An almost sure central limit theorem for products of sums of partial sums under association

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Let \( \{X_n, n \geq 1\} \) be a strictly stationary positively or negatively associated sequence of positive random variables with \( E X_1 = \mu > 0 \), and \( \text{Var} X_1 = \sigma^2 < \infty \). Denote \( S_n = \sum_{i=1}^{n} X_i \), \( T_n = \sum_{i=1}^{n} S_i \) and \( \gamma = \sigma / \mu \) the coefficient of variation. Under suitable conditions, we show that

\[
\forall x \lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left( \frac{2^{k}}{k!} \frac{\prod_{j=1}^{k} T_j}{(k+1)! \mu^k} \right)^{1/(\gamma \sigma_1 \sqrt{k})} \leq x = F(x) \quad \text{a.s.,}
\]

where \( \sigma_1^2 = 1 + \frac{2}{\sigma^2} \sum_{j=2}^{\infty} \text{Cov}(X_1, X_j) \). \( F(\cdot) \) is the distribution function of the random variables \( e^{\sqrt{10/3} N} \) and \( N \) is a standard normal random variable.

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1. Introduction and main results

Let \( \{X_n, n \geq 1\} \) be a sequence of identically distributed random variables and define the partial sums \( S_n = \sum_{i=1}^{n} X_i \), the sums of partial sums \( T_n = \sum_{i=1}^{n} S_i = \sum_{i=1}^{n} (n+1-i)X_i \) for \( n \geq 1 \). In the past century, the partial sums \( S_n \) has been the most popular topic for study. Such well-known classic CLT, ASCLT, and LIL are known for describing the asymptotic behavior of the partial sums.

The limit theorems of products of \( \prod_{j=1}^{n} S_j \) was initiated by Arnold and Villaseñor [1] who obtained the following version of the CLT for a sequence \( \{X_n, n \geq 1\} \) of i.i.d. exponential r.v.’s with the mean equal to one

\[
\left( \prod_{j=1}^{n} S_j \right)^{1/\sqrt{n}} \xrightarrow{d} e^{\sqrt{2} \mathcal{N}}.
\]

Here and in the sequel, \( \mathcal{N} \) is a standard normal random variable. Their proof was heavily based on a very special property of exponential distributions. Later on, Rempala and Wesolowski [16] proved the following

**Theorem A.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. positive square integrable random variables with \( E X_1 = \mu > 0 \), and \( \text{Var} X_1 = \sigma^2 < \infty \) and the coefficient of variation \( \gamma = \sigma / \mu \). Then

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Theorem 1.2. Then \( EH \) \( \text{Var} \) variables. So it is necessary to study the sum of partial sums.

Theorem 1.1. Expressions. The following are our main results.

Starting with Brosamler [3], Khurelbaatar [7] and Schatte [18], in the past decade, many authors investigated the almost sure central limit theorem (ASCLT) for partial sums of random variables. Very recently, Khurelbaatar and Rempala [6] proved the following ASCLT of products \( \prod_{j=1}^{n} S_j \) for i.i.d. sequence.

\[
\left( \prod_{j=1}^{n} \frac{S_j}{n! \mu^n} \right)^{1/\sqrt{n}} \xrightarrow{d} e^{\sqrt{2N}}.
\]  

(1.1)

Recently, Qi [14] and Lu and Qi [11] extended (1.1) by assuming that the underlying distribution \( F \) is in the domain of attraction of a stable law with exponent \( \alpha \in (1, 2) \) and \( \alpha = 1 \), respectively.

Theorem B. Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. positive square integrable random variables with \( EX_1 = \mu > 0 \), and \( Var \) \( X_1 = \sigma^2 < \infty \) and the coefficient of variation \( \gamma = \sigma/\mu \). Then

\[
\forall x \lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left\{ \left( \prod_{j=1}^{k} \frac{S_j}{k! \mu^k} \right)^{1/(\sqrt{2k})} \leq x \right\} = F(x) \quad \text{a.s.}
\]

Here and in the sequel, \( I[\cdot] \) denotes indicator function and \( F(x) \) is the distribution function of the random variables \( e^{\sqrt{2N}} \).

The study of the sum of partial sums was initiated by Resnick [15] and Arnold and Villaseñor [1] who obtained the CLT for sums of records. As we know, the sum of exponential records is the sum of partial sums of exponential random variables. So it is necessary to study the sum of partial sums.

In this paper, we shall study the ASCLT of products \( \prod_{j=1}^{n} T_j \) under association assumption. For a finite index set \( I \), the r.v.s \( \{X_t, t \in I\} \) are said to be negatively associated (NA) if for any disjoint nonempty subsets \( A \) and \( B \) of \( I \), and any coordinatewise increasing function \( G \) and \( H \) with \( G : R^A \to R \) and \( H : R^B \to R \) and \( EG^2(Z_i, i \in A) < \infty \), \( EH^2(Z_j, j \in B) < \infty \), it holds that \( Cov(G(X_i, i \in A), H(X_j, j \in B)) \leq 0 \). These r.v.s are said to be positively associated (PA), if for any real-valued coordinatewise increasing function \( G \) and \( H \) defined on \( R^1 \), \( Cov(G(X_i, i \in I), H(X_j, j \in I)) \geq 0 \), provided \( EG^2(Z_i, i \in I) < \infty \), and \( EH^2(Z_i, i \in I) < \infty \). If \( I \) is not finite, the r.v.s \( \{X_t, t \in I\} \) are said to be NA or PA, if any finite sub-collection is a set of NA or PA r.v.s, respectively. The first definition is due to Joag-Dev and Proschan [5], the second to Esary, Proschan and Walkup [4]. Association has been found application in reliability theory, in statistical mechanics and in multivariate statistical analysis, the interested reader is referred to Roussas [17].

Two random variables \( X \) and \( Y \) are said to be negative (resp. positive) quadrant dependent (NQD) (resp. (PQD)), if \( P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y) \leq 0 \) (resp. \( \geq 0 \)) for all \( x, y \in R \). A sequence \( \{X_k, k \in Z\} \) is said to be linear negative (resp. positive) quadrant dependent (LNQD) (resp. (LPQD)) if for any disjoint finite subsets \( A, B \subset Z \) and any positive real numbers \( r_j, \sum_{j \in A} r_j X_j \) and \( \sum_{j \in B} r_j X_j \) are NQD (resp. (PQD)). The definition of LNQD (LPQD) can be found in Li and Wang [9]. Throughout the paper, \( C \) denotes a positive constant, which may take different values whenever it appears in different expressions. The following are our main results.

Theorem 1.1. Let \( \{X_n, n \geq 1\} \) be a strictly stationary NA (PA) sequence of positive random variables with \( EX_1 = \mu > 0 \), and \( Var \) \( X_1 = \sigma^2 < \infty \). Denote \( S_n = \sum_{i=1}^{n} X_i, T_n = \sum_{i=1}^{n} S_i \) and \( \gamma = \sigma/\mu \) the coefficient of variation. Assume that

(C1) \( |Cov(X_1, X_{n+1})| = O(n^{-1}(\log n)^{-2-\epsilon}) \), for some \( \epsilon > 0 \),

(C2) \( \sigma_1^2 = 1 + \frac{2}{\sigma^2} \sum_{j=1}^{\infty} \text{Cov}(X_1, X_j) > 0 \).

Then

\[
\forall x \lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left\{ \left( \frac{2k^{1/2} T_j}{k!(k+1)! \mu^k} \right)^{1/(\sqrt{2k})} \leq x \right\} = F(x) \quad \text{a.s.}
\]  

(1.2)

Here and in the sequel, \( F(x) \) is the distribution function of the random variables \( e^{\sqrt{2N}} \) and \( N \) is a standard normal random variable.

Theorem 1.2. Let \( \{X_n, n \geq 1\} \) be a strictly stationary LNQD (LPQD) sequence of positive random variables with \( EX_1 = \mu > 0 \), and \( Var \) \( X_1 = \sigma^2 < \infty \). Denote \( S_n = \sum_{i=1}^{n} X_i, T_n = \sum_{i=1}^{n} S_i \) and \( \gamma = \sigma/\mu \) the coefficient of variation. Assume that (C1) and (C2) hold. Then (1.2) holds.

The following corollary is the special case of Theorem 1.1.

Corollary 1.3. Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. positive square integrable random variables with \( EX_1 = \mu > 0 \), and \( Var \) \( X_1 = \sigma^2 < \infty \) and the coefficient of variation \( \gamma = \sigma/\mu \). Then
\[
\forall x \lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k!} \left\{ \frac{\sum_{j=1}^{k} T_j}{\sqrt{k!(k+1)!\mu^k}} \right\}^{1/(\gamma \sqrt{k})} \leq x = F(x) \text{ a.s.}
\]

2. Proof

Let \( b_{k,n} = \sum_{i=1}^{n-k} \frac{1}{i} \), \( c_{k,n} = \sum_{i=1}^{n-k} \frac{2(i+1-k)}{(i+1)\mu} \), \( d_{k,n} = \frac{n+1-k}{n+1} \), \( k \leq n \) with \( b_{k,n} = c_{k,n} = d_{k,n} = 0 \), \( k > n \). Let \( S_n = \sum_{k=1}^{n} Y_k \) and \( S_{n,n} = \sum_{k=1}^{n} c_{k,n} Y_k \), where \( Y_k = (X_k - \mu)/\sigma \), \( k \geq 1 \). Let \( \sigma_n^2 = \text{Var}(S_{n,n}) \). Note that
\[
c_{k,n} = 2(b_{k,n} - d_{k,n}).
\]

We will need the following two properties.

(H1) Increasing functions defined on disjoint subsets of a set of NA (resp. PA) random variables are NA (resp. PA);
(H2) (Hoeffding equality) For any absolutely continuous functions \( f \) and \( g \) on \( \mathbb{R}^1 \) and for any random variables \( X \) and \( Y \) satisfying \( EF^2(X) + EG^2(Y) < \infty \), we have
\[
\text{Cov}(f(X), g(Y)) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f'(x)g'(y) \left\{ P(X > x, Y > y) - P(X > x)P(Y > y) \right\} \, dx \, dy.
\]

The proof of Theorem 1.1 is based on the following lemmas.

**Lemma 2.1.** Let \( \{X_n, \ n \geq 1\} \) be an associated sequence of random variables with \( EX_n = 0 \), and \( \{a_{ni}, \ 1 \leq i \leq n, \ n \geq 1\} \) be an array of real numbers such that \( \sup_n \sum_{i=1}^{n} a_{ni}^2 < \infty \) and \( \max_{1 \leq i \leq n} |a_{ni}| \to 0 \) as \( n \to \infty \). Assume that \( \sum_{j \geq 1} \text{Cov}(X_k, X_j) / j \to 0 \) as \( n \to \infty \) uniformly for \( k \geq 1 \). If \( \max_{1 \leq i \leq n} a_{ni} X_i \to 0 \) and \( \{X_n^2\} \) is an uniformly integrable family, then \( \sum_{i=1}^{n} a_{ni} X_i \to N(0, 1) \).

**Proof.** If \( \{X_n, \ n \geq 1\} \) are NA, see Theorem 3.1 on P386 from Liang et al. [8], if \( \{X_n, \ n \geq 1\} \) are PA, see Theorem 2.3 on P446 from Peligrad and Utev [13]. \( \square \)

**Lemma 2.2.** Let \( \{X_n, \ n \geq 1\} \) be a strictly stationary associated associated sequence of random variables with \( EX_1 = 0 \), and \( EX_1^2 < \infty \). Assume that \( S_n = \sum_{i=1}^{n} X_i \), \( T_n = \sum_{i=1}^{n} S_i \), and \( 0 < \sigma_2 = EX_1^2 + 2 \sum_{j=2}^{\infty} EX_1 X_j < \infty \), then for \( 0 < p < 2/3 \),
\[
n^{-1/p} T_n \to 0 \quad \text{a.s. as} \ n \to \infty.
\]

**Proof.** Under the conditions of this lemma, it is easy to obtain that
\[
\lim_{n \to \infty} \frac{ET_n^2}{n^2} = \frac{\sigma_2^2}{3}
\]

Let \( n_k = \alpha^k \), where \( \alpha > \max\{1, p/(2-3p)\} \). By (2.2), we have
\[
\sum_{k=1}^{\infty} P\left\{ |T_n| \geq \varepsilon n_k^{1/p} \right\} \leq \sum_{k=1}^{\infty} \frac{ET_n^2}{n_k^{2/p}} \leq \sum_{k=1}^{\infty} C/(\varepsilon^2 k^{2(2/p-3)}) < \infty.
\]

By Borel–Cantelli lemma, we know
\[
n_k^{-1/p} T_n \to 0 \quad \text{a.s. as} \ k \to \infty.
\]

In addition, we have
\[
\sum_{k=1}^{\infty} P\left\{ \max_{n \leq n < n_k+1} |T_n - T_{n_k}| \geq \varepsilon \right\} \leq \sum_{k=1}^{\infty} \frac{E \max_{n \leq n < n_k+1} |T_n - T_{n_k}|^2}{\varepsilon^2 n_k^{2/p}}
\]
\[
= \sum_{k=1}^{\infty} \frac{E \max_{n \leq n < n_k+1} \left\{ \sum_{i=n_k+1}^{n} (n+1-i)X_i + (n-n_k)S_{n_k} \right\}^2}{\varepsilon^2 n_k^{2/p}}
\]
\[
\leq C \sum_{k=1}^{\infty} \frac{\sum_{i=n_k+1}^{n} (n+1-i)^2}{\varepsilon^2 n_k^{2/p}} + C \sum_{k=1}^{\infty} \frac{\max_{n \leq n < n_k+1} (n-n_k)^2 S_{n_k}^2}{\varepsilon^2 n_k^{2/p}}
\]
\[
\leq C \sum_{k=1}^{\infty} \frac{\sum_{i=n_k+1}^{n} (n+1-i)^2 \rho_k}{\varepsilon^2 n_k^{2/p}} + C \sum_{k=1}^{\infty} \frac{(n_k+1-n_k)^2 \rho_k}{\varepsilon^2 n_k^{2/p}}
\]

\[
\leq C \sum_{k=1}^{\infty} \frac{\rho_k}{\varepsilon^2 n_k^{2/p}} + C \sum_{k=1}^{\infty} \frac{(n_k+1-n_k)^2 \rho_k}{\varepsilon^2 n_k^{2/p}}
\]

\[
\leq C \sum_{k=1}^{\infty} \frac{\rho_k}{\varepsilon^2 n_k^{2/p}} + C \sum_{k=1}^{\infty} \frac{(n_k+1-n_k)^2 \rho_k}{\varepsilon^2 n_k^{2/p}}
\]

\[
\leq C \sum_{k=1}^{\infty} \frac{\rho_k}{\varepsilon^2 n_k^{2/p}} + C \sum_{k=1}^{\infty} \frac{(n_k+1-n_k)^2 \rho_k}{\varepsilon^2 n_k^{2/p}}
\]

\[
\leq C \sum_{k=1}^{\infty} \frac{\rho_k}{\varepsilon^2 n_k^{2/p}} + C \sum_{k=1}^{\infty} \frac{(n_k+1-n_k)^2 \rho_k}{\varepsilon^2 n_k^{2/p}}
\]

\[
\leq C \sum_{k=1}^{\infty} \frac{\rho_k}{\varepsilon^2 n_k^{2/p}} + C \sum_{k=1}^{\infty} \frac{(n_k+1-n_k)^2 \rho_k}{\varepsilon^2 n_k^{2/p}}
\]
By Borel–Cantelli lemma, we know
\[ \max_{n_k < n < n_{k+1}} \frac{|T_n - T_{nk}|}{n^{1/p}} \to 0 \quad \text{a.s. as } k \to \infty. \]

For \( n_k < n < n_{k+1} \), we have
\[ \frac{|T_n|}{n^{1/p}} \leq \frac{|T_{nk}|}{n^{1/p}} + \max_{n_k \leq n < n_{k+1}} \frac{|T_n - T_{nk}|}{n^{1/p}} \to 0 \quad \text{a.s. } n \to \infty. \]

Therefore, Lemma 2.2 can be concluded at once. \( \square \)

**Lemma 2.3.** Under the conditions of Theorem 1.1, we have
\[ \frac{n!}{\left( \frac{1}{3} - 4b_{1,n} + \frac{10n}{3(n+1)} \right)^2} \to \sigma_1^2 \quad \text{a.s. } n \to \infty. \] (2.3)

**Proof.** By (H1) and the stationarity of \( \{X_n, \ n \geq 1\} \), it is easy to see that \( \{Y_n\} \) is a strictly stationary associated sequence with \( EY_1 = 0 \) and \( EY_1^2 = 1 \). By Lemma 1 on \( P_{193} \) from Khurelbaatar and Rempala [6], we know
\[ \sum_{i=1}^{n} b_{i,n}^2 = 2n - b_{1,n}. \] (2.4)

It is easy to prove that
\[ \sum_{i=1}^{n} d_{i,n}^2 = \frac{n}{3} - \frac{n}{6(n+1)}, \quad \sum_{i=1}^{n} b_{i,n}d_{i,n} = \frac{3n}{4} - \frac{n}{2(n+1)}, \] (2.5)
\[ \sum_{i=1}^{n} c_{i,n}^2 = \frac{10n}{3} - 4b_{1,n} + \frac{10n}{3(n+1)}. \] (2.6)

By (2.1), we have
\[ \begin{align*}
\sigma_n^2 &= \text{Var}(S_{n,n}) = \text{Var}\left(2 \sum_{i=1}^{n} (b_{i,n} - d_{i,n})Y_i\right) = 4 \text{Var}\left(\sum_{i=1}^{n} b_{i,n}Y_i\right) - 8 \text{Cov}\left(\sum_{i=1}^{n} b_{i,n}Y_i, \sum_{j=1}^{n} d_{j,n}Y_j\right) + 4 \text{Var}\left(\sum_{i=1}^{n} d_{i,n}Y_i\right).
\end{align*} \] (2.7)

By (2.4)–(2.7), note that (2.3) is equivalent to the following
\[ \begin{align*}
\frac{\text{Var}(\sum_{i=1}^{n} b_{i,n}Y_i)}{2n - b_{1,n}} &\to \sigma_1^2 \quad \text{a.s. } n \to \infty, \quad \text{(2.8)}
\end{align*} \]
\[ \frac{\text{Cov}(\sum_{i=1}^{n} b_{i,n}Y_i, \sum_{j=1}^{n} d_{j,n}Y_j)}{\frac{3n}{4} - \frac{n}{2(n+1)}} &\to \sigma_1^2 \quad \text{a.s. } n \to \infty, \quad \text{(2.9)}
\end{align*} \]
\[ \frac{\text{Var}(\sum_{i=1}^{n} d_{i,n}Y_i)}{\frac{n}{3} - \frac{n}{6(n+1)}} &\to \sigma_1^2 \quad \text{a.s. } n \to \infty. \quad \text{(2.10)} \]

By Lemma 2.3 on \( P_{370} \) from Li and Wang [10], we can get (2.8) immediately. Now we estimate (2.10).
Thus by (2.5), we have

\[
\left| \frac{\text{Var}(\sum_{i=1}^{n} d_{i,n} Y_i)}{\binom{n}{2} / \binom{n+1}{2}} - \sigma^2 \right| \leq \frac{2}{n} \sum_{j=2}^{n} \sum_{i=1}^{j-1} \frac{(n+1-i)^2}{(n+1)^2} |\text{Cov}(Y_1, Y_j)|
\]

\[
+ \frac{2}{n} \sum_{j=2}^{n} \sum_{i=1}^{j} d_{i,n} \frac{j-1}{n+1} |\text{Cov}(Y_1, Y_j)| + 2 \sum_{j=n+1}^{\infty} |\text{Cov}(Y_1, Y_j)|
\]

\[=: I_1 + I_2 + I_3. \tag{2.11}\]

By (C1), for some \(\epsilon > 0\), we have

\[
I_1 = \frac{2}{n} \sum_{j=2}^{n} \sum_{i=1}^{j-1} \frac{(n+1-i)^2}{(n+1)^2} |\text{Cov}(Y_1, Y_j)|
\]

\[
\leq \frac{C}{n} \frac{n}{(n+1)^2} \frac{1}{j(\log j)^{2+\epsilon}}
\]

\[
\leq \frac{C}{n} \frac{n}{(n+1)^2 \log n} \leq \frac{C}{\log n} \to 0, \quad \text{as } n \to \infty. \tag{2.13}\]

\[
I_3 \leq C (\log n)^{-1-\epsilon} \to 0, \quad \text{as } n \to \infty. \tag{2.14}\]

and note that \(d_{i,n} \leq 1\),

\[
I_2 \leq \frac{C}{n} \frac{n}{(n+1)^2} \sum_{j=2}^{n} \sum_{i=1}^{n+1-j} \frac{j-1}{n+1} \frac{1}{j(\log j)^{2+\epsilon}}
\]

\[
\leq \frac{C}{n} \frac{n}{(n+1)^2} \frac{1}{(\log j)^{2+\epsilon}}
\]

\[
\leq \frac{C}{n} \frac{n}{(n+1)^2} \frac{1}{\log n} \leq \frac{C}{\log n} \to 0, \quad \text{as } n \to \infty. \tag{2.15}\]

Then by (2.11)-(2.15), we can immediately obtain (2.10).

Finally we estimate (2.9), as the same argument of (2.11), we get

\[
\frac{\text{Cov}(\sum_{i=1}^{n} b_{i,n} Y_i, \sum_{j=1}^{n} d_{i,n} Y_j)}{\binom{n}{2} / \binom{n+1}{2}} = 1 + \frac{1}{n} \sum_{j=2}^{n} \sum_{i=1}^{n} b_{i,n} d_{i,n} \text{Cov}(Y_1, Y_j)
\]

\[- \frac{1}{n} \sum_{j=2}^{n} \sum_{i=1}^{n} b_{i,n} d_{i,n} \text{Cov}(Y_1, Y_j)
\]

\[- \frac{1}{n} \sum_{j=2}^{n} \sum_{i=1}^{n} b_{i,n} \frac{j-1}{n+1} \text{Cov}(Y_1, Y_j)
\]

\[+ \frac{1}{n} \sum_{j=1}^{n} \sum_{i=2}^{j} d_{j,n} b_{j,n} \text{Cov}(Y_1, Y_i)\]
Thus by (2.5), we have
\[
\left| \text{Cov}\left( \sum_{i=1}^{n} b_{i,n} Y_i, \sum_{j=1}^{n} d_{j,n} Y_j \right) - \sigma_i^2 \right| \leq \frac{1}{n^{(n+1)}} \sum_{j=2}^{n} \sum_{i=2}^{n} b_{i,n} d_{j,n} |\text{Cov}(Y_1, Y_j)|
\]
\[
+ \frac{1}{n^{(n+1)}} \sum_{j=2}^{n} \sum_{i=2}^{n} b_{j,n} \frac{j-1}{n+1} |\text{Cov}(Y_1, Y_j)|
\]
\[
+ \frac{1}{n^{(n+1)}} \sum_{j=2}^{n} \sum_{i=2}^{n} d_{j,n} b_{j,n} |\text{Cov}(Y_1, Y_j)|
\]
\[
+ \frac{1}{n^{(n+1)}} \sum_{j=2}^{n} \sum_{i=2}^{n} d_{j,n} b_{i,j-n-i+2} |\text{Cov}(Y_1, Y_j)| + 2 \sum_{j=n+1}^{\infty} |\text{Cov}(Y_1, Y_j)|
\]
\[
J_1 + J_2 + J_3 + J_4 + J_5.
\]

Thus by (2.5), we have
\[
\left| \text{Cov}\left( \sum_{i=1}^{n} b_{i,n} Y_i, \sum_{j=1}^{n} d_{j,n} Y_j \right) - \sigma_i^2 \right| \leq \frac{1}{n^{(n+1)}} \sum_{j=2}^{n} \sum_{i=2}^{n} b_{i,n} d_{j,n} |\text{Cov}(Y_1, Y_j)|
\]
\[
+ \frac{1}{n^{(n+1)}} \sum_{j=2}^{n} \sum_{i=2}^{n} b_{j,n} \frac{j-1}{n+1} |\text{Cov}(Y_1, Y_j)|
\]
\[
+ \frac{1}{n^{(n+1)}} \sum_{j=2}^{n} \sum_{i=2}^{n} d_{j,n} b_{j,n} |\text{Cov}(Y_1, Y_j)|
\]
\[
+ \frac{1}{n^{(n+1)}} \sum_{j=2}^{n} \sum_{i=2}^{n} d_{j,n} b_{i,j-n-i+2} |\text{Cov}(Y_1, Y_j)| + 2 \sum_{j=n+1}^{\infty} |\text{Cov}(Y_1, Y_j)|
\]
\[
:= J_1 + J_2 + J_3 + J_4 + J_5.
\]

Note that $b_{i,n} \leq \log n$, $d_{i,n} \leq 1$ and (C1), we have
\[
J_1 \leq \frac{1}{n^{(n+1)}} \sum_{j=2}^{n} (j-1) \log n \frac{1}{j!(\log j)^{2+\epsilon}} \leq \frac{Cn}{n^{(n+1)}} \frac{1}{(\log n)^{\epsilon}} \to 0, \quad \text{as } n \to \infty,
\]
\[
J_3 \leq \frac{C}{n^{(n+1)}} \sum_{i=2}^{n} i \log n \frac{1}{i!(\log i)^{2+\epsilon}} \leq \frac{Cn}{n^{(n+1)}} \frac{1}{(\log n)^{\epsilon}} \to 0, \quad \text{as } n \to \infty,
\]
\[
J_5 \leq C (\log n)^{-1-\epsilon} \to 0, \quad \text{as } n \to \infty,
\]
\[
J_2 \leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{j=1}^{n} \log n \frac{j-1}{n+1} (\log j)^{2+\epsilon}
\]
\[
\leq \frac{Cn}{n^{(n+1)}} \frac{1}{(\log n)^{\epsilon}} \to 0, \quad \text{as } n \to \infty,
\]
\[
J_4 = \frac{1}{n^{(n+1)}} \sum_{i=2}^{n} \sum_{j=1}^{n} \sum_{k=1}^{i+j-2} \frac{1}{k!} |\text{Cov}(Y_1, Y_i)|
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \frac{1}{k!} |\text{Cov}(Y_1, Y_i)|
\]
\[
= \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} |\text{Cov}(Y_1, Y_i)|
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} (\log j)^{2+\epsilon}
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} (\log j)^{2+\epsilon}
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} (\log j)^{2+\epsilon}
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} (\log j)^{2+\epsilon}
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} (\log j)^{2+\epsilon}
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} (\log j)^{2+\epsilon}
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} (\log j)^{2+\epsilon}
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} (\log j)^{2+\epsilon}
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} (\log j)^{2+\epsilon}
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} (\log j)^{2+\epsilon}
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} (\log j)^{2+\epsilon}
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} (\log j)^{2+\epsilon}
\]
\[
\leq \frac{C}{n^{(n+1)}} \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j=1}^{i+k} \sum_{j=1}^{i+j-2} \frac{1}{k!} (\log j)^{2+\epsilon}
\]
\[
\leq C (\log n)^{-1-\epsilon} \to 0, \quad \text{as } n \to \infty.
\]

Then by (2.16)–(2.22), we can immediately obtain (2.9). Therefore the proof of this lemma is completed. \(\square\)
Lemma 2.4. Let \( \{\xi_n, n \geq 1\} \) be a sequence of zero mean, uniformly bounded random variables. Assume that \( |E\xi_n\xi_l| \leq C(k/l)^{\varepsilon} \) for some \( \varepsilon > 0 \). Then
\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \xi_k \to 0 \quad \text{a.s.}
\]

Proof. See Lemma 2 on \( P_{193} \) from Khurelbaatar and Rempala [6]. \( \square \)

Lemma 2.5. Under the conditions of Theorem 1.1, we have
\[
\forall \xi \text{ such that } S_{k,k} \to N(0,1) \text{ as } n \to \infty.
\]

Proof. Note that \( \{Y_n\} \) is a strictly stationary associated sequence with \( EY_1 = 0 \) and \( EY_1^2 = 1 \). We first prove
\[
\frac{S_{k,k}}{\sqrt{\frac{10k}{3}\sigma_1}} \to N(0,1) \quad \text{as } n \to \infty.
\]

Let \( a_{ni} = c_i/n^{\sigma_i}, 1 \leq i \leq n, n \geq 1 \). Obviously, \( \text{Var}(\sum_{i=1}^{n} a_{ni} Y_i) = 1 \) and \( \sum_{i=1}^{n} |\text{Cov}(Y_1, Y_i)| \to 0 \) as \( n \to \infty \) by (C1). Note that \( \sigma_n^2 = (\frac{10n}{3} - 4b_1n + \frac{10n}{3\sigma_1})(1 + o(1)) \) from Lemma 2.3. Hence by (2.6), we have \( \sup_n \sum_{i=1}^{n} a_{ni}^2 < \infty \) and \( \max_{1 \leq i \leq n} |a_{ni}| \to 0 \) as \( n \to \infty \). Therefore (2.24) is satisfied by Lemmas 2.1 and 2.3. Let \( f(x) \) be a bounded Lipschitz function and have a Radon–Nikodym derivative \( f'(x) \) bounded by \( f' \). By (2.24), we have
\[
E f\left( \frac{S_{k,k}}{\sqrt{\frac{10k}{3}\sigma_1}} \right) \to E f\left( N(0,1) \right) \quad \text{as } n \to \infty.
\]

On the other hand, note that (2.23) is equivalent to
\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} f\left( \frac{S_{k,k}}{\sqrt{\frac{10k}{3}\sigma_1}} \right) = E f\left( N(0,1) \right) \quad \text{a.s.}
\]

from Section 2 of Peligrad and Shao [12] and Theorem 7.1 on \( P_{42} \) from Billingsley [2]. Hence, to prove (2.23), it suffices to show that
\[
T_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left[ f\left( \frac{S_{k,k}}{\sqrt{\frac{10k}{3}\sigma_1}} \right) - E f\left( \frac{S_{k,k}}{\sqrt{\frac{10k}{3}\sigma_1}} \right) \right] \to 0 \quad \text{a.s. as } n \to \infty.
\]

Note that, for \( l > k \),
\[
S_{l,l} - S_{k,k} = \left( \sum_{i=1}^{l} c_i Y_i \right) - \left( \sum_{i=1}^{k} c_i Y_i \right) = \sum_{i=k+1}^{l} (c_i - c_k) Y_i + \sum_{i=k+1}^{l} c_i Y_i.
\]

Let \( \xi_k = f\left( \frac{S_{k,k}}{\sqrt{\frac{10k}{3}\sigma_1}} \right) - E f\left( \frac{S_{k,k}}{\sqrt{\frac{10k}{3}\sigma_1}} \right), 1 \leq k \leq n \). For \( l > k \), we have
\[
|E\xi_k\xi_l| = \left| \text{Cov}\left( f\left( \frac{S_{k,k}}{\sqrt{\frac{10k}{3}\sigma_1}} \right), f\left( \frac{S_{l,l}}{\sqrt{\frac{10k}{3}\sigma_1}} \right) \right) \right|
\leq \left| \text{Cov}\left( f\left( \frac{S_{k,k}}{\sqrt{\frac{10k}{3}\sigma_1}} \right), f\left( \frac{S_{l,l}}{\sqrt{\frac{10k}{3}\sigma_1}} \right) - f\left( \frac{S_{l,l} - S_{k,k} - \sum_{i=k+1}^{l} (c_i - c_k) Y_i}{\sqrt{\frac{10k}{3}\sigma_1}} \right) \right) \right|
+ \left| \text{Cov}\left( f\left( \frac{S_{k,k}}{\sqrt{\frac{10k}{3}\sigma_1}} \right), f\left( S_{l,l} - S_{k,k} - \sum_{i=k+1}^{l} (c_i - c_k) Y_i \right) \right) \right|
= \left| \text{Cov}\left( f\left( \frac{S_{k,k}}{\sqrt{\frac{10k}{3}\sigma_1}} \right), f\left( \frac{S_{l,l}}{\sqrt{\frac{10k}{3}\sigma_1}} \right) - f\left( \sum_{i=k+1}^{l} c_i Y_i \right) \right) \right|
+ \left| \text{Cov}\left( f\left( \frac{S_{k,k}}{\sqrt{\frac{10k}{3}\sigma_1}} \right), f\left( \sum_{i=k+1}^{l} c_i Y_i \right) \right) \right|.
\]
If \( \{X_n, n \geq 1\} \) are NA, by (H1), we know \( \{S_{k,k}\} \) and \( \{\sum_{i=k+1}^{l} c_{i,j}Y_i\} \) are NA. Therefore

\[
H(x, y) := P \left( S_{k,k} > x \frac{10k}{3} \sigma_1, \sum_{i=k+1}^{l} c_{i,j}Y_i > y \frac{10l}{3} \sigma_1 \right)
- P \left( S_{k,k} > x \frac{10k}{3} \sigma_1 \right) P \left( \sum_{i=k+1}^{l} c_{i,j}Y_i > y \frac{10l}{3} \sigma_1 \right) \leq 0.
\]

for every \( x, y \in \mathbb{R}^1 \). By (H2), note that \( c_{i,j} \leq 2b_{l,i} \), \( \sum_{j=1}^{k} b_{j,k} = k \),

\[
\left| \text{Cov} \left( f \left( \frac{S_{k,k}}{\frac{10k}{3} \sigma_1} \right), f \left( \frac{\sum_{i=k+1}^{l} c_{i,j}Y_i}{\frac{10l}{3} \sigma_1} \right) \right) \right| \leq \left| \iint f'(x)f'(y)H(x, y)\,dx\,dy \right|
\]

\[
\leq -R^2 \int \int H(x, y)\,dx\,dy = \text{Cov} \left( \frac{S_{k,k}}{\frac{10k}{3} \sigma_1}, \sum_{i=k+1}^{l} c_{i,j}Y_i \right)
\]

\[
= -R^2 \frac{1}{\sqrt{\frac{10k}{3} \sigma_1}} \frac{1}{\sqrt{\frac{10l}{3} \sigma_1}} \sum_{j=1}^{k} C_{j,k} \sum_{i=k+1}^{l} c_{i,l} \text{Cov}(Y_j, Y_i)
\]

\[
\leq C \frac{\log(\frac{1}{l})}{k^{1/2} \sigma_1} \sum_{j=1}^{k} b_{j,k} \sum_{l=1}^{k} \left| \text{Cov}(Y_j, Y_i) \right| \leq C \frac{\log(\frac{1}{l})}{k^{1/2} \sigma_1} \sum_{j=1}^{k} b_{j,k} \sum_{l=1}^{k} \left| \text{Cov}(Y_j, Y_i) \right| \leq C \frac{\log(\frac{1}{l})}{k^{1/2} \sigma_1} \sum_{j=1}^{k} b_{j,k} \sum_{l=1}^{k} \left| \text{Cov}(Y_j, Y_i) \right| \leq C \left( \frac{k}{l} \right)^{1/2} \log(\frac{1}{k}). \tag{2.29}
\]

If \( \{X_n, n \geq 1\} \) are PA, we similarly have

\[
\left| \text{Cov} \left( f \left( \frac{S_{k,k}}{\frac{10k}{3} \sigma_1} \right), f \left( \frac{\sum_{i=k+1}^{l} c_{i,j}Y_i}{\frac{10l}{3} \sigma_1} \right) \right) \right| \leq C \left( \frac{k}{l} \right)^{1/2} \log(\frac{1}{k}), \tag{2.30}
\]

Since \( f \) is a bounded Lipschitz function, by Hölder’s inequality and the assumptions of Theorem 1.1, note that for \( n \geq j \), \( \sum_{i=1}^{n} \log^2(n/i) \leq C j (1 + \log^2(n/j)) \), we have

\[
\left| \text{Cov} \left( f \left( \frac{S_{k,k}}{\frac{10k}{3} \sigma_1} \right), f \left( \frac{S_{j,l}}{\frac{10l}{3} \sigma_1} \right) - f \left( \frac{\sum_{i=k+1}^{l} c_{i,j}Y_i}{\frac{10l}{3} \sigma_1} \right) \right) \right|
\]

\[
\leq C \left( \frac{l}{l^{1/2}} \right) \sum_{i=1}^{k} c_{i,j}Y_i \leq C \left( \frac{l}{l^{1/2}} \right) \sum_{i=1}^{k} b_{i,j}Y_i + C \left( \frac{l}{l^{1/2}} \right) \sum_{i=1}^{k} d_{i,j}Y_i
\]

\[
\leq C \left( \frac{l}{l^{1/2}} \right) \left( \sum_{i=1}^{k} b_{i,j}Y_i \right)^{1/2} + C \left( \frac{l}{l^{1/2}} \right) \left( \sum_{i=1}^{k} d_{i,j}Y_i \right)^{1/2}
\]

\[
\leq C \left( \frac{l}{l^{1/2}} \right) \left( \sum_{i=1}^{k} b_{i,j} \left( 1 + \sum_{j=2}^{\infty} |\text{Cov}(Y_j, Y_i)| \right)^{1/2} + C \left( \frac{l}{l^{1/2}} \right) \left( \sum_{i=1}^{k} d_{i,j} \left( 1 + \sum_{j=2}^{\infty} |\text{Cov}(Y_j, Y_i)| \right) \right)^{1/2}
\]

\[
\leq C \left( \frac{l}{l^{1/2}} \right) \left( \sum_{i=1}^{k} \left( \log \frac{1}{l} \right)^{2} \right)^{1/2} + C \left( \frac{l}{l^{1/2}} \right) \left( \sum_{i=1}^{k} \left( \frac{l+1-j}{l} \right)^{2} \right)^{1/2}
\]

\[
\leq C \left( \frac{l}{l^{1/2}} \right) \left( \sum_{i=1}^{k} \left( \log \frac{1}{l} \right) \right)^{1/2} \leq C \left( \frac{k}{l} \right)^{1/2} \left( \left( 1 + \log \frac{1}{k} \right) \right) + C \left( \frac{k}{l} \right)^{1/2}. \tag{2.31}
\]

Hence if \( l > k \), by (2.28)-(2.31), for some \( 0 < \varepsilon < 1/2 \), \( |E_{k,l}^{c,j}| \leq C(k/l)^{\varepsilon} \), then by Lemma 2.4, (2.27) holds. The proof is completed. \( \square \)

**Proof of Theorem 1.1.** Let \( C_i = \frac{T_i}{(i+1)\mu_2} \), we have
\[
\frac{1}{\sqrt{2k\sigma_1}} \sum_{i=1}^{k} (C_i - 1) = \frac{1}{\sqrt{2k\sigma_1}} \sum_{i=1}^{k} \left( \frac{T_i}{i(i+1)\mu/2} - 1 \right) = \frac{1}{\sqrt{2k\sigma_1}} \sum_{i=1}^{k} \sum_{j=1}^{i} \frac{(i+j)(x_j-\mu)}{i(i+1)\mu/2} \\
= \frac{1}{\sqrt{2k\sigma_1}} \sum_{j=1}^{k} \sum_{i=j}^{k} \frac{(i+j)(x_j-\mu)}{i(i+1)\mu/2} = \frac{1}{\sqrt{2k\sigma_1}} \sum_{j=1}^{k} C_{j,k} = \frac{1}{\sqrt{2k\sigma_1}} \sum_{j=1}^{k} \sqrt{k} Y_j = -\frac{S_{k,k}}{\sqrt{2k\sigma_1}}.
\]

Hence (2.23) is equivalent to
\[
\forall x \lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I \left\{ \frac{1}{\sqrt{2k\sigma_1}} \sum_{i=1}^{k} (C_i - 1) \leq x \right\} = \Phi(x) \quad \text{a.s.} \quad (2.32)
\]

On the other hand, to prove (1.2), it suffices to show that
\[
\forall x \lim_{n \to \infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{1}{k} I \left\{ \frac{1}{\sqrt{2k\sigma_1}} \sum_{i=1}^{k} \log C_i \leq x \right\} = \Phi(x) \quad \text{a.s.} \quad (2.33)
\]

By Lemma 2.2, for enough large i, for some \(4/7 < p < 2/3\), we have
\[
|C_i - 1| \leq C_1 p^{-2} \quad \text{a.s.}
\]

Since \(|x| < 1\), we have \(\log(1+x) = x - R(x)\) with \(\lim_{x \to 0} R(x)/x^2 = 1/2\), thus
\[
\left| \sum_{i=1}^{k} \log C_i - \sum_{i=1}^{k} (C_i - 1) \right| \leq C \sum_{i=1}^{k} (C_i - 1)^2 \leq Ck^2/3 \quad \text{a.s.}
\]

Hence for almost every event \(\omega\) and for arbitrary small \(\varepsilon > 0\) there exists \(n_0 = n_0(\omega, \varepsilon, x)\) such that for \(k > n_0\)
\[
I \left\{ \frac{1}{\sqrt{2k\sigma_1}} \sum_{i=1}^{k} (C_i - 1) \leq x - \varepsilon \right\} \leq I \left\{ \frac{1}{\sqrt{2k\sigma_1}} \sum_{i=1}^{k} \log C_i \leq x \right\} \leq I \left\{ \frac{1}{\sqrt{2k\sigma_1}} \sum_{i=1}^{k} (C_i - 1) \leq x + \varepsilon \right\},
\]

and thus (2.32) implies (2.33), as desired. \(\square\)

**Proof of Theorem 1.2.** The proof is similar to that of Theorem 1.1 with some corresponding lemmas found in Li and Wang [9]. So we omit it here. \(\square\)

**References**


