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Equivalence of Hardy Submodules Generated by Polynomials[1]

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polynomials, I has a greatest common divisor p. So, I can be uniquely written as $I = p\,L$ which is called the Beurling form of I. Let $I_1 = p_1 L_1$, $I_2 = p_2 L_2$. We prove that $[I_1]$ and $[I_2]$ are unitarily equivalent if and only if there are polynomials q_1 and q_2 with $Z(q_1) \cap D^n = Z(q_2) \cap D^n = \varnothing$ such that $|p_1 q_1| = |p_2 q_2|$ on T^n, and $[p_1 L_1] = [p_1 L_2]$. Consequently, two principal submodules $[p_1]$ and $[p_2]$ are unitarily equivalent if and only if there are polynomials q_1 and q_2 with $Z(q_1) \cap D^n = Z(q_2) \cap D^n = \varnothing$ such that $|p_1 q_1| = |p_2 q_2|$ on T^n. Furthermore, we give a complete similarity classification for submodules generated by homogeneous ideals. Finally, we point out that in the case of the Hardy module on the unit ball, $[I_1]$ and $[I_2]$ are unitarily equivalent if and only if they are equal. If I_1 and I_2 are homogeneous ideals, then $[I_1]$ and $[I_2]$ are quasi-similar if and only if $I_1 = I_2$. © 2000 Academic Press

1. INTRODUCTION

Let T be the unit circle in the complex plane, and let $L^2(T)$ be the Hilbert space of square integrable functions, with respect to arc-length measure. Recall that the Hardy space $H^2(D)$ over the open unit disk D is the closed subspace of $L^2(T)$ spanned by the non-negative powers of the coordinate function z. If M is a (closed) subspace of $H^2(D)$ that is invariant for the multiplication operator M_z, then Beurling's theorem says that there exists an inner function η such that $M = \eta H^2(D)$. In the language of Douglas and Paulsen [DP], each submodule M of the Hardy module

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$H^2(D)$ over the disk algebra $A(D)$ has the form $M=\eta H^2(D)$. For the Hardy module $H^2(D^n)$, the multivariable version, a natural problem is to consider the structure of submodules. However, one quickly sees that a Beurling-like characterization is impossible, and hence attention is directed to investigating equivalence classes of submodules of $H^2(D^n)$ under some kind of equivalence relation.

DEFINITION. Let M_1, M_2 be two submodules of $H^2(D^n)$. We say that

(1) they are unitarily equivalent if there exists a unitary module map $X: M_1 \to M_2$, that is, X is a unitary operator and for any polynomial p, $X(p\,h) = p\,X(h)$, $\forall h \in M_1$;

(2) they are similar if there exists an invertible module map $X: M_1 \to M_2$;

(3) they are quasi-similar if there exist module maps $X: M_1 \to M_2$ and $Y: M_2 \to M_1$ with dense ranges.

Agrawal, Clark and Douglas studied unitary equivalence among some special Hardy submodules on the polydisk in [ACD]. The extension of their results to general domains in C^n was considered by Agrawl and Salinas [AS]. Furthermore, for Hardy submodules on the polydisk, some deep results were obtained by Douglas and Yan in [DY]. Most of these results require the codimension of the zero variety of the submodule in question to be at least 2. Based on results and ideas from algebraic geometry, under mild restrictions, Douglas *et al.* [DPSY] showed that submodules obtained from the closures of ideals are quasi-similar if and only if the ideals coincide. Analytic submodules in several variables exhibit this phenomenon, called "rigidity," for several reasons. From an analytic point of view, the appearance of rigidity is natural because of the Hartogs phenomenon in several variables. From an algebraic point of view, the reason may be that the submodules are not singly generated. However, K. Yan [Yan] proved that two submodules of $H^2(D^n)$ each of which is singly generated by homogeneous polynomial are unitarily equivalent if and only if the ratio of the moduli of corresponding homogeneous polynomials is constant on T^n. In the case of the Hardy module $H^2(B_n)$ on the unit ball B_n, the corresponding problem was considered by Chen and Douglas [CD]. They showed that two homogeneous principal submodules are quasi-similar if and only if the corresponding homogeneous polynomials coincide. Hong and Guo [HG] studied equivalence of homogeneous principal submodules on bounded complete Reinhardt domains.

In the present paper, using the characteristic space theory developed by the author [Guo1], we obtain a complete classification under unitary equivalence for Hardy submodules on the polydisk that are generated by ideals of polynomials. Let I be an ideal of polynomials in n variables. Since I is generated by finitely many polynomials, I has a greatest common divisor p, which is unique except for a constant factor. So, I can be uniquely written as $I = pL$, the ideal L having the greatest common divisor 1. We call $I = pL$ the Beurling form of I. For an ideal I, we let $[I]$ is the submodule of $H^2(D^n)$ generated by I. It is easy to see that $[I]$ equals the closure of I in $H^2(D^n)$. Let $I_1 = p_1 L_1$, $I_2 = p_2 L_2$. We prove that $[I_1]$ and $[I_2]$ are unitarily equivalent if and only if there are polynomials q_1 and q_2 with $Z(q_1) \cap D^n = Z(q_2) \cap D^n = \emptyset$ such that $|p_1 q_1| = |p_2 q_2|$ on T^n, and $[p_1 L_1] = [p_1 L_2]$. A straightforward corollary is that two principal submodules $[p_1]$ and $[p_2]$ are unitarily equivalent if and only if there are polynomials q_1 and q_2 with $Z(q_1) \cap D^n = Z(q_2) \cap D^n = \emptyset$ such that $|p_1 q_1| = |p_2 q_2|$ on T^n. Furthermore, we give a complete classification under similarity for submodules generated by homogeneous ideals.

In the case of Hardy submodules on the unit ball B_n, we point out that $[I_1]$ and $[I_2]$ are unitarily equivalent only if they are equal. If I_1 and I_2 are homogeneous ideals, then $[I_1]$ and $[I_2]$ are quasi-similar only if $I_1 = I_2$.

This paper is organized as follows. In Section 2, we develop some basic properties of analytic submodules with finite rank by using the characteristic space theory in [Guo1]. These results will be used in the proof of the classification theorem in Section 3. In Section 4, we use some techniques developed in Sections 2 and 3 to study similarity and quasi-similarity of submodules generated by polynomials. In the last Section, we consider the case of Hardy submodules on the unit ball.

2. SOME BASIC PROPERTIES OF ANALYTIC SUBMODULES WITH FINITE RANK

To prove the classification theorem in Section 3, we develop some basic properties of analytic submodules with finite rank. First let us recall some terminology from [DPSY, Guo1] which will be used throughout this section.

Let Ω be a bounded nonempty open subset of C^n, and let $Hol(\Omega)$ denote the ring of analytic functions on Ω. We use $\mathscr{C}$ to denote the polynomial ring on C^n. Let X be Banach space contained in $Hol(\Omega)$. We call X a reproducing Ω-space if X contains 1 and if for each $w \in \Omega$ the evaluation functional $E_w(f) = f(w)$ is a continuous linear functional on X. We say that X is a reproducing $\mathscr{C}$-module on Ω if X is a reproducing Ω-space, and for each $p \in \mathscr{C}$ and each $x \in X$, px is in X. Note that, by a simple application of the closed graph theorem, the operator T_p defined to be multiplication by p is bounded on X. Note also that $\mathscr{C} \subset X$ follows from the fact that 1 is in X. For $\lambda \in C^n$, one says that λ is a virtual point of X provided that the homomorphism $p \mapsto p(\lambda)$ defined on $\mathscr{C}$ extends to a bounded linear functional on X. Since X is a reproducing Ω-space, every point in Ω is a virtual point. We use $vp(X)$ to denote the collection of virtual points; then $vp(X) \supseteq \Omega$. Finally we say that X is an analytic Hilbert module on Ω if the following conditions are satisfied:

(1) X is a reproducing $\mathscr{C}$-module on Ω;

(2) $\mathscr{C}$ is dense in X;

(3) $vp(X) = \Omega$.

Remark. Notice that the conditions (2) and (3) are equivalent to the following statement: for each $\lambda \notin \Omega$, $\mathscr{U}_\lambda$, the maximal ideal of functions in $\mathscr{C}$ that vanish at λ, is dense in X. In fact, if for each $\lambda \notin \Omega$, $\mathscr{U}_\lambda$ is dense in X, then the condition (2) is immediate. If there is a $\lambda_0 \notin \Omega$, with $\lambda_0 \in vp(X)$, then there exists a constant c_0 such that for any polynomial p, $|p(\lambda_0)| \leqslant c_0 \|p\|$. Since the maximal ideal $\mathscr{U}_{\lambda_0}$ is dense in X, it follows that there exists a sequence $\{p_n\} (\subset \mathscr{U}_{\lambda_0})$ converges to 1 in the norm of X. However,

$$1 = |p_n(\lambda_0) - 1| \leqslant c_0 \|p_n - 1\|.$$

This contradiction says that $vp(X) = \Omega$. In the opposite direction, suppose that there is a $\lambda_0 \notin \Omega$ such that $\mathscr{U}_{\lambda_0}$ is not dense in X. Then there exists a bounded linear functional x^* on X that annihilates $\mathscr{U}_{\lambda_0}$. Therefore for any polynomial p,

$$x^*(p) = x^*(p - p(\lambda_0)) + p(\lambda_0)\, x^*(1) = p(\lambda_0)\, x^*(1).$$

By condition (2), $x^*(1) \neq 0$. This insures that there exists a constant c_0 such that $|p(\lambda_0)| \leqslant c_0 \|p\|$ for each polynomial p. This contradicts condition (3). We thus achieve the opposite implication.

Most "natural" reproducing Ω-spaces are analytic Hilbert modules on Ω. The basic examples are the Hardy module and the Bergman module on the polydisk (and on the unit ball). In the following we will use submodule to mean a closed subspace of X that is invariant under the multiplications of polynomials. Let M be a submodule of an analytic Hilbert module X on Ω. The zero variety of M is defined by

$$Z(M) = \{z \in \Omega : f(z) = 0, \forall f \in M\}.$$

For $\lambda \in \Omega$, set

$$M_\lambda = \{q \in \mathscr{C} : q(D)\, f|_\lambda = 0, \forall f \in M\},$$

where $q(D)$ is the linear partial differential operator $\sum a_{m_1 \cdots m_n} (\partial^{m_1+m_2+\cdots+m_n}/\partial z_1^{m_1} \partial z_2^{m_2} \cdots \partial z_n^{m_n})$ if $q = \sum a_{m_1 \cdots m_n} z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$. From [Guo1], we know that M_λ is invariant under the action by the basic partial differential operators $\{\partial/\partial z_1, \partial/\partial z_2, ..., \partial/\partial z_n\}$, and M_λ is called the characteristic space of M at λ. The envelope of M at λ, M_λ^e, is defined by

$$M_\lambda^e = \{ f \in X : q(D) f|_\lambda = 0, \forall q \in M_\lambda \}.$$

Then M_λ^e is a submodule of X, and $M_\lambda^e \supseteq M$.

It is well known that "rank" is one of important invariants of Hilbert modules. Recall that a submodule M of the analytic Hilbert module X is finitely generated if there exists a finite set of vectors $x_1, x_2, ..., x_n$ in M such that $\mathscr{C} x_1 + \mathscr{C} x_2 + \cdots + \mathscr{C} x_n$ is dense in M. The minimum cardinality of such a set is called the rank of M, and denoted by $rank(M)$. If $rank(M) = 1$, we call M a principal submodule. By Beurling's theorem, each submodule of $H^2(D)$ is principal. Even more generally, for any analytic Hilbert module X on a domain in the complex plane C, the submodule generated by some set of polynomials is principal because every ideal of polynomials of one variable is principal. However, for the Hardy module $H^2(D^n)$ in several variables, there exist submodules with any rank.

Let $\lambda \in C^n$. We denote by $\mathcal{O}_\lambda$ the ring of all germs of analytic functions at λ. For detailed information about $\mathcal{O}_\lambda$ we refer to [DY, Kr]. We summarize some properties of $\mathcal{O}_\lambda$. First $\mathcal{O}_\lambda$ is a unique factorization domain (UFD), and the units of $\mathcal{O}_\lambda$ are those germs which are nonvanishing at λ. Second, $\mathcal{O}_\lambda$ is a Noetherian local ring of Krull dimension n.

Let I be an ideal of $\mathcal{O}_\lambda$. As in the case of analytic submodules, we define the characteristic space of I by

$$I_\lambda = \{ q \in \mathscr{C} : q(D) f|_\lambda = 0, \forall f \in I \}.$$

The envelope of I, I_λ^e is defined by

$$I_\lambda^e = \{ f \in \mathcal{O}_\lambda : q(D) f|_\lambda = 0, \forall q \in I_\lambda \}.$$

It is easy to check that I_λ^e is an ideal of $\mathcal{O}_\lambda$, and $I_\lambda^e \supseteq I$. Furthermore, by the reasoning in the proof of Theorem 2.1 in [Guo1], we have

$$I_\lambda^e = \bigcap_{j \geqslant 1} (I + \mathscr{M}_\lambda^j),$$

where $\mathscr{M}_\lambda$ is the maximal ideal of $\mathcal{O}_\lambda$, that is, $\mathscr{M}_\lambda = \{ f \in \mathcal{O}_\lambda : f(\lambda) = 0 \}$. Now using Krull's Theorem [ZS, Vol. (I), p.217, Theorem 12'] or Lemma 2.11 in [DPSY], we have the following proposition.

Proposition 2.1. *Let I be an ideal of $\mathcal{O}_\lambda$. Then*

$$I = I_\lambda^e.$$

Hence I is completely determined by its characteristic space.

Let X be an analytic Hilbert module on $\Omega(\subset C^n)$, and let $\lambda \in \Omega$. For $f \in X$ we denote by f_λ the element of $\mathcal{O}_\lambda$ defined by the restriction of f to a neighborhood of λ. For a submodule M of X, we denote by $M^{(\lambda)}$ the ideal of $\mathcal{O}_\lambda$ generated by $\{f_\lambda : f \in M\}$. Let $f_1, f_2, ..., f_m$ be in X. Write $[f_1, f_2, ..., f_m]$ for the submodule generated by $f_1, f_2, ..., f_m$.

Lemma 2.2. *Let $\lambda \in \Omega$. Then*

$$[f_1, f_2, ..., f_m]^{(\lambda)} = f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda.$$

Proof. From the inclusion

$$f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda \subset [f_1, f_2, ..., f_m]^{(\lambda)},$$

we see that

$$\{[f_1, f_2, ..., f_m]^{(\lambda)}\}_\lambda \subset (f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda)_\lambda.$$

For $f \in [f_1, f_2, ..., f_m]$, there exist polynomials $p_n^{(1)}, p_n^{(2)}, ..., p_n^{(m)}$ such that

$$f = \lim_{n\to\infty} (p_n^{(1)} f_1 + p_n^{(2)} f_2 + \cdots + p_n^{(m)} f_m)$$

in the norm of X. It is easy to check that for every polynomial q and each $w \in \Omega$, the linear functional on X, $f \mapsto q(D) f|_w$, is continuous. Let q be in $(f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda)_\lambda$. Since

$$q(D)(p_n^{(1)} f_1 + p_n^{(2)} f_2 + \cdots + p_n^{(m)} f_m)|_\lambda = 0,$$

this implies that

$$q(D) f|_\lambda = 0.$$

It follows that

$$(f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda)_\lambda \subset [f_1, f_2, ..., f_m]_\lambda.$$

It is easy to see that

$$[f_1, f_2, ..., f_m]_\lambda = \{[f_1, f_2, ..., f_m]^{(\lambda)}\}_\lambda,$$

and hence

$$\{[f_1, f_2, ..., f_m]^{(\lambda)}\}_\lambda = (f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda)_\lambda.$$

Applying Proposition 2.1, we obtain that

$$[f_1, f_2, ..., f_m]^{(\lambda)} = f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda.$$

THEOREM 2.3. *Let M be the submodule of X generated by $f_1, f_2, ..., f_m$. Then for each $f \in M$, there are $g_1, g_2, ..., g_m$ in $Hol(\Omega)$ such that*

$$f = f_1 g_1 + f_2 g_2 + \cdots + f_m g_m.$$

Proof. The proof of Theorem 2.3 is based on sheaf theory (see [Kr, Chaps., 6, 7]). Let $\mathcal{O}$ denote the sheaf of germs of analytic functions on Ω. The sheaf $\mathcal{F} = \mathcal{F}(M)$ generated by M is defined as follows. For $\lambda \in \Omega$, $\mathcal{F}_\lambda = M^{(\lambda)}$. From Lemma 2.2, we see that $\mathcal{F}$ is the subsheaf of $\mathcal{O}$ generated by $f_1, f_2, ..., f_m$. Consider the exact sequence of sheafs

$$0 \to \mathcal{R} \xrightarrow{i} \mathcal{O}^m \xrightarrow{\alpha} \mathcal{F} \to 0,$$

where $\alpha(g_{1\lambda}, g_{2\lambda}, ..., g_{m\lambda}) = \sum_{i=1}^m f_{i\lambda} g_{i\lambda}$, $\mathcal{R}$ is the kernel sheaf and i is the inclusion. The Oka Coherence Theorem [Kr, Theorem 7.1.8] implies that $\mathcal{R}$ is coherent, so by Theorem B of Cartan [Kr, Theorem 7.1.7], $H^1(\Omega, \mathcal{R}) = 0$. Now the long exact cohomology sequence [Kr, Theorem 6.2.22] gives

$$H^0(\Omega, \mathcal{O}^m) \xrightarrow{\alpha^*} H^0(\Omega, \mathcal{F}) \xrightarrow{\delta^*} H^1(\Omega, \mathcal{R}) = 0.$$

So α^* is surjective. This says that for every $f \in M$, there exist $g_1, g_2, ..., g_m$ in $Hol(\Omega)$ such that

$$f = f_1 g_1 + f_2 g_2 + \cdots + f_m g_m.$$

The proof is complete.

In Douglas and Paulsen's book [DP, p. 42, Problem 2.23], it is asked when a principal submodule of $H^2(D^n)$ is the closure of an ideal of polynomials. Combining Theorem 2.3 with the characteristic space theory of polynomials (see [Guo1]), one can characterize when a principal submodule is generated by polynomials.

Let X be an analytic Hilbert module on Ω. For an ideal I of polynomials, we let $[I]$ denote the closure of I in X and let $Z(I)$ be the zero variety of I, that is, $Z(I) = \{z \in C^n : p(z) = 0, \forall p \in I\}$.

THEOREM 2.4. *Let $I = p\,L$ be the Beurling form of I. If $[I]$ is principal, then $Z(L) \cap \Omega = \emptyset$. Equivalently, if $Z(L) \cap \Omega \neq \emptyset$, then rank$([I]) \geqslant 2$.*

Proof. Let $\{p_1, p_2, ..., p_k\}$ be a set of generators of L. Then the greatest common divisor $GCD\{p_1, p_2, ..., p_k\} = 1$. Now suppose that there exists $\lambda \in \Omega$ such that $p_i(\lambda) = 0$ for $i = 1, 2, ..., k$. Decompose $p_i = p_i' p_i''$ such that each prime factor of p_i' vanishes at λ, and $p_i''(\lambda) \neq 0$, and $p = q_1 q_2$ such that each prime factor of q_1 vanishes at λ, and $q_2(\lambda) \neq 0$. Since $[I]$ is principal, this says that there exists some f in X such that

$$[f] = [pp_1, pp_2, ..., pp_k].$$

By Theorem 2.3, there exist analytic functions on Ω, $g_1, g_2, ..., g_k$, $h_1, h_2, ..., h_k$ such that

$$pp_1 = fg_1, \qquad pp_2 = fg_2, ..., pp_k = fg_k, \qquad \text{and} \qquad f = \sum_{i=1}^{k} h_i pp_i.$$

Therefore,

$$\sum_{i=1}^{k} h_i g_i = 1.$$

So, the functions $g_1, g_2, ..., g_k$ have no common zero in Ω. This implies that there is some g_s such that $g_s(\lambda) \neq 0$. From the equality $pp_s = fg_s$ one has

$$[pp_s]_\lambda = [f]_\lambda.$$

According to Theorem 2.1 in [Guo1],

$$q_1 p_s' \mathscr{C} = [pp_s]_\lambda^e = [f]_\lambda^e.$$

However, for each i, $[f]_\lambda \subset [pp_i]_\lambda$ and hence for every i,

$$[f]_\lambda^e \supset [pp_i]_\lambda^e = q_1 p_i' \mathscr{C}.$$

So, for every i,

$$q_1 p_s' \mathscr{C} \supset q_1 p_i' \mathscr{C}.$$

Thus each p_i' is divisible by p_s'. So, every p_i is divisible by p_s'. This is impossible. Thus, $p_1, p_2, ..., p_k$ have no common zero in Ω, that is, $Z(L) \cap \Omega = \emptyset$. The proof is complete.

COROLLARY 2.5. *Let $I = pL$ be the Beurling form of the ideal I. If every algebraic component of $Z(I)$ has a nonempty intersection with Ω, then $[I]$*

is principal if and only if $I = p\mathscr{C}$. *In particular, if* I *is prime, and* $Z(I) \cap \Omega \neq \emptyset$, *then* $[I]$ *is principal if and only if* $I = p\mathscr{C}$, *and* p *is prime.*

Proof. If $[I]$ is principal, then by Theorem 2.4, $Z(L) \cap \Omega = \emptyset$. So for each $\lambda \in \Omega$,

$$[I]_\lambda = [p\mathscr{C}]_\lambda.$$

From Corollary 2.3 in [Guo1], we see that $I \supset p\mathscr{C}$, and hence $I = p\mathscr{C}$. The opposite direction is obvious.

When $n = 2$, one can obtain a more detailed result. We will need a lemma due to Yang (see [Yang]).

LEMMA 2.6. *Suppose* $p_1, p_2, ..., p_k$ *are polynomials in two variables such that the greatest common divisor* $GCD(p_1, p_2, ..., p_k) = 1$. *Then the ideal* $(p_1, p_2, ..., p_k)$ *generated by* $p_1, p_2, ..., p_k$ *is finite codimensional.*

Therefore on C^2, every ideal $I = pL$ is "almost principal" because L is of finite codimension.

COROLLARY 2.7. *Let* X *be an analytic Hilbert module on* $\Omega (\subset C^2)$, *and let* $I = pL$ *be the Beurling form of* I. *Then* $[I]$ *is principal if and only if* $[I] = [p]$.

Proof. Let $[I]$ be principal. By Theorem 2.4, $Z(L) \cap \Omega = \emptyset$. Using Lemma 2.6, we see $[L] = X$. So $[I] = [p]$.

Let X be an analytic Hilbert module on Ω. Then naturally, the ring $\mathscr{C}$ of polynomials is endowed with the topology induced by X. In this topology, an ideal I is closed if and only if $[I] \cap \mathscr{C} = I$. Following the language in [DPSY], closed ideals are also called contracted. For "natural" analytic Hilbert modules it would be interesting to classify the contracted ideals. In particular, for $H^2(D^n)$, Douglas and Paulsen conjectured that if I is contracted, then each algebraic component of $Z(I)$ has a nonempty intersection with D^n (see [DPSY, DP]). When $n = 2$, this conjecture was affirmed by R. Gelca [Ge1].

Let Ω be a bounded complete Reinhart domain (i.e., a bounded domain with the property that for $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \Omega$, if $|z_i| \leqslant 1$, $i = 1, 2, ..., n$, then $(z_1\lambda_1, z_2\lambda_2, ..., z_n\lambda_n) \in \Omega$.) For a polynomial p, we denote by $d(p)$ the sum of the degrees of p in each variable. The following lemma is due to R. Gelca [Ge1, Ge2].

LEMMA 2.8. *Let* Ω *be a bounded complete Reinhart domain. If* p *is a polynomial having no zeros in* Ω, *then for every* r *in* $(1/2, 1)$ *and* $(z_1, z_2, ..., z_n) \in \bar{\Omega}$, $|p(z_1, z_2, ..., z_n)/p(rz_1, rz_2, ..., rz_n)| \leqslant 2^{d(p)}$.

The next proposition will be used in Section 3 several times. It first was proved for $n=2$ by R. Gelca [Ge1]. Here we give a simple proof by using the Dominated Convergence Theorem.

PROPOSITION 2.9. *Let p be a polynomial having no zeros in D^n. Then $p\mathscr{C}$ is dense in $H^2(D^n)$.*

Proof. Let

$$f_r(z_1, z_2, ..., z_n) = p(z_1, z_2, ..., z_n)/p(rz_1, rz_2, ..., rz_n).$$

Then for $1<r<1/2$, $f_r \in pH^2(D^n)$. Let $\{r_n\}_n$ be a sequence such that $r_n \to 1$ as $n \to \infty$. Now by Lemma 2.8, the family f_{r_n} is uniformly bounded, and $f_{r_n} \to 1$ almost everywhere on T^n. By the Dominated Convergence Theorem, f_{r_n} converges to 1 in the norm of $H^2(D^n)$. The conclusion follows because the ring of polynomials is dense in $H^2(D^n)$.

Remark. By the same as the proof, we have that if $Z(p) \cap B_n = \emptyset$, then $p\mathscr{C}$ is dense in the Hardy space $H^2(B_n)$ on the unit ball B_n.

PROPOSITION 2.10. *Let I be an ideal of polynomials on C^2. If $Z(I) \cap D^2 = \emptyset$, then there exists a polynomial $q \in I$ such that $Z(q) \cap D^2 = \emptyset$.*

Proof. Let $I = p\,L$ be the Beurling form of I. Obviously, $Z(p) \cap D^2 = \emptyset$. To complete the proof, we will find a polynomial h in L such that $Z(h) \cap D^2 = \emptyset$. Since L is of finite codimension, and $Z(L) \cap D^2 = \emptyset$, there exist finitely many zeros $\lambda_1, \lambda_2, ..., \lambda_k$ of L, which lie in $C^2 \backslash D^2$. Now decompose L as

$$L = \bigcap_{i=1}^{k} L_k,$$

where L_i are $\mathscr{U}_{\lambda_i}$-primary for $i = 1, 2, ..., k$. From [AM], we see that there is a positive integer m such that

$$\mathscr{U}_{\lambda_i}^m \subset L_i, \qquad i = 1, 2, ..., k.$$

Since $\lambda_i = (\lambda_i', \lambda_i'') \notin D^2$, $|\lambda_i'| \geqslant 1$ or $|\lambda_i''| \geqslant 1$. For each i, we may suppose that $|\lambda_i'| \geqslant 1$. This implies that $\prod_{i=1}^k (z - \lambda_i')^m$ has no zero in D^2, and $\prod_{i=1}^k (z - \lambda_i')^m \in L$. Set

$$q = p \prod_{i=1}^{k} (z - \lambda_i')^m,$$

which gives the desired conclusion.

However, for $n > 2$, we do not know if the same conclusion is true.

Conjecture. Let $n > 2$, and I be an ideal of polynomials on C^n. If $Z(I) \cap D^n = \emptyset$, then there exists a polynomial q in I such that $Z(q) \cap D^n = \emptyset$.

Remark. The preceding Conjecture implies Douglas and Paulsen's conjecture [DPSY, DP]. That is, if I is contracted in $H^2(D^n)$, then each algebraic component of I meets D^n nontrivially. In fact, first let $I = \bigcap_{j=1}^m I_j$ be an irredundant primary decomposition of I. We may suppose that there are $I_1, I_2, ..., I_k$ such that $Z(I_j) \cap D^n = \emptyset$ for $j = 1, 2, ..., k$, and $Z(I_j) \cap D^n \neq \emptyset$ for $j = k+1, ..., m$. By the preceding Conjecture, there exists a polynomial $q \in I_1 I_2 \cdots I_k$ such that $Z(q) \cap D^n = \emptyset$. Therefore Proposition 2.9 implies that

$$H^2(D^n) = [q\mathscr{C}] = [I_1 I_2 \cdots I_k].$$

From the inclusions

$$I_1 I_2 \cdots I_k \left(\bigcap_{j=k+1}^{m} I_j \right) \subset I \subset \bigcap_{j=k+1}^{m} I_j,$$

we see that

$$[I] = \left[\bigcap_{j=k+1}^{m} I_j \right].$$

By [DPSY, Theorem 2.7], $[I] \cap \mathscr{C} = \bigcap_{j=k+1}^m I_j$. This contradicts the contractedness of I. It follows that the preceding Conjecture implies Douglas and Paulsen's conjecture.

3. UNITARY EQUIVALENCE OF HARDY SUBMODULES GENERATED BY POLYNOMIALS

In this section we will prove the classification theorem for Hardy submodules on the polydisk generated by polynomials. Let p_1 and p_2 be two polynomials in n variables. We say that p_1, p_2 are modulus equivalent if there exist two polynomials q_1, q_2 with $Z(q_1) \cap D^n = Z(q_2) \cap D^n = \emptyset$ such that $|p_1 q_1| = |p_2 q_2|$ on T^n. If p_1 and p_2 are modulus equivalent we will write $p_1 \asymp p_2$.

THEOREM 3.1. *Let $I_1 = p_1 L_1$, $I_2 = p_2 L_2$ be two ideals of polynomials. Then $[I_1]$ and $[I_2]$ are unitarily equivalent if and only if $p_1 \asymp p_2$ and $[p_1 L_1] = [p_1 L_2]$.*

To prove Theorem 3.1, we will need several lemmas.

LEMMA 3.2. *Let $f=p/q$ be a rational function, where p and q are without common factors. If f is analytic on D^n, then $Z(q)\cap D^n=\emptyset$.*

Proof. In fact, if there is a $\lambda\in D^n$ such that $q(\lambda)=0$, then $p(\lambda)=f(\lambda)\,q(\lambda)=0$. Let $p_i^{m_i}$ be the primary factors of p with $\lambda\in Z(p_i)$, $i=1, 2, ..., s$. Then by [Guo1, Corollary 2.2], we see that

$$(p)_\lambda^e=p_1^{m_1}p_2^{m_2}\cdots p_s^{m_s}\mathscr{C},$$

where (p) is the ideal of $\mathscr{C}$ generated by p, and $(p)_\lambda^e$ is the envelope of (p) at λ. Let $(p)_\lambda$ and $(fp)_\lambda$ denote the characteristic spaces of (p) and (fp) at λ (cf. [Guo1]), respectively, where (fp) is the ideal of $Hol(D^n)$ generated by fp. Since

$$(p)_\lambda=(fq)_\lambda\supset(q)_\lambda,$$

we have

$$(p)_\lambda^e\subset(q)_\lambda^e.$$

So,

$$p_1^{m_1}p_2^{m_2}\cdots p_s^{m_s}\mathscr{C}\subset q_1^{n_1}q_2^{n_2}\cdots q_t^{n_t}\mathscr{C},$$

where $q_i^{n_i}$ are the primary factors of q with $\lambda\in Z(q_i)$, $i=1, 2, ..., t$. Therefore there exists a polynomial r such that

$$p_1^{m_1}p_2^{m_2}\cdots p_s^{m_s}=r\,q_1^{n_1}q_2^{n_2}\cdots q_t^{n_t}.$$

This is contradictory to our assumption.

Let $f\in Hol(D^n)$. For each $w\in T^n$, the slice function f_w on D is defined by $f_w(z)=f(zw)$, $\forall z\in D$.

We will now give a modification of Theorem 5.2.2 in Rudin's book [Ru1]; the proof is similar to that of Rudin [Ru1].

LEMMA 3.3. *Let f be in the Nevanlinna class on D^n, and let the slice functions f_w be rational (in one variable) for almost all $w\in T^n$. Then f is a rational function (of n variables).*

Proof. First of all, for a rational function $r=p/q$ in one variable, we define the degree of r is the maximum of $deg\ p$, $deg\ q$, provided that the common factors of p, q have first been cancelled. If for almost all $w\in T^n$, $deg\ f_w=0$, it is easy to verify that $f=c$ for some constant c. Thus, without a loss of generality, we assume that there exist a subset E of T^n with

$m_n(E)>0$ and a natural number k such that $deg\, f_w = k$ for $w \in E$ (here m_n is the Lebesgue measure on T^n, divided by $(2\pi)^n$ in order to have $m_n(T^n)=1$.) Let $f=\sum_{i=0}^{\infty} F_i$ be the homogeneous expansion of f. Notice that f_w is analytic on the unit disk D. Thus by Lemma 3.2, for each $w \in E$, $f_w(z)$ is uniquely written as

$$f_w(z)=\frac{\beta_k(w)\, z^k+\beta_{k-1}(w)\, z^{k-1}+\cdots+\beta_0(w)}{\alpha_k(w)\, z^k+\alpha_{k-1}(w)\, z^{k-1}+\cdots+1}.$$

This implies that for every $w \in E$, the infinite system of linear equations

$$F_m(w)+F_{m-1}(w)\, x_1+\cdots+F_{m-k}(w)\, x_k=0, \qquad (m>k) \qquad (\bigstar)$$

has a unique solution $(\alpha_1(w), \alpha_2(w), ..., \alpha_k(w))$. This uniqueness ensures that the vectors

$$v_m(w)=(F_{m-1}(w), ..., F_{m-k}(w)) \qquad (m>k)$$

span all of C^k if $w \in E$. Now let $w_0 \in E$. It follows that there exist vectors $v_{m_1}(w_0), v_{m_2}(w_0), ..., v_{m_k}(w_0)$ which are linearly independent. Consider the determinant $r(w)$ of these k vectors $v_{m_1}(w), v_{m_2}(w), ..., v_{m_k}(w)$. Then $r(w)$ is a polynomial and $r(w_0)\neq 0$. Notice that $Z(r)\cap T^n$ is a null-measurable subset of T^n. If we write E' for $E-Z(r)\cap T^n$, then $m_n(E')=m_n(E)>0$. On E', we can use the corresponding k equations

$$F_{m_t}(w)+F_{m_t-1}(w)\, x_1+\cdots+F_{m_t-k}(w)\, x_k=0 \qquad (t=1, 2, ..., k),$$

to solve for the α_i. By Cramer's rule, there are rational functions $h_1, h_2, ..., h_k$, whose denominators have no zeros on E', such that $\alpha_i(w)=h_i(w)$ for all $w \in E'$, $i=1, 2, ..., k$. The equalities

$$h_i'=F_i+F_{i-1}h_1+\cdots+F_0 h_i \qquad (i=0, 1, ..., k), \qquad (\bigstar\bigstar)$$

then define rational functions $h_0', h_1', ..., h_k'$, whose denominators have no zeros on E', such that

$$f_w(z)=f(zw)=\frac{h_0'(w)+h_1'(w)\, z+\cdots+h_k'(w)\, z^k}{1+h_1(w)\, z+\cdots+h_k(w)\, z^k}$$

for $w \in E'$. Since f is in the Nevanlinna class, $f(w)=\lim_{r\to 1} f(rw)$ exist for almost all $w \in T^n$. It follows that there exists a subset E'' of E' such that $m_n(E'')>0$, and on E'',

$$f(w)(1+h_1(w)+\cdots+h_k(w))=h_0'(w)+h_1'(w)+\cdots+h_k'(w).$$

Since h_i, h'_j are rational functions for all i and j, we multiply the two sides of the above equality by a polynomial p so that the functions $p(w)(1+h_1(w)+\cdots+h_k(w))$ and $p(w)(h'_0(w)+h'_1(w)+\cdots+h'_k(w))$ become polynomials. Therefore, there exist polynomials q_1 and q_2 such that on E'',

$$f(w)\,q_1(w)=q_2(w).$$

By Rudin [Ru1, Theorem 3.3.5], we see that E'' is a determining set for Nevanlinna functions. Therefore for almost all $w \in T^n$,

$$f(w)\,q_1(w)=q_2(w).$$

So, for every $z \in D^n$, we have

$$f(z)\,q_1(z)=q_2(z).$$

Now let us assume that the common factors of q_1 and q_2 have been cancelled. Thus by Lemma 3.2, $f(z)=q_2(z)/q_1(z)$ is a rational function, and $Z(q_1)\cap D^n=\emptyset$, completing the proof.

Let P be a prime ideal in the polynomial ring $\mathscr{C}$. The height of P is defined to be the maximal length l of all properly increasing chains of prime ideals

$$0=P_0\subset P_1 \cdots \subset P_l=P.$$

Since the ring $\mathscr{C}$ is Noetherian every prime ideal has finite height and the height of an arbitrary ideal is defined to be the minimum of the heights of its associated prime ideals. For an ideal I, one has $dim_C\, Z(I)=n-l$, where l is the height of I, $dim_C\, Z(I)$ the complex dimension of the zero variety of I (cf. [DPSY]). It is well known that analytic varieties in C^n of codimension at least 2 are removable singularities for analytic functions, (cf. [KK]). This says that if *height* $I\geqslant 2$, then $Z(I)$ is a removable singularity.

Lemma 3.4. *Assume that I is an ideal of polynomials such that the greatest common divisor $GCD(I)$ of I is equal to* 1. *Then height* $I\geqslant 2$.

Proof. Let $I=\bigcap_{i=1}^{l} I_i$ be an irredundant primary decomposition of I with its associated prime ideals $P_1, P_2, ..., P_l$. For a prime ideal P, it is not difficult to verify that *height* $P=1$ if and only if P is principal. Now suppose that some P_i is principal, that is, there is some prime polynomial p_i such that $P_i=p_i\mathscr{C}$. Since

$$I=\bigcap_{i=1}^{l} I_i\subseteq I_i\subseteq P_i=p_i\mathscr{C},$$

every polynomial in I is divisible by p_i. This is impossible. We thus conclude that *height* $P_i \geqslant 2$ for $i=1, 2, ..., l$. So, *height* $I \geqslant 2$.

A straightforward corollary of Lemma 3.4 is

LEMMA 3.5. *Let $I=pL$ be the Beurling form of the ideal I. If there is a function $\varphi \in L^\infty(T^n)$ such that $\varphi I \subset H^2(D^n)$, then φp is in $H^\infty(D^n)$.*

Proof. By Lemma 3.4, we see that the submodule $[L]$ satisfies the condition (*)in [DY](see [DY, Definition 5]). Since $\varphi p[L] \subset H^2(D^n)$, using Theorem 1 in [DY], we obtain $\varphi p \in H^\infty(D^n)$.

Proof of Theorem 3.1. $(\Rightarrow)$ By assumption, there is a unimodular function η such that

$$p_1 q_1 = \eta p_2 q_2.$$

Since each q_i is a generator of $H^2(D^n)$ (see Proposition 2.9), we see that

$$[I_1]=[p_1 L_1]=[p_1 L_2]=[p_1 q_1 L_2]=\eta[p_2 q_2 L_2]=\eta[p_2 L_2]=\eta[I_2],$$

and hence $[I_1]$ and $[I_2]$ are unitarily equivalent.

$(\Leftarrow)$ Suppose that $[I_1]$ and $[I_2]$ are unitarily equivalent. Thus there exists a unimodular function η such that

$$\eta[I_1]=[I_2].$$

Let $\{p_2 q_1, ..., p_2 q_k\}$ be a set of generators of I_2, and hence a set of generators of $[I_2]$. By Theorem 2.3, every function $g(z)$ in $[I_2]$ has form $g(z)=p_2(z)\,\gamma(z)$, where $\gamma(z)$ is analytic on D^n. From Lemma 3.5, $\eta p_1 \in H^\infty(D^n)$. This implies that for each $f \in L_1$, there is an unique analytic function h_f on D^n such that

$$(\eta p_1)(z)\, f(z) = p_2(z)\, h_f(z).$$

Now for $f_1 \in L_1$, we define an analytic function on $D^n \backslash Z(f_1)$ by

$$\phi_{f_1}(z) = \frac{h_{f_1}(z)}{f_1(z)}, \qquad \forall z \in D^n \backslash Z(f_1).$$

For another $f_2 \in L_1$, we also define an analytic function on $D^n \backslash Z(f_2)$ by

$$\phi_{f_2}(z) = \frac{h_{f_2}(z)}{f_2(z)}, \qquad \forall z \in D^n \backslash Z(f_2).$$

Since

$$(\eta p_1)(z)\, f_1(z)\, f_2(z) = p_2(z)\, h_{f_1}(z)\, f_2(z) = p_2(z)\, h_{f_2}(z)\, f_1(z), \qquad \forall z \in D^n,$$

we have

$$\phi_{f_1}(z) = \phi_{f_2}(z), \qquad \forall z \in D^n \backslash Z(f_1) \cup Z(f_2).$$

The above argument shows that for any $z \in D^n \backslash Z(L_1)$, we can define $\phi(z) = h_f(z)/f(z)$ for any $f \in L_1$ with $f(z) \neq 0$ and ϕ is independent of f, and $\phi(z)$ is analytic on $D^n \backslash Z(L_1)$.

From Lemma 3.4, one sees that *height* $L_1 \geqslant 2$, and hence by [KK], $D^n \cap Z(L_1)$ is a removable singularity for analytic functions. This shows that $\phi(z)$ extends to an analytic function on all of D^n. Now we regard $\phi(z)$ as an analytic function on D^n, and notice that for $f \in L_1$,

$$(\eta p_1)(z)\, f(z) = p_2(z)\, \phi(z)\, f(z).$$

It follows that

$$(\eta p_1)(z) = p_2(z)\, \phi(z).$$

For $0 < r < 1$,

$$\log |(\eta p_1)(rz)| = \log |p_2(rz)| + \log |\phi(rz)|.$$

Because $(\eta p_1)(z)$, $p_2(z)$ are bounded analytic functions on D^n, we conclude that $\{\log |\phi_r|\}_{0<r<1}$ is a bounded set in $L^1(dm_n)$. Thus ϕ is in the Nevanlinna class.

Also notice that

$$\bar{\eta}[I_2] = [I_1].$$

Just as in the above discussion, there is a Nevanlinna class function $\psi(z)$ on D^n such that

$$(\bar{\eta} p_2)(z) = p_1(z)\, \psi(z).$$

Since

$$(\eta p_1)(\bar{\eta} p_2) = p_2 p_1 \phi \psi$$

on T^n, one sees that $\phi\psi = 1$ on T^n. By the same reasoning, one shows that $\phi(z)\, \psi(z)$ is also in the Nevanlinna class. This implies that $\phi(z)\, \psi(z) = 1$. It follows that $\frac{1}{\phi(z)}$ is in the Nevanlinna class. Now by [Ru1], for almost all

$w \in T^n$, the slice function $\phi_w(z) = \phi(zw)$ is in the Nevanlinna class on D, and hence by [Gar], there are singular inner functions $\eta_1^{(w)}, \eta_2^{(w)}$ and outer functions $f_1^{(w)}, f_2^{(w)}$ in $H^2(D)$ such that

$$\phi_w = \frac{f_1^{(w)} \eta_1^{(w)}}{f_2^{(w)} \eta_2^{(w)}}.$$

So,

$$(\eta p_1)_w(z)\, \eta_2^{(w)}(z)\, f_2^{(w)}(z) = \eta_1^{(w)}(z)\, f_1^{(w)}(z)\, p_{2w}(z), \qquad \forall z \in D.$$

Since the inner factor of $p_{2w}(z)$ is a finite Blaschke product, and $\eta_2^{(w)}(z)$ is singular, $\eta_1^{(w)}(z)$ is divisible by $\eta_2^{(w)}(z)$. Since $\frac{1}{\phi(z)}$ is in the Nevanlinna class, and by Lemma 3.5, $\bar{\eta} p_2 \in H^\infty(D^n)$. From the equality $\bar{\eta} p_2 = \frac{1}{\phi} p_1$, as above, $\eta_2^{(w)}(z)$ is divisible by $\eta_1^{(w)}(z)$. So, there exists a unimodular constant c_w such that

$$\phi_w = c_w \frac{f_1^{(w)}}{f_2^{(w)}}.$$

Now let $p(z, w)$ be the Poisson kernel for D^n. Then

$$(\eta p_1)(z) = \int_{T^n} p(z, w)(\eta p_1)(w)\, dm_n(w).$$

This implies that

$$|(\eta p_1)(z)| = \left| \int_{T^n} p(z, w)(\eta p_1)(w)\, dm_n(w) \right| \leqslant \int_{T^n} p(z, w)\, |p_1(w)|\, dm_n(w).$$

Set

$$\widetilde{p_1}(z) = \int_{T^n} p(z, w)\, |p_1(w)|\, dm_n(w).$$

Then $\widetilde{p_1}(z)$ extends to a continuous function on $\bar{D}^n$, and $\widetilde{p_1}(w) = |p_1(w)|$ on T^n. For $w \in T^n$, let $(\eta p_1)_w^*$ be the radial limit of $(\eta p_1)_w(z)$. Thus, one has

$$|(\eta p_1)_w^*(e^{i\theta})| \leqslant |p_{1w}(e^{i\theta})|.$$

We denote the outer factor of $(\eta p_1)_w(z)$ by $\widetilde{f}_w(z)$, and the inner factor by $\widetilde{\eta_w}(z)$. Let $p_1^{(w)}, p_2^{(w)}$ be the outer factors of p_{1w}, p_{2w}, and let $\eta_{p_1}^{(w)}, \eta_{p_2}^{(w)}$ be inner factors of p_{1w}, p_{2w}, respectively. It is easy to see that $p_1^{(w)}$, $p_2^{(w)}$ are polynomials in one variable, and $\eta_{p_1}^{(w)}$, $\eta_{p_2}^{(w)}$ are finite Blaschke products. By the equality

$$\tilde{f}_w(z)\,\tilde{\eta}_w(z)\,f_2^{(w)}(z)=c_w f_1^{(w)}(z)\,p_2^{(w)}(z)\,\eta_{p_2}^{(w)}(z),$$

we see that there is a unimodular constant c'_w such that

$$\tilde{f}_w(z)\,f_2^{(w)}(z)=c'_w f_1^{(w)}(z)\,p_2^{(w)}(z).$$

Since

$$|\tilde{f}_w\,(e^{i\theta})|=|(\eta p_1)_w^*(e^{i\theta})|\leqslant|p_{1w}(e^{i\theta})|=|p_1^{(w)}(e^{i\theta})|,$$

by [Gar], we obtain

$$|\tilde{f}_w(z)|\leqslant|p_1^{(w)}(z)|,\qquad \forall z\in D.$$

It follows that

$$|c'_w f_1^{(w)}(z)\,p_2^{(w)}(z)|=|\tilde{f}_w(z)\,f_2^{(w)}(z)|\leqslant|p_1^{(w)}(z)\,f_2^{(w)}(z)|,\qquad \forall z\in D.$$

This implies that

$$|\phi_w(z)|\leqslant\frac{|p_1^{(w)}(z)|}{|p_2^{(w)}(z)|}.$$

From the equality $(\bar{\eta}p_2)(z)=\frac{1}{\phi(z)}\,p_1(z)$, and the fact $\bar{\eta}p_2\in H^\infty(D^n)$, similar reasoning shows that

$$\frac{1}{|\phi_w(z)|}\leqslant\frac{|p_2^{(w)}(z)|}{|p_1^{(w)}(z)|}.$$

We thus conclude that

$$|\phi_w(z)|=\frac{|p_1^{(w)}(z)|}{|p_2^{(w)}(z)|}.$$

Because $\phi_w(z)=c_w(f_1^{(w)}(z)/f_2^{(w)}(z))$, and $f_1^{(w)}(z)$ and $f_2^{(w)}(z)$ are outer, there is a unimodular constant c''_w such that

$$\phi_w(z)=c_w\,\frac{f_1^{(w)}(z)}{f_2^{(w)}(z)}=c''_w\,\frac{p_1^{(w)}(z)}{p_2^{(w)}(z)}.$$

Hence, for almost all $w \in T^n$, ϕ_w is rational function in one variable. From Lemma 3.2 and Lemma 3.3, we see that there exist polynomials $q_1(z)$, $q_2(z)$ with $Z(q_1) \cap D^n = Z(q_2) \cap D^n = \emptyset$ such that

$$\phi(z) = \frac{q_2(z)}{q_1(z)}.$$

Since $\eta p_1 = \phi p_2$, we conclude that

$$|p_1 q_1| = |p_2 q_2|.$$

Next we will show that $[p_1 L_1] = [p_1 L_2]$. By the equality $|p_1 q_1| = |p_2 q_2|$, there is a unimodular function η', such that $p_2 q_2 = \eta' p_1 q_1$. Because each q_i is a generator of $H^2(D^n)$ for $i = 1, 2$ (see Proposition 2.9), we have

$$[I_2] = [p_2 L_2] = [p_2 q_2 L_2] = \eta' [p_1 q_1 L_2] = \eta' [p_1 L_2].$$

Since $[I_1]$ and $[I_2]$ are unitarily equivalent, $[p_1 L_1]$ and $[p_1 L_2]$ are unitarily equivalent. Thus, there is a unimodular function η such that

$$\eta [p_1 L_1] = [p_1 L_2].$$

As in the above proof, we see that there exists a Nevanlinna class function $\phi(z)$ which does not vanish at any point in D^n, with $\frac{1}{\phi(z)}$ also in the Nevanlinna class, such that $\eta p_1 = \phi p_1$. Also as above, for almost all $w \in T^n$, there exists a unimodular constant c_w such that

$$\phi_w(z) = c_w \frac{p_1^{(w)}(z)}{p_1^{(w)}(z)} = c_w.$$

So, $\phi_w(z) = \phi(0)$. Since for almost all $w \in T^n$,

$$\phi(w) = \lim_{r \to 1} \phi(rw) = \lim_{r \to 1} \phi_w(r) = \phi(0),$$

we conclude that $\phi(z)$ is a nonzero constant. So, η is a unimodular constant. This gives that $[p_1 L_1] = [p_1 L_2]$. The proof of Theorem 3.1 is completed.

COROLLARY 3.6. *Let p_1, p_2 be two polynomials. Then $[p_1]$ and $[p_2]$ are unitarily equivalent if and only if $p_1 \asymp p_2$.*

Let p, q be two polynomials such that the zero set of each of their prime factors meets D^n nontrivially. For two such polynomials, in [Guo1], the author conjectured that if $[p]$ and $[q]$ are unitarily equivalent, then there exists an invertible analytic function f in $H^\infty(D^n)$ such that $\frac{|p|}{|q|}=|f|$ on T^n. The next example shows that this conjecture is not true in general. Let

$$p(z_1, z_2)=z_1+z_2+2z_1z_2, \qquad q(z_1, z_2)=z_1+z_2-2z_1z_2$$

be two polynomials on C^2. Since on T^2,

$$\frac{|z_1+z_2+2z_1z_2|}{|z_1+z_2-2z_1z_2|}=\frac{|z_1+z_2+2|}{|z_1+z_2-2|},$$

one sees that $[p]$ and $[q]$ are unitarily equivalent. However, because

$$\lim_{(z_1, z_2)\to(-1, -1)} \frac{|z_1+z_2+2|}{|z_1+z_2-2|}=0,$$

there is not any invertible analytic function f in $H^\infty(D^n)$ such that $\frac{|p|}{|q|}=|f|$ on T^n.

LEMMA 3.7. *Let $f\in Hol(D^n)$. If for almost all $w\in T^n$, the slice function $f_w(z)=f(zw)$ is a polynomial, then f is a polynomial.*

Proof. The proof is similar to that of Corollary 3.6 in [Guo1]. For completeness, we give the details of the proof. Let $f=F_0+F_1+\cdots$ be f's homogeneous expression. For almost all $w\in T^n$, since

$$f_w(z)=\sum_{n\geqq 0} F_n(zw)=\sum_{n\geqq 0} F_n(w)\, z^n,$$

there exists a measurable subset $\tilde{T}^n$ of T^n with $m_n(\tilde{T}^n)=1$ such that, for each $w\in\tilde{T}^n$, there is a natural number $n(w)$ which satisfies $F_n(w)=0$ if $n\geqslant n(w)$. Assume that there exist infinitely many $F_{k_1}, ..., F_{k_n}, ...$ that are not zero. Since

$$\tilde{T}^n\subseteq\bigcup_{i=1}^{\infty}(Z(F_{k_i})\cap T^n)$$

and each $Z(F_{k_i})\cap T^n$ is null-measurable on T^n, this leads to a contradiction. We therefore conclude that there exist only finitely many F_i's such that $F_i\neq 0$, that is, f is a polynomial. This completes the proof of Lemma 3.7.

COROLLARY 3.8. *Let p be a polynomial, and let q be a homogeneous polynomial. Then $[p]$ and $[q]$ are unitarily equivalent if and only if there* *exists a polynomial r with $Z(r) \cap D^n = \emptyset$ such that $|p| = |rq|$ on T^n. In particular, if p, q are homogeneous, then $[p]$ and $[q]$ are unitarily equivalent if and only if there exists a constant c such that $|p| = c\,|q|$ on T^n.*

Proof. By Corollary 3.6, sufficiency is obvious. If $[p]$ and $[q]$ are unitarily equivalent, Corollary 3.6 says that there exist two polynomials q_1, q_2 with $Z(q_1) \cap D^n = Z(q_1) \cap D^n = \emptyset$ such that on T^n,

$$\frac{|p|}{|q|} = \frac{|q_1|}{|q_2|}.$$

For any $z \in D^n$, set

$$r(z) = \frac{q_1(z)}{q_2(z)}.$$

Since q_{1w}, q_{2w} are outer functions in $H^2(D)$, there exists a unimodular constant c_w such that for any $z \in D$,

$$q_{2w}(z)\, p^{(w)}(z) = c_w\, q_{1w}(z)\, q(w),$$

where $p^{(w)}(z)$ is the outer factor of $p_w(z)$. Thus,

$$r_w(z) = \frac{q_{1w}(z)}{q_{2w}(z)} = \bar{c}_w\, \frac{p^{(w)}(z)}{q(w)}$$

for almost all $w \in T^n$. Since $p^{(w)}(z)$ is a polynomial, Lemma 3.7 implies that $r(z)$ is a polynomial, and $Z(r) \cap D^n = \emptyset$. The remaining case is obvious. This completes the proof of Corollary 3.8.

Combining Theorem 3.1 with Corollary 3.8, we immediately obtain

COROLLARY 3.9. *Let $I = pL$ be the Beurling form of the ideal I. Then $[I]$ and $H^2(D^n)$ are unitarily equivalent if and only if there exists a polynomial r with $Z(r) \cap D^n = \emptyset$ such that $|p| = |r|$ on T^n, and L is dense in $H^2(D^n)$.*

Remark. In Corollary 3.9, the condition $|p| = |r|$ is a very restrictive condition on p. Here, we give an exact characterization for those p for which there exists a polynomial r with $Z(r) \cap D^n = \emptyset$ such that $|p| = |r|$ on T^n. Decompose $p = p_1 p_2$ such that the zero set of each prime factor of p_1 meets D^n nontrivially, and each of p_2 does not. We call p_1 the D^n-factor of p. From Proposition 2.9 and Corollary 3.9, we see that p has the property mentioned above if and only if p_1 has. Thus, for simplicity, we may assume that the zero set of each prime factor of p meets D^n nontrivially. Let q be a polynomial. We use $\tilde{q}$ to denote the polynomial whose coefficients are the complex conjugates of the coefficients of q. Therefore on T^n, $\overline{q(w)} = \tilde{q}(\bar{w})$. Now for a polynomial p with the property mentioned above, by [Ru1, Theorem 5.2.4], there exist a polynomial q with $Z(q) \cap D^n = \emptyset$, and a monomial $\tau(z)$ such that $\tau(z)\, \tilde{q}(\frac{1}{z})$ is a polynomial and

$$\frac{p(z)}{r(z)} = \frac{\tau(z)\, \tilde{q}(1/z)}{q(z)}, \qquad \forall z \in D^n.$$

So,

$$p(z)\, q(z) = \tau(z)\, \tilde{q}\left(\frac{1}{z}\right) r(z), \qquad \forall z \in D^n.$$

Since the zero set of each prime factor of p meets D^n nontrivially, and $Z(r) \cap D^n = \emptyset$, $r \mid pq$ implies $r \mid q$. It follows that there is a polynomial h with $Z(h) \cap D^n = \emptyset$ such that

$$p(z)\, h(z) = \tau(z)\, \tilde{q}\left(\frac{1}{z}\right) \qquad \forall z \in D^n.$$

Therefore p is the D^n-factor of $\tau(z)\, \tilde{q}(\frac{1}{z})$. Conversely, it is easy to check that if p is the D^n-factor of some $\tau(z)\, \tilde{q}(\frac{1}{z})$, then p has the property mentioned above. We thus conclude that a polynomial p has the property mentioned above if and only if there exists a polynomial q with $Z(q) \cap D^n = \emptyset$ such that $p(z)$ and $\tau(z)\, \tilde{q}(\frac{1}{z})$ have the same D^n-factor, where $\tau(z)$ is a monomial such that $\tau(z)\, \tilde{q}(\frac{1}{z})$ is a polynomial.

Definition. An ideal I is said to be homogeneous if the relation $p \in I$ implies that all homogeneous components of p are in I. Equivalently, an ideal I is homogeneous if and only if I is generated by homogeneous polynomials.

Let I be homogeneous, and $I = qL$ be the Beurling form of I. Then it is easy to check that both q and L are homogeneous.

The next corollary generalizes Theorem 2 in [Yan].

Corollary 3.10. *Let I_1, I_2 be homogeneous, and $I_1 = p_1 L_1$, $I_2 = p_2 L_2$ be their Beurling forms. Then $[I_1]$ and $[I_2]$ are unitarily equivalent if and only if there exists a constant c such that $|p_1| = c|p_2|$ on T^n and $L_1 = L_2$.*

Proof. From Theorem 3.1, the sufficiency is immediate. Now assume that $[I_1]$ and $[I_2]$ are unitarily equivalent. Then as in the proof of Corollary 3.8, one sees that there exists a constant c such that $|p_1| = c\, |p_2|$

on T^n. Note that both $p_1 L_1$ and $p_1 L_2$ are homogeneous, and hence by [ZS, Vol. (II), p. 153, Theorem 9 and its Corollary], each of their associated prime ideals is homogeneous. Now combining Theorem 3.1 with [DPSY, Theorem 2.7], the equality $[p_1 L_1] = [p_1 L_2]$ implies that $p_1 L_1 = p_1 L_2$, and therefore $L_1 = L_2$, completing the proof.

Now let us endow the ring $\mathscr{C}$ with the topology induced by the Hardy space $H^2(D^n)$. It is easy to see that studying the unitary equivalence of submodules generated by ideals and by their closures is the same thing. For an ideal I, we write $\bar{I}$ for the closure of I under the Hardy topology.

The following theorem is an equivalent form of Theorem 3.1.

THEOREM 3.11. *Let I_1, I_2 be two ideals of polynomials, and let $\bar{I}_1 = p_1 L_1$, $\bar{I}_2 = p_2 L_2$ be the Beurling forms of their closures. Then $[I_1]$ and $[I_2]$ are unitarily equivalent if and only if $p_1 \asymp p_2$ and $L_1 = L_2$.*

Proof. It is easy to see that the sufficiency is obvious. Now suppose that $[I_1]$ and $[I_2]$ are unitarily equivalent, that is, $[\bar{I}_1]$ and $[\bar{I}_2]$ are unitarily equivalent. By Theorem 3.1, one immediately obtains that $p_1 \asymp p_2$, and $[p_1 L_1] = [p_1 L_2]$, $[p_2 L_2] = [p_2 L_1]$. Since $p_1 L_1$, $p_2 L_2$ are closed ideals, we have

$$p_1 L_1 \supseteq p_1 L_2, \qquad p_2 L_2 \supseteq p_2 L_1.$$

The above inclusions imply that $L_1 = L_2$.

When $n = 2$ we can give simpler conditions for $[I_1]$ and $[I_2]$ to be unitarily equivalent.

THEOREM 3.12. *Let I_1 and I_2 be ideals in two variables, and let $I_1 = p_1 L_1$, $I_2 = p_2 L_2$ be their Beurling forms. Then the submodules $[I_1]$, $[I_2]$ of $H^2(D^2)$ are unitarily equivalent if and only if $p_1 \asymp p_2$ and $\overline{L_1} = \overline{L_2}$.*

Proof. It is easy to see that the sufficiency is obvious. Now we decompose p_1 as the product of p_1' and p_1'' such that the zero set of each of the prime factors of p_1' meets D^2 nontrivially, and each of p_1'' does not. Combining Lemma 2.6 with [DPSY, Proposition 2.9], we see that $\overline{p_1 L_1} = p_1' \overline{L_1}$, and $\overline{p_1 L_2} = p_1' \overline{L_2}$. Thus, if $[I_1]$ and $[I_2]$ are unitarily equivalent, then by Theorem 3.1 one immediately obtains that $p_1 \asymp p_2$, and

$$p_1' \overline{L_1} = [p_1 L_1] \cap \mathscr{C} = [p_1 L_2] \cap \mathscr{C} = p_1' \overline{L_2}.$$

This gives that $\overline{L_1} = \overline{L_2}$, completing the proof.

4. SIMILARITY OF HARDY SUBMODULES GENERATED BY POLYNOMIALS

In this section, we consider the similarity problem by the methods used in Sections 2 and 3.

Let $I_1 = p_1 L_1$, $I_2 = p_2 L_2$ be two ideals of polynomials, and let both module maps $X: [I_1] \to [I_2]$ and $Y: [I_2] \to [I_1]$ have dense range. Then by [DY], there exist $\phi, \psi \in L^\infty(T^n)$ such that $X = M_\phi$, $Y = M_\psi$. By Lemma 3.5, $f = \phi p_1$, $g = \psi p_2$ are in $H^\infty(D^n)$. As in the proof of Theorem 3.1, there are analytic functions r_1 and r_2 such that $f = r_1 p_2$, $g = r_2 p_1$.

LEMMA 4.1. *Under the above statements, both r_1 and r_2 have no zero points in D^n.*

Proof. Suppose that there exists a point $z_0 \in D^n$ such that $r_1(z_0) = 0$. From Lemma 2.2, we obtain

$$[fL_1]^{(z_0)} = [r_1 p_2 L_1]^{(z_0)} = r_{1z_0} p_{2z_0} L_1^{(z_0)} = [p_2 L_2]^{(z_0)} = p_{2z_0} L_2^{(z_0)},$$

and

$$[gL_2]^{(z_0)} = [r_2 p_1 L_2]^{(z_0)} = r_{2z_0} p_{1z_0} L_2^{(z_0)} = [p_1 L_1]^{(z_0)} = p_{1z_0} L_1^{(z_0)},$$

where $L_i^{(z_0)}$ denote the ideals of $\mathcal{O}_{z_0}$ generated by $\{p_{z_0} : p \in L_i\}$, and r_{iz_0}, p_{iz_0} denote the elements of $\mathcal{O}_{z_0}$ defined by the restriction of r_i, p_i to neighborhoods of z_0, $i = 1, 2$. From the above equalities, we obtain

$$L_2^{(z_0)} = r_{1z_0} r_{2z_0} L_2^{(z_0)}.$$

By Nakayama's lemma (see [AM, Proposition 2.6]), this is impossible, and hence r_1 has no zeros in D^n. For the same reason, r_2 has no zeros in D^n.

PROPOSITION 4.2. *Let $I_1 = p_1 L_1$, $I_2 = p_2 L_2$ be two ideals of polynomials such that each of their algebraic components meets D^n nontrivially. If $[I_1]$ and $[I_2]$ are quasi-similar, then $L_1 = L_2$.*

Proof. By the assumptions, there exist module maps $X: [I_1] \to [I_2]$ and $Y: [I_2] \to [I_1]$ with dense ranges. Thus, there exist $\phi, \psi \in L^\infty(T^n)$ such that $X = M_\phi$, $Y = M_\psi$. Then by Lemma 4.1, there are analytic functions r_1 and r_2, each of which has no zeros in D^n such that $\phi p_1 = r_1 p_2$, $\psi p_2 = r_2 p_1$. Note that ϕp_1, ψp_2 are in $H^\infty(D^n)$. We thus have that for $\lambda \in D^n$,

$$(p_2 L_2)_\lambda = [p_2 L_2]_\lambda = [\phi p_1 L_1]_\lambda = [r_1 p_2 L_1]_\lambda = [p_2 L_1]_\lambda = (p_2 L_1)_\lambda$$

and

$$(p_1 L_1)_\lambda = [p_1 L_1]_\lambda = [\psi p_2 L_2]_\lambda = [r_2 p_1 L_2]_\lambda = [p_1 L_2]_\lambda = (p_1 L_2)_\lambda,$$

where $(p_1 L_1)_\lambda$ and $(p_2 L_2)_\lambda$ are the characteristic spaces of the ideals $p_1 L_1$ and $p_2 L_2$ at λ, respectively (cf. [Guo1]). Using [Guo1, Corollary 2.3], we see that

$$p_2 L_2 \supset p_2 L_1, \qquad p_1 L_1 \supset p_1 L_2$$

and hence $L_1 = L_2$, which proves the assertion.

The following theorem strengths [Yan, Theorem 1].

THEOREM 4.3. *Let $I_1 = p_1 L_1$, $I_2 = p_2 L_2$ be homogeneous ideals. Then the following are equivalent*:

(1) *$[I_1]$ and $[I_2]$ are similar*;

(2) *$[I_1]$ and $[I_2]$ are quasi-similar*;

(3) *there exists a constant c such that $c < |p_1|/|p_2| < c^{-1}$ on T^n, and $L_1 = L_2$.*

Proof. (1) ⇒ (2). This is obvious.

(2) ⇒ (3). Because $p_1 L_1$ and $p_2 L_2$ are homogeneous ideals, each of their associated prime ideals is homogeneous (cf. [ZS, Vol. (II), p. 153, Theorem 9 and its Corollary]). We apply Proposition 4.2 to obtain $L_1 = L_2$. We use some techniques in [Yan]. Let both module maps $X: [I_1] \to [I_2]$ and $Y: [I_2] \to [I_1]$ have dense range. Then there exist $\phi, \psi \in L^\infty(T^n)$ such that $X = M_\phi$, $Y = M_\psi$. By Lemma 3.5, $f = \phi p_1$, $g = \psi p_2$ are in $H^\infty(D^n)$. Since p_1 is homogeneous, from the equality $f = \phi p_1$, we see that

$$|f(w)| \leqslant \|\phi\|_\infty \, |p_1(w)| \qquad \text{a.e. on } T^n.$$

Therefore,

$$|f_w(e^{i\theta})| \leqslant \|\phi\|_\infty \, |p_1(w)|,$$

for almost all $w \in T^n$. This leads to the inequality

$$|f_w(z)| \leqslant \|\phi\|_\infty \, |p_1(w)|, \qquad z \in D,$$

for almost all $w \in T^n$, and hence for all $w \in T^n$, we have the inequality

$$|f_w(z)| \leqslant \|\phi\|_\infty \, |p_1(w)|, \qquad z \in D.$$

Suppose there is a sequence $w_n \to w_0$ such that

$$\frac{|p_1(w_n)|}{|p_2(w_n)|} \to 0.$$

By Lemma 4.1, $f = r_1 p_2$. Now for every fixed $z \neq 0$, $z \in D$, we have

$$|r_1(zw_n)| = \frac{|f(zw_n)|}{|p_2(zw_n)|} \leqslant \frac{\|\phi\|_\infty \, |p_1(w_n)|}{|p_2(w_n)| \, |z|^k} \to 0,$$

where $k = deg\ p_2$. This implies that $r_1(zw_0) = 0$. From Lemma 4.1, this is impossible, and hence there exists a positive constant c' such that

$$c' < \frac{|p_1|}{|p_2|}$$

on T^n. Similarly, there exists a positive constant c'' such that

$$c'' < \frac{|p_2|}{|p_1|}$$

on T^n. Thus, there exists a constant c such that

$$c < \frac{|p_1|}{|p_2|} < c^{-1}$$

on T^n.

(3) ⇒ (1). Set $\phi = p_2/p_1$ and $\psi = p_1/p_2$. It is easy to see that module maps

$$M_\phi\colon [I_1] \to [I_2], \qquad M_\psi\colon [I_2] \to [I_1]$$

give similarity between $[I_1]$ and $[I_2]$.

From Theorem 4.3, one sees that under the conditions of Theorem 4.3, quasi-similarity implies similarity. We do not know if there exists an example of two submodules generated by polynomials that are quasi-similar, but not similar.

Furthermore, from Proposition 4.2 and Theorem 4.3, one sees that question about the similarity of submodules can be reduced to question about the similarity of principal submodules. Then one wants to know when two principal submodules are similar.

PROPOSITION 4.4. *Let p_1 and p_2 be two polynomials. If there exist polynomials q_1, q_2 with $Z(q_1) \cap D^n = Z(q_2) \cap D^n = \emptyset$ such that $c < |p_1 q_1| / |p_2 q_2| < c^{-1}$ for some constant c, then $[p_1]$ and $[p_2]$ are similar.*

Proof. Set $\phi = p_1 q_1 / p_2 q_2$. Then $c < |\phi| < c^{-1}$. Since $p_i q_i$ is a generator of $[p_i]$ for $i = 1, 2$,

$$\phi[p_2] \subset [p_1] \qquad \text{and} \qquad \phi^{-1}[p_1] \subset [p_2].$$

It follows that the maps defined by

$$X_\phi : [p_2] \to [p_1], \quad X_\phi f = \phi f; \qquad X_{\phi^{-1}} : [p_1] \to [p_2], \quad X_{\phi^{-1}} f = \phi^{-1} f$$

are module maps. It is easy to see

$$X_\phi X_{\phi^{-1}} = 1, \qquad X_{\phi^{-1}} X_\phi = 1.$$

So, $[p_1]$ and $[p_2]$ are similar.

The preceding Proposition 4.4 and Corollary 3.6 thus suggest the following conjecture.

Conjecture. If $[p_1]$ and $[p_2]$ are similar, then there exist polynomials q_1, q_2 with $Z(q_1) \cap D^n = Z(q_2) \cap D^n = \emptyset$ such that $c < |p_1 q_1| / |p_2 q_2| < c^{-1}$ for some constant c. That is, the conditions in Proposition 4.4 are also necessary.

5. THE CASE OF HARDY SUBMODULES ON THE UNIT BALL B_n

We conclude this paper with a look at the case of Hardy submodules on the unit ball B_n. In [CD], X. M. Chen and R. G. Douglas proved that two homogeneous principal submodules of $H^2(B_n)$ are quasi-similar if and only if the corresponding homogeneous polynomials are equal. Combining this fact with Yan's work [Yan], one finds that the classification of submodules depends heavily on the geometric properties of domains.

By the methods of Section 3, we can obtain corresponding results in the case of Hardy submodules on the unit ball B_n. However, on the unit ball B_n, one has a special conclusion for polynomials.

PROPOSITION 5.1. *Let p_1, p_2 be two polynomials on $C^n (n > 1)$. If $|p_1| = |p_2|$ on ∂B_n (the boundary of B_n), then there is a unimodular constant c such that $p_1 = c p_2$.*

The proof of Proposition 5.1 is based on the remarkable Theorem 14.3.3 in Rudin's book [Ru2]. Assume $n>1$. Let Ω be a bounded domain in C^n, and let $A(\Omega)=C(\bar{\Omega})\cap Hol(\Omega)$ be the so called Ω-algebra. If $f\in A(\Omega)$, $g\in A(\Omega)$, and $|f(\lambda)|\leqslant|g(\lambda)|$ for each boundary point λ of Ω, then $|f(z)|\leqslant|g(z)|$ for every $z\in\Omega$. Also notice that if p is a polynomial, and $Z(p)\cap B_n=\emptyset$, then p is a generator of $H^2(B_n)$ (see the remark following Proposition 2.9). Thus, combining Proposition 5.1 with the methods in Section 3, in case of Hardy submodules on the unit ball B_n, our result is

THEOREM 5.2. *Let $I_1=p_1L_1$, $I_2=p_2L_2$ be the Beurling forms of I_1 and I_2. Then the following are equivalent*:

(1) *$[I_1]$ and $[I_2]$ are unitarily equivalent*;

(2) *there exist polynomials q_1 and q_2 with $Z(q_1)\cap B_n=Z(q_2)\cap B_n=\emptyset$ such that $p_1q_1=p_2q_2$, and $[p_1L_1]=[p_1L_2]$*;

(3) $[I_1]=[I_2]$.

COROLLARY 5.3. *Let p_1, p_2 be two polynomials. Then $[p_1]$ and $[p_2]$ are unitarily equivalent if and only if there exist polynomials q_1 and q_2 with $Z(q_1)\cap B_n=Z(q_2)\cap B_n=\emptyset$ such that $p_1q_1=p_2q_2$. In particular, if the zero set of each of the prime factors of p_i meets B_n nontrivially for $i=1, 2$, then $[p_1]$ and $[p_2]$ are unitarily equivalent if and only if there is a constant c such that $p_1=cp_2$.*

From Theorem 5.2, we see that there is more rigidity among submodules of $H^2(B_n)$ than in the case of $H^2(D^n)$. Furthermore, From the proof of Theorem 4.3 and Proposition 5.1, we have

THEOREM 5.4. *Let I_1, I_2 be homogeneous ideals. Then the following are equivalent*:

(1) *$[I_1]$ and $[I_2]$ are unitarily equivalent*;

(2) *$[I_1]$ and $[I_2]$ are similar*;

(3) *$[I_1]$ and $[I_2]$ are quasi-similar*;

(4) $I_1=I_2$.

Based on Theorems 5.2 and 5.4, one thus conjectures.

Conjecture. Let I_1 and I_2 be two ideals of polynomials, and let $[I_1]$, $[I_2]$ be the submodules of $H^2(B_n)$ generated by I_1, I_2 respectively. If $[I_1]$ and $[I_2]$ are quasi-similar, then $[I_1]=[I_2]$.

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REFERENCES

[ACD] O. P. Agrawal, D. N. Clark, and R. G. Douglas, Invariant subspaces in the polydisk, *Pacific J. Math.* **121** (1986), 1–11.

[AC] P. Ahern and D. N. Clark, Invariant subspaces and analytic contiuation in several variables, *J. Math. Mech.* **19** (1970), 963–969.

[AM] M. F. Atiyah and I. G. MacDonald, "Introduction to Commutative Algebra," Addison–Wesley, Menlo Park, CA, 1969.

[AS] O. P. Agrawal and N. Salinas, Sharp kernels and canonical subspaces, *Amer. J. Math.* **109** (1987), 23–48.

[CD] X. M. Chen and R. G. Douglas, Rigidity of Hardy submodules on the unit ball, *Houston J. Math.* **18** (1992), 117–125.

[DP] R. G. Douglas and V. I. Paulsen, "Hilbert Modules over Function Algebras," Vol. 217, Longman Research Notes in Math., Longman, Harlow, 1989.

[DPSY] R. G. Douglas, V. I. Paulsen, C. H. Sah, and K. R. Yan, Algebraic reduction and rigidity for Hilbert modules, *Amer. J. Math.* **117** (1995), 75–92.

[DY] R. G. Douglas and K. Yan, On the rigidity of Hardy submodules, *Integral Equations Operator Theory* **13** (1990), 350–363.

[GAR] J. B. Garnett, "Bounded Analytic Functions," Academic Press, New York, 1981.

[Ge1] R. Gelca, Rings with topologies induced by spaces of functions, *Houston J. Math.* **21** (1995), 395–405.

[Ge2] R. Gelca, Topological Hilbert nullstellensatz for Bergman spaces, *Integral Equations Operator Theory* **28** (1997), 191–195.

[Guo1] K. Y. Guo, Characteristic spaces and rigidity for analytic Hilbert modules, *J. Funct. Anal.* **163** (1999), 133–151.

[Guo2] K. Y. Guo, Algebraic reduction for Hardy submodules over polydisk algebras, *J. Operator Theory* **41** (1999), 127–138.

[HG] X. J. Hong and K. Y. Guo, Polynomial principal submodules on the strongly pseudoconvex domains, *J. Fudan Univ. Natur. Sci.* **37** (1998), 669–673. [In Chinese]

[KK] L. Kaup and B. Kaup, "Holomorphic Functions," de Gruyter, Berlin, 1983.

[Kr] S. G. Krantz, "Function Theory of Several Complex Variables," Wiley, New York, 1982.

[Ru1] W. Rudin, "Function Theory in Polydisks," Benjamin, New York, 1969.

[Ru2] W. Rudin, "Function Theory in the Unit Ball of C^n," Springer-Verlag, New York, 1980.

[Yan] K. Yan, Equivalence of homogeneous principle Hardy submodules, *Indiana Univ. Math. J.* **41** (1992), 875–883.

[Yang] R. W. Yang, The Berger–Shaw theorem in the Hardy module over the bidisk, *J. Operator Theory* **42** (1999), 379–404.

[ZS] O. Zariski and P. Samuel, "Commutative Algebra," Vols. I, II, Van Nostrand, Princeton, NJ, 1958/1960.