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Some Properties of a Class of Analytic Functions

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Let $\mathcal{A}$ be the class of functions $f(z)$, which are analytic in the unit disc $U = \{z : |z| < 1\}$, with $f(0) = f'(0) - 1 = 0$. In this paper we introduce a new subclass $Q_\alpha(\beta)$ of $\mathcal{A}$ and study some properties of $Q_\alpha(\beta)$. Hence we extend the results of MacGregor, Chen, and Chichra. We also get a new univalent criterion and some interesting properties of Hadamard products.

1. INTRODUCTION

Let

$$P_\beta = \{p(z) : p(z) \text{ is analytic in } U \text{ and } p(0) = 1, \operatorname{Re}(p(z)) > \beta, \beta < 1\},$$

and define $\phi(a, c; z)$ by

$$\phi(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \qquad z \in U, \qquad c \neq 0, -1, -2, \ldots,$$

and we shall use notation $\phi(a, c) = \phi(a, c; z)$ for convenience. We define $L(a, c)$ by

$$L(a, c) f = \phi(a, c) * f(z), \qquad f(z) \in \mathcal{A}, \tag{1}$$

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where $(\lambda)_n = \Gamma(n+\lambda)/\Gamma(\lambda)$. "$*$" denotes the Hadamard product.

The operator $L(a, c)$ is called the Carlson–Shaffer operator and was applied extensively to the theory of generalized hypergeometric functions in [8] and [9] where some very strong results were achieved. It is known in [1] that $L(a, c)$ maps A into itself. If $a \neq 0, -1, -2, \ldots$, then $L(a, c)$ has a continuous inverse $L(c, a)$ and $L(c, a)$ maps $\mathcal{A}$ into $\mathcal{A}$ injectively. Clearly, $L(a, a)$ is the unit operator and

$$L(a,c) = L(a,b)L(b,c) = L(b,c)L(a,b), \qquad b, c \neq 0, -1, -2, \ldots.$$

Also, if $c > a > 0$, we have

$$L(a,c)f = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-2}(1-u)^{c-a-1} f(uz)\, du. \tag{2}$$

If a, b, and c are real numbers other than $0, -1, -2, \ldots$, the hypergeometric series

$$F(a,b,c;z) = 1 + \frac{a \cdot b}{1 \cdot c} z + \cdots + \frac{a \cdot (a+1) \cdot b \cdot (b+1)}{1 \cdot 2 \cdot c \cdot (c+1)} z^2 + \cdots$$

represents an analytic function in U [2, p. 281], and if $c > b > 0$, we have

$$F(a,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt. \tag{3}$$

Also, in the theory of Gauss hypergeometric functions, it is well-known that

$$\begin{aligned} F(1,c,c+1;-1) &= c \sum_{n=0}^{\infty} \frac{(-1)^n}{c+n} \\ &= \frac{1}{2}\left[\Psi\left(\frac{1}{2}c + \frac{1}{2}\right) - \Psi\left(\frac{1}{2}c\right)\right] \end{aligned}$$

in terms of the psi (or digamma) function

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log \Gamma(z) = \int_1^z \Psi(\zeta)\, d\zeta.$$

Now, we introduce the following new family:

$$Q_\alpha(\beta) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right] > \beta, \alpha \geq 0, \beta < 1 \right\}.$$

It is easy to see that $Q_{\alpha_1}(\beta) \subset Q_{\alpha_2}(\beta)$ for $\alpha_1 > \alpha_2 \geq 0$. Thus, for $\alpha \geq 1$, $0 \leq \beta < 1$, $Q_\alpha(\beta) \subset Q_1(\beta) = \{f \in \mathcal{A}: \mathrm{Re}(f'(z)) > \beta, 0 \leq \beta < 1\}$, and hence $Q_\alpha(\beta)$ is a univalent class. MacGregor [3] proved that $Q_0(0)$ was a univalent class in $|z| < \sqrt{2} - 1$. Ming-Po Chen [4] obtained the sharp radius of univalency for $Q_0(\beta)$ $(0 \leq \beta < 1)$. Later, Chichra [5] got the sharp univalent radius for $Q_\alpha(0)$ $(0 < \alpha < 1)$.

In our present paper, we extend the above results. We also get a new univalent criterion. Finally, we obtain some interesting properties of Hadamard product for $Q_\alpha(\beta)$.

2. Preliminaries

We get the following Lemma 1 from [6] and Lemma 2 from [4].

Lemma 1. *If the function $f(z)$ is analytic in $|z| < R$ and $\mathrm{Re}(f'(z)) > 0$ for $|z| < R$, then $f(z)$ is univalent in $|z| < R$.*

Lemma 2. *Let the function $f(z) \in Q_0(0)$; then*

$$\mathrm{Re}(f'(z)) \geq \frac{1 - 2r - r^2}{(1 + r)^2} \quad \text{for} \quad 0 \leq r < \frac{1}{2}, \tag{4}$$

and

$$\mathrm{Re}(f'(z)) \geq \frac{-r^4}{(1 - r^2)^2} \quad \textit{for} \quad \frac{1}{2} \leq r < 1, \tag{5}$$

where $r = |z|$. The results are sharp.

Lemma 3. $z[\phi(c, c + 1)]' = c\phi(c + 1, c + 1) - (c - 1)\phi(c, c + 1)$.

Proof. Since

$$\phi(c, c + 1) = \sum_{n=0}^{\infty} \frac{(c)_n}{(c + 1)_n} z^{n+1} = \sum_{n=0}^{\infty} \frac{c}{n + c} z^{n+1},$$

we have

$$z[\phi(c, c+1)]' = \sum_{n=0}^{\infty} \frac{c}{n+c}(n+1)z^{n+1}$$
$$= c\sum_{n=0}^{\infty} \frac{n+c-(c-1)}{n+c} z^{n+1}$$
$$= c\phi(c+1, c+1) - (c-1)\phi(c, c+1).$$

LEMMA 4. *If $p(z) \in P(\beta)$ $(\beta < 1)$, then*

$$\operatorname{Re}[p(z) - p(uz)] \geq -2(1-\beta)\frac{(1-u)r}{(1+r)(1+ur)},$$

where $r = |z|$, $0 < u < 1$. This result is sharp.

Proof. Note

$$\operatorname{Re}\left(\frac{1}{1-z} - \frac{1}{1-uz}\right) \geq \frac{-(1-u)r}{(1+r)(1+ur)}. \tag{6}$$

In fact, let $z = r\cos\theta + ir\sin\theta$, $\theta \in [0, 2\pi]$; then

$$\operatorname{Re}\left(\frac{1}{1-z} - \frac{1}{1-uz}\right) = \operatorname{Re}\left[\frac{1}{1-r\cos\theta - ir\sin\theta} - \frac{1}{1-ur\cos\theta - iur\sin\theta}\right]$$
$$= \frac{1-r\cos\theta}{1-2r\cos\theta+r^2} - \frac{1-ur\cos\theta}{1-2ur\cos\theta+u^2r^2}$$
$$= \frac{(1-u)[r\cos\theta - r^2(1+u) + ur^3\cos\theta]}{(1-2r\cos\theta+r^2)(1-2ur\cos\theta+u^2r^2)}$$
$$\geq \frac{(1-u)[r\cos\pi - r^2(1+u) + ur^3\cos\pi]}{(1-2r\cos\pi+r^2)(1-2ur\cos\pi+u^2r^2)}$$
$$= \frac{-(1-u)r}{(1+r)(1+ur)}.$$

And if $p(z) \in P_\beta$, by the Herglotz formula

$$p(z) = \int_{|x|=1} \frac{1+(1-2\beta)xz}{1-xz}\, d\mu(x), \tag{7}$$

where $\mu(x)$ is a probability measure on $|x| = 1$. Thus, by (6) and (7), we have

$$\begin{aligned}
&\operatorname{Re}[p(z) - p(uz)]\\
&\quad = \operatorname{Re}\left[\int_{|x|=1} \frac{1 + (1 - 2\beta)xz}{1 - xz}\, d\mu(x) - \int_{|x|=1} \frac{1 + (1 - 2\beta)xuz}{1 - xuz}\, d\mu(x)\right]\\
&\quad = \int_{|x|=1} \operatorname{Re}\left[2(1 - \beta)\left(\frac{1}{1 - xz} - \frac{1}{1 - xuz}\right)\right] d\mu(x)\\
&\quad \geq -2(1 - \beta)\frac{(1 - u)r}{(1 + r)(1 + ur)}.
\end{aligned}$$

Finally, the result is sharp for the function $p_0(z)$ defined by $p_0(z) = (1 + (1 - 2\beta)z)/(1 - z)$.

3. Main Results

THEOREM 1. *Let* $\alpha \geq 0$, $\beta < 1$. *If* $f(z) \in Q_\alpha(\beta)$, *then*

(i)

$$\operatorname{Re}(f'(z)) \geq \frac{1 + (4\beta - 2)r + (2\beta - 1)r^2}{(1 + r)^2}, \qquad 0 \leq r < \frac{1}{2}, \tag{8}$$

and

$$\operatorname{Re}(f'(z)) \geq \frac{\beta - 2\beta r^2 + (2\beta - 1)r^4}{(1 - r^2)^2}, \qquad \frac{1}{2} \leq r < 1, \tag{9}$$

for $\alpha = 0$, $\beta < 1$;

(ii)

$$\operatorname{Re}(f'(z)) \geq (2\beta - 1) + 2(1 - \beta)\frac{c}{1 + r} + 2(1 - c)(1 - \beta)F(1, c, c + 1; -r) \tag{10}$$

for $\alpha > 0$, $\beta < 1$. *Where* $c = 1/\alpha$, $r = |z|$. *The results are sharp.*

Proof. (i) Let $f(z) \in Q_0(\beta)$, $\beta < 1$, it is easy to know that $F(z) = (f(z) - \beta z)/(1 - \beta) \in Q_0(0)$. By Lemma 2, we see that (8) and (9) are true.

(ii) Let $f(z) \in Q_\alpha(\beta)$, $\alpha > 0$, $\beta < 1$, and

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = p(z). \tag{11}$$

We know that $p(z) \in P_\beta$. Using (1) and (11), we get

$$\begin{aligned} zp(z) &= (1-\alpha)f(z) + \alpha z f'(z) \\ &= L\left(\frac{1}{\alpha}+1, \frac{1}{\alpha}\right) f(z), \end{aligned}$$

that is,

$$f(z) = L\left(\frac{1}{\alpha}, \frac{1}{\alpha}+1\right)[zp(z)].$$

Let $c = 1/\alpha$; then

$$f(z) = L(c, c+1)[zp(z)]. \tag{12}$$

By Lemma 3, we have

$$\begin{aligned} zf'(z) &= \{z[\phi(c, c+1)]'\} * [zp(z)] \\ &= cL(c+1, c+1)[zp(z)] - (c-1)L(c, c+1)[zp(z)] \\ &= czp(z) - (c-1)L(c, c+1)[zp(z)]. \end{aligned} \tag{13}$$

If $c > 1$, from (2), (13), and Lemma 4, we get

$$\begin{aligned} \operatorname{Re}(f'(z)) &= \operatorname{Re}\left[cp(z) - c(c-1)\int_0^1 u^{c-1} p(uz)\, du\right] \\ &= \operatorname{Re}[p(z)] + c(c-1)\int_0^1 u^{c-1} \operatorname{Re}[p(z) - p(uz)]\, du \\ &\geq \frac{1-(1-2\beta)r}{1+r} + c(c-1)\int_0^1 u^{c-1} 2(1-\beta)\frac{-(1-u)r}{(1+r)(1+ur)}\, du \\ &= \frac{1-(1-2\beta)r}{1+r} + 2(1-\beta)c(c-1)\int_0^1 u^{c-1}\left[\frac{1}{1+r} - \frac{1}{1+ur}\right] du \\ &= (2\beta-1) + 2(1-\beta)\frac{c}{1+r} + 2c(1-c)(1-\beta)\int_0^1 \frac{u^{c-1}}{1+ur}\, du. \end{aligned} \tag{14}$$

If $0 < c \le 1$, from (2) and (13), we have

$$\begin{aligned}\operatorname{Re}(f'(z)) &= \operatorname{Re}\left[cp(z) - c(c-1)\int_0^1 u^{c-1}p(uz)\,du\right]\\ &= c\operatorname{Re}[p(z)] + c(1-c)\int_0^1 u^{c-1}\operatorname{Re}[p(uz)]\,du \qquad (15)\\ &\ge c\frac{1-(1-2\beta)r}{1+r} + c(1-c)\int_0^1 u^{c-1}\frac{1-(1-2\beta)ur}{1+ur}\,du\\ &= (2\beta-1) + 2(1-\beta)\frac{c}{1+r} + 2c(1-c)(1-\beta)\int_0^1 \frac{u^{c-1}}{1+ur}\,du.\end{aligned}$$

By using (3), from (14) and (15), we deduce (10).

That the result is sharp follows from

$$f_0(z) = L\left(\frac{1}{\alpha}, \frac{1}{\alpha} + 1\right)\frac{z + (1-2\beta)z^2}{1-z} \in Q_\alpha(\beta).$$

This completes the proof of our theorem.

From [4] we know that there exists $f(z) \in Q_0(\beta)$ ($\beta < 1$) such that $f(z)$ is not univalent. But in the following theorem, we shall prove that $f(z) \in Q_\alpha(\beta)$ is univalent for $0 < \alpha < 1$ and $\beta > \beta_0$, where $0 < \beta_0 < 1$. And we shall also prove that $f(z) \in Q_\alpha(\beta)$ is univalent for the case that $\alpha > 1$ and β is larger than some negative number.

THEOREM 2. *If $\alpha > 0$ and $\beta_0 < \beta < 1$, then $f(z) \in Q_\alpha(\beta)$ is univalent, where*

$$\beta_0 = 1 - \frac{1}{2 - c - 2(1-c)F(1, c, c+1; -1)} \qquad \text{for } c = \frac{1}{\alpha}, \tag{16}$$

and the constant β_0 cannot be replaced by any smaller one.

Proof. We note that the right sides in both (14) and (15) are decreasing with respect to r. So if $f(z) \in Q_\alpha(\beta)$ and $\alpha > 0$, $\beta < 1$, by (9) we have

$$\begin{aligned}\operatorname{Re}(f'(z)) &\ge (2\beta-1) + (1-\beta)c + 2(1-c)(1-\beta)F(1, c, c+1; -1)\\ &= 1 - (1-\beta)[2 - c - 2(1-c)F(1, c, c+1; -1)], \qquad (17)\end{aligned}$$

where $c = 1/\alpha$. But

$$F(1, c, c+1; -1) = c\int_0^1 \frac{u^{c-1}}{1+u}\,du$$
$$= \frac{1}{2} + \int_0^1 \frac{u^c}{(1+u)^2}\,du.$$

From

$$0 < \int_0^1 \frac{u^c}{(1+u)^2}\,du < \int_0^1 \frac{1}{(1+u)^2}\,du = \frac{1}{2},$$

we get

$$\frac{1}{2} < F(1, c, c+1; -1) < 1,$$

so

$$2 - c - 2(1-c)F(1, c, c+1; -1) > 0. \qquad (18)$$

Thus, by (17) we have

$$\mathrm{Re}(f'(z)) > 1 - (1-\beta_0)[2 - c - 2(1-c)F(1, c, c+1; -1)] = 0$$

for $\beta > \beta_0$. By Lemma 1, we have proved that $f(z) \in Q_\alpha(\beta)$ is univalent for $\alpha > 0$ and $\beta > \beta_0$. Note $\frac{1}{2} < F(1, c, c+1; -1) < 1$; it is easy to see that $f(z) \in Q_\alpha(\beta)$ is univalent in the case either $0 < \alpha < 1$ and $0 < \beta < 1$ or $\alpha \geq 1$ and $\beta \leq 0$.

In order to show that the result is sharp, we consider the following function

$$f_0(z) = L(c, c+1)\left[\frac{z + (1-2\beta)z^2}{1-z}\right], \qquad \alpha > 0, \beta < 1, c = \frac{!}{\alpha}.$$

It is easy to see that $f_0(z) \in Q_\alpha(\beta)$. By (13), we get

$$zf_0'(z) = \frac{z + (1-2\beta)z^2}{1-z} - (c-1)L(c, c+1)\left[\frac{z + (1-2\beta)z^2}{1-z}\right],$$

so

$$f_0'(-1) = 1 - (1 - \beta)[2 - c - 2(1 - c)F(1, c, c + 1; -1)].$$

If $\beta < \beta_0$, by (18) we have

$$f_0'(-1) < 1 - (1 - \beta_0)[2 - c - 2(1 - c)F(1, c, c + 1; -1)] = 0,$$

and $f_0'(0) = 1$. So there exists a point $z = -r_0$, such that $f'(-r_0) = 0$. Then $f_0(z)$ is not univalent when $\beta < \beta_0$ as required.

COROLLARY 1. *If $f(z) \in \mathcal{A}$ and*

$$\operatorname{Re}\left[\frac{f(z)}{z} + f'(z)\right] > 2\left[1 - \frac{1}{4(1 - \ln 2)}\right] \simeq 0.372,$$

then $f(z)$ is univalent. The result is sharp.

COROLLARY 2. *If $f(z) \in \mathcal{A}$ and*

$$\operatorname{Re}\left[2f'(z) - \frac{f(z)}{z}\right] > 1 - \frac{2}{3 - 2F(1, \frac{1}{2}, \frac{3}{2}; -1)},$$

then $f(z)$ is univalent. The result is sharp.

By using Lemma 1 and Theorem 1, we have the following two theorems.

THEOREM 3. *If $f(z) \in Q_0(\beta)$ and $\beta < 0$, then $f(z)$ is univalent in $|z| < (\sqrt{4\beta^2 - 6\beta + 2} + 2\beta - 1)/(1 - 2\beta)$. The result is sharp.*

THEOREM 4. *Let $\alpha > 0$ and $\beta < \beta_0$. If $f(z) \in Q_\alpha(\beta)$, then $f(z)$ is univalent in $|z| < r_0$. Here β_0 is defined by* (16) *and r_0 is the smallest positive root of the equation*

$$2\beta - 1 + 2(1 - \beta)\frac{c}{1 + r} + 2(1 - c)(1 - \beta)F(1, c, c + 1; -r) = 0,$$

where $c = 1/\alpha$. The result is sharp.

Remark. Taking $\beta = 0$, $0 < \alpha < 1$ in Theorem 4, we have the result of [5]. From [7], we have the following lemma.

LEMMA 5. *If $p_1(z) \in P_\alpha$, $p_2(z) \in P_\beta$, $\alpha, \beta < 1$, then $p_1 * p_2(z) \in P_t$, where $t = 1 - 2(1 - \alpha)(1 - \beta)$. The result is sharp.*

THEOREM 5. *Let $\alpha > 0$, $\beta_1, \beta_2 < 1$; if $f(z) \in Q_\alpha(\beta_1)$, $g(z) \in Q_\alpha(\beta_2)$, then $f * g(z) \in Q_\alpha(t)$, where*

$$t = 1 - 4(1-\beta_1)(1-\beta_2) + 4(1-\beta_1)(1-\beta_2)F(1, c, c+1; -1) \quad \text{for } c = \frac{1}{\alpha}.$$

The result is sharp.

Proof. Let $f(z) \in Q_\alpha(\beta_1)$, $g(z) \in Q_\alpha(\beta_2)$; by (12) we have

$$f(z) = L(c, c+1)[zp_1(z)], \qquad g(z) = L(c, c+1)[zp_2(z)],$$

where $c = 1/\alpha$, $p_1(z) \in P_{\beta_1}$, $p_2(z) \in P_{\beta_2}$. Then, using Lemma 5 we obtain

$$\begin{aligned} f(z) * g(z) &= L(c, c+1)[zp_1(z)] * L(c, c+1)[zp_2(z)] \\ &= L(c, c+1)[L(c, c+1)(zp_1(z) * zp_2(z))] \\ &= L(c, c+1)[L(c, c+1)(z(p_1(z) * p_2(z)))] \\ &= L(c, c+1)[L(c, c+1)(zp_\lambda(z))], \end{aligned} \tag{19}$$

where $\lambda = 1 - 2(1-\beta_1)(1-\beta_2)$, $p_\lambda(z) = p_1(z) * p_2(z)$.

Note that

$$\begin{aligned} \operatorname{Re}\left[\frac{L(c, c+1)(zp_\lambda(z))}{z}\right] &= \operatorname{Re}\left[c\int_0^1 u^{c-1} p_\lambda(uz)\,du\right] \\ &= c\int_0^1 u^{c-1} \operatorname{Re}[p_\lambda(uz)]\,du \\ &> c\int_0^1 u^{c-1} \frac{1-(1-2\lambda)u}{1+u}\,du \\ &= 2\lambda - 1 + 2(1-\lambda)F(1, c, c+1; -1) \qquad (20) \\ &= 1 - 4(1-\beta_1)(1-\beta_2) \\ &\quad + 4(1-\beta_1)(1-\beta_2)F(1, c, c+1; -1) \\ &= t. \end{aligned}$$

From (19) and (20), we have

$$f * g(z) = L(c, c+1)[zp_t(z)],$$

where $p_t(z) = L(c, c+1)(zp_\lambda(z))/z$ and $\operatorname{Re}(p_t(z)) > t$. So we know that $f * g(z) \in Q_\alpha(t)$.

That the result is sharp is obtained by $f(z) = L(c, c+1)[z + (1 - 2\beta_1)z^2)/(1 - z)$ and $g(z) = L(c, c+1)(z + (1 - 2\beta_2)z^2)/(1 - z)$. The proof is completed.

COROLLARY 3. *Let $\alpha > 0$ and $\beta < 1$. If $f(z) \in Q_\alpha(\beta)$, $g(z) \in Q_\alpha(\beta)$, and $\beta \geq \beta_0$, where*

$$\beta_0 = 1 - \frac{1}{4[1 - F(1, c, c+1; -1)]} \qquad \text{for } c = \frac{1}{\alpha},$$

then $f * g(z) \in Q_\alpha(\beta)$, that is, $Q_\alpha(\beta)$ is closed under the Hadamard product when $\beta > \beta_0$.

Proof. By Theorem 5 and

$$1 - 4(1-\beta)^2 + 4(1-\beta)^2 F(1, c, c+1; -1) \geq \beta \quad \text{if and only if} \quad \beta \geq \beta_0,$$

we get the required result immediately.

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