cdots & c
\end{pmatrix}
A
\begin{pmatrix}
\varphi_{J(0)}^\vee \\
\varphi_{J(1)}^\vee \\
\vdots \\
\varphi_{J(m)}^\vee
\end{pmatrix}
\]
with the same \( c \) as in (4.4).

4.6. Corollary. Assume the notation of (4.5). Then the values of \( \mathcal{F}(\varphi_{J(m)})(e_{J(i)}^\vee) \) (0 \( \leq i \leq m = |J(I)| - 1 \)) are given as follows.

For \((A_{l''-1,l''-1}, l')\) with \( 1 \leq l' \leq l'' \),
\[
(-1)^{l''} q^{l''(i)} (1 - q^{l''-l'-1})(1 - q^{l''-l'+2}) \cdots (1 - q^{l''-i}).
\]

For \((^2A_{l-1,l}, l')\),
\[
(-1)^l q^{l' \choose 2} (1 + (-1)^l q)(1 + (-1)^l q^2) \cdots (1 + (-1)^l q^{l'-i}).
\]

For \((D_1, 1)\) and \((^2D_1, 1)\) with \( l \geq 4, \)
\[
\begin{cases}
\varepsilon q^{l - 2}(1 - q)(1 - \varepsilon q^{l - 1}) & (i = 0), \\
\varepsilon q^{l - 2}(1 - q) & (i = 1), \\
\varepsilon q^{l - 2} & (i = 2),
\end{cases}
\]
where \( \varepsilon = 1 \) for \((D_1, 1)\) and \( \varepsilon = -1 \) for \((^2D_1, 1)\).

For \((D_1, l)\) with \( l = 2m \geq 4, \)
\[
(-1)^m q^{2(l+1)/2}(1 - q)(1 - q^3) \cdots (1 - q^{2(m-i)-1}).
\]

For \((D_1, l)\) with \( l = 2m + 1 \geq 5, \)
\[
(-1)^m q^{2(l+1)/2}(1 - q^3)(1 - q^5) \cdots (1 - q^{2(m-i)+1}).
\]

For \((E_6, 1), \)
\[
\begin{cases}
q^3(1 - q^3)(1 - q^5) & (i = 0), \\
q^3(1 - q^5) & (i = 1), \\
q^3 & (i = 2),
\end{cases}
\]

For \((E_7, 7), \)
\[
\begin{cases}
-q^{12}(1 - q)(1 - q^5)(1 - q^9) & (i = 0), \\
-q^{12}(1 - q)(1 - q^5) & (i = 1), \\
-q^{12}(1 - q) & (i = 2), \\
-q^{12} & (i = 3).
\end{cases}
\]

The above formulas can be obtained either by a direct calculation based on (4.5) or by using the technique given in [4, §4].

4.7. Observation. There is an obvious pattern for the values of \( \mathcal{F}(\varphi_{J(m)})(e_{J(i)}^\vee) \) (0 \( \leq i \leq m \)) which enables us to deduce them from the values for \( i = 0. \)

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Therefore let us look at the values of $\mathcal{F}(\varphi_{J(m)})(e_{J(0)}^\vee)$. We restrict ourselves to the untwisted cases (i.e., the cases except for $(2A_{2l-1}, l)$ and $(2D_l, 1)$). In the case $(X, i_0) = (A_{l'+l''-1}, l')$, $(D_l, 1)$, etc., let

$$b_{i_0}(s) = \prod_{j=1}^{d} (s + \alpha_j)$$

be the $b$-function of the associated semi-invariant. (See [5] for the definition. In particular, if $(G_l, U')$ is a regular prehomogeneous vector space, this is equal to the $b$-function of the irreducible relative invariant.) Let us list $d$ and $\{\alpha_1, \cdots, \alpha_d\}$ in this order. (Cf. [5].)

- $d = l'$, $\{1, 2, \cdots, l'\}$, for $(A_{l'+l''-1}, l')$ with $l'' \geq l' \geq 1$.
- $d = 2$, $\{1, l - 1\}$, for $(D_l, 1)$ with $l \geq 4$.
- $d = m$, $\{1, 3, \cdots, 2m - 1\}$, for $(D_l, l)$ with $l = 2m \geq 4$.
- $d = m$, $\{1, 3, \cdots, 2m - 1\}$, for $(D_l, l)$ with $l = 2m + 1 \geq 5$.
- $d = 2$, $\{1, 4\}$, for $(F_4, 1)$.
- $d = 3$, $\{1, 5, 9\}$, for $(E_7, 7)$.

Comparing the above table with (4.6), we get

$$\mathcal{F}(\varphi_{J(m)})(e_{J(0)}^\vee) = (-1)^d q \sum_{j=1}^{d} (\alpha_j - 1) \prod_{j=1}^{d} (1 - q^{\alpha_j - 1 + c}),$$

where $c = \dim O_{J(0)} - \dim O_{J(1)}$, i.e., the codimension of the complement of the open $G_l$-orbit in $U'$. (Recall that $\varphi_{J(m)}$ is the characteristic function of the open orbit, and $e_{J(0)}^\vee$ is the origin of the vector space $U'^\vee$.)

**4.8. Remark.** The formulas in (4.5) and (4.6) are already obtained by Tsao [13], [14] by an elementary method. Therefore our result can be regarded as a geometric counterpart of the result of Tsao.

**REFERENCES**


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