Reflections on dyadic compacta

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Abstract

Let $M$ be an elementary submodel of the universe of sets, and $(X, T)$ a topological space in $M$. Let $X_M$ be $X \cap M$ with topology generated by $\{U \cap M : U \in T \cap M\}$. Let $D$ be the two-point discrete space. Suppose the least cardinal $\kappa$ of a basis for $X_M$ is a member of $M$, and $X_M$ is an uncountable continuous image of $D^\kappa$. Then $X = X_M$ if either $0^\theta$ does not exist or $\kappa$ is less than the first inaccessible cardinal. A corollary is that if $G_M$ is a compact group and the least cardinal of a basis for $G_M$ is in $M$, then $G = G_M$.

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As usual, we formalize the notion of an elementary submodel of $V$, the universe of all sets, by considering elementary submodels of $H(\theta)$, the set of all sets of hereditary cardinality less than $\theta$, for $\theta$ a “sufficiently large” regular cardinal. For a careful discussion of this point and for the basic facts about elementary submodels, we refer to [12]. Now let $(X, T)$ be a topological space which is a member of $M$, an elementary submodel of some $H(\theta)$ as above. We define $X_M$ to be $X \cap M$ with topology generated by $\{U \cap M : U \in T \cap M\}$. The basic results about $X_M$ are explored in [10]. In [17] we initiated the study of when $X_M$ characterizes $X$, and proved among other things that if $X_M$ is homeomorphic to the Cantor set, then $X = X_M$. In [18] and [11] this was extended to get

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Theorem 1. Let $D$ be the two-point discrete space. Suppose $X_M$ is homeomorphic to $D^\kappa$, where $\kappa \in M$ is less than the first inaccessible cardinal. Then $X = X_M$. If $0^\#$ does not exist, the inaccessibility restriction on $\kappa$ can be removed.

Here, the assertion that $0^\#$ does not exist is a consequence of Gödel’s Axiom of Constructibility and is often stated equivalently as “The Covering Lemma for $L$”. For more information on this, see [13], but in fact the only consequence of this assumption that we use is given in

Lemma 2 [14]. If $0^\#$ does not exist, then if $M$ is an elementary submodel of some $H(\theta)$ and $|M| \geq \kappa$, then $\kappa \subseteq M$.

We will implicitly assume all spaces are $T_2$. It is easy to see that if $X_M$ is $T_2$, so is $X$ [17]. In this note, we considerably improve Theorem 1 by weakening “homeomorphic” to “continuous”. More precisely, a space is said to be a dyadic compactum if it is a continuous image of $D^\kappa$ for some $\kappa$. We shall prove

Theorem 3. Suppose $X_M$ is an uncountable dyadic compactum and $w(X_M)$ (the least cardinal of a base for $X_M$) is less than the first inaccessible cardinal and is a member of $M$. Then $X_M = X$. If $0^\#$ does not exist, the inaccessibility restriction on $w$ can be removed.

An interesting consequence of this is

Corollary 4. Suppose $X_M$ is a compact topological group and $w(X_M) \in M$ is less than the first inaccessible cardinal. Then $X_M = X$. If $0^\#$ does not exist, the inaccessibility restriction on $w$ can be removed.

This is because compact groups are dyadic [15] and, by homogeneity and Baire category, uncountable.

Uncountability is needed, however, in Theorem 3, for consider $X = \omega_1 + 1$ and $M$ a countable elementary submodel. Then $X_M$ is homeomorphic to $\alpha + 1$ for some $\alpha < \omega_1$, $X_M \neq X$; claim $X_M$ is dyadic. Let $f: \omega \to \alpha + 1$ be a bijection. Let $\{C_n\}_{n<\omega}$ be a sequence of disjoint copies of the Cantor set. The one-point compactification of $\bigcup_{n<\omega} C_n$ is dyadic [4] and the map that sends $C_n$ to $\{f(n)\}$ and the point at infinity to $\alpha$ is continuous, so $\alpha + 1$ is dyadic.

In [11] we explored the general question of what additional properties on a compact $X_M$ imply that $X = X_M$; the proof of Theorem 3 uses the methods of [18] and [11] in conjunction with the technology for dyadic compacta developed by B. Efimov in a series of papers, in particular [4–7]. The following result is key.

Lemma 5 [7]. If $Y$ is dyadic and either $Y$ is uncountable and $w(Y) = \aleph_0$, or else $\text{cf}(w(Y)) > \omega$, then $Y$ maps onto $I^{w(Y)}$, where $I = [0, 1]$.

Let $\kappa = w(X_M)$. Then, unless $\omega = \text{cf}(\kappa) < \kappa$, $X_M$ maps onto $I^\kappa$. As in Efimov’s work, the case when $\text{cf}(\kappa)$ is countable and less than $\kappa$ requires considerably more care and
effort, so we first consider the case when \( cf(\kappa) \) is uncountable. (The case when \( \kappa = \aleph_0 \) is done in [17] and [11].) As in [11], under the hypotheses of Theorem 3, we can conclude that \( 2^\kappa \subseteq M \). Sketching the proof of that for the convenience of the reader, we first note that \( X \) is compact.

**Lemma 6** [9]. If \( Y_M \) is compact, so is \( Y \), and \( Y_M \) is a continuous image of \( Y \) under the map \( \Pi \) defined by \( \Pi^{-1}(y) = \bigcap \{ U \in T \cap M : y \in U \} \).

This gives us a map from \( X \) onto \( I^* \), whence by elementarity we get a map from \( X_M \) onto \( (I^*)_M \) and indeed one onto \( I_M \). Since \( I_M \) is dense in \( I \), the compactness of \( I_M \) implies \( I = I_M \). It follows, again by compactness, that \( (I^*)_M = I^* \cap M \). We need only prove that \( \kappa \subseteq M \) to conclude that \( I^* \cap M = I^* \) and hence \( 2^\kappa \subseteq M \). This can be proved by induction for \( \kappa \) less than the first inaccessible [18,11]. It will also follow from Lemma 2 if we assume \( 0^\kappa \) does not exist, for if not, repeating the proof in [11] of Theorem 1, there is a least \( \alpha < \kappa \) such that \( \alpha \subseteq M \) but \( \notin M \). Since \( |M| \geq |I^* \cap M| \geq 2^{\kappa \cap M}, |M| \geq 2^{\alpha |} \geq |\alpha|^+ \). Then \( |\alpha|^+ \subseteq M \), so \( \alpha \in M \). Once we have \( 2^\kappa \subseteq M \), by Theorem 8 below we will get \( X = X_M \).

To establish Theorem 8 we need:

**Lemma 7** [11]. Suppose \( Y_M \) is compact, \( \chi(Y) \leq \lambda \), and \( \lambda \subseteq M \). Then \( Y_M = Y \).

To prove this, one observes that the map \( \Pi \) defined above in this case is one-to-one and hence is the identity homeomorphism.

We will use Lemma 7 in proving the following generalization of Theorem 5.1 of [11], where the case of \( \lambda \) countable was proved.

**Theorem 8.** Suppose \( Y_M \) is compact and \( \chi(Y_M) \leq \lambda \). If \( 2^\lambda \subseteq M \), then \( Y = Y_M \).

**Proof.** By Lemma 7, it suffices to prove \( \chi(Y) < \lambda \). Suppose not. Then by, for example, [3], \( Y \) has a subspace \( Z \) of size \( \lambda^+ \) such that \( \chi(Z) > \lambda \). By elementarity, since \( 2^\lambda \subseteq M \) implies \( \lambda \subseteq M \), we may take \( Z \subseteq M \). As in [11], it suffices to show \( \chi(Z_M) \geq \chi(Y_M) \). Without loss of generality then, we may take \( Y = Z \).

Recall \( t(Y) \) is the least cardinal \( \tau \) such that whenever \( y \in A \subseteq Y \), there is a \( B \subseteq A \), \( |B| \leq \tau \), such that \( y \in B \). Again the case of \( \lambda \) countable in the following result was proved in [11]. \( \square \)

**Lemma 9.** Suppose \( Y_M \) is compact and \( t(Y_M) \leq \lambda \), where \( \lambda^+ \subseteq M \). Then \( t(Y) \leq \lambda \).

The proof is as in [11], using elementarity and the characterization of \( t \) in compact spaces in terms of free sequences. We can now prove Theorem 8 as in [11]. For since \( 2^{\lambda} \subseteq M \), Lemma 9 applies, so \( Y = Z = \bigcup \{ \overline{E} : E \in [Z]^\lambda \} \). By elementarity, since \( Z \cap M = Z \), \( Y_M = \bigcup \{ \overline{E}_M : E \in [Z]^\lambda \cap M \} \). It suffices to show \( \overline{E}_M = \overline{E} \), for then \( Y_M = \bigcup \{ \overline{E} : E \in [Z]^\lambda \cap M \} \). But \( Z \subseteq M \) and \( \lambda \subseteq M \), so \( [Z]^\lambda \subseteq M \). \( \|Z\|^\lambda \leq \lambda^+ \leq 2^\lambda \), so \( [Z]^\lambda \subseteq M \) so \( [Z]^\lambda \cap M = [Z]^\lambda \) and then \( Y_M = Y \).

To prove \( \overline{E}_M = \overline{E} \), we need to generalize yet another result from [11] by proving...
Lemma 10. Suppose $Y_M$ is compact and $d(Y_M) \leq \lambda$ (where $d$ of a space $Z$ is the least cardinal of a dense subspace of $Z$). If $2^\lambda \subseteq M$, then $Y = Y_M$.

Proof. As in [11], the proof divides into two cases. First assume $Y$ maps onto $I^{2^\lambda}$. Then, as usual, we conclude $2^\lambda \subseteq M$. Since $d(Y_M) \leq \lambda$, $w(Y_M) \leq 2^\lambda$. Since $2^\lambda \subseteq M$, as in [17] we conclude that $Y$ has no left- or right-separated subspace of size $(2^\lambda)^+$ since $Y_M$ does not, so $|Y| \leq 2^\lambda$. But then by compactness, $w(Y) \leq 2^\lambda$, whence by Lemma 7, $Y = Y_M$.

The other case is when $Y$ does not map onto $I^{2^\lambda}$. We need a result of Šapirovskiǐ, proved, e.g., in [8]. Let $\rho(Y)$ be the number of regular open subsets of $Y$.

Lemma 11. If $Y$ is compact and does not map onto $I^{\kappa^+}$, then $\rho(Y) \leq \kappa^{c(X)}$.

Dyadic compacta satisfy the countable chain condition, i.e., $c(Y_M) = \aleph_0$, since powers of $D$ do. Since $\omega_1 \subseteq M$, by elementarity $Y$ also satisfies the countable chain condition. Since $Y$ does not map onto $I^{2^\lambda}$, it certainly does not map onto $I^{(2^\lambda)^+}$, so $\rho(Y) \leq (2^\lambda)^0 = 2^\lambda$. But then $w(Y) \leq 2^\lambda$, so by Lemma 7, we are done with the proof of Lemma 10. □

This completes the proof of Theorem 3 for the case when $X_M$ has countable weight or weight of uncountable cofinality. In dealing with the case of $\kappa > \text{cf}(\kappa) = \omega$, where $\kappa = w(X)$, we are handicapped by the fact that $X_M$ and hence $X$ may not be assumed to map onto $D^{w(X)}$. However we will still be able to use the ideas of the uncountable cofinality proof, thanks to Efimov’s analysis of dyadic spaces with countably cofinal uncountable weight. Efimov [6] provided the first sentence of the following lemma:

Lemma 12. If $w(X) > \text{cf}(w((X))) = \omega$, where $X$ is dyadic, then either $X$ maps onto $I^{w(X)}$ or $X = \bigcup_{n<\omega} F_n$, where each $F_n$ is closed, $w(F_n) < w(X)$, and the $F_n$’s are increasing via inclusion. Moreover, we may take the $F_n$’s so that

$$\sum_{n<\omega} w(\text{int} F_n) = w(X).$$

The first alternative given by Lemma 12 is dealt with exactly as for uncountable cofinality, so we consider the second alternative. Before dealing with dyadic compacta in general, it is interesting to note that we already have enough information to deduce Corollary 4, because we can show the second alternative cannot occur for compact groups. For suppose it did. By Lemma 12 and the Baire Category Theorem, there is a closed $F \subseteq G$ with $\text{int} F \neq 0$ and $w(F) < w(G)$. Let $g \in \text{int} F$. Then $\chi(g) \leq w(F)$. Groups are homogeneous, so $\chi(G) = \chi(g)$. But in his survey [2], Comfort notes (p. 1158), that for a compact group $G$, $\pi\chi(G) = w(G)$. ($\pi\chi(x) = \min\{\lambda: \text{there is a collection } U \text{ of open sets, } |U| = \lambda, \text{ such that every open set containing } x \text{ includes a member of } U\}$). $\pi\chi(X) = \sup\{\pi\chi(x): x \in X\}$. $U$ is called a local $\pi$-base for $x$. Since $\pi\chi(G) \leq \chi(G) \leq w(G)$, $\chi(G) = w(G)$, which yields a contradiction.

The advantage of having the $\text{int} F_n$’s in Lemma 12 is that closures of open subsets of dyadic compacta are themselves dyadic compacta [4]. On the way to proving the new part of Lemma 12, note that by the Baire Category Theorem, some $F_n$, hence all $F_n$
from some point on, has non-empty interior. Thus without loss of generality, we may assume each of the $F_n$’s has non-empty interior. Claim then $U = \bigcup_{n<\omega} \text{int} F_n$ is dense in $X$. If not, there is a non-empty open $V \subseteq X - U$. Then $V = \bigcup_{n<\omega} F_n \cap V$. Again by Baire Category, for some $n$, $F_n \cap V$ has non-empty interior in $V$ and hence in $X$. But $\text{int}(F_n \cap V) \subseteq \text{int} F_n \subseteq X - V$, contradiction. Now $\pi$-weight = weight for dyadic compacta [16], so it suffices to show $\sum_{n<\omega} \pi(\text{int} F_n) = \pi(X)$. ($\pi(X) = \min \{\lambda: \text{there is a collection } \mathcal{U} \text{ of non-empty open sets, } |\mathcal{U}| = \lambda, \text{ such that each non-empty open set includes}\}$ a limit of smaller cardinals, each of which is included in $X$ by breaking up $X$ into a union of smaller compacta, and then using these to obtain

Lemma 13. Suppose $X$ is a dyadic compactum and $w = cf(w(X)) < w(X)$. Then either $X$ maps onto $I^{w(X)}$ or there exist dyadic compacta $(X_n)_{n<\omega}$ such that each $X_n \subseteq X$, $\sum_{n<\omega} w(X_n) = w(X)$, and each $X_n$ maps onto $I^{w(X_n)}$.

Proof. By induction. This is clear for $\aleph_0$ by Lemmas 5 and 12. Suppose true for all dyadic compacta with weight $< w(X)$. If $X$ does not map onto $I^{w(X)}$, apply Lemma 12 to decompose $X$. Let $X_n = \text{int} F_n - X$. Each $X_n$ either maps onto $I^{w(X_n)}$ or else, by induction hypothesis, there exist dyadic bicompacta $(X_{nk})_{k<\omega}$ included in $X_n$ such that $\sum_{k<\omega} w(X_{nk}) = w(X_n)$ and each $X_{nk}$ maps onto $I^{w(X_{nk})}$. Either there is a sequence $(X_{ni})_{i<\omega}$ with supremum $w(X_n)$: $i < \omega = w(X)$ satisfying the first alternative, or there is one satisfying the second. In either case, we obtain the desired conclusion.$\square$

Assume then that there are dyadic compacta $H_n$, $n < \omega$, such that $H_n \subseteq X_M$, $w(X_M) = \sum_{n<\omega} w(H_n)$ and $H_n$ maps onto $I^{w(H_n)}$. Since $X$ maps onto $X_M$, for each $n$, some closed subset of $X$ maps onto $H_n$. So for each $n$, some closed subset of $X$ maps onto $I^{w(H_n)}$. By the usual argument, it follows that if either $0^\sharp$ does not exist or $w(X_M)$ is less than the first inaccessible, then $2^{w(H_n)} \subseteq M$. Thus $\sum_{n<\omega} 2^{w(H_n)} = 2^{w(X_M)} \subseteq M$.

With the preliminary work done, we now can start directly attacking the countable cofinality case of Theorem 3. Our aim is to get $X \subseteq M$. That will suffice, since $X_M$ is a weaker $T_2$ topology on the compact space $X$. We will get $|X| \subseteq M$ and hence $X \subseteq M$ by breaking up $X$ into a union of smaller compacta, and then using these to obtain $|X|$ as a limit of smaller cardinals, each of which is included in $M$. Now for the details.

We will be able to get $X \subseteq M$ if we can show that $|X| \leq 2^{w(X_M)}$. Let

$S_\kappa = \{x \in X: \pi_X(x) < \kappa\}$.

Since we may assume $X_M$ and hence $X$ does not map onto $I^\kappa$, where $\kappa = w(X_M)$, the following lemma is relevant.

Lemma 14. If $X$ is compact and does not map onto $I^\kappa$, then $S_\kappa$ is $G_\delta$-dense in $X$, i.e., $S_\kappa$ meets every non-empty $G_\delta$ in $X$. 


This is proved in [8, 3.20] for the case of open sets rather than $G_δ$’s but the only use of openness is that every non-empty open set includes a non-empty closed $G_δ$. But the same is true for non-empty $G_δ$’s.

Since $κ ∈ M$ and $cf(κ) = ω ⊆ M$, there is a sequence in $M$ of smaller cardinals in $M$, $\{κ_n\}_{n<ω}$, cofinal in $κ$. Let

$$K_n = \{x : πχ(x) ⩽ κ_n\}.$$ 

Then $X = \bigcup_{n<ω} K_n$. For suppose not. Then $G = \bigcap_{n<ω}(X - K_n) ≠ 0$, so $G \cap S_κ ≠ 0$, contradiction.

We will show that in fact each $|K_n| ⩽ 2^{<κ}$ by showing that $K_n = (K_n)_M ⊆ X_M$, and that $|(K_n)_M| ⩽ 2^{<κ}$. Note that $K_n \in M$ since it is definable from $X$ and $κ_n$. In order to prove that $K_n = (K_n)_M$, we first establish two general results about the elementary submodel topology.

**Lemma 15.** Suppose $Y$ is a topological space in $M$. If $πχ(y, Y) < λ$, for each $y ∈ Z ⊆ Y$, where $Z$ and $λ ∈ M$, then also $πχ(y, Y_M) < λ$, for all $y ∈ Z_M$.

**Proof.** Note $Z_M$ is a subspace of $Y_M$. Let $y ∈ Z_M$. There is a local $π$-base $P$ for $y$ in $Y$, with $|P| < λ$. We may take $P ∈ M$. Then claim $\{P \cap M : P ∈ P\}$ is a local $π$-base for $y$ in $Y_M$. For given a basic open $U \cap M$, $U ∈ M$, $U$ open in $Y$ with $y ∈ U \cap M$, there is a $P ∈ P$ with $P ⊆ U$. By elementarity, we may take $P ∈ M$. Then $P \cap M \subseteq U \cap M$ as required. □

**Lemma 16.** If $K ∈ M$, $K$ a subset of a space $Y ∈ M$, then $K_M = K \cap M = K_M$ where the second and third closures are taken in $Y_M$.

**Proof.** The last equality is clear. To prove the first, $y ∈ K_M$ if and only if $y ∈ K \cap M$ if and only if $y ∈ M$ and every open set about $y$ in $M$ meets $K$. But by elementarity, since $y$ and $K ∈ M$, that’s if and only if every open set containing $y$ which lies in $M$ meets $K$ in $M$, which is if and only if $y ∈ K \cap M$ in $Y_M$. □

Thus, getting back to the $K_n$’s, we have $(K_n)_M = K_n \cap M$. Now since $X_M$ is compact, so is $(K_n)_M$. We will show that $2^{w((K_n)_M)} ⩽ 2^{<κ}$ for then, by Lemma 8 it will follow that $(K_n)_M = K_n$, as required.

Again, we need results of Efimov:

**Lemma 17** [4]. If $Y$ is a dyadic bicom pact with a dense subspace of points of first countability, then $Y$ is metrizable.

**Lemma 18** [5]. If $Y$ is a dyadic bicom pact and $Z ⊆ Y$ and $πχ(z, Y) ⩽ λ ≥ 80$ for all $z ∈ Z$, then $w(Z) ⩽ λ$.

Let $I_n = \{x ∈ K_n \cap M : πχ(x, X_M) < 80\}$. Then each point in $I_n$ is isolated. Taking closures in $X_M$, $K_n \cap M = (K_n - I_n) \cap M \cup I_n$. By Lemma 18, the weight of the first term is $⩽ κ_n$; on the other hand, the weight of the second is countable. The point is that as the
closure of an open subspace of a dyadic compactum, it is dyadic, whence it is metrizable by Lemma 17. By Arhangel’skiǐ’s addition theorem [1] then, \( w(K_n \cap M) \leq \kappa_n + \aleph_0 = \kappa_n < \kappa \). But then \( 2^{w(K_n \cap M)} \leq 2^{\kappa_n} \leq 2^{\kappa} \). Thus \((K_n)_M = K_n \cap M = K_n\) and \( |K_n| \leq 2^{w(K_n)} \leq 2^{\kappa}\), and so \( |X| \leq 2^{\kappa} \). This completes the proof of Theorem 3.

Remarks. It perhaps would be more natural to simply assume that some \( D^\kappa \) maps onto \( X_M \), rather than \( w(X_M) \in M \), but I do not see how to prove this. However, note that if \( w(X_M) \) is definable, e.g., \( N_1, N_{753} \), etc., then \( w(X_M) \in M \). Some restriction on \( \kappa \) is required in Theorem 3, since in [18] I showed that if there is a 2-huge cardinal, there are \( \kappa \) and \( M \) such that \((D^\kappa)_M \) is compact but not equal to \( D^\kappa \).

After this paper was completed, Kunen [14] considerably improved Theorem 1, replacing “inaccessible” by “measurable” and in fact larger cardinals. Since the non-existence of \( 0^\# \) implies there are no measurable cardinals, the final sentence of the theorem is rendered superfluous. Since the cardinality restrictions in Theorem 3 and Corollary 4 depend only on those of Theorem 1, they can be improved correspondingly.

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References