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L^1-Factorization for C_{00}-Contractions with Isometric Functional Calculus

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Let T be an absolutely continuous contraction acting on a Hilbert space $\mathscr{H}$. For $x, y \in \mathscr{H}$, define $x \overset{T}{\cdot} y \in L^1(\mathbb{T})$ by its Fourier coefficients: $x \overset{T}{\cdot} y\,\hat{}\,(n) = (T^{*n}x, y)$ if $n \geqslant 0$ and $x \overset{T}{\cdot} y\,\hat{}\,(n) = (T^{-n}x, y)$ if $n < 0$. The main technical result of the paper is that the vanishing condition $\lim_{n\to\infty} (\|x_n \overset{T}{\cdot} w\|_{L^1/H^1_0} + \|w \overset{T}{\cdot} x_n\|_{L^1/H^1_0}) = 0, \quad w \in \mathscr{H}$ implies that $\lim_{n\to\infty} \|x_n \overset{T}{\cdot} w\|_{L^1} = 0$, $w \in \mathscr{H}$. Using known factorization techniques, we exhibit a Borel set σ_T such that for any $f \in L^1(\sigma_T)$, there exist $x, y \in \mathscr{H}$ such that $f = (x \overset{T}{\cdot} y)_{|\sigma_T}$. In the case where $T \in \mathbb{A} \cap C_{00}$, this leads to a simple proof of the fact that for every $f \in L^1(\mathbb{T})$ there exists $x, y \in \mathscr{H}$ such that $f = x \overset{T}{\cdot} y$. In this case we also show, using dilation theory in the unit disk, that every strictly positive lower semicontinuous function $\varphi \in L^1(\mathbb{T})$ can be written in the form $\varphi = x \overset{T}{\cdot} x$. Examples show that this is the best possible result for the class $\mathbb{A} \cap C_{00}$.

1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite-dimensional complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. A contraction $T \in \mathscr{L}(\mathscr{H})$ is absolutely continuous if T is completely nonunitary or if the spectral measure of its unitary part is absolutely continuous with respect to Lebesgue measure. If $T \in \mathscr{L}(\mathscr{H})$ is an absolutely continuous

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contraction, then, for any $x, y \in \mathscr{H}$, there exists a function $x \overset{T}{\cdot} y \in L^1$ such that the Fourier coefficients of $x \overset{T}{\cdot} y$ satisfy:

$$(x \overset{T}{\cdot} y)^\wedge (n) = \begin{cases} (T^{*n}x, y) & n \geqslant 0 \\ (T^{-n}x, y) & n < 0. \end{cases}$$

We write $\mathbb{D}$ for the open unit disk in the complex plane $\mathbb{C}$, and $\mathbb{T}$ for the unit circle. The spaces $L^p = L^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$ are the usual Lebesgue function spaces relative to Lebesgue measure m on $\mathbb{T}$. The spaces $H^p = H^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$ are the usual Hardy spaces. It is well-known [13] that the dual space of L^1/H^1_0, where $H^1_0 = \{f \in L^1 : \int_0^{2\pi} f(e^{it})\, e^{int}\, dt = 0, n = 0, 1, ...\}$, can be identified by H^∞. If we denote by $[g]$ the class of $g \in L^1$ in L^1/H^1_0, the duality is given by the pairing:

$$\langle f, [g] \rangle = \int_{\mathbb{T}} fg\, dm, \qquad f \in H^\infty, \quad g \in L^1.$$

We denote by $\mathbb{A} = \mathbb{A}(\mathscr{H})$ the class of all absolutely continuous contractions $T \in \mathscr{L}(\mathscr{H})$ for which the Sz.-Nagy-Foias functional calculus $\Phi_T : H^\infty \to \mathscr{L}(\mathscr{H})$ is an isometry. In 1988, H. Bercovici and B. Chevreau have proved independently that if $T \in \mathbb{A}$, then, for any function $f \in L^1$, there exists $x, y \in \mathscr{H}$ such that $[f] = [x \overset{T}{\cdot} y]$, that is, $f^\wedge(-n) = (T^n x, y)$, $n \geqslant 0$ (see [3], [10]). Notice that if we take $f = 1$, we obtain a nontrivial invariant subspace for T.

In this paper, we study the possibility of solving exactly equations of the form $f = x \overset{T}{\cdot} y$ (that is, for any $n \in \mathbb{Z}$, $(x \overset{T}{\cdot} y)^\wedge (n) = f^\wedge(n)$) where f is a given function in L^1 and where $T \in \mathscr{L}(\mathscr{H})$ is an absolutely continuous contraction.

In the case where $T = S$, where S is the usual shift on H^2 (so that $x \overset{S}{\cdot} y = x\bar{y}$ for $x, y \in H^2$), this question has been solved by J. Bourgain in 1986, [9]: The above factorization holds for $f \in L^1$, $f \neq 0$ if and only if $\log |f| \in L^1$. Notice that if $\log |f| \notin L^1$, we can nevertheless find $x, y \in H^2$ such that we have $f^\wedge(n) = (x \overset{S}{\cdot} y)^\wedge (n)$, $n \neq 1$, whereas the Bercovici–Chevreau Theorem gives $f^\wedge(n) = (x \overset{S}{\cdot} y)^\wedge (n)$, $n \leqslant 0$.

Recall that if T is an absolutely continuous contraction on $\mathscr{H}$ and if σ is a Borel subset of $\mathbb{T}$, then σ is said to be essential for T (cf., Definition 3.1 in [10]) if:

$$\|f(T)\| \geqslant \|f_{|\sigma}\|_\infty, \qquad f \in H^\infty(\mathbb{T}).$$

We will denote by $Ess(T)$ the maximal essential Borel subset for T (see Proposition 3.3 of [10]). Also denote by Σ_T (resp. Σ_{*T}) the support of the spectral measure of the unitary part of the minimal isometric dilation (resp.

minimal coisometric extension) of T and by σ_T the Borel set $Ess(T)\setminus(\Sigma_T \cup \Sigma_{*T})$.

We show (Theorem 4.3) that if $f \in L^1(\sigma_T)$, the equation $f = (x \overset{T}{\cdot} y)_{|\sigma_T}$ has a solution in $\mathscr{H} \times \mathscr{H}$. More precisely, for any infinite array $(f_{i,j})_{i,j\geqslant 1}$ consisting of elements of $L^1(\sigma_T)$, there exist sequences $(x_i)_{i\geqslant 1}$, $(y_j)_{j\geqslant 1}$ of $\mathscr{H}$ such that $f_{i,j} = (x_i \overset{T}{\cdot} y_j)_{|\sigma_T}$, $i, j \geqslant 1$.

Recall that $C_{0\cdot} = C_{0\cdot}(\mathscr{H})$ is the class of all contractions $T \in \mathscr{L}(\mathscr{H})$ such that the sequence $(\|T^n x\|)_{n\geqslant 1}$ converges to zero for every $x \in \mathscr{H}$, and that $C_{\cdot 0}$ and C_{00} are defined by $C_{\cdot 0} = (C_{0\cdot})^*$, $C_{00} = C_{\cdot 0} \cap C_{0\cdot}$. In the case of the usual shift on H^2, $\Sigma_T = \varnothing$, but if $T \in \mathbb{A} \cap C_{00}$, then $\sigma_T = \mathbb{T}$, and so T has the following factorization property:

(1.1) For any infinite array $(f_{i,j})_{i,j\geqslant 1}$ consisting of elements of L^1, there exist sequences $(x_i)_{i\geqslant 1}$, $(y_j)_{j\geqslant 1}$ of $\mathscr{H}$ such that $f_{i,j} = (x_i \overset{T}{\cdot} y_j)$, $i, j \geqslant 1$.

This property of the class $\mathbb{A} \cap C_{00}$ was never explicitly stated in the literature, but it can be deduced immediately from Corollary 6.9 in [7] and Proposition 4.2 in [6] (see Remark 1, Section 5). The main new technical result of the paper is given by Theorem 3.2 which says that if $T \in \mathscr{L}(\mathscr{H})$ is an absolutely continuous contraction and if $(u_n)_n$ is a sequence of elements of $\mathscr{H}$ verifying $\lim_{n\to\infty}(\|[u_n \overset{T}{\cdot} w]\| + \|[w \overset{T}{\cdot} u_n]\|) = 0$ for every $w \in \mathscr{H}$, then $\lim_{n\to\infty} \|u_n \overset{T}{\cdot} w\|_1 = 0$ for every $w \in \mathscr{H}$. Using this theorem, we can prove that every $T \in \mathbb{A} \cap C_{00}$ satisfies (1.1) following the standard Scott Brown's approximation scheme developed in [8] (see Theorem B, below).

In the last section we discuss factorizations of the form $f = x \overset{T}{\cdot} x$ for positive functions $f \in L^1$. We show that, if f is strictly positive and lower semi-continuous, such a factorization holds for every $T \in \mathbb{A} \cap C_{00}$ (examples show that this result is the best possible for the class $\mathbb{A} \cap C_{00}$).

The theory of contractions in the class $\mathbb{A} \cap C_{00}$ is based on the following three results.

Theorem A (Theorem 10 in [4]). *Let $T \in \mathscr{L}(\mathscr{H})$ be an absolutely continuous contraction. For any $f \in L^1(Ess(T))$ there exist some sequences $(x_n)_n$, $(y_n)_n$ in $\mathscr{H}$ which converge weakly to* 0 *and such that*:

$$\begin{cases} \lim\limits_{n\to\infty} \|f - x_n \overset{T}{\cdot} y_n\|_1 = 0 \\ \|x_n\| \, \|y_n\| \leqslant \|f\|_1, \qquad n \geqslant 1. \end{cases}$$

Theorem B (Corollary of Proposition 7.2 in [8]). *Let E, F, G be complex Banach spaces and let $\varphi: E \times F \to G$ be a bilinear map. Suppose that there exists $K > 0$ such that for any $z \in G$, there exists a sequence $(x_n, y_n)_n$ of elements of $E \times F$ verifying*:

$$\begin{cases} \lim_{n\to\infty} \|\varphi(x_n, y_n) - z\| = 0 & \\ \|x_n\| \, \|y_n\| \leqslant K \, \|z\| & n \geqslant 1 \\ \lim_{n\to\infty} (\|\varphi(x, y_n)\| + \|\varphi(x_n, y)\|) = 0 & x \in E, \quad y \in F. \end{cases}$$

Then, for any infinite array $(z_{i,j})_{i,j\geqslant 1}$ *of elements of* G *and any* $\varepsilon > 0$, *there exist sequences* $(u_i)_{i\geqslant 1}$ *in* E *and* $(v_j)_{j\geqslant 1}$ *in* F *such that*:

$$\begin{cases} \varphi(u_i, v_j) = z_{i,j} & i, j \geqslant 1 \\ \sum_{i,j\geqslant 1} \|u_i\| \, \|v_j\| \leqslant (K+\varepsilon) \sum_{i,j\geqslant 1} \|z_{i,j}\|. & \end{cases}$$

THEOREM C (Proposition 2.7 in [12]). *Let* $T \in \mathscr{L}(\mathscr{H})$ *be a contraction in the class* C_{00} *and let* $(x_n)_n$ *be a sequence of elements of* $\mathscr{H}$ *which converges weakly to* 0. *Then, for any* $w \in \mathscr{H}$, *we have*:

$$\lim_{n\to\infty} (\|[x_n \overset{T}{\cdot} w]\| + \|[w \overset{T}{\cdot} x_n]\|) = 0.$$

The proof of our main result, Theorem 3.2, is based on approximations of functions in BMOA by functions in H^∞ due to S. V. Kisliakov ([17, 18]) and rediscovered independently by J. Bourgain (Lemma 1 in [9]) and also on the classical functional model of absolutely continuous contractions (cf. [20]):

For a separable Hilbert space $\mathscr{D}$, we denote by $L^2(\mathscr{D})$ the classes of measurable functions $u: \mathbb{T} \to \mathscr{D}$ such that:

$$\|u\|_2 := \left(\frac{1}{2\pi} \int_0^{2\pi} \langle u(e^{it}), u(e^{it}) \rangle_{\mathscr{D}} \, dt\right)^{1/2} < \infty,$$

where $\langle \cdot, \cdot \rangle_{\mathscr{D}}$ denotes the scalar product in $\mathscr{D}$. We denote by $L^\infty(\mathscr{D})$ the set of all essentially bounded functions in $L^2(\mathscr{D})$. If $f \in L^\infty = L^\infty(\mathbb{T})$, we can define the multiplication operator M_f on $L^2(\mathscr{D})$ by:

$$(M_f u)(\xi) = f(\xi) \, u(\xi), \qquad u \in L^2(\mathscr{D}), \qquad \xi \in \mathbb{T}.$$

In particular, if z denotes the identity map of $\mathbb{T}$ ($z(\xi) = \xi$, $\xi \in \mathbb{T}$), then M_z is a unitary operator. It follows from [20] that for every absolutely continuous contraction T, there exists a Hilbert space $\mathscr{D}$ and a subspace $\mathscr{H} \subset L^2(\mathscr{D})$ such that:

- $\mathscr{H}$ is semi-invariant for M_z, i.e. $P_{\mathscr{H}} M^n_{z \mid \mathscr{H}} = (P_{\mathscr{H}} M_{z \mid \mathscr{H}})^n$, $n \geqslant 1$ and
- $P_{\mathscr{H}} M_{z \mid \mathscr{H}}$ is unitarily equivalent to T.

In this situation, for every x and y in $\mathscr{H}$, we have:

$$(x \overset{T}{\cdot} y)(\xi) = \langle x(\xi), y(\xi) \rangle_{\mathscr{D}}, \qquad \text{a.e. on } \mathbb{T}.$$

Also, it follows from Proposition 5 in [2] that $\mathscr{D}$ and $\mathscr{H}$ can be chosen so that $L^\infty(\mathscr{D}) \cap \mathscr{H}$ is dense in $\mathscr{H}$.

The fact that $L^\infty(\mathscr{D}) \cap \mathscr{H}$ is dense in $\mathscr{H}$ allows us to reduce the proof of Theorem 3.2 to the case where $w \in L^\infty(\mathscr{D}) \cap \mathscr{H}$. We can then use approximation results of Section 2. In this section we present, in a slightly more general form, the method of Kisliakov–Bourgain to approximate, with respect to L^2-norm, functions in H^p, $p > 4$ (and thus functions in BMOA) by functions in H^∞ with some control on the H^∞-norm.

The results of Section 5 are based on the theory of compressions for contractions in the class $\mathbb{A}_{\aleph_0}$ (see [6]). Recall that $\mathscr{K}$ is said to be a semi-invariant subspace for $T \in \mathscr{L}(\mathscr{H})$ (write $K \in SI(T)$) if $\mathscr{K} = \mathscr{M} \cap \mathscr{N}^\perp$ where $\mathscr{N} \subset \mathscr{M}$ and $\mathscr{M}, \mathscr{N} \in Lat(T)$ ($Lat(T)$ denotes, as usual the lattice of all invariant subspaces for T). If $\mathscr{K} \in SI(T)$, the operator $T_{\mathscr{K}} := P_{\mathscr{K}} T_{|\mathscr{K}}$, where $P_{\mathscr{K}}$ denotes the orthogonal projection of $\mathscr{H}$ onto $\mathscr{K}$, is called the compression of T to $\mathscr{K}$ and we also say that T dilates $T_{\mathscr{K}}$. Remark that if $\mathscr{K} \in SI(T)$ and if $x, y \in \mathscr{K}$, then $x \overset{T}{\cdot} y = x \overset{T_{\mathscr{K}}}{\cdot} y$. In [6] it is proved that if $T \in \mathbb{A}_{\aleph_0}$, then T has a compression which is unitarily equivalent to an arbitrary diagonal operator with eigenvalues in the open unit disk.

2. APPROXIMATION BY H^∞ IN H^p AND BMOA

We denote by $N^+ = N^+(\mathbb{D})$ the Smirnoff class, which can be defined as the algebra of all holomorphic functions f in $\mathbb{D}$ such that $f = Ag$ where A is an inner function and where g is an outer function (see, for example, Theorem 4.14 in [19]).

LEMMA 2.1. *Let f be a function of N^+ and let $\delta \geqslant 1$. Then there exists a function $g \in H^\infty$, $\|g\|_\infty \leqslant \delta$, such that:*

$$\|f - g\|_2 \leqslant \sqrt{\frac{2+\delta^2}{\pi}} \left(\int_{E_\delta} |f(e^{it})|^2 \, dt \right)^{1/2} \qquad \textit{where}$$

$$E_\delta = \{ t \in [0, 2\pi); \, |f(e^{it})| > \delta \}.$$

Proof. Consider the analytic function g in $\mathbb{D}$ defined by $g = fG$ where G is the outer function

$$G(z) = \exp \left\{ (1/2\pi) \int_{E_\delta} (\log \delta - \log |f(e^{it})|) \left(\frac{e^{it} + z}{e^{it} - z} \right) dt \right\}.$$

If we set $F_\delta = \{ t \in [0, 2\pi); \, |f(e^{it})| \leqslant \delta \}$, then we get:

$$\begin{cases} |g(e^{it})| = |f(e^{it})|, & t \in F_\delta \\ |g(e^{it})| = \delta, & \text{elsewhere.} \end{cases}$$

Thus, the function g belongs to H^∞ and $\|g\|_\infty \leqslant \delta$. It is clear that:

$$2\pi \|f - g\|_2^2 = \int_{E_\delta} |(f-g)(e^{it})|^2 \, dt + \int_{F_\delta} |(f-g)(e^{it})|^2 \, dt.$$

Remark that:

$$\int_{F_\delta} |(f-g)(e^{it})|^2 \, dt = \int_{F_\delta} |f(e^{it})|^2 \, |1 - G(e^{it})|^2 \, dt.$$

If we define the function φ in $\mathbb{T}$ by:

$$\begin{cases} \varphi(e^{it}) = 0, & t \in F_\delta \\ \varphi(e^{it}) = \log \delta - \log |f(e^{it})| & \text{elsewhere,} \end{cases}$$

then $G(e^{it}) = \exp\{\varphi(e^{it}) + i\tilde{\varphi}(e^{it})\}$, $t \in [0, 2\pi)$ where $\tilde{\varphi}$ denotes the Hilbert transform of φ. Since $G(e^{it}) = \exp\{i\tilde{\varphi}(e^{it})\}$, $t \in F_\delta$, we get:

$$\int_{F_\delta} |(f-g)(e^{it})|^2 \, dt \leqslant \delta^2 \int_{F_\delta} |1 - \exp i\tilde{\varphi}(e^{it})|^2 \, dt.$$

Using the inequality $|1 - e^{ix}|^2 \leqslant 2x^2$, $x \in \mathbb{R}$, we obtain that:

$$\int_{F_\delta} |(f-g)(e^{it})|^2 \, dt \leqslant 2\delta^2 \int_{F_\delta} |\tilde{\varphi}(e^{it})|^2 \, dt \leqslant 2\delta^2 \int_{\mathbb{T}} |\tilde{\varphi}(e^{it})|^2 \, dt.$$

Since the Hilbert transform is an isometry with respect to the L^2-norm and since $\varphi(e^{it}) = 0$, $t \in F_\delta$, it follows that:

$$\int_{F_\delta} |(f-g)(e^{it})|^2 \, dt \leqslant 2\delta^2 \int_{E_\delta} |\varphi(e^{it})|^2 \, dt.$$

Since $\delta \geqslant 1$, it is clear that $|\varphi(e^{it})|^2 \leqslant |f(e^{it})|^2$ for any $t \in E_\delta$, which implies that:

$$\int_{F_\delta} |(f-g)(e^{it})|^2 \, dt \leqslant 2\delta^2 \int_{E_\delta} |f(e^{it})|^2 \, dt.$$

Moreover, we easily get that:

$$\int_{E_\delta} |(f-g)(e^{it})|^2 \, dt \leqslant 4 \int_{E_\delta} |f(e^{it})|^2 \, dt,$$

and the lemma follows. ∎

PROPOSITION 2.2. 1. *Let $p>4$ and let $f \in H^p$. For any $\varepsilon \in (0,1]$ there exists a function $g \in H^\infty$ such that*:

$$\begin{cases} \|f-g\|_2 < \varepsilon \, \|f\|_p \\ \|g\|_\infty < c_p \varepsilon^{2/(4-p)} \, \|f\|_p \qquad \text{where} \quad 0 < c_p \leqslant 6^{1/(p-4)}. \end{cases}$$

2. *Let $\varepsilon \in (0, 1/2]$ and let $f \in BMOA$. Then there exists a function $g \in H^\infty$ and a numerical constant $d>0$ such that*:

$$\begin{cases} \|f-g\|_2 \leqslant \varepsilon \, \|f\|_{BMO} \\ \|g\|_\infty \leqslant d \log\left(\dfrac{1}{\varepsilon}\right) \|f\|_{BMO}. \end{cases}$$

Proof. For the first assertion, we may suppose that $\|f\|_p \leqslant 1$. Since $H^p \subset N^+$ for $p>0$, there exists a function $g \in H^\infty$, $\|g\|_\infty \leqslant \delta$, such that:

$$\|f-g\|_2 \leqslant \sqrt{\frac{2+\delta^2}{\pi}} \left(\int_{E_\delta} |f(e^{it})|^2 \, dt \right)^{1/2} \tag{1}$$

where $E_\delta = \{t \in [0, 2\pi); |f(e^{it})| > \delta\}$. Applying Hölder's inequality, we obtain that:

$$\int_{E_\delta} |f(e^{it})|^2 \, dt \leqslant \left(\int_{E_\delta} |f(e^{it})|^p \, dt \right)^{2/p} m(E_\delta)^{1-2/p}. \tag{2}$$

Moreover, since $\|f\|_p^p \geqslant (1/2\pi) \int_{E_\delta} |f(e^{it})|^p \, dt \geqslant (1/2\pi) \, \delta^p m(E_\delta)$, we obtain $m(E_\delta) \leqslant 2\pi/\delta^p$. Hence, we get, for $p>4$:

$$\|f-g\|_2 \leqslant \frac{\sqrt{4+2\delta^2}}{\delta^{p/2-1}} \leqslant \frac{\sqrt{6}}{\delta^{(p-4)/2}}.$$

For $\varepsilon \in (0,1]$, set $\delta = (\sqrt{6}/\varepsilon)^{2/(p-4)}$. The first assertion follows.

For the second assertion, we may suppose that $\|f\|_{\mathrm{BMO}} \leqslant 1$. By using (1) and (2) for $p=3$ (for example) and since $\mathrm{BMOA} \subset \bigcap_{p>0} H^p$, we get:

$$\|f-g\|_2 \leqslant K \delta^{1/2} m(E_\delta)^{1/6}$$

for some positive constant $K>0$. Moreover, by the John–Nirenberg Theorem (see [16]), there exists a numerical constant $k>0$ such that:

$$m(E_\delta)<\frac{1}{k}\exp(-k\delta).$$

Hence, we have for some constant $c_0>0$:

$$\begin{cases} \|f-g\|_2 \leqslant c_0\delta^{1/2}\exp\left(\dfrac{-k\delta}{6}\right) \\ \|g\|_\infty \leqslant \delta. \end{cases}$$

For ε small enough set $\delta=6/k\log(1/\varepsilon^2)$. We easily get that:

$$\begin{cases} \|f-g\|_2 \leqslant \varepsilon \\ \|g\|_\infty \leqslant d\log\left(\dfrac{1}{\varepsilon}\right) \end{cases}$$

for some positive constant $d>0$, which completes the proof of the lemma. ∎

COROLLARY 2.3 ([9, 17, 18]). *Let $\varepsilon\in(0,1/2]$ and let $f\in L^\infty$. Then there exist $g^+\in H^\infty$, $g^-\in\overline{H_0^\infty}$ such that:*

$$\begin{cases} \|f-(g^+ + g^-)\|_2 \leqslant \varepsilon\|f\|_\infty \\ \|g^+\|_\infty + \|g^-\|_\infty \leqslant c\log\left(\dfrac{1}{\varepsilon}\right)\|f\|_\infty \end{cases}$$

where c is a numerical constant.

Proof. For $f\in L^2$ denote by $\hat{f}(n)$ the nth Fourier coefficient of f and set $P_+(f)(e^{it})=\sum_{n\geqslant 0}\hat{f}(n)\,e^{int}$, $P_-(f)(e^{it})=\sum_{n<0}\hat{f}(n)\,e^{int}$. Since $P_+(H^\infty)\subset$ BMOA and since L^∞ embeds continuously in BMO (see [15, p. 223]), the corollary follows immediately from Proposition 2.2. ∎

3. VANISHING CONDITIONS

Recall that if T is an absolutely continuous contraction, there exists a w*-w* continuous L^∞-functional calculus $\Psi_T\colon L^\infty\to\mathscr{L}(\mathscr{H})$. This functional calculus is defined by the formula:

$$(f(T)\,x, y)=\langle f, x \overset{T}{\cdot} y\rangle, \qquad x, y\in\mathscr{H},\ f\in L^\infty.$$

It is easy to check that this functional calculus is not multiplicative unless T is a unitary operator. For H^∞, we obtain the usual Sz.-Nagy-Foias functional calculus Φ_T. Also, for $\varphi \in \overline{H^\infty}$, $\Psi_T(\varphi) = \Phi_{T^*}(\tilde{\varphi})$ where $\tilde{\varphi}(z) = \sum_{n \geqslant 0} \hat{\varphi}(-n)\, z^n$ (see [5], p. 12).

In this section we use Corollary 2.3 to obtain "vanishing conditions." Remark that we do not use the full strength of BMO estimates.

Let T be an absolutely continuous contraction. We use the same notations as in the introduction. We thus identify $\mathscr{H}$ with a closed subspace of $L^2(\mathscr{D})$ semi-invariant for the multiplication operator M_z such that $L^\infty(\mathscr{D}) \cap \mathscr{H}$ is dense in $\mathscr{H}$, and we identify T with the compression of M_z to $\mathscr{H}$. For x in $L^\infty(\mathscr{D})$, set $\|x\|_\infty = \operatorname{ess\,sup}_{\xi \in \mathbb{T}} \|x(\xi)\|$.

Lemma 3.1. *Let $T \in \mathscr{L}(\mathscr{H})$ be an absolutely continuous contraction. Let $f \in L^\infty$ and let $x \in L^\infty(\mathscr{D}) \cap \mathscr{H}$. Then we have:*

$$\|f(T)\, x\| \leqslant \|f_2\|\ \|x\|_\infty.$$

Proof. We have for $y \in \mathscr{H}$,

$$\begin{aligned} |(f(T)\, x, y)| &= |\langle f, x \overset{T}{\cdot} y \rangle| \\ &= \frac{1}{2\pi} \left| \int_0^{2\pi} f(e^{it}) \langle x(e^{it}), y(e^{it}) \rangle_{\mathscr{D}}\, dt \right| \\ &\leqslant \|f\|_2\ \|x\|_\infty\ \|y\|. \end{aligned}$$

The lemma follows. ∎

Theorem 3.2. *Let $T \in \mathscr{L}(\mathscr{H})$ be an absolutely continuous contraction and let $(x_n)_n$ be a sequence of elements of $\mathscr{H}$. The following assertions are equivalent:*

$$\text{(i)}\quad \lim_{n \to \infty} \|x_n \overset{T}{\cdot} w\|_1 = 0, \qquad w \in \mathscr{H}$$

$$\text{(ii)}\quad \lim_{n \to \infty} (\|[x_n \overset{T}{\cdot} w]\| + \|[w \overset{T}{\cdot} x_n]\|) = 0, \qquad w \in \mathscr{H}.$$

Remark. Since $y \overset{T}{\cdot} z = \overline{z \overset{T}{\cdot} y}$, $y, z \in \mathscr{H}$, we have $\|y \overset{T}{\cdot} z\|_1 = \|z \overset{T}{\cdot} y\|_1$.

Proof. We only have to prove that if $\|[x_n \overset{T}{\cdot} w]\| + \|[w \overset{T}{\cdot} x_n]\| \to 0$ for every $w \in \mathscr{H}$, then $\lim_{n \to \infty} \|x_n \overset{T}{\cdot} w\|_1 = 0$ for every $w \in \mathscr{H}$. Assume that the sequence $(x_n)_{n \geqslant 1}$ satisfies Condition (ii) and let $w \in \mathscr{H}$. We have

$$|(x_n, w)| = |\langle 1, [x_n \overset{T}{\cdot} w] \rangle| \leqslant \|[x_n \overset{T}{\cdot} w]\|,$$

and so the sequence $(x_n)_{n\geqslant 1}$ converges weakly to 0. Let $w \in L^\infty(\mathscr{D}) \cap \mathscr{H}$. Then there exists a function $\varphi_n \in L^\infty$, $\|\varphi_n\|_\infty = 1$ such that $\|x_n \overset{T}{\cdot} w\|_1 = (\varphi_n(T)\, w, x_n)$. Assume that $\limsup_{n\to\infty} \|x_n \overset{T}{\cdot} w\|_1 > 0$. Without loss of generality, we may suppose that for $n \geqslant 1$, $\|x_n \overset{T}{\cdot} w\|_1 \geqslant \tau > 0$, $\|x_n\| \leqslant 1$, $\|w\|_\infty \leqslant 1$. By Corollary 2.3, there exist $g_n^+ \in H^\infty$, $g_n^- \in \overline{H_0^\infty}$, such that:

$$\begin{cases} \|\varphi_n - g_n\|_2 \leqslant \dfrac{\tau}{3} \qquad \text{where} \quad g_n = g_n^+ + g_n^- \\[2ex] \|g_n^+\|_\infty + \|g_n^-\|_\infty \leqslant c \log\left(\dfrac{3}{\tau}\right). \end{cases}$$

We have:

$$(\varphi_n(T)\, w, x_n) = ((\varphi_n - g_n)(T)\, w, x_n) + (g_n(T)\, w, x_n).$$

By Lemma 3.1 and Schwartz inequality, we get $|((\varphi_n - g_n)(T)\, w, x_n)| \leqslant \tau/3$. Also,

$$\begin{aligned} (g_n(T)\, w, x_n) &= (g_n^+(T)\, w, x_n) + (g_n^-(T)\, w, x_n) \\ &= \langle g_n^+, [w \overset{T}{\cdot} x_n] \rangle + \overline{\langle \overline{g_n^-}, [x_n \overset{T}{\cdot} w] \rangle}. \end{aligned}$$

Since the sequences $(g_n^+)_n$ and $(g_n^-)_n$ are bounded in H^∞ and in $\overline{H_0^\infty}$ respectively, and since $\lim_{n\to\infty} (\|[x_n \overset{T}{\cdot} w]\| + \|[w \overset{T}{\cdot} x_n]\|) = 0$, we have $|(g_n(T)\, w, x_n)| \leqslant \tau/3$ if n is large enough. Hence we obtain $|(\varphi_n(T)\, w, x_n)| \leqslant 2\tau/3 < \tau$ if n is large enough, contradicting the assertion $\|x_n \overset{T}{\cdot} w\|_1 \geqslant \tau$. The theorem follows then from the fact that $L^\infty(\mathscr{D}) \cap \mathscr{H}$ is dense in $\mathscr{H}$. ∎

The next Corollary yields information about the continuity of the L^∞-functional calculus Ψ_T in the particular case where $T \in C_{00}$.

Corollary 3.3. *Let T be in the class C_{00}. Then, for any sequence $(\varphi_n)_{n\geqslant 1}$ in L^∞ such that $\varphi_n \xrightarrow{w*} 0$, we have* $\lim_{n\to\infty} \|\varphi_n(T)\, x\| = 0$, $x \in \mathscr{H}$.

Proof. We will prove the corollary by showing that if $(\varphi_n)_{n\geqslant 1}$ is a bounded sequence in L^∞ such that $\limsup_{n\to\infty} \|\varphi_n(T)\, x\| > 0$ for some $x \in \mathscr{H}$, then the sequence $(\varphi_n)_{n\geqslant 1}$ is not w*-convergent to 0. In this situation, there exists a sequence $(y_n)_{n\geqslant 1}$ of elements of $\mathscr{H}$ satisfying $\|y_n\| = 1$ and

$$\|\varphi_n(T)\, x\| = |(\varphi_n(T)\, x, y_n)| = |\langle \varphi_n, x \overset{T}{\cdot} y_n \rangle|.$$

Recall (see, for example, [12, Proposition 2.7]) that if $T \in C_{0\cdot}$ (resp. $T \in C_{\cdot 0}$) and if $(z_n)_{n\geqslant 1}$ converges weakly to 0, then $\lim_{n\to\infty} \|[w \overset{T}{\cdot} z_n]\| = 0$ (resp. $\lim_{n\to\infty} \|[z_n \overset{T}{\cdot} w]\| = 0$) for any $w \in \mathscr{H}$. We can assume, without loss of generality, that $\delta = \inf_{n \geqslant 1} \|\varphi_n(T)\, x\| > 0$ and that there exists $y \in \mathscr{H}$ such that $(y_n)_{n \geqslant 1}$ converges weakly to y. Set $z_n = y - y_n$. Since $T \in C_{00}$, it follows then from Theorem 3.2 that $\lim_{n \to \infty} \|x \overset{T}{\cdot} z_n\|_1 = 0$. Since the sequence $(\varphi_n)_{n \geqslant 1}$ is bounded in L^∞, we have:

$$\lim_{n \to \infty} |\langle \varphi_n, x \overset{T}{\cdot} z_n \rangle| = 0.$$

Also,

$$(\varphi_n(T)\, x, y) = \langle \varphi_n, x \overset{T}{\cdot} y_n \rangle + \langle \varphi_n, x \overset{T}{\cdot} z_n \rangle.$$

Hence,

$$\liminf_{n \to \infty} |(\varphi_n(T)\, x, y)| = \liminf_{n \to \infty} |\langle \varphi_n, x \overset{T}{\cdot} y_n \rangle|$$

with

$$\liminf_{n \to \infty} |\langle \varphi_n, x \overset{T}{\cdot} y_n \rangle| = \liminf_{n \to \infty} \|\varphi_n(T)\, x\| \geqslant \delta > 0.$$

Since the L^∞-functional calculus $\Psi_T: f \to f(T)$ is w*-w* continuous from L^∞ into $\mathscr{L}(\mathscr{H})$, the sequence $(\varphi_n)_{n \geqslant 1}$ is not w*-convergent to 0 in L^∞, and the corollary follows. ▮

4. L^1-FACTORIZATION

We discuss here factorizations of the form $f = x \overset{T}{\cdot} y$ where f is a given function in L^1 and where T is an absolutely continuous contraction.

The notation and terminology employed herein agree with those in [11, 20]. Recall that the minimal unitary dilation $U \in \mathscr{L}(\mathscr{U})$ of an absolutely continuous contraction T is also absolutely continuous.

The minimal isometric dilation U_+ of T is the restriction of $U \in \mathscr{L}(\mathscr{U})$ to the subspace $\mathscr{U}_+ = \operatorname{Span}\{U^n \mathscr{H}, n \geqslant 0\}$, which is invariant for U. The operator U_+ has a Wold decomposition $U_+ = S_* \oplus R$ corresponding to a decomposition of $\mathscr{U}_+$ as $\mathscr{S}_* \oplus \mathscr{R}$, where $S_* \in \mathscr{L}(\mathscr{S}_*)$ is a unilateral shift of some multiplicity and $R \in \mathscr{L}(\mathscr{R})$ is an absolutely continuous unitary operator.

The minimal coisometric extension B of T is the compression of U to the subspaces $\mathscr{B} = \operatorname{Span}\{U^n \mathscr{H}, n \leqslant 0\} = \operatorname{Span}\{U^{*n} \mathscr{H}, n \geqslant 0\}$, invariant for U^* (hence semi-invariant for U). The operator B has a Wold decomposition $B = S^* \oplus R_*$ corresponding to a decomposition of $\mathscr{B}$ as $\mathscr{S} \oplus \mathscr{R}_*$, where $S \in \mathscr{L}(\mathscr{S})$ is a unilateral shift of some multiplicity and where $R_* \in \mathscr{L}(\mathscr{R}_*)$

is an absolutely continuous unitary operator. We denote by Q, Q_*, A, A_* the orthogonal projections of $\mathcal{U}$ onto $\mathcal{S}$, $\mathcal{S}_*$, $\mathcal{R}$, $\mathcal{R}_*$ respectively.

Throughout this section, expressions such as maximality, uniqueness, and equality of Borel subsets of $\mathbb{T}$ are to be interpreted as satisfied up to Borel subsets of Lebesgue measure zero.

We denote by Σ_T (resp. Σ_{*T}) the support of the spectral measure of R (resp. R_*).

The following lemma is a direct consequence of Theorem 3.2.

LEMMA 4.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction and let $(x_n)_n$ be a sequence of elements of $\mathcal{H}$ which converges to 0 in the weak topology. Assume that $\lim_{n\to\infty}(\|Ax_n\| + \|A_* x_n\|) = 0$. Then $\lim_{n\to\infty} \|x_n \overset{T}{\cdot} w\|_1 = 0$ for every $w \in \mathcal{H}$.*

Proof. By Proposition 2.7 in [12], we know that:

$$\lim_{n\to\infty} \|[Qw \overset{B}{\cdot} Qx_n]\| = 0 = \lim_{n\to\infty} \|[Q_* x_n \overset{U_+}{\cdot} Q_* w]\|.$$

Also, for $w \in \mathcal{H}$,

$$[w \overset{T}{\cdot} x_n] = [Qw \overset{B}{\cdot} Qx_n] + [A_* w \overset{B}{\cdot} A_* x_n] \qquad \text{and,}$$

$$[x_n \overset{T}{\cdot} w] = [Q_* x_n \overset{U_+}{\cdot} Q_* w] + [Ax_n \overset{U_+}{\cdot} A_w].$$

Hence $\lim_{n\to\infty}(\|[x_n \overset{T}{\cdot} w]\| + \|[w \overset{T}{\cdot} x_n]\|) = 0$ for every $w \in \mathcal{H}$. The result follows then from Theorem 3.2. ∎

Recall that if T is an absolutely continuous contraction on $\mathcal{H}$ and if σ is a Borel subset of $\mathbb{T}$, then σ is said to be essential for T (cf., Definition 3.1 in [10]) if:

$$\|f(T)\| \geqslant \|f_{|\sigma}\|_\infty, \qquad f \in H^\infty(\mathbb{T}).$$

We will denote by $Ess(T)$ the maximal essential Borel subset for T (see Proposition 3.3 in [10]).

LEMMA 4.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction. For any function $f \in L^1(Ess(T) \setminus (\Sigma_T \cup \Sigma_{*T}))$, there exist two sequences of elements of $\mathcal{H}$, $(x_n)_n$ and $(y_n)_n$ bounded by $\|f\|_1^{1/2}$ such that:*

$$\begin{cases} \lim\limits_{n\to\infty} \|f - x_n \overset{T}{\cdot} y_n\|_1 = 0 \\ \lim\limits_{n\to\infty} (\|x_n \overset{T}{\cdot} w\|_1 + \|y_n \overset{T}{\cdot} w\|_1) = 0, \qquad w \in \mathcal{H}. \end{cases}$$

Proof. Using the standard functional process of approximation of Bercovici (see [3, 11]), we see that if $f \in L^1(Ess(T))$ there exist in $\mathscr{H}$ two sequences $(u_n)_n$ and $(v_n)_n$ which converge to 0 in the weak topology and such that:

$$\begin{cases} \lim_{n\to\infty} \|f - u_n \overset{T}{\cdot} v_n\|_1 = 0 \\ \|u_n\| \leqslant \|f\|_1^{1/2} \quad \text{and} \quad \|v_n\| \leqslant \|f\|_1^{1/2}, \quad n \geqslant 1. \end{cases}$$

By Lemma 4.1, it is sufficient to prove that, if $f = 0$ a.e. on $\Sigma_T \cup \Sigma_{*T}$, we have:

$$\lim_{n\to\infty} (\|A_* u_n\| + \|A_* v_n\| + \|Au_n\| + \|Av_n\|) = 0.$$

Set $\sigma = Ess(T) \backslash (\Sigma_T \cup \Sigma_{*T})$ and denote by χ_σ the characteristic function of σ. We have:

$$u_n \overset{T}{\cdot} v_n = Qu_n \overset{B}{\cdot} Qv_n + A_* u_n \overset{B}{\cdot} A_* v_n.$$

Identifying again $\mathscr{H}$ to a closed subspace of $L^2(\mathscr{D})$, we obtain $\chi_\sigma A_* u_n = 0$. Hence $\chi_\sigma(u_n \overset{T}{\cdot} v_n) = \chi_\sigma Qu_n \overset{B}{\cdot} \chi_\sigma Qv_n$.

Moreover, we have

$$\|\chi_\sigma Qu_n\| \leqslant \|Qu_n\| \leqslant \|u_n\| \leqslant \|f\|_1^{1/2}, \tag{3}$$

$$\|\chi_\sigma Qv_n\| \leqslant \|Qv_n\| \leqslant \|v_n\| \leqslant \|f\|_1^{1/2}. \tag{4}$$

Given that $f \in L^1(\sigma)$, it is clear that $\lim_{n\to\infty} \|f - \chi_\sigma(u_n \overset{T}{\cdot} v_n)\|_1 = 0$, that is, $\lim_{n\to\infty} \|f - \chi_\sigma Qu_n \overset{B}{\cdot} \chi_\sigma Qv_n)\|_1 = 0$. It follows from this that

$$\lim_{n\to\infty} \|\chi_\sigma Qu_n\| = \|f\|_1^{1/2} = \lim_{n\to\infty} \|\chi_\sigma Qv_n\|,$$

which implies that, by (3):

$$\begin{cases} \lim_{n\to\infty} \|u_n\| = \lim_{n\to\infty} \|Qu_n\| \\ \lim_{n\to\infty} \|v_n\| = \lim_{n\to\infty} \|Qv_n\|. \end{cases}$$

It follows from the equalities

$$\|u_n\|^2 = \|A_* u_n\|^2 + \|Qu_n\|^2, \qquad \|v_n\|^2 = \|A_* v_n\|^2 + \|Qv_n\|^2,$$

that $\lim_{n\to\infty} \|A_* u_n\| = 0 = \lim_{n\to\infty} \|A_* v_n\|$.

The proof of $\lim_{n\to\infty} \|Au_n\| = 0 = \lim_{n\to\infty} \|Av_n\|$ uses similar arguments and is left to the reader. The starting point is the equality:

$$u_n \overset{T}{\cdot} v_n = Q_* u_n \overset{U_+}{\cdot} Q_* v_n + Au_n \overset{U_+}{\cdot} Av_n. \quad \blacksquare$$

Let $f = (f_{i,j})_{i,j \geqslant 1}$ be an infinite array of functions in L^1. We define $\|f\|_1 \in [0, \infty]$ by the formula $\|f\|_1 = \sum_{i,j \geqslant 1} \|f_{i,j}\|_1$. We can now formulate our main result:

THEOREM 4.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction and let $\varepsilon > 0$.*

Then, for any infinite array $(f_{i,j})_{i,j \geqslant 1}$ of functions in $L^1(\sigma_T)$ where $\sigma_T = Ess(T) \setminus (\Sigma_T \cup \Sigma_{*T})$, there exist some sequences $(x_i)_{i \geqslant 1}$ and $(y_j)_{j \geqslant 1}$ of elements of $\mathcal{H}$, bounded by $(1+\varepsilon)\|f\|_1^{1/2}$, such that $f_{i,j} = (x_i \overset{T}{\cdot} y_j)_{|\sigma_T}$ $(i \geqslant 1, j \geqslant 1)$.

In particular, for any function $f \in L^1(\sigma_T)$, there exist $x \in \mathcal{H}$, $y \in \mathcal{H}$ such that $f = (x \overset{T}{\cdot} y)_{|\sigma_T}$ and $\|x\|\,\|y\| \leqslant (1+\varepsilon)\|f\|_1$.

Proof. By Lemma 4.2 we know that for any $f \in L^1(\sigma_T)$, there exist two sequences of elements of $\mathcal{H}$, $(x_n)_n$ and $(y_n)_n$ bounded by $\|f\|_1^{1/2}$ such that:

$$\begin{cases} \lim_{n\to\infty} \|f - x_n \overset{T}{\cdot} y_n\|_1 = 0 \\ \lim_{n\to\infty} (\|x_n \overset{T}{\cdot} w\|_1 + \|y_n \overset{T}{\cdot} w\|_1) = 0, \qquad w \in \mathcal{H}. \end{cases}$$

In particular, we get:

$$\begin{cases} \lim_{n\to\infty} \|f - (x_n \overset{T}{\cdot} y_n)_{|\sigma_T}\|_1 = 0 \\ \lim_{n\to\infty} (\|(x_n \overset{T}{\cdot} w)_{|\sigma_T}\|_1 + \|(y_n \overset{T}{\cdot} w)_{|\sigma_T}\|_1) = 0, \qquad w \in \mathcal{H}. \end{cases}$$

The proof of the theorem is now an immediate consequence of Proposition 7.2 of [8] applied to the sesquilinear map $\Lambda: \mathcal{H} \times \mathcal{H} \to L^1(\sigma_T)$ defined by the formula $\Lambda(x, y) = (x \overset{T}{\cdot} y)_{|\sigma_T}$. $\blacksquare$

In the case where $T \in \mathbb{A} \cap C_{00}$, we have $\Sigma_T = \emptyset = \Sigma_{*T}$ and $Ess(T) = \mathbb{T}$. Indeed, $T \in C_{0\cdot}$ (resp. $T \in C_{\cdot 0}$) if and only if $\Sigma_{*T} = \emptyset$ (resp. $\Sigma_T = \emptyset$) and $T \in \mathbb{A}$ if and only if $\mathbb{T} = Ess(T)$. We obtain the following corollary.

COROLLARY 4.4. *Let $T \in \mathcal{L}(\mathcal{H})$ be in the class $\mathbb{A} \cap C_{00}$ and let $\varepsilon > 0$.*

Then, for any infinite array $(f_{i,j})_{i,j \geqslant 1}$ of functions in L^1, there exist some sequences $(x_i)_{i \geqslant 1}$ and $(y_j)_{j \geqslant 1}$ of elements of $\mathcal{H}$, bounded by $(1+\varepsilon)\|f\|_1^{1/2}$, such that $f_{i,j} = x_i \overset{T}{\cdot} y_j$ $(i \geqslant 1, j \geqslant 1)$.

In particular, for any function $f \in L^1$, *there exist* $x \in \mathcal{H}$, $y \in \mathcal{H}$ *such that* $f = x \overset{T}{\cdot} y$ *and* $\|x\| \, \|y\| \leqslant (1+\varepsilon) \, \|f\|_1$.

Recall that the class $\mathbb{A}_{\aleph_0}$ consists in those absolutely continuous contractions T for which given any family $(f_{i,j})_{i,j \geqslant 1}$ of elements of L^1/H^1_0, there exist two sequences $(x_i)_{i \geqslant 1}$ and $(y_j)_{j \geqslant 1}$ of elements of $\mathcal{H}$ such that $f_{i,j} = [x_i \overset{T}{\cdot} y_j]$ $(i, j \geqslant 1)$. By Proposition 4.2 of [6], if $T \in \mathbb{A}_{\aleph_0}$, there exists a compression $T_{\mathcal{M}}$ of T which is in the class $\mathbb{A} \cap C_{00}$, see below. Using Corollary 4.4 and the equality $x \overset{T_{\mathcal{M}}}{\cdot} y = x \overset{T}{\cdot} y$ for any $x, y \in \mathcal{M}$, we see that the assertion of Corollary 4.4 is still true under the relaxed hypothesis $T \in \mathbb{A}_{\aleph_0}$. In the other direction, Corollary 4.4 can be deduced immediately from [6, 7] since $\mathbb{A} \cap C_{00} \subset \mathbb{A}_{\aleph_0}$ (see Remark 1, Section 5).

5. SPATIAL FACTORIZATIONS FOR THE CLASS $\mathbb{A} \cap C_{00}$

We discuss here factorizations of the form $f = x \overset{T}{\cdot} x$, where $T \in \mathbb{A} \cap C_{00}$. If $\Lambda = (\lambda_n)_n$ is a sequence of complex numbers, we will say that an operator $T \in \mathcal{L}(\mathcal{H})$ is Λ-diagonal if there exists an orthonormal basis $(e_n)_{n \geqslant 1}$ of $\mathcal{H}$ such that $Te_n = \lambda_n e_n$ $(n \geqslant 1)$. Now we state as a lemma a basic result from [6].

LEMMA 5.1 ([6], Proposition 4.2). *Let* $T \in \mathbb{A}_{\aleph_0}$. *Then for every sequence* $\Lambda = (\lambda_n)_{n \geqslant 1}$ *of elements of* $\mathbb{D}$, *there exists a compression* $T_{\mathcal{M}}$ *of* T *which is* Λ*-diagonal.*

For $r \in [0, 1)$ denote by $P_r(t) = (1 - r^2)/|1 - re^{it}|^2$ the usual Poisson kernel, and for $\lambda = re^{i\theta} \in \mathbb{D}$, set:

$$P_\lambda(e^{it}) = P_r(\theta - t) = \frac{1 - |\lambda|^2}{|1 - \bar{\lambda} e^{it}|^2}.$$

We will need the following standard fact.

LEMMA 5.2. *Let* $f \geqslant 0$ *be a continuous function on* $\mathbb{T}$. *Then for every* $\varepsilon > 0$ *there exists* $c_1, ..., c_n \geqslant 0$ *and* $\lambda_1, ..., \lambda_n \in \mathbb{D}$ *such that*:

$$\left\| f - \sum_{k=1}^{n} c_k P_{\lambda_k} \right\|_\infty < \varepsilon.$$

Proof. For $r \in [0, 1)$, set:

$$f_r(e^{it}) = \sum_{n \in \mathbb{Z}} r^{|n|} \hat{f}(n) \, e^{int} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{is}) \, P_r(t - s) \, ds.$$

Then $f_r = 1/2\pi \int_0^{2\pi} f(e^{is})\, P_{re^{is}}\, ds$, the integral being computed in the Bochner sense in $\mathscr{C}(\mathbb{T})$, and there exists $\varepsilon > 0$ such that $\|f - f_r\|_\infty < \varepsilon/2$. Now there exists a Riemann sum $g = 1/2\pi \sum_{j=1}^{n} (s_{j+1} - s_j)\, f(e^{is_j})\, P_{re^{is_j}}$ such that $\|f_r - g\|_\infty < \varepsilon/2$, and the lemma follows.

Recall that a sequence $\Lambda = (\lambda_n)_{n \geqslant 1}$ of elements of $\mathbb{D}$ is dominating if $\|u\|_\infty = \sup_{n \geqslant 1} |u(\lambda_n)|$ for every $u \in H^\infty(\mathbb{D})$. Denote by ℓ^1_+ the set of all sequences $(c_n)_{n \geqslant 1}$ of non-negative real number such that $\sum_{n \geqslant 1} c_n < \infty$. We identify as usual functions on $\mathbb{T}$ which agree almost everywhere.

THEOREM 5.3. *Let $f \in L^1 \backslash \{0\}$. Then the following conditions imply each other.*

1. *f is lower semi-continuous (l.s.c.) and strictly positive on $\mathbb{T}$.*
2. *For every $T \in \mathbb{A}_{\aleph_0}$, there exists $x \in \mathscr{H}$ such that $x \overset{T}{\cdot} x = f$.*
3. *For every dominating sequence $\Lambda = (\lambda_n)_{n \geqslant 1}$ of elements of $\mathbb{D}$, there exists $(c_n)_{n \geqslant 1} \in \ell^1_+$ such that $f = \sum_{n=1}^\infty c_n P_{\lambda_n}$.*

Proof. Denote by $\mathscr{F}$ the set of all finite sums $\sum_{k=1}^n c_k P_{\lambda_k}$, where $c_1, ..., c_n \geqslant 0$, $\lambda_1, ..., \lambda_n \in \mathbb{D}$ and denote by $\mathscr{G}$ the set of all functions $\varphi \in L^1$ which can be written under the form $\varphi = \sum_{n=1}^\infty c_n P_{\lambda_n}$ where $(c_n)_{n \geqslant 1} \in \ell^1_+$ and where $\lambda_n \in \mathbb{D}$ $(n \geqslant 1)$. Clearly $\sum_{n=1}^\infty \varphi_n \in \mathscr{G}$ if $\varphi_n \in \mathscr{G}$ for $n \geqslant 1$ and if $\sum_{n=1}^\infty \varphi_n \in L^1$. Let f be a strictly positive continuous function on $\mathbb{T}$. Then $\delta = \inf \{ f(\xi), \xi \in \mathbb{T} \} > 0$. It follows from Lemma 5.2 that there exists $\varphi_1 \in \mathscr{F}$ such that $-\delta/2 \leqslant f - \varphi_1 - \delta \leqslant \delta/2$, so that $\delta/2 \leqslant f - \varphi_1 \leqslant 2\delta$. By using the same argument, we can construct by induction a sequence φ_n of elements of $\mathscr{F}$ such that:

$$\frac{\delta}{2^n} \leqslant f - (\varphi_1 + \cdots + \varphi_n) \leqslant \frac{\delta}{2^{n-2}}, \qquad n \geqslant 1.$$

Hence we have:

$$f(\xi) - \frac{\delta}{2^{n-2}} \leqslant \varphi_1(\xi) + \cdots + \varphi_n(\xi) \leqslant f(\xi) \qquad \text{for} \quad n \geqslant 1, \quad \xi \in \mathbb{D}.$$

So, we have proved that $f = \sum_{n=1}^\infty \varphi_n \in \mathscr{G}$. Now assume that $f \in L^1$ is strictly positive and l.s.c. on $\mathbb{T}$. Then $\delta = \min_{\xi \in \mathbb{T}} f(\xi) > 0$. There exists a sequence $(f_n)_n$ of non-negative continuous functions on $\mathbb{T}$ such that $f - \delta = \sum_{n=1}^\infty f_n$. Set $g_n = f_n + \delta/2^n$ $(n \geqslant 1)$. Then $g_n \in \mathscr{G}$, and so $f = \sum_{n=1}^\infty g_n \in \mathscr{G}$. Hence there exists a sequence $\Lambda = (\lambda_n)_{n \geqslant 1}$ of elements of $\mathbb{D}$ and $(c_n)_{n \geqslant 1} \in \ell_1^+$ such that $f = \sum_{n \geqslant 1} c_n P_{\lambda_n}$.

Let $T \in \mathbb{A}_{\aleph_0}$. It follows from Lemma 5.1 that there exists a closed subspace $\mathscr{M}$ of $\mathscr{H}$ semi-invariant for T such that the compression $S = T_{\mathscr{M}}$ is

Λ-diagonal. Let $(e_n)_{n\geqslant 1}$ be an orthonormal basis of $\mathscr{M}$ such that $Se_n=\lambda_n e_n$ $(n\geqslant 1)$. An immediate computation shows that $e_n \overset{S}{\cdot} e_m=\delta_{n,m}P_{\lambda_n}$ for $n\geqslant 1$, $m\geqslant 1$. Hence we have:

$$x \overset{T}{\cdot} x = x \overset{S}{\cdot} x = \sum_{n=1}^{\infty} |(x, e_n)|^2 P_{\lambda_n}, \qquad x \in \mathscr{M}.$$

Taking $(x, e_n)=c_n^{1/2}$ for $n\geqslant 1$, we see that $x \overset{T}{\cdot} x=f$, and Condition (2) is satisfied.

Now, let $f\in L^1\backslash\{0\}$ satisfying Condition (2), let $\Lambda=(\lambda_n)_{n\geqslant 1}$ be a dominating sequence in $\mathbb{D}$ and set $Te_n=\lambda_n e_n$ where $(e_n)_{n\geqslant 1}$ is an orthonormal basis of the separable Hilbert space $\mathscr{H}$. Then $T\in C_{00}$ and since $u(T)\,e_n=u(\lambda_n)\,e_n$ for $u\in H^\infty$, we see that $T\in\mathbb{A}$. Since f satisfies Condition (2), there exists $x\in\mathscr{H}$ such that $x \overset{T}{\cdot} x=f$. The same computation as above shows that $x \overset{T}{\cdot} x=\sum_{n=1}^{\infty} c_n P_{\lambda_n}$ where $(c_n)_{n\geqslant 1}=(|(x, e_n)|^2)_{n\geqslant 1}\in\ell^1_+$ and so f satisfies Condition (3).

Now, if $f=\sum_{n=1}^{\infty} c_n P_{\lambda_n}$ with $(c_n)_{n\geqslant 1}\in\ell^1_+$, $(\lambda_n)_{n\geqslant 1}\subset\mathbb{D}$, we can assume that $c_1>0$. Set $f_p=\sum_{n=1}^{p} c_n P_{\lambda_n}$. Then $f(\xi)=\lim_{p\to\infty} f_p(\xi)$ for $\xi\in\mathbb{T}$ and f_p is strictly positive and continuous for every $p\geqslant 1$. Hence f is strictly positive and lower semi-continuous on $\mathbb{T}$, which concludes the proof of the theorem. ∎

Remarks. 1. As in the introduction, we say that an absolutely continuous contraction T has Property (1.1) if for any infinite array $(f_{i,j})_{i,j\geqslant 1}$ consisting of elements of L^1, there exist sequences $(x_i)_{i\geqslant 1}$, $(y_j)_{j\geqslant 1}$ of $\mathscr{H}$ such that $f_{i,j}=x_i \overset{T}{\cdot} y_j$, $i, j\geqslant 1$. First notice that T has Property (1.1) if and only if there exists a compression of T, say $T_{\mathscr{M}}$, which has Property (1.1). Let $(\mu_n)_{n\geqslant 1}\subset\mathbb{D}$ be a dominating sequence for $\mathbb{T}$ and let $\Lambda=(\lambda_n)_{n\geqslant 1}\subset\mathbb{D}$ be a sequence such that the set $\{n\geqslant 1; \lambda_n=\mu_m\}$ is infinite for $m\geqslant 1$. Let S be a Λ-diagonal operator. It follows immediately from Lemma 5.1 that every $T\in\mathbb{A}_{\aleph_0}$ dilates S. Since the essential spectrum of S is dominating for $\mathbb{T}$, we have $S\in(BCP)=\bigcap_{0\leqslant\theta<1}(BCP)_\theta$ (see the definitions in [7], p. 354). Applying Corollary 6.9 of [7], we obtain that S has Property (1.1). So every $T\in\mathbb{A}_{\aleph_0}$ has Property (1.1) whereas the definition of the class $\mathbb{A}_{\aleph_0}$ only gives sequences $(u_i)_{i\geqslant 1}$ and $(v_j)_{j\geqslant 1}$ of vectors in $\mathscr{H}$ satisfying the much weaker condition $[u_i \overset{T}{\cdot} v_j]=[f_{ij}]$ $(i, j\geqslant 1)$. The present paper gives a new proof of this fact. By means of Theorem 3.2, we deduce from Theorem A, B and C that every $T\in\mathbb{A}\cap C_{00}$ has Property (1.1). The fact that Property (1.1) holds for any $T\in\mathbb{A}_{\aleph_0}$ follows then immediately from Lemma 5.1 (consider a sequence $\Lambda=(\lambda_n)_{n\geqslant 1}\subset\mathbb{D}$ which is dominating for $\mathbb{T}$). A partial result in this direction was given in [14].

2. For $x=(x_n)_{n\geqslant 1}\in\ell^2$, $y=(y_n)_{n\geqslant 1}\in\ell^2$, set $x\cdot y=(x_n\cdot y_n)_{n\geqslant 1}$. An easy computation shows that if a sequence $(x^{(p)})_{p\geqslant 1}$ of elements of ℓ^2

converges to 0 in the weak* topology, then $\lim_{p\to\infty} \|x^{(p)} \cdot y\|_1 = 0$ for every $y \in \ell^2$. Now, let $\Lambda = (\lambda_n)_{n\geqslant 1} \subset \mathbb{D}$ be dominating for $\mathbb{T}$. For $g \in L^\infty(\mathbb{T})$ denote by $P(g)$ the Poisson integral of g. Then the non-tangential limits of $P(g)$ agree with g almost everywhere on $\mathbb{T}$, and so for $p \geqslant 1$:

$$\|g\|_\infty = \sup_{n\geqslant p} |P(g)(\lambda_n)| = \sup_{n\geqslant p} \left|\int_{\mathbb{T}} P_{\lambda_n} g \, dm\right|.$$

Since $\|P_\lambda\|_1 = 1$ for every $\lambda \in \mathbb{D}$, this implies as well known that the closed absolutely convex hull of the sequence $(P_{\lambda_n})_{n\geqslant p}$ equals the closed unit ball of L^1 for every $p \geqslant 1$. Let $f \in L^1$. We deduce from the above observation that for every $\varepsilon > 0$ and every $p \geqslant 1$, there exists a sequence $(c_n)_{n\geqslant p} \in \ell_1$ such that:

$$\begin{cases} \sum\limits_{n\geqslant p} |c_n| < \|f_1\| + \varepsilon \\ f = \sum\limits_{n\geqslant p} c_n P_{\lambda_n}. \end{cases}$$

In other terms, we can construct a sequence $(c^{(p)})_{p\geqslant 1}$ (where $c^{(p)} = (c_n^{(p)})_{n\geqslant 1}$) of elements of ℓ^1 such that $c_n^{(p)} = 0$ for $n \leqslant p$, $\lim_{p\to\infty} \sum_{n\geqslant 1} |c_n^{(p)}| = \|f\|_1$ and $f = \sum_{n\geqslant 1} c_n^{(p)} P_{\lambda_n}$ for every $p \geqslant 1$.

Let T be a Λ-diagonal operator and let $(e_n)_{n\geqslant 1}$ be an orthonormal basis of $\mathscr{H}$ such that $Te_n = \lambda_n e_n$, $n \geqslant 1$. For $p \geqslant 1$, $n \geqslant 1$, let $\alpha_n^{(p)}$ and let $\beta_n^{(p)}$ be complex numbers such that $|\alpha_n^{(p)}| = |\beta_n^{(p)}| = |c_n^{(p)}|^{1/2}$ and such that $\alpha_n^{(p)} \cdot \beta_n^{(p)} = c_n^{(p)}$. Set $\alpha^{(p)} = (\alpha_n^{(p)})_{n\geqslant 1}$, $\beta^{(p)} = (\beta_n^{(p)})_{n\geqslant 1}$, $x_p = \sum_{n\geqslant 1} \alpha_n^{(p)} e_n$ and $y_p = \sum_{n\geqslant 1} \beta_n^{(p)} e_n$. We easily check that:

$$\begin{cases} x_p \overset{T}{\cdot} y_p = \sum\limits_{n\geqslant 1} c_n^{(p)} P_{\lambda_n} = f \\ \lim\limits_{p\to\infty} \|x_p\| = \lim\limits_{p\to\infty} \|\alpha_n^{(p)}\|_2 = \|f\|_1^{1/2} \\ \lim\limits_{p\to\infty} \|y_p\| = \lim\limits_{p\to\infty} \|\beta_n^{(p)}\|_2 = \|f\|_1^{1/2}. \end{cases}$$

Clearly, the sequence $(\alpha^{(p)})_{p\geqslant 1}$ converges to 0 for the weak* topology on ℓ^2, and so $\|w \overset{T}{\cdot} x_p\|_1 \leqslant \sum_{n\geqslant 1} |(w, e_n)|\, |\alpha_n^{(p)}| \to 0$ $(p \to \infty)$ for every $w \in \mathscr{H}$. By analogous computations we show that $\lim_{p\to\infty} \|w \overset{T}{\cdot} y_p\|_1 = 0$ for every $w \in \mathscr{H}$. We can now apply directly Proposition 7.2 of [8] to show that T has Property (1.1).

It follows from Lemma 5.1 that every contraction $T \in \mathbb{A}_{\aleph_0}$ has a Λ-diagonal compression and we obtain an other proof of the fact that T has Property (1.1), based on dilation theory, which does no depend on the elaborated construction of [7].

3. Here are other examples of contractions $T \in \mathbb{A} \cap C_{00}$ such that $x \overset{T}{\cdot} x$ is l.s.c. and strictly positive for every non-zero $x \in \mathscr{H}$. Let $x = (w(n))_{n \geqslant 1}$ be a strictly decreasing sequence of positive real numbers such that $\lim_{n \to \infty} w(n) = 0$ and $\lim_{n \to \infty} w(n)^{1/n} = 1$. For $f \in Hol(\mathbb{D})$, denote by $\hat{f}(n)$ the nth Taylor coefficient of f at the origin. Consider the weighted Hardy space

$$H_w = \left\{ f \in Hol(\mathbb{D});\ \|f\|_w := \left(\sum_{n \geqslant 0} |\hat{f}(n)|^2\, w^2(n) \right)^{1/2} < \infty \right\}.$$

Denote by $\alpha: z \to z$ the identity map on $\mathbb{D}$ and denote by $S: f \to \alpha f$ the usual shift operator on H_w. Clearly, $\sigma(S) = \overline{\mathbb{D}}$ and $S \in \mathbb{A} \cap C_{00}$. An easy computation shows that:

$$\begin{cases} S^{*p} \alpha^n = \dfrac{w^2(n)}{w^2(n-p)}\, \alpha^{n-p} & \text{for} \quad n \geqslant p \\ S^{*p} \alpha^n = 0 & \text{for} \quad n < p. \end{cases}$$

Set $\Delta = (1 - SS^*)^{1/2}$, $D = (1 - S^*S)^{1/2}$. We obtain for any $f \in H_w$:

$$\begin{cases} Df = \displaystyle\sum_{n \geqslant 0} \left(1 - \frac{w^2(n+1)}{w^2(n)} \right)^{1/2} \hat{f}(n)\, \alpha^n \\ \Delta f = \hat{f}(0) + \displaystyle\sum_{n \geqslant 1} \left(1 - \frac{w^2(n)}{w^2(n-1)} \right)^{1/2} \hat{f}(n)\, \alpha^n. \end{cases}$$

For $|z| < 1$, set

$$\tilde{f}(z) = \sum_{n \geqslant 0} z^n D S^n f = \sum_{n \geqslant 0} \left(1 - \frac{w^2(n+1)}{w^2(n)} \right)^{1/2} \left(\sum_{p=0}^{n} \hat{f}(p)\, z^{n-p} \right) \alpha^n$$

and

$$\begin{aligned} \tilde{f}_*(z) &= \sum_{n \geqslant 0} z^n \Delta S^{*n} f \\ &= \sum_{p \geqslant 0} \hat{f}(p)\, w^2(p)\, z^p \\ &\quad + \sum_{n \geqslant 1} \frac{1}{w(n)} \left(\frac{1}{w^2(n)} - \frac{1}{w^2(n-1)} \right)^{1/2} \left(\sum_{p \geqslant 0} \hat{f}(n+p)\, w^2(n+p)\, z^p \right) \alpha^n. \end{aligned}$$

Then $\tilde{f}$ and $\tilde{f}_*$ belong to the vector-valued Hardy space $H^2(\mathbb{D}, H_w)$, the maps $f \to \tilde{f}$ and $f \to \tilde{f}_*$ are isometries from H_w onto closed subspaces $\mathscr{M}$ and $\mathscr{N}$ of $H^2(\mathbb{D}, H_w)$ which are invariant for the backward shift T on $H^2(\mathbb{D}, H_w)$. Also S is unitarily equivalent to $T_{|\mathscr{M}}$ and S^* is unitarily equivalent to $T_{|\mathcal{N}}$. Let $h \in H^2(\mathbb{D}, H_w)$. It is a standard fact that the non-tangential limit $h(e^{it})$ exists almost everywhere on $\mathbb{T}$, that $\int_0^{2\pi} \|h(e^{it})\|^2 \, dt < \infty$, and that we have for $h, l \in H^2(\mathbb{D}, H_w)$:

$$\langle h, l \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle h(e^{it}), l(e^{it}) \rangle \, dt.$$

Hence $(h \overset{T}{\cdot} l)(e^{it}) = \langle h(e^{-it}), l(e^{-it}) \rangle$ almost everywhere on $\mathbb{T}$. Since $f \overset{S}{\cdot} g = \tilde{f} \overset{T}{\cdot} \tilde{g}$ and since $f \overset{S^*}{\cdot} g = \tilde{f}_* \overset{T}{\cdot} \tilde{g}_*$ for $f, g \in H_w$, we obtain, almost everywhere on $\mathbb{T}$, for any $f \in H_w$:

$$\begin{aligned} (f \overset{S}{\cdot} f)(e^{it}) &= \sum_{n \geqslant 0} (w^2(n) - w^2(n+1)) \left| \sum_{p=0}^{n} \hat{f}(p) \, e^{ipt} \right|^2 \\ &= \left| \sum_{p \geqslant 0} \hat{f}(p) \, w^2(p) \, e^{ipt} \right|^2 \\ &\quad + \sum_{n \geqslant 1} \left(\frac{1}{w^2(n)} - \frac{1}{w^2(n-1)} \right) \left| \sum_{p \geqslant 0} \hat{f}(n+p) \, w^2(n+p) \, e^{ipt} \right|^2. \end{aligned}$$

It follows then immediately from the first equality that if f is a non-zero function of H_w, then $f \overset{S}{\cdot} f$ is l.s.c. and strictly positive on $\mathbb{T}$.

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