A systematic method of smooth switching LPV controllers design for a morphing aircraft

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Abstract This paper is concerned with a systematic method of smooth switching linear parameter-varying (LPV) controllers design for a morphing aircraft with a variable wing sweep angle. The morphing aircraft is modeled as an LPV system, whose scheduling parameter is the variation rate of the wing sweep angle. By dividing the scheduling parameter set into subsets with overlaps, output feedback controllers which consider smooth switching are designed and the controllers in overlapped subsets are interpolated from two adjacent subsets. A switching law without constraint on the average dwell time is obtained which makes the conclusion less conservative. Furthermore, a systematic algorithm is developed to improve the efficiency of the controllers design process. The parameter set is divided into the fewest subsets on the premise that the closed-loop system has a desired performance. Simulation results demonstrate the effectiveness of this approach.

1. Introduction

A morphing aircraft can adaptively alter its aerodynamic configuration to obtain optimal flight performance and adapt to different flight environments and combat missions.\textsuperscript{1–4} However, due to the change of configuration, its aerodynamic parameters vary dramatically, and that will make it a complicated system with strong nonlinearity and uncertainty. Therefore, analysis and control for morphing aircraft are more challenging than those for traditional flight vehicles.\textsuperscript{5–8}

As a powerful tool to study this class of complicated systems, switched linear systems have received considerable attention in recent years.\textsuperscript{9–13} Especially, numerous significant advances have been achieved in the switched linear parameter-varying (LPV) systems theory.\textsuperscript{14–21} However, a potential shortcoming of switching LPV controllers proposed in Refs.\textsuperscript{14–16} is that they may cause transient responses when switching occurs, and that leads to the research of smooth switching LPV control.\textsuperscript{22–26} A single state feedback controller which considers smooth switching is devised for an aircraft dynamic system in Ref.\textsuperscript{22}. In Refs.\textsuperscript{23,24}, smooth switching LPV controllers are designed by means of interpolating LPV controllers in overlapped regions from two adjacent subregions. In Ref.\textsuperscript{25}, the smooth switching controllers design...
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2. Modeling and control of switched LPV systems

2.1. Switched LPV systems

The generalized state-space representation of an LPV system to be studied is described as

\[
\begin{bmatrix}
\dot{x} \\
\dot{z} \\
\dot{y} \\
\end{bmatrix} =
\begin{bmatrix}
A(\rho) & B_1(\rho) & B_2(\rho) \\
C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\
C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \\
\end{bmatrix}
\begin{bmatrix}
x \\
w \\
u \\
\end{bmatrix},
\]

where the vectors \(x \in \mathbb{R}^n\), \(z \in \mathbb{R}^m\), and \(y \in \mathbb{R}^p\) denote the plant state, the controlled output, and the measured output, respectively; \(w \in \mathbb{R}^q\) and \(u \in \mathbb{R}^r\) denote the exogenous input and the control input, respectively. The state-space matrices \(A, B_1, B_2, C_1, D_{11}, D_{12}, D_{21},\) and \(D_{22}\) in system (1) are supposed to be continuous functions of the scheduling parameter \(\rho\), which is measurable in real-time. It is assumed that \(\rho\) is in a compact set \(\Theta = \{\rho | \underline{\rho} \leq \rho \leq \bar{\rho}\}\) with its variation rate bounded by \(\gamma \leq \dot{\rho} \leq \bar{\gamma}\). The following assumptions also apply.\(^{14}\)

**Assumption 1.** \((A(\rho), B_2(\rho), C_2(\rho))\) triple is parameter-dependent stabilizable and detectable for all \(\rho\).

**Assumption 2.** The matrix functions \([B_1(\rho) \quad D_{12}(\rho)]\) and \([C_2(\rho) \quad D_{22}(\rho)]\) have full row ranks for all \(\rho\).

**Assumption 3.** \(D_{22}(\rho) = 0\).

Suppose that the entire parameter set \(\Theta\) is divided into a finite number of subsets \(\Theta_i, i \in Z_i, Z_i = \{1, 2, \ldots, I\}\), and there is no overlapped region among the subsets, as shown in Fig. 1. That is

\[
\Theta_i = \{\rho \in \Theta | \underline{\rho}_i \leq \rho \leq \bar{\rho}_i\}, \quad i \in Z_I
\]

\[
\bigcup_{i=1}^{I} \Theta_i = \Theta.
\]

Accordingly, we can obtain \(I\) subsystems from system (1).

\[
\begin{bmatrix}
\dot{x} \\
\dot{z} \\
\dot{y} \\
\end{bmatrix} =
\begin{bmatrix}
A_i(\rho) & B_1(\rho) & B_2(\rho) \\
C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\
C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \\
\end{bmatrix}
\begin{bmatrix}
x \\
w \\
u \\
\end{bmatrix}, \quad i \in Z_I
\]

We design an output feedback gain-scheduling LPV controller for each subsystem as

\[
\begin{bmatrix}
x_k \\
u \\
\end{bmatrix} = K_i(\rho)\begin{bmatrix}
x_k \\
z \\
y \\
\end{bmatrix}, \quad \begin{bmatrix}
x_k \\
z \\
y \\
\end{bmatrix} = \begin{bmatrix}
A_{i,\sigma}(\rho) & B_{i,\sigma}(\rho) \\
C_{i,\sigma}(\rho) & D_{i,\sigma}(\rho) \\
\end{bmatrix}\begin{bmatrix}
x_k \\
w \\
\end{bmatrix}, \quad i \in Z_I
\]

where \(x_k \in \mathbb{R}^n\) is the controller state vector. The closed-loop switched LPV system can be given as

\[
\begin{bmatrix}
x_k \\
z \\
y \\
\end{bmatrix} =
\begin{bmatrix}
A_{i,\sigma}(\rho) & B_{i,\sigma}(\rho) \\
C_{i,\sigma}(\rho) & D_{i,\sigma}(\rho) \\
\end{bmatrix}
\begin{bmatrix}
x_k \\
w \\
\end{bmatrix}, \quad i \in \Theta
\]

Fig. 1 Parameter set is partitioned without overlaps.
multiple parameter-dependent Lyapunov functions can be defined as
\[ V_i(t) = V_i(x_i(t), \rho) = x_i^T(t)P_i(x_i(t)) \]
where \( P_i(\rho), i \in Z_\ell \) denote a family of positive definite matrix functions, and each of them is smooth over the corresponding parameter subset \( \Theta_i \).

**Lemma 1.** Given a closed-loop LPV system (7), scalars \( \beta > 0, \mu > 1 \), the parameter set \( \Theta \) and its partition \( \Theta_i, i \in Z_\ell \), as shown in Eqs. (2)-(4), if there exist positive definite matrix functions \( P_i(\rho) \) and positive scalars \( \gamma_i, i \in Z_\ell \), so that for any \( \rho \in \Theta_i, i \in Z_\ell \),

\[
\begin{bmatrix}
\Pi_{d,i} & P_i(\rho)B_{d,i}(\rho) & C_{d,i}^T(\rho) \\
* & -\gamma_i I_{n_\rho} & D_{d,i}(\rho) \\
* & * & -\gamma_i I_{n_\rho}
\end{bmatrix} < 0
\]  

(9)

where the asterisk (*) denotes a term that is induced by symmetry in symmetric block matrices, \( I_{n_\rho} \) denotes an \( n_\rho \times n_\rho \) dimensional identity matrix, and

\[
\Pi_{d,i} = A_{d,i}^T(\rho)P_i(\rho) + P_i(\rho)A_{d,i}(\rho) + \{\Xi, \bar{v}\} \partial P_i(\rho) \over \partial \rho + \beta P_i(\rho),
\]

Besides, when \( \rho \) reaches the switching surface between \( \Theta_i \) and \( \Theta_{i+1} \), that is, \( \rho = \bar{\rho}_i = \rho_{i+1} \),

\[
\frac{1}{\mu} P_{i+1}(\rho) \leq P_i(\rho) \leq \mu P_{i+1}(\rho), \quad i \in Z_{\ell-1}
\]

(10)

where

\[
\begin{cases}
P_i(\rho) = P_{i+1}(\rho) | \rho = \bar{\rho}_i, \\
P_{i+1}(\rho) = P_{i+1}(\rho) | \rho = \rho_{i+1},
\end{cases}
\]

Then the closed-loop LPV system (7) is exponentially stabilized for every switching signal \( \sigma \) with the average dwell time

\[
\tau_\beta > \tau_\gamma = \frac{\ln \mu}{\beta}
\]

(11)

and its performance is maintained as \( \|z\|_2 < \gamma \|w\|_2 \) with \( \gamma = \max \{\gamma_i\}_{i \in Z_\ell} \).

**Lemma 2.** Given an open-loop LPV system (5), scalars \( \beta > 0, \mu > 1 \), the parameter set \( \Theta \) and its partition \( \Theta_i, i \in Z_\ell \), as shown in Eqs. (2)-(4), if there exist positive definite matrix functions \( R_i(\rho), S_i(\rho) \), and positive scalars \( \gamma_i, i \in Z_\ell \), so that for any \( \rho \in \Theta_i, i \in Z_\ell \),

\[
\begin{bmatrix}
\Phi_i & R_i(\rho)C_i^T(\rho) & B_i(\rho) \\
* & -\gamma_i I_{n_\rho} & D_{i1,1}(\rho) \\
* & * & -\gamma_i I_{n_\rho}
\end{bmatrix} N_{R,i}(\rho) < 0
\]

(12)

\[
\begin{bmatrix}
\Psi_i & S_i(\rho)B_i(\rho) & C_i^T(\rho) \\
* & -\gamma_i I_{n_\rho} & D_{i1,1}(\rho) \\
* & * & -\gamma_i I_{n_\rho}
\end{bmatrix} N_{S,i}(\rho) < 0
\]

(13)

\[
\begin{bmatrix}
R_i(\rho) & I_n \\
I_n & S_i(\rho)
\end{bmatrix} \geq 0
\]

(14)

where

\[
\begin{align*}
N_{R,i}(\rho) &= \text{Ker} \left[ B_i^T(\rho) \quad D_{i1,1}^T(\rho) \quad 0 \right] \\
N_{S,i}(\rho) &= \text{Ker} \left[ C_i(\rho) \quad D_{i1,1}(\rho) \quad 0 \right]
\end{align*}
\]

(15)

\[
\begin{align*}
\Phi_i &= R_i(\rho)A_i^T(\rho) + A_i(\rho)R_i(\rho) - \{\Xi, \bar{v}\} \over \partial R_i(\rho) \over \partial \rho + \beta R_i(\rho) \\
\Psi_i &= A_i^T(\rho)S_i(\rho) + S_i(\rho)A_i(\rho) + \{\Xi, \bar{v}\} \over \partial S_i(\rho) \over \partial \rho + \beta S_i(\rho)
\end{align*}
\]

(16)

and for any \( \rho = \bar{\rho}_i = \rho_{i+1} \),

\[
\begin{align*}
\frac{1}{\mu} R_{i+1}(\rho) & \leq \bar{R}_i(\rho) \leq \mu R_{i+1}(\rho) \\
\frac{1}{\mu} (S_{i+1}(\rho) - R_{i+1}^{-1}(\rho)) & \leq \bar{S}_i(\rho) - \bar{R}_i^{-1}(\rho) \leq \mu (S_{i+1}(\rho) - R_{i+1}^{-1}(\rho))
\end{align*}
\]

(17)

Then the closed-loop LPV system (7) is exponentially stabilized for every switching signal \( \sigma \) with the average dwell time

\[
\tau_\beta > \tau_\gamma = \frac{\ln \mu}{\beta}
\]

(18)

and its performance is maintained as \( \|z\|_2 < \gamma \|w\|_2 \) with \( \gamma = \max \{\gamma_i\}_{i \in Z_\ell} \).

**Proof.** The proof is the same as that in Ref.14 and the detailed process is omitted here. The key point is partitioning the Lyapunov function \( P_i(\rho) \) as

\[
\begin{bmatrix}
P_i(\rho) = S_i(\rho)N_i(\rho) \\
M_i(\rho)
\end{bmatrix}
\]

(19)

where \( M_i(\rho)N_i^T(\rho) = I - R_i(\rho)S_i(\rho) \) and “?” means that the elements can be neglected. □

**Remark 1.** Guarantee the stability of switched systems, the derivative of Lyapunov functions \( V_i(t) \) with respect to time must satisfy \( V_i(t) \leq -\mu V_i(t), i \in Z_\ell \) within each subset and inequality (10) holds for each switching surface.14 It should be noted that there is no limit to the monotony of the continuous Lyapunov function \( V_i(t) \) along with the scheduling parameter \( \rho \), so the switching direction of the system can be bidirectional.

**Remark 2.** The constraints (17) are equivalent to the following ones. Readers may refer to Ref.14 for more details.

\[
\begin{align*}
\frac{1}{\mu} S_{i+1}(\rho) & \leq \bar{S}_i(\rho) \leq \mu S_{i+1}(\rho) \\
\frac{1}{\mu} (R_{i+1}(\rho) - S_{i+1}^{-1}(\rho)) & \leq \bar{R}_i(\rho) - S_i^{-1}(\rho) \leq \mu (R_{i+1}(\rho) - S_{i+1}^{-1}(\rho))
\end{align*}
\]

(20)

After solving Lemma 2, matrix functions \( R_i(\rho) \) and \( S_i(\rho) \) can be obtained and the gains of switching LPV controllers can be constructed as

\[
K_i(\rho) = \begin{bmatrix}
A_i(\rho) & B_i(\rho) \\
C_i(\rho) & D_i(\rho)
\end{bmatrix} = \text{Func} \left( R_i(\rho), S_i(\rho) \right), \quad i \in Z_\ell
\]

(21)

where \( \text{Func}(R_i(\rho), S_i(\rho)) \) represents a function of \( R_i(\rho) \) and \( S_i(\rho) \), which can be defined as
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\[ A_{k_i}(\rho) = -N_{1i}^{-1}(\rho) \times \{ A^T(\rho) - S_i(\rho) \frac{dR_i}{dt} \} \]
\[ -N(\rho) \frac{dM_T^1}{dt} + S_i(\rho) A(\rho) + B(\rho) F(p) \]
\[ + L_i(p) C_{i_1}(\rho) R_i(\rho) + \frac{1}{\gamma} S_i(\rho) C_{i_1}(\rho) \]
\[ + A_{k_{i1}}(\rho) F(p) R_i(\rho) \times M_T^1(\rho) \]
\[ B_i(\rho) = N_{1i}^{-1}(\rho) S_i(\rho) L_i(p) \]
\[ C_{k_i}(\rho) = F(p) R_i(\rho) M_T^1(\rho) \]
\[ D_{k_i}(\rho) = 0 \]

\[ F(p) = -\left( D_{12,i}(\rho) \right)^{-1} \left[ \gamma_1 R_{i1}(\rho) C_{i_1}^T(\rho) + D_{12,i}(\rho) C_{i_1}(\rho) \right] \]
\[ L_i(p) = -\gamma_1 S_i(\rho) C_{i_1}(\rho) + B_{i1}(\rho) D_{12,i}(\rho)[D_{12,i}(\rho) D_{12,i}^T(\rho)]^{-1} \]

3. A systematic method of smooth switching LPV controllers design

3.1. Smooth switching LPV controllers design

Generally speaking, for the closed-loop system (7), switching between multiple controllers may produce transient responses, which are undesirable in practical applications. To alleviate the jumping problem and enhance system performance, a class of smooth switching LPV controllers will be designed.

Suppose that the scheduling parameter set $\Theta$ is divided into $I$ subsets $\Theta_i, i \in Z_I$, and every two adjacent subsets have an overlapped region. The overlapped partition method is depicted in Fig. 2. The overlapped region of the two adjacent subsets $\Theta_i$ and $\Theta_{i+1}$ is defined as an overlapped subset $\Theta_{i,i+1}$, and the subset that $\Theta_i$ dislodges the overlaps is defined as a non-overlapped subset $\Theta_{i,j}$.

The smooth switching LPV controllers are designed as follows: the controller in the non-overlapped subset $\Theta_{i,j}$ is the same as that in subset $\Theta_i$, and the controller in the overlapped subset $\Theta_{i,i+1}$ is smoothly scheduled between the two controllers in adjacent subsets $\Theta_i$ and $\Theta_{i+1}$. Therefore, the smooth switching LPV controllers can be described as

\[ K(\rho) = \begin{cases} K_i(\rho) = \text{Func} \left( R_i(\rho), S_i(\rho) \right), & i \in Z_I \\ K_{i,i+1}(\rho) = \text{Func} \left( R_{i,i+1}(\rho), S_{i,i+1}(\rho) \right), & i \in Z_{I-1} \end{cases} \]

where

\[ R_{i,i+1}(\rho) = \frac{\partial \rho}{\partial i} R_i(\rho) + \frac{\partial \rho}{\partial \rho} R_i(\rho) \]
\[ S_{i,i+1}(\rho) = \frac{\partial S_i(\rho)}{\partial i} S_i(\rho) + \frac{\partial S_i(\rho)}{\partial \rho} S_i(\rho) \]
\[ z_i(\rho) = \frac{\partial z_i(\rho)}{\partial i} + \frac{\partial z_i(\rho)}{\partial \rho} + \frac{\partial z_i(\rho)}{\partial \rho} \]
\[ z_i(\rho) = \frac{\partial z_i(\rho)}{\partial i} + \frac{\partial z_i(\rho)}{\partial \rho} + \frac{\partial z_i(\rho)}{\partial \rho} \]

Fig. 2 Overlapped partition method for scheduling parameter set.
Then the closed-loop LPV system \((7)\) is exponentially stabilized for every signal \(w\), and its performance is maintained as \(\|z\|_2 < \gamma \|w\|_2\) with \(\gamma = \max \{\gamma_i \} \in \mathbb{Z}_+^s\). After solving matrix functions \(R(p)\) and \(S(p)\), the gains of the switching LPV controllers can be constructed by Eq. \((24)\).

**Proof.** Obviously, dividing \(\Theta\) into \(J\) subsets \(\Theta_j, j \in \mathbb{Z}_+^s\), with overlaps is equivalent to dividing \(\Theta\) into \(J = 2J - 1\) subsets \(\Theta_j, j \in \mathbb{Z}_+^s\), \(\mathbb{Z}_+^s = \{1, 2, \ldots, 2J - 1\}\) without overlaps. The correspondences between the two partition methods can be described as

\[
\Theta_1 = \Theta_{1,1}, \quad \Theta_{(2)} = \Theta_{1,2}, \ldots, \Theta_{(2J-1,2J-2)} = \Theta_{1,J}; \quad \Theta_{(2J-1,2J-2)} = \Theta_{1,1}
\]

(37)

As follows, the proof is performed in three steps.

1. According to the overlapped partition method and the correspondences shown in Eq. \((37)\), we have

\[
\Theta_{(2J-1,2J-2)} = \Theta_{(2J-1,2J-2)} \subset \Theta_j
\]

From Eqs. \((27), (28)\) and \((33)\), it is easy to know that for any \(j = 2i - 1, \ i \in \mathbb{Z}_+^s\)

\[
\mathbf{z}_i = \mathbf{z}_i < 0
\]

\[
\Gamma_j = \Gamma_j < 0
\]

\[
\begin{bmatrix}
R_{ij}(\rho) & I_n \\
I_n & S_{ij}(\rho)
\end{bmatrix} \geq 0
\]

(40)

hold, where the expressions of \(\mathbf{z}_i\) and \(\Gamma_j\) are the same as those in Eqs. \((12)\) and \((13)\), respectively, and the controller coefficient matrices are

\[
K_{ij}(\rho) = K_{ij}(\rho) = K_{ij}(\rho) = \text{Func}(R_{ij}(\rho), S_{ij}(\rho)), i \in \mathbb{Z}_+^s
\]

(42)

2. From Eqs. \((29)\) and \((30)\), it can be inferred that the convex combination of Eqs. \((29)\) and \((30)\)

\[
z_{ij}(\rho)z_{ij+1} + z_{ij}(\rho)z_{ij+1} < 0, \quad i \in \mathbb{Z}_+^s
\]

holds. Define

\[
R_{ij+1}(\rho) = z_{ij}(\rho)R_{ij+1}(\rho) + z_{ij}(\rho)R_{ij}(\rho)
\]

From Eqs. \((26)\) and \((44)\), the upper and lower bounds of the derivative of \(R_{ij+1}(\rho)\) with respect to time can be represented as

\[
\left\{ \mathbf{z}, \overline{\mathbf{z}} \right\} \frac{\partial R_{ij+1}(\rho)}{\partial \rho} = \left\{ \mathbf{z}, \overline{\mathbf{z}} \right\} \left( \frac{1}{\rho_i - \rho_{ij}} R_{ij+1}(\rho) + \frac{\rho - \rho_{ij}}{\rho_i - \rho_{ij}} \frac{\partial R_{ij+1}}{\partial \rho} \right) + \frac{1}{\rho_i - \rho_{ij}} R_{ij}(\rho) + \frac{\rho - \rho_{ij}}{\rho_i - \rho_{ij}} \frac{\partial R_{ij}}{\partial \rho}
\]

\[
= z_{ij}(\rho) \left\{ \mathbf{z}, \overline{\mathbf{z}} \right\} \frac{\partial R_{ij+1}}{\partial \rho} + \frac{\mathbf{z} \mathbf{v}}{\rho - \rho_{ij}} R_{ij+1}(\rho)
\]

\[
+ z_{ij}(\rho) \left\{ \mathbf{z}, \overline{\mathbf{z}} \right\} \frac{\partial R_{ij}}{\partial \rho} + \frac{\mathbf{z} \mathbf{v}}{\rho - \rho_{ij}} R_{ij}(\rho)
\]

(45)

It can be obtained from Eqs. \((25), (44), \) and \((45)\) that Eq. \((43)\) is equivalent to

\[
\mathbf{z}(\rho) = \mathbf{z}_{ij+1} < 0, \quad j = 2i, \ i \in \mathbb{Z}_+^s
\]

where

\[
\begin{bmatrix}
\Phi_{ij+1} R_{ij+1}(\rho)C_{ij+1}(\rho) \quad B_{ij+1}(\rho) \\
* & -\gamma \mathbf{I}_n \\
* & * & -\gamma \mathbf{I}_n
\end{bmatrix}
\]

\[
\times \mathbf{N}_{ij+1}(\rho)
\]

\[
\begin{bmatrix}
\Phi_{ij} R_{ij+1}(\rho)A_{ij+1}^T(\rho) \quad A_{ij+1}(\rho)R_{ij+1}(\rho)
\end{bmatrix}
\]

\[
\mathbf{S}_{ij+1}(\rho) < 0
\]

Similarly, define

\[
\mathbf{S}_{ij+1}(\rho) = z_{ij}(\rho)\mathbf{S}_{ij+1}(\rho) + z_{ij}(\rho)\mathbf{S}_{ij+1}(\rho)
\]

It also can be obtained that

\[
\Gamma_j = \Gamma_j < 0, \quad j = 2i, \ i \in \mathbb{Z}_+^s
\]

(48)

Moreover, for any \(j = 2i, \ i \in \mathbb{Z}_+^s\), we get

\[
\begin{bmatrix}
R_{ij}(\rho) & I_n \\
I_n & S_{ij}(\rho)
\end{bmatrix} \geq 0
\]

(49)

Combining Eqs. \((46), (48), \) and \((49)\), we know that for any \(j = 2i, \ i \in \mathbb{Z}_+^s\),

\[
\mathbf{z}_i = \mathbf{z}_i < 0
\]

\[
\Gamma_j = \Gamma_j < 0
\]

(51)

\[
\begin{bmatrix}
R_{ij}(\rho) & I_n \\
I_n & S_{ij}(\rho)
\end{bmatrix} \geq 0
\]

(52)

hold and the controller coefficient matrices are

\[
K_{ij}(\rho) = K_{ij+1}(\rho) = \text{Func}(R_{ij+1}(\rho), S_{ij+1}(\rho))
\]

(53)

3. According to the definitions in Eqs. \((44)\) and \((47)\), we know that when \(\rho = \rho_{ij} = \mathbf{z}_{ij+1}\),

\[
\mathbf{z}_{ij}(\rho) = \mathbf{z}_{ij+1}(\rho)
\]

\[
\mathbf{S}_{ij}(\rho) = \mathbf{S}_{ij+1}(\rho) - \mathbf{R}_{ij+1}^T(\rho)
\]

(54)

Thus, from Eqs. \((17)\) and \((18)\), it can be inferred that \(\mu = 1\) and the lower bound of the average dwell time \(\gamma_d^* = 0\).

In conclusion, since the conditions \((39), (42), \) and \((50) - (54)\) are satisfied, Theorem 1 can be obtained from Lemma 2. This completes the proof of Theorem 1.
Remark 3. Since the matrix functions \( R_i(\rho) \) and \( S_i(\rho) \) are defined as in Eqs. (44) and (47), it can be deduced from Eq. (19) that the values of Lyapunov functions will not change when switching occurs. Therefore, the smoothness of switching can be ensured under the controllers (24). Meanwhile, a switching law without constraint on the average dwell time is obtained in Theorem 1, which makes the conclusion less conservative.

Remark 4. Notice that the LMI conditions (27)–(33) correspond to an infinite-dimensional convex problem due to their parametric dependence. The gridding technique and the approximate basis function can be used to obtain a finite-dimensional problem.31 Thus, choose appropriate basis functions \( \sum f_i \), so that

\[
\begin{align*}
R_i(\rho) &= \sum_{i=1}^{n_f} f_i R_i(\rho) > 0 \\
S_i(\rho) &= \sum_{i=1}^{n_f} f_i S_i(\rho) > 0
\end{align*}
\]  

Corollary 1. Given a scalar \( \beta > 0 \), the open-loop LPV system (5), the suboptimal robust \( H_\infty \) controllers can be obtained by solving the following convex optimization problem

\[
\begin{align*}
\min_{\gamma} & \quad \gamma \\
\text{s.t.} & \quad \text{Eqs. (27)–(33), (55)}
\end{align*}
\]

and the gains of switching LPV controllers can be constructed by Eq. (24).

3.2. A systematic algorithm to design controllers

In the available literature, such as Refs.14,25, to design the switching LPV controllers, the scheduling parameter set is divided into predetermined subsets. A shortcoming of these traditional methods is that the parameter set division process depends more on the experience of a designer, or even by trial and error. In other words, if the predetermined subsets lead to unsatisfactory controllers, we need to tune them heuristically. To solve these problems, a systematic algorithm to design switching LPV controllers satisfying a desired performance will be proposed via a parameter set automatic partition method.

To present the main idea of the algorithm without so much technical detail, we give some preparation for this algorithm. Firstly, the desired performance \( \gamma_c \) can be selected as

\[ \gamma_{\min} \leq \gamma_c \leq \gamma_{\max} \]

where \( \gamma_{\min} \) and \( \gamma_{\max} \) are the minimal and maximal performances that can be achieved by switching gain-scheduling control method, respectively. \( \gamma_{\min} \) can be given by

\[ \gamma_{\min} = \max_{\rho \in \Theta} \gamma_e \]

where \( \gamma_e \) is the performance when \( \rho = \rho_c \) and \( \rho_c \) can be any fixed value within \( \Theta \). \( \gamma_{\max} \) is the performance which can be calculated by utilizing a common Lyapunov function. Intuitively speaking, \( \gamma_{\min} \) corresponds to dividing the entire set \( \Theta \) into an infinite number of subsets and \( \gamma_{\max} \) corresponds to not dividing.

Secondly, suppose that subset \( \Theta_i \) is characterized by its lower bound \( \rho_i \) and width \( \tau_i > 0 \).

\[ \Theta_i(\rho_i, \tau_i) = \{ \rho \in \Theta | 0 \leq \rho - \rho_i \leq \tau_i \} \]  

When we design the \( i \)-th subset \( \Theta_i(\rho, \tau) \), the lower bound \( \rho_i \) will be selected so that subset \( \Theta \) overlaps with the adjacent designed subset \( \Theta_{i-1} \) and the width of the overlapped subset \( \tau_{i-1} \) is limited to be a constant percent \( \omega \% \) of the width \( \tau_{i-1} \).

Algorithm 1.

\begin{enumerate}
\item [Step 1.] Set a desired performance \( \gamma_c \).
\item [Step 2.] As an initialization, set \( I = 1 \), the unpartitioned subsets \( \Theta^{unp} = \Theta \), and the partitioned \( \Theta^p = \emptyset \).
\item [Step 3.] Design the \( i \)-th subset \( \Theta_i(\rho, \tau) \) in \( \Theta^{unp} \). After the lower bound \( \rho_i \) has been decided, the width \( \tau_i \) and the matrix functions \( R_i(\rho), S_i(\rho), i \in \Omega \) can be obtained by solving the following optimization problem

\[
\begin{align*}
\max_{\omega} & \quad \tau_i \\
\text{s.t.} & \quad \text{Eq. (56) and } \gamma \leq \gamma_c
\end{align*}
\]

and the controllers can be constructed by Eq. (24). Then, we reset

\[
\begin{align*}
\Theta^p &= \Theta^p \cup \Theta_i \\
\Theta^{unp} &= \Theta - \Theta^p
\end{align*}
\]

\item [Step 4.] If \( \Theta^p = \emptyset \), terminate. Otherwise, reset \( I = I + 1 \), and go back to Step 3.
\end{enumerate}

\[
\text{Fig. 3 illustrates the division process using the above algorithm and the algorithm terminates after three iterations, i.e., } I = 3. \text{ It can be known that } \tau_{1,2} = \omega \% \times \tau_1 \text{ and } \tau_{2,3} = \omega \% \times \tau_2.
\]

Remark 5. In the aforementioned discussion, we only consider the situation that the scheduling parameter is one-dimensional. Actually, further research on smooth switching LPV control and parameter set automatic partition can be done, so that the proposed systematic method can be extended to the situation...
that the scheduling parameter is multi-dimensional. For instance, it can be applied to flight control for variable-span and variable-sweep morphing aircraft, large-envelope flight control for hypersonic aircraft, etc.

4. Application to morphing aircraft

4.1. LPV model of morphing aircraft

In what follows, an application to the flight control of Teledyne Ryan BQM-34 “Firebee”\(^{32}\), depicted in Fig. 4, is presented to show the effectiveness of the proposed method. The Firebee, first flown in 1985, is an unmanned remotely controlled aircraft originally constructed as a target drone for training intercept pilots and for missile targeting practice. Suppose that the wing sweep angle \(\lambda\) of Firebee can be changed to accommodate mission requirements such as subsonic targeting, supersonic targeting, reconnaissance, and attack configurations. Accordingly, some parameters, such as mean aerodynamic chord, span, and wing area, will change with the wing sweep angle. It is assumed that the wing sweep angle can change continuously from 15° to 60°, which are corresponding to loiter and dash configurations, respectively. Define \(\xi = (\lambda - \lambda_0)/\lambda_0\) as the variation rate of the wing sweep angle, where \(\lambda_0 = 15°\) is the minimum wing sweep angle corresponding to the loiter configuration. We get \(\xi \in [0, 3]\).

The longitudinal short-period nonlinear dynamic model of Firebee can be described as\(^{32}\)

\[
\begin{align*}
&m_T V_T (\dot{z} - q) = g m_T (\cos \theta \cos \alpha \sin \theta \sin \alpha) \\
&\quad - S \dot{q} C_L (\lambda, H, M_a, \delta_e, q, \alpha) + (m_u \dot{x}_u + m_v \dot{x}_v) \sin \alpha \\
&\quad - 2(m_u \dot{x}_u + m_v \dot{x}_v) q \cos \alpha \\
\end{align*}
\]

\[
J_q = S \dot{q} C_m (\lambda, H, M_a, \delta_e, q, \alpha) \\
\quad - (m_u \dot{x}_u^2 + m_v \dot{x}_v^2) q \\
\quad - 2(m_u \dot{x}_u \dot{x}_u + m_v \dot{x}_v \dot{x}_v) q
\]

(62)

where \(z\) and \(q\) denote the angle of attack and the pitch rate, respectively. \(\delta_e\) denotes the elevator deflection. \(V_T\), \(H\), \(M_a\), and \(\theta\) denote the velocity, the altitude, the Mach number, and the flight path angle, respectively. \(m_T\), \(S\), \(J_f\), and \(\epsilon\) denote the mass, the wing area, the y-axis inertia, and the mean aerodynamic chord, respectively. \(m_u\), \(m_v\), \(x_u\), and \(x_v\) denote the mass of the wing, the mass of the counterweight, the position of the mass center of the wing, and the position of the mass center of the counterweight, respectively. \(C_L\) and \(C_m\) denote coefficients of aerodynamic force and aerodynamic moment, respectively, which can be approximately expressed as

\[
\begin{align*}
C_L &= C_{L_{\alpha=0}} + C_{L_{\lambda}} \lambda + C_{L_{\delta_e}} \delta_e \\
C_m &= C_{m_{\alpha=0}} + C_{m_{\lambda}} \lambda + C_{m_{\delta_e}} \delta_e \\
\end{align*}
\]

The flight condition of interest is selected as the altitude \(H = 9144\) m, the Mach number \(M_a = 0.5\), and other parameters for loiter and dash configurations as listed in Table 1.\(^{32}\)

Choosing 11 reference points as \(\xi = 0, 0.3, 0.6, \ldots, 3.0\), the aerodynamic parameters for different variation rates can be calculated through computational fluid dynamics (CFD). Then the aerodynamic parameters of the morphing aircraft during the wing-transforming process can be interpreted by those of static configurations with the help of MATLAB.

\[
\begin{align*}
C_{L_{\alpha=0}} &= 0.05437\xi^3 - 0.208\xi^2 + 0.1476\xi - 0.1036 \\
C_{L_{\lambda}} &= -0.894\xi + 5.538 \\
C_{L_{\delta_e}} &= -0.0053\xi + 0.0065 \\
C_{M_{\alpha=0}} &= 0.583\xi^3 - 2.223\xi^2 + 1.639\xi - 0.00035 \\
C_{M_{\lambda}} &= 0.0474\xi^3 - 0.3003 \\
C_{M_{\delta_e}} &= 0.0066\xi - 0.0222 \\
\end{align*}
\]

In general, there are three LPV modeling approaches that can be used to transform the nonlinear model (62) of the morphing aircraft into an LPV model.\(^{33}\) They are Jacobian linearization, state transformation, and function substitution. We adopt the most widespread one, Jacobian linearization, in this paper. The Jacobian linearization approach is valid for any nonlinear system that can be linearized at its equilibrium points. The resulted model is a local approximation to the dynamics of the nonlinear plant around this set of equilibrium points.

Firstly, to cover the whole work area, the equilibrium points are selected as \(\xi = 0, 0.3, 0.6, \ldots, 3\). Based on the nonlinear dynamic model (62), we define

\[
\begin{align*}
f_1 &= \frac{1}{m_T V_T} [g m_T (\cos \theta \cos \alpha \sin \theta \sin \alpha) \\
&\quad - S \dot{q} C_L (\lambda, H, M_a, \delta_e, q, \alpha) \sin \alpha \\
&\quad - 2(m_u \dot{x}_u + m_v \dot{x}_v) q \cos \alpha + q] \\
\end{align*}
\]

\[
\begin{align*}
f_2 &= \frac{S \dot{q} C_m - 2(m_u \dot{x}_u x_u + m_v \dot{x}_v x_v)}{J_f + (m_u \dot{x}_u^2 + m_v \dot{x}_v^2)} \\
\end{align*}
\]

Table 1: Morphing aircraft parameters for loiter and dash configurations.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>(\lambda) (°)</th>
<th>(J_f) (kg · m²)</th>
<th>(m_T) (kg)</th>
<th>(m_u) (kg)</th>
<th>(m_v) (kg)</th>
<th>(x_u) (m)</th>
<th>(x_v) (m)</th>
<th>(S) (m²)</th>
<th>(\epsilon) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loiter</td>
<td>15</td>
<td>3107.5</td>
<td>907.8</td>
<td>272</td>
<td>26.36</td>
<td>0</td>
<td>-3.2090</td>
<td>4.3621</td>
<td>0.7101</td>
</tr>
<tr>
<td>Dash</td>
<td>60</td>
<td>3107.5</td>
<td>907.8</td>
<td>272</td>
<td>26.36</td>
<td>-0.6072</td>
<td>3.0656</td>
<td>6.0792</td>
<td>1.9117</td>
</tr>
</tbody>
</table>
Note that the angular accelerations of $x$ and $q$ are zero when the vehicle is in equilibrium, that is, $f_1 = 0$, $f_2 = 0$. Then, the values of $x$, $q$, and $\delta_e$ on the equilibrium points, denoted as $x_0(\xi)$, $q_0(\xi)$, and $\delta_{e0}(\xi)$, can be worked out by Eq. (65).

We define the deviation variables as

$$\Delta x = x - x_0(\xi)$$
$$\Delta q = q - q_0(\xi)$$
$$\Delta \delta_e = \delta_e - \delta_{e0}(\xi)$$

and we can obtain 11 linear small perturbation equations by linearizing the nonlinear model (62) at the 11 equilibrium points as follows:

$$\begin{bmatrix} \Delta \dot{x} \\ \Delta \dot{q} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta q \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} \Delta \delta_e$$

where

$$A_{11} = \frac{\partial f_1}{\partial x}, A_{12} = \frac{\partial f_1}{\partial q}, A_{21} = \frac{\partial f_2}{\partial x}, A_{22} = \frac{\partial f_2}{\partial q}$$
$$B_{11} = \frac{\partial f_1}{\partial \Delta \delta_e}, B_{21} = \frac{\partial f_2}{\partial \Delta \delta_e}$$

Lastly, based on the 11 linear small perturbation equations, the longitudinal short-period LPV model of Firebee can be obtained by numerical fitting as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A(\xi) & 0_{2\times 2} & B_1(\xi) \\ C_1(\xi) & 0_{3\times 2} & D_1(\xi) \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} (66)$$

where the state variables $x = [\Delta x \ \Delta q]^T$, the control input $u = \Delta \delta_e$, and

$$A(\xi) = \begin{bmatrix} 0.2255\xi - 1.3967 & 1 \\ 0.0876\xi^2 - 0.4889\xi - 0.3775 & 0.4489\xi - 0.8229 \end{bmatrix}$$
$$B_1(\xi) = \begin{bmatrix} 0.00034\xi^2 - 0.001638 \\ 0.000053\xi^2 - 0.044151\xi - 0.143984 \end{bmatrix}$$
$$C_1(\xi) = \begin{bmatrix} I_2 \\ 0_{1\times 2} \end{bmatrix}, \ D_1(\xi) = \begin{bmatrix} 0_{2\times 1} \\ 1 \end{bmatrix}$$

Remark 6. As shown in Eqs. (64) and (66), the dynamic characteristics of morphing aircraft vary dramatically following their wing sweep angles. Conventional gain-scheduling techniques can be used to handle this kind of complex systems. However, it can’t theoretically guarantee the robustness, performance, or even stability of the close-loop systems.\(^{3,21}\)

To overcome those problems and capture the wing transition phase’s complex behavior, LPV control is adopted in this paper. The parameter-varying dynamic characteristics are simplified and transformed to the LPV plant model of morphing aircraft.

4.2. Nonlinear simulation

Considering Corollary 1, we pick two basis functions in Eq. (55) as $f_1 = 1$ and $f_2 = \xi$. Suppose that the variation rate of the sweep angle is less than $3(\degree)/s$, that is, $\bar{\gamma} = 0.2$ and $\bar{v} = 0.2$. The whole range of $\Theta$ will be divided into 50 gridding cells by the gridding technique. Setting $\beta = 0.8$, we can calculate the minimal performance $\gamma_{\text{min}} = 0.7407$ and the maximal performance $\gamma_{\text{max}} = 2.6619$. In this simulation, the desired performance is chosen as $\gamma_0 = 1.8$, and the constant percent $\omega\% = 10\%$. Then the parameter set will be divided into $I = 3$ subsets after three iterations by Algorithm 1, as shown in the second column of Table 2. Meanwhile, the matrix functions $R(\rho)$ and $S(\rho)$ ($i = 1, 2, 3$) can be obtained and we can construct the controllers by Eq. (24).

It is assumed that the variation rate of the wing sweep angle is

$$\rho(t) = \xi(t) = 1.5 + 1.5\sin((1/7.5)t - 0.5\pi - (5/7.5)), 0 \leq t \leq 60$$

as shown in Fig. 5. Hence, using the smooth switching controllers, there are 8 times of switching that occur at 15.38, 16.08, 20.76, 21.48, 35.64, 36.36, 41.04, and 41.75 s, respectively, also being marked in Fig. 5 by small circles.

Comparative simulations are developed to verify the smoothness of the proposed controllers. We also divide the scheduling parameter set into three subsets without overlaps as shown in the third column of Table 2. There are 4 times of switching that occur at 15.38, 20.76, 36.36, and 41.75 s, respectively, also being marked in Fig. 5 by small circles.

From the curves of the angle of attack and pitch rate as shown in Fig. 6(a) and (b), it can be observed that the actual

<table>
<thead>
<tr>
<th>Subsets</th>
<th>Smooth switching controller</th>
<th>Non-smooth switching controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Theta_1$</td>
<td>[0.136]</td>
<td>[0.122]</td>
</tr>
<tr>
<td>$\Theta_2$</td>
<td>[1.22, 2.38]</td>
<td>[1.22, 2.26]</td>
</tr>
<tr>
<td>$\Theta_3$</td>
<td>[2.26, 3.00]</td>
<td>[2.26, 3.00]</td>
</tr>
</tbody>
</table>

Fig. 5 Variation rate of wing sweep angle.
angle of attack and pitch rate can track the reference signals well even the wing sweep angle is changing. Besides, in Fig. 6, the subplots are the zoomed views of responses around the switching times 15.38 s and 36.36 s, which are the first and third switching times between the non-smooth switching controllers. It can be seen that, using the non-smooth switching controllers, the system states and the control input have sudden undesirable transient behavior when switching occurs. In contrast, using the smooth switching controllers, the closed-loop system can switch smoothly and has a better performance.

Furthermore, suppose that there exist ±15% parameter uncertainties in aerodynamic forces and moments of the non-linear model. Using the smooth switching controllers, we do 50 Monte Carlo simulations on the nonlinear model. Due to the limit of space, we take the response curves of the angle of attack for example, as shown in Fig. 7. $\alpha_{\text{ref}}$ and $\alpha_{\text{act}}$ represent the reference and actual angles of attack, respectively. It is apparent that the angle of attack has good tracking performance even there are parameter uncertainties.

Taken together, through the comparative simulations and Monte Carlo simulations, the stability, smoothness, and robustness of the proposed controllers have been commendably validated.

5. Conclusions

In this paper, a systematic method of smooth switching LPV controllers design is explored for a morphing aircraft with a variable wing sweep angle.

(1) The LPV model of the morphing aircraft is developed and it can characterize the wing transition phase’s complex behavior.

(2) A sufficient condition to ensure that the switched LPV systems has exponential stability and a certain robust performance is presented. A switching law without constraint on the average dwell time is obtained which makes the conclusion less conservative.

(3) An algorithm is designed so that the scheduling parameter set can be partitioned into overlapped subsets automatically and the output feedback smooth switching controllers, which have a desired performance, can be constructed efficiently.

(4) Simulation results illustrate that, using the proposed smooth switching controllers, the morphing aircraft flight system not only has excellent stability and robustness, but also can switch smoothly.

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References


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