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Applied Mathematics Letters 17 (2004) 677–682
Applied Mathematics Letters
www.elsevier.com/locate/aml

Some New Variants of Newton's Method

A. Y. ÖZBAN
Department of Mathematics
Atilim University
TR-06836, İncek-Ankara, Turkey
ozbany@yahoo.com

(Received June 2002; revised and accepted August 2003)

Abstract—Some new variants of Newton's method based on harmonic mean and midpoint integration rule have been developed and their convergence properties have been discussed. The order of convergence of the proposed methods are three. In addition to numerical tests verifying the theory, a comparison of the results for the proposed methods and some of the existing ones have also been given.

Keywords—Newton's method, Iteration function, Order of convergence, Function evaluations.

1. INTRODUCTION

We consider the problem of finding a real zero α of a function $f : I \subseteq R \to R$. This zero can be determined as a fixed point of some iteration function g by means of the one-point iteration method

$$x_{n+1} = g(x_n), \qquad n = 0, 1, \dots,$$

where x_0 is the starting value. The best known and the most widely used example of these types of methods is the classical Newton's method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, \dots. \tag{1}$$

It converges quadratically to simple zeros and linearly to multiple zeros. In the literature, some of its modifications have been introduced in order to accelerate it or to get a method with a higher order of convergence (see, e.g., [1–3]). The method developed by Fernando *et al.* [1], which can be called as trapezoidal Newton's or arithmetic mean Newton's method, suggests some other variants of Newton's method. In Section 2, we briefly give some definitions and concepts. In Section 3, we introduce some new variants of Newton's method in addition to some of the known ones, and a detailed convergence analysis of the proposed methods has been supplied in Section 4. Numerical results and comparisons have been given in the last section.

The author is grateful to the anonymous referee for many helpful comments and constructive criticism.

0893-9659/04/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX
doi:10.1016/j.aml.2003.08.009

2. DEFINITIONS

DEFINITION 2.1. *(See [4].) If the sequence $\{x_n \mid n \geq 0\}$ tends to a limit α in such a way that*

$$\lim_{x_n \to \alpha} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = C$$

for some $C \neq 0$ and $p \geq 1$, then the order of convergence of the sequence is said to be p, and C is known as the asymptotic error constant.

When $p = 1$ the *convergence is linear*, while for $p = 2$ and $p = 3$ the sequence is said to *converge quadratically* and *cubically*, respectively. The value of p is called the *order of convergence* of the method which produces the sequence $\{x_n \mid n \geq 0\}$. Let $e_n = x_n - \alpha$. Then the relation

$$e_{n+1} = Ce_n^p + O\left(e_n^{p+1}\right)$$

is called the *error equation* for the method, p being the order of convergence.

DEFINITION 2.2. *(See [1].) Let α be a zero of the function f and suppose that x_{n-1}, x_n, and x_{n+1} are three successive iterations closer to the zero α. Then the computational order of convergence ρ can be approximated using the formula*

$$\rho \approx \frac{\ln|(x_{n+1} - \alpha)\,/\,(x_n - \alpha)|}{\ln|(x_n - \alpha)\,/\,(x_{n-1} - \alpha)|}.$$

3. DESCRIPTION OF THE METHODS

3.1. Classical Newton's (CN) Method and Arithmetic Mean Newton's (AN) Method

Let α be a simple zero of a sufficiently differentiable function f and consider the numerical solution of the equation $f(x) = 0$. It is clear that

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)\,dt. \tag{2}$$

Suppose we interpolate f' on the interval $[x_n, x]$ by the constant $f'(x_n)$. Then $(x - x_n)f'(x_n)$ provides an approximation for the indefinite integral in (2) and by taking $x = \alpha$ we obtain

$$0 \approx f(x_n) + (\alpha - x_n)f'(x_n),$$

and hence, a new approximation x_{n+1} to α is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

which is Newton's method for $n = 0, 1, \ldots$. On the other hand, if we approximate the indefinite integral in (2) by the trapezoidal rule and take $x = \alpha$, we obtain

$$0 \approx f(x_n) + \frac{1}{2}(\alpha - x_n)\left(f'(x_n) + f'(\alpha)\right),$$

and therefore, a new approximation x_{n+1} to α is given by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_{n+1})}.$$

If the $(n + 1)^{\text{th}}$ value of Newton's method is used on the right-hand side of the above equation to overcome the implicity problem, then

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_{n+1})}, \qquad \text{where } z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3}$$

is obtained which is, for $n = 0, 1, 2, \ldots$, the trapezoidal Newton's method of Fernando *et al.* [1]. Let us rewrite equation (3) as

$$x_{n+1} = x_n - \frac{f(x_n)}{\left(f'(x_n) + f'(z_{n+1})\right)/2}, \qquad n = 0, 1, \ldots. \tag{4}$$

So, this variant of Newton's method can be viewed as obtained by using arithmetic mean of $f'(x_n)$ and $f'(z_{n+1})$ instead of $f'(x_n)$ in Newton's method defined by (1). Therefore, we call it *arithmetic mean Newton's* (AN) *method*.

3.2. New Variants of Newton's Method

In (4), if we use the harmonic mean instead of the arithmetic mean we get

$$x_{n+1} = x_n - \frac{f(x_n)\left(f'(x_n) + f'(z_{n+1})\right)}{2f'(x_n)f'(z_{n+1})}, \qquad n = 0, 1, \dots, \tag{5}$$

which we call *harmonic mean Newton's* (HN) *method*. On the other hand, if we approximate the indefinite integral in (2) by the midpoint integration rule instead of the trapezoidal rule and take $x = \alpha$, we obtain

$$0 \approx f(x_n) + (\alpha - x_n) f'\left(\frac{x_n + \alpha}{2}\right)$$

and in this case a new approximation x_{n+1} to α is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'((x_n + x_{n+1})/2)}.$$

Again using the $(n+1)^{\text{th}}$ value of Newton's method on the right-hand side of the last equation to overcome the implicity problem, we obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f'((x_n + z_{n+1})/2)}, \qquad \text{where } z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{6}$$

which, for $n = 0, 1, 2, \dots$, we call *midpoint Newton's* (MN) *method*.

4. CONVERGENCE ANALYSIS

THEOREM 4.1. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq R \to R$ for an open interval I. If x_0 is sufficiently close to α, then the methods defined by (5) and (6) converge cubically.*

PROOF. Let α be a simple zero of f. Since f is sufficiently differentiable, by expanding $f(x_n)$ and $f'(x_n)$ about α we get

$$f(x_n) = f'(\alpha)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots\right], \tag{7}$$

and

$$f'(x_n) = f'(\alpha)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + \cdots\right], \tag{8}$$

where $c_k = (1/k!) f^{(k)}(\alpha)/f'(\alpha)$, $k = 2, 3, \dots$ and $e_n = x_n - \alpha$. Since the terms in square brackets are polynomials in terms of e_n, direct division gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2\left(c_2^2 - c_3\right) e_n^3 + O\left(e_n^4\right),$$

and hence, for z_{n+1} given in (6) we have

$$z_{n+1} = \alpha + c_2 e_n^2 + 2\left(c_3 - c_2^2\right) e_n^3 + O\left(e_n^4\right). \tag{9}$$

Again expanding $f'(z_{n+1})$ about α and using (9) we obtain

$$\begin{aligned} f'(z_{n+1}) &= f'(\alpha) + (z_{n+1} - \alpha) f''(\alpha) + \frac{(z_{n+1} - \alpha)^2}{2!} f'''(\alpha) + \cdots \\ &= f'(\alpha) + \left[c_2 e_n^2 + 2\left(c_3 - c_2^2\right) e_n^3 + O\left(e_n^4\right)\right] f''(\alpha) + O\left(e_n^4\right) \\ &= f'(\alpha)\left[1 + 2c_2^2 e_n^2 + 4\left(c_2 c_3 - c_2^3\right) e_n^3 + O\left(e_n^4\right)\right]. \end{aligned} \tag{10}$$

From (8) and (10), we get

$$f'(x_n)f'(z_{n+1}) = (f'(\alpha))^2 \left(1 + 2c_2e_n + \left(2c_2^2 + 3c_3\right) e_n^2 + 4(c_2c_3 + c_4)e_n^3 + O\left(e_n^4\right)\right)$$

and

$$f'(x_n) + f'(z_{n+1}) = 2f'(\alpha) \left(1 + c_2e_n + \left(c_2^2 + \frac{3}{2} c_3\right) e_n^2 + 2\left(c_2c_3 - c_2^3 + c_4\right) e_n^3 + O\left(e_n^4\right)\right).$$

So, using (7)

$$f(x_n)\left(f'(x_n) + f'(z_{n+1})\right) = 2\left(f'(\alpha)\right)^2 \left(e_n + 2c_2e_n^2 + \left(2c_2^2 + \frac{5}{2} c_3\right) e_n^3 + O\left(e_n^4\right)\right).$$

Hence,

$$\frac{f(x_n)\left(f'(x_n) + f'(z_{n+1})\right)}{2f'(x_n)f'(z_{n+1})} = e_n - \frac{1}{2} c_3e_n^3 + O\left(e_n^4\right),$$

and therefore,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)\left(f'(x_n) + f'(z_{n+1})\right)}{2f'(x_n)f'(z_{n+1})} \\ &= x_n - \left(e_n - \frac{1}{2} c_3e_n^3 + O\left(e_n^4\right)\right), \end{aligned}$$

or subtracting α from both sides of this equation we get

$$e_{n+1} = \frac{1}{2} c_3e_n^3 + O\left(e_n^4\right),$$

which shows that harmonic Newton's method is of third order for simple zeros.

Now consider the midpoint Newton's method defined by (6). It is clear that

$$\frac{x_n + z_{n+1}}{2} - \alpha = \frac{e_n}{2} + \frac{1}{2} \left[c_2e_n^2 + \left(2c_3 - 2c_2^2\right) e_n^3\right] + O\left(e_n^4\right). \tag{11}$$

By expanding $f'((x_n + z_{n+1})/2)$ about α and using (11) we obtain

$$\begin{aligned} f'\left(\frac{x_n + z_{n+1}}{2}\right) &= f'(\alpha) + \left\{\frac{e_n}{2} + \frac{1}{2} \left[c_2e_n^2 + \left(2c_3 - 2c_2^2\right) e_n^3\right]\right\} f''(\alpha) \\ &+ \frac{1}{8} \left\{e_n^2 + 2c_2e_n^3\right\} f'''(\alpha) + \frac{1}{48} e_n^3 f''''(\alpha) + O\left(e_n^4\right). \end{aligned} \tag{12}$$

Hence, direct division of (7) by (12) gives us

$$\frac{f(x_n)}{f'((x_n + z_{n+1})/2)} = e_n + \left(\frac{1}{4} c_3 - c_2^2\right) e_n^3 + O\left(e_n^4\right).$$

Using this in (6) and subtracting α from both sides we get

$$x_{n+1} - \alpha = x_n - \alpha - \left(e_n + \left(\frac{1}{4} c_3 - c_2^2\right) e_n^3 + O\left(e_n^4\right)\right)$$

or

$$e_{n+1} = \left(c_2^2 - \frac{1}{4} c_3\right) e_n^3 + O\left(e_n^4\right),$$

which shows that the midpoint Newton's method converges cubically.

Table 1.

$f(x)$	x_0	N				COC				NOFE			
		CN	AN	HN	MN	CN	AN	HN	MN	CN	AN	HN	MN
a	−0.5	97	6	52	10	2.00	3.00	3.00	3.00	194	18	156	30
	1	5	3	3	3	2.00	3.00	3.00	3.00	10	9	9	9
	2	5	4	3	4	2.00	3.00	3.00	3.00	10	12	9	12
b	1	6	4	3	4	2.00	3.00	ND	3.00	12	12	9	12
	3	6	3	3	4	2.00	ND	ND	3.00	12	9	9	12
c	2	5	4	4	3	2.00	3.00	3.00	3.01	10	12	12	9
	3	6	4	4	4	2.00	3.00	3.00	ND	12	12	12	12
d	1	4	2	3	3	2.00	2.75	3.00	3.00	8	6	9	9
	1.7	4	3	3	3	2.00	3.01	3.00	3.00	8	9	9	9
	−0.3	5	4	4	4	2.00	3.00	3.00	3.00	10	12	12	12
e	0	9	15	5	6	2.00	3.00	3.00	3.00	18	45	15	18
	1.5	7	5	4	5	2.00	3.00	3.00	3.00	14	15	12	15
	2.5	6	4	3	4	2.00	3.00	3.01	3.00	12	12	9	12
	3	6	4	4	4	2.00	3.00	3.00	3.00	12	12	12	12
	3.5	7	5	4	5	2.00	3.00	3.00	3.00	14	15	12	15
f	1.5	16	467	7	59	2.00	3.00	3.00	3.00	32	1401	21	177
	2.5	7	5	5	5	2.00	3.00	3.00	3.00	14	15	15	15
	3.0	9	6	5	6	2.00	3.00	3.00	3.00	18	18	15	18
	3.5	10	7	6	6	2.00	3.00	3.00	3.00	20	21	18	18
g	1.5	27	NC	13	NC	2.00	–	3.00	–	54	–	39	–
	2.5	8	5	5	5	2.00	3.00	3.00	3.00	16	15	15	15
	3.0	9	6	5	6	2.00	3.00	3.00	3.00	18	18	15	18
	3.5	12	8	7	7	2.00	3.00	3.00	3.00	24	24	21	21
h	−2	8	6	5	5	2.00	3.00	3.00	3.00	16	18	15	15
	−3	14	9	8	9	2.00	3.00	3.00	3.00	28	27	24	27
i	3.5	12	8	7	7	2.00	3.00	3.00	3.00	24	24	21	21
	3.25	8	6	5	5	2.00	3.00	3.00	3.00	16	18	15	15
j	−0.5	16	11	9	10	2.00	3.00	3.00	3.00	32	33	27	30
k	−2	11	7	6	7	2.00	3.00	3.00	3.00	22	21	18	21
l	1.4	84	55	44	52	1.00	1.00	1.00	1.00	168	165	132	156
	−1	119	79	63	74	1.00	1.00	1.00	1.00	238	237	189	222

5. NUMERICAL RESULTS AND CONCLUSIONS

In this section, we present the results of some numerical tests to compare the efficiencies of the methods. We employed CN method, AN method of Fernando *et al.* [1], and HN and MN methods that we developed. Numerical computations reported here have been carried out in a MAPLE V environment rounding to 64 significant digits. The stopping criterion has been taken as $|x_{n+1} - \alpha| + |f(x_{n+1})| < 10^{-14}$, and the following test functions have been used (see [1,5]).

(a) $x^3 + 4x^2 - 10$, $\alpha = 1.365230013414097$,
(b) $\sin^2 x - x^2 + 1$, $\alpha = 1.404491648215341$,
(c) $x^2 - e^x - 3x + 2$, $\alpha = 0.2575302854398608$,
(d) $\cos x - x$, $\alpha = 0.7390851332151607$,
(e) $(x-1)^3 - 1$, $\alpha = 2$,
(f) $(x-1)^6 - 1$, $\alpha = 2$,
(g) $(x-1)^8 - 1$, $\alpha = 2$,
(h) $xe^{x^2} - \sin^2 x + 3\cos x + 5$, $\alpha = -1.207647827130919$,

(i) $e^{x^2+7x-30} - 1$, $\alpha = 3$,
(j) $\prod_{m=0}^{4}(x - (1 + 0.1m))$, $\alpha = 1$,
(k) $\prod_{m=0}^{5}(x - (m + 1))$, $\alpha = 1$,
(l) $(x - 2)^3(x + 2)^4$, $\alpha_1 = 2$, and $\alpha_2 = -2$.

In Table 1, we give the number of iterations (N) and the number of function evaluations (NOFE) required to satisfy the stopping criterion, and the computational order of convergence (COC) taken as: ρ_n if $100|\rho_n - \rho_{n-1}|/\min\{\rho_n, \rho_{n-1}\} \leq 10$ and ρ_1 if $N = 2$ where $\rho_n = \ln|(x_{n+1} - \alpha)/(x_n - \alpha)|/\ln|(x_n - \alpha)/(x_{n-1} - \alpha)|$. The notation ND (not defined) has been used if $100|\rho_n - \rho_{n-1}|/\min\{\rho_n, \rho_{n-1}\} > 10$.

All numerical results are in accordance with the theory and the basic advantage of the variants of Newton's method based on *means* or *integration methods* that they do not require the computation of second- or higher-order derivatives although they are of third order (for simple zeros).

As far as the numerical results are considered, for most of the cases HN method requires the least number of function evaluations while AN and MN methods require almost the same number of function evaluations. The AM and MN methods did not converge (NC) to the root for the function in (g) for $x_0 = 1.5$ in 1000 iterations. Although the roots of the function in (l) are not simple, and hence all the methods converge linearly, we have used it to see the effectiveness of the methods in case of multiple roots. While the CN and HN methods converge quite slowly for the function in (a) for $x_0 = -0.5$, the AN and MN methods, especially AN method, converge quite slowly for the function in (f) for $x_0 = 1.5$, which shows the importance of the initial approximations and the functions although all the methods stem from the same fact: approximation to the same indefinite integral.

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