Delbrück scattering and the $g$-factor of a bound electron

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Received 25 September 2002; received in revised form 25 October 2002; accepted 29 October 2002

Abstract

The leading contribution of the light-by-light scattering effects to $g$-factor of a bound electron is derived. The corresponding amplitude is expressed in terms of low-energy Delbrück scattering of a virtual photon. The result reads $\Delta g = \frac{7}{216} \alpha (Z \alpha)^5$. © 2002 Elsevier Science B.V. PACS: 12.20.-m; 31.30.Jv; 32.10.Dk

Keywords: Quantum electrodynamics; $g$-factor; Delbrück scattering

1. Introduction

Recently a significant progress in measurements of the $g$-factor of a bound electron in a hydrogen-like ion with a spinless nucleus has been achieved [1,2]. At the same time an accurate theory was successfully developed [3–10]. Comparison of theory and experiment [1] for the hydrogen-like carbon ion $^{12}\text{C}^{5+}$ allows to determine the most accurate values of the electron mass [7]

$$m_e = 0.000 \, 548 \, 579 \, 909 \pm 2(4)\text{u} \quad (1)$$

and the proton-to-electron mass ratio

$$m_p/m_e = 1836.152 \, 673 \pm 3(14). \quad (2)$$

Those are three times more accurate than the recommended CODATA values [11] based on study of protons and electrons in Penning trap [12].

Theory and experiment equally contribute into uncertainty of the values in Eqs. (1) and (2). An essential part of theoretical uncertainty (maybe even the dominant part) is due to the light-by-light scattering effects. A part of them is related to the Wichmann–Kroll potential (so-called the “electric-loop” term presented in Fig. 1, where the contribution of a vacuum polarization by free electrons known analytically for a point-like nucleus [6] is not included). Its leading contribution ($\propto \alpha (Z \alpha)^6$) has been found analytically [4]:

$$\Delta g_{(EL)} = 2 \left( \frac{38}{45} - \frac{2 \pi^2}{27} \right) \frac{\alpha (Z \alpha)^6}{\pi}. \quad (3)$$

Here $Z$ is the nuclear charge number, $\alpha = e^2$ is the fine-structure constant, $\hbar = c = 1$. The other part (so-called the “magnetic-loop” term) presented in Fig. 2...
has not been known. Even the order of magnitude of the leading term has not been clarified. Some rough estimations for it were included into evaluations in Refs. [3,6,7]. We claim that the magnetic-loop effects contribute in order $\alpha(Z\alpha)^5$. Therefore, potentially they could strongly affect the $g$-factor of a bound electron. For instance, with

$$\Delta g(\text{ML}) = C_{\text{ML}} \alpha(Z\alpha)^5$$

one can find a shift of $g$-factor up to $2C_{\text{ML}} \times 10^{-9}$ in the case of the hydrogen-like ion of carbon, and $C_{\text{ML}} \times 10^{-8}$ in the case of oxygen. That may be bigger than both experimental and theoretical uncertainty. In this Letter we study the leading contribution of the magnetic-loop effects and determine coefficient $C_{\text{ML}}$.

2. Low-energy Delbrück scattering and the contribution of “magnetic loop”

First we note that we need to study a vacuum polarization loop with the two lines of external virtual photons in the Coulomb field of a nucleus. One of these photons is related to an homogeneous magnetic field and the other connects the electron loop and the atomic electron line. Because of the Furry theorem the leading effects of the electric field are related to the diagram with the two Coulomb lines. Then, we apply the block of the vacuum polarization when both external photon lines transfer momenta $k_1$ and $k_2$ significantly smaller than the electron mass, $|k_2| \sim Z \alpha m_e$. The rigorous analysis shows that the contribution to $g$-factor of the region $|k_2| \sim m_e$ is of order of $\alpha(Z\alpha)^6$.

Delbrück scattering (see [13,14]) is a process in which the initial photon turns into a virtual electron–positron pair that scatters in the electric field of an atom and then transforms into the final photon. The Feynman diagram corresponding to the amplitude in the lowest in $Z\alpha$ order is shown in Fig. 3. Let us consider the scattering of a virtual photon with the initial momentum $k_1 = (\omega, k_1)$ and the final momentum $k_2 = (\omega, k_2)$, with $\omega, |k_i| \ll m_e$. Due to the gauge invariance, the Delbrück scattering amplitude reads

$$T_{\mu\nu} = \frac{\alpha(Z\alpha)^2}{m_e^3} \left\{ C_1 \cdot \left[ g^{\mu\nu}(k_1 \cdot k_2) - k_2^\mu k_1^\nu \right] 
+ C_2 \cdot \left[ \omega^2 g^{\mu\nu} - \omega (n^\mu k_1^\nu + k_2^\mu n^\nu) + (k_1 \cdot k_2) n^\mu n^\nu \right] \right\},$$

(4)
where \( n^\mu = g^00 \), \( C_1 \) and \( C_2 \) are some constants. This form of the amplitude follows from the relations
\[ k_{1\mu}T^{\mu\nu} = T^{\nu\mu}k_{2\mu} = 0 \]
and from the linearity of \( T^{ij} \) with respect to \( k_1 \) and \( k_2 \). To calculate the magnetic loop contribution we need only the amplitude \( T^{ij} \) at \( \omega = 0 \). We can find it since the amplitude in Eq. (4) was derived for the real photons. For \( \omega = |k_1| = |k_2| \) the amplitude \( T_D = e_1^{(1)}T^{\mu\nu}e_2^{(2)*} (e_1^{(1,2)} \) are the polarization vectors) is of the form
\[
T_D = \frac{\alpha(Z\alpha)^2}{m_e^2} \left\{ -(C_1 + C_2)\omega^2 (e_1 \cdot e_2^*) + C_1 [e_1 \times k_1][e_2^* \times k_2] \right\}. \tag{5}
\]
The amplitude \( T_D \) (5) was obtained in Ref. [15] (see also [16]). Using the results of [15] and Eq. (5), we obtain
\[
C_1 = 7/16 \cdot 72 \quad \text{and} \quad C_2 = -73/32 \cdot 72. \tag{6}
\]
Let us consider now the amplitude \( T_M \) of interaction of the magnetic field with the spin part of the magnetic moment of the atomic electron. In the zero approximation it reads
\[
T_M^{(0)} = -\frac{ie}{m_e} \int \frac{d^3k}{(2\pi)^3} \left( A_k \cdot |k \times s| / \rho_k \right)
= \frac{e}{m_e} \int \frac{d^3k}{(2\pi)^3} B_k \cdot s / \rho_k, \tag{7}
\]
where \( A_k \) is the vector potential, \( s \) is the spin operator, and \( \rho_k \) is the Fourier transform of the electron density \(|\psi(\mathbf{r})|^2\). \( \psi(\mathbf{r}) \) is the bound electron wave function. Note that a sign of \( T_M \) is opposite to that of Hamiltonian. In the case of the homogeneous magnetic field \( B \), we have
\[
B_k = i [k \times A_k] = (2\pi)^3 \delta(\mathbf{k}) B,
\]
and we can replace in Eq. (7) \( T_M^{(0)} \) by \( \rho_0 = 1 \). Using (4) we can represent the correction \( T_M^{(1)} \) as follows:
\[
T_M^{(1)} = -\frac{ie\alpha(Z\alpha)^2}{m_e^2} C_1 \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{4\pi}{q^4} \times \left\{ (k \cdot q) A_k - (A_k \cdot q)k \right\} \cdot |q \times s| / \rho_q
= -\frac{ie\alpha(Z\alpha)^2}{m_e^2} C_1 \int \frac{d^3k}{(2\pi)^3} A_k \cdot |k \times s|
\times \int \frac{d^3q}{(2\pi)^3} \frac{8\pi}{3} \rho_q
\]
\[
\times \int \frac{d^3k}{(2\pi)^3} \frac{8\pi}{3} \rho_q
= -\frac{ie\alpha(Z\alpha)^2}{m_e^2} C_1
\times \int \frac{d^3k}{(2\pi)^3} A_k \cdot |k \times s|
\times \int \frac{d^3q}{(2\pi)^3} \frac{8\pi}{3} \rho_q
\times \int \frac{d^3k}{(2\pi)^3} \frac{8\pi}{3} \rho_q
\]
\[
\times \int \frac{d^3k}{(2\pi)^3} \frac{8\pi}{3} \rho_q
\]
Thus, our final result for the relative correction of the magnetic loop reads
\[
\Delta g(\text{ML}) = \frac{7}{216} \alpha(Z\alpha)^5. \tag{10}
\]
For the interesting cases of hydrogen-like carbon (\( Z = 6 \)) and oxygen (\( Z = 8 \)) the numerical values are \( 0.4 \times 10^{-10} \) and \( 1.6 \times 10^{-10} \) respectively. The obtained correction has the same order of magnitude as results in Ref. [3]. However, a quantitative comparison of our analytic result with numerical calculations there is not helpful because of lack of accurate numerical data for \( Z \leq 10 \).

3. Conclusions

If one applies Eq. (10) to the \( g \)-factor for hydrogen-like carbon and oxygen, then the result in some sense is controversial. First, it has lower order of magnitude than expected, i.e., \( \alpha(Z\alpha)^5 \) versus \( \alpha(Z\alpha)^6 \). On the other side, the correction (10) is consistent with the preliminary estimate (see, e.g., [6]) because of its very small numerical coefficient. Next, it is not clear if the leading in \( Z\alpha \) term gives a dominant contribution. E.g., in the case of recoil correction for carbon, the leading term is smaller than the next-to-leading term because of the small coefficients in it [8,9].

To estimate uncertainty related to the higher-order terms, we note that the coefficients in the contribution of the free vacuum polarization [6]
\[
\Delta g(\text{VP}) = 2 \cdot \frac{\alpha}{\pi} \left[ \frac{8}{15}(Z\alpha)^4 + \frac{5\pi}{18}(Z\alpha)^5 \right]
\]
\[
\simeq -0.34 \times \alpha(Z\alpha)^5 + 0.56 \times \alpha(Z\alpha)^6 \tag{11}
\]
are about unity in contrast to the light-by-light contributions (3) and (10).
\[
\Delta g(\text{ML}) \simeq 0.032 \times \alpha(Z\alpha)^5,
\]
Table 1

The bound electron $g$-factor in low-$Z$ hydrogen-like ions with spinless nucleus. The uncertainty for the two-loop contribution is taken from [6]. Ref. (one-loop) is related to the one-loop result for the self-energy contribution. For lighter atoms it is taken from [17] based on fitting data of [3], while for heavier isotopes we use the results of [10].

<table>
<thead>
<tr>
<th>Ion</th>
<th>$g$</th>
<th>Ref. (one-loop)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^4_{\text{He}}\text{+}$</td>
<td>2.0021774067(1)</td>
<td>[17]</td>
</tr>
<tr>
<td>$^{10}_{\text{Be}}\text{+}$</td>
<td>2.0017515745(4)</td>
<td>[17]</td>
</tr>
<tr>
<td>$^{12}_{\text{C}}\text{+}$</td>
<td>2.0010415901(4)</td>
<td>[10]</td>
</tr>
<tr>
<td>$^{16}_{\text{O}}\text{+}$</td>
<td>2.000470201(8)</td>
<td>[10]</td>
</tr>
</tbody>
</table>

\[
\Delta g(\text{EL}) \simeq 0.072 \times \alpha(Z\alpha)^6. \quad (12)
\]

We consider an estimation $\pm \alpha(Z\alpha)^6$ for higher order magnetic-loop effects as a conservative one. It is below $10^{-10}$ at $Z < 6$, leads to $\Delta g \sim 1 \times 10^{-10}$ for carbon and $\Delta g \sim 6 \times 10^{-10}$ for oxygen. Our results are collected in the Table 1. The other sources of theoretical uncertainty, as explained in Refs. [6,17], are numerical error of evaluation of the one-loop self-energy contribution and estimation of unknown higher-order two-loop terms.

The experimental results for $g$ are available for the ions of carbon and oxygen with a relative uncertainty of $2 \times 10^{-9}$ [1,2] being limited by our knowledge of the electron mass [12]. However, their ratio is free of this uncertainty

\[
g(^{12}_{\text{C}}\text{+})/g(^{16}_{\text{O}}\text{+}) = 1.0004972731(15). \quad (13)
\]

and is in a fair agreement with the theoretical prediction

\[
g(^{12}_{\text{C}}\text{+})/g(^{16}_{\text{O}}\text{+}) = 1.0004972733(3). \quad (14)
\]

Calculation of the next-to-leading term of the magnetic-loop effects is in progress.

Acknowledgements

We are grateful to Peter Mohr, Günther Werth, Vladimir Ivanov for stimulating discussions. We thank the School of Physics at the University of New South Wales for their warm hospitality during the stay when this work has been done. The work of S.G.K. was supported in part by RFBR grant 00-02-16718, and the work of A.I.M. by UNSW.

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