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A characterization of graphic matroids using non-separating cocircuits

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ABSTRACT

In this paper, we settle a conjecture made by Wu. We show that a 3-connected binary matroid M is graphic if and only if each element avoids exactly $r(M) - 1$ non-separating cocircuits of M. This result is a natural companion to the following theorem of Bixby and Cunningham: a 3-connected binary matroid M is graphic if and only if each element belongs to exactly 2 non-separating cocircuits of M.

1. Introduction

In this paper, for matroid notation and terminology, we follow Oxley [9]. A cocircuit C^* of a connected matroid M is said to be *non-separating* provided $M \backslash C^*$ is connected. Bixby and Cunningham [1] wrote the first paper dealing with non-separating cocircuits in binary matroids. In that article, Bixby and Cunningham proved three conjectures due to Edmonds, namely:

Theorem 1.1. *If M is a 3-connected binary matroid such that $|E(M)| \geqslant 4$, then*:

(i) *Each element of M belongs to at least two non-separating cocircuits of M.*
(ii) *The set of non-separating cocircuits of M spans the cocycle space of M.*
(iii) *M is a graphic matroid if and only if each element is contained in at most two non-separating cocircuits of M.*

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When M is a cographic matroid, each item of the previous result is a theorem due to Tutte [11].

Let M be a 3-connected binary matroid. For $A \subseteq E(M)$, we denote the set of non-separating cocircuits of M avoiding A by $\mathcal{R}^*_A(M)$. The dimension of the subspace spanned by $\mathcal{R}^*_A(M)$ in the cocycle space of M is denoted by $\dim_A(M)$. When $|A| = 1$, say $A = \{a\}$, we use $\mathcal{R}^*_a(M)$ and $\dim_a(M)$ instead of $\mathcal{R}^*_A(M)$ and $\dim_A(M)$ respectively. Lemos [4] proved the next result:

Theorem 1.2. *Let M be a 3-connected binary matroid such that $r(M) \geqslant 1$. If a is an element of M, then* $\dim_a(M) = r(M) - 1$.

Wu [14] made the following conjecture which is closed related to Theorems 1.1 and 1.2 (see [2]):

Conjecture 1.1. *Let M be a 3-connected binary matroid such that $r(M) \geqslant 1$. Then, M is graphic if and only if each element avoids exactly $r(M) - 1$ non-separating cocircuits of M.*

In this note, we prove this conjecture:

Theorem 1.3. *If M is a 3-connected binary matroid such that $|E(M)| \geqslant 4$, then:*

(i) *M is a graphic matroid if and only if each element avoids exactly $r(M) - 1$ non-separating cocircuits of M.*
(ii) *M is a graphic matroid if and only if $\mathcal{R}^*_a(M)$ is linearly independent in the cocycle space of M, for every element a of M.*

In Theorem 1.3(ii), we cannot replace the word "every" by "some" because, for example, $\mathcal{R}^*_a(S_8)$ is linearly independent in the cocycle space of S_8, where a is the unique element of S_8 satisfying $S_8 \backslash a \cong F^*_7$.

Note that the "only if" part of the proof of each item of this theorem is straightforward. We need to show only the "if" part. Observe that Theorem 1.3(ii) and Theorem 1.2 implies Theorem 1.3(i). Thus it is enough to prove Theorem 1.3(ii). Note that Theorem 1.3(ii) is a consequence of the next result:

Proposition 1.1. *Let M be a 3-connected binary matroid. If M is not graphic, then there is an element f of M such that $\mathcal{R}^*_f(M)$ is linearly dependent in the cocycle space of M.*

We prove Proposition 1.1 in the last section of this paper. For other results about non-separating cocircuits in connected binary matroids see [2,3,5–8].

2. A preliminary lemma

For a 3-connected binary matroid M, we denote the set of non-separating cocircuits of M by $\mathcal{R}^*(M)$.

Lemma 2.1. *Suppose that e is an element of a 3-connected binary matroid M such that $\mathrm{co}(M \backslash e)$ is 3-connected, say $\mathrm{co}(M \backslash e) = M \backslash e / \{b_1, \dots, b_n\}$, where n is a non-negative integer, and $r^*(M) \geqslant 4$. For $i \in \{1, \dots, n\}$, let T^*_i be the triad of M that contains $\{e, b_i\}$, say $T^*_i = \{e, a_i, b_i\}$. If $C^* \in \mathcal{R}^*(\mathrm{co}(M \backslash e))$, then either:*

(i) *C^* is a cocircuit of M and:*
  (a) *$C^* \in \mathcal{R}^*(M)$; or*
  (b) *$\{a_1, \dots, a_n\} \subseteq C^*$ and $C := \{e, b_1, \dots, b_n\}$ is a circuit of M; or*
(ii) *$C^* \cup e$ is a cocircuit of M and:*
  (a) *$C^* \cup e \in \mathcal{R}^*(M)$; or*
  (b) *$C^* \cap \{a_1, \dots, a_n\} \neq \emptyset$ and, for each $i \in \{1, \dots, n\}$ such that $a_i \in C^*$, $C^* \triangle \{a_i, b_i\} \in \mathcal{R}^*(M)$; or*
  (c) *$\{a_1, \dots, a_n\} \subseteq C^*$ and, for each $i \in \{1, \dots, n\}$,*

(1) *$C^* \triangle \{a_i, b_i\} \in \mathcal{R}^*(M)$; or*
(2) *$C_i := \{e, b_1, \ldots, b_n\} \triangle \{a_i, b_i\}$ is a circuit of M.*

Moreover, at most one of the sets $C, C_1, \ldots, C_n$ is a circuit of M.

Proof. Assume that $C^* \cap \{a_1, \ldots, a_n\} = \{a_1, \ldots, a_m\}$, for a non-negative integer m. By definition, $H' = E(M) - [\{e, b_1, \ldots, b_n\} \cup C^*]$ is a connected hyperplane of $\mathrm{co}(M\backslash e)$. Therefore $H = H' \cup \{b_1, \ldots, b_n\} = E(M) - [e \cup C^*]$ is a hyperplane of $M\backslash e$. As $T_i^* \cap H = \{b_i\}$, for each $i \in \{1, \ldots, m\}$, it follows that $b_1, \ldots, b_m$ are coloops of $M \mid H$. Note that $[M \mid H]\backslash\{b_1, \ldots, b_m\}$ is connected because $[M \mid H]\backslash\{b_1, \ldots, b_m\}$ is obtained from $\mathrm{co}(M\backslash e) \mid H'$ by adding b_i in series with a_i, for each $i \in \{m+1, \ldots, n\}$. (Observe that $|H'| \geqslant 2$ because $r^*(\mathrm{co}(M\backslash e)) \geqslant 3$ and so $r(\mathrm{co}(M\backslash e)) \geqslant 3$. In particular, $r_{\mathrm{co}(M\backslash e)}(H') \geqslant 2$.) We have two cases to deal with.

Suppose that H spans e in M. Hence $H \cup e$ is a hyperplane of M and so C^* is a cocircuit of M. If $C^* \in \mathcal{R}^*(M)$, then (i)(a) follows. Assume that $C^* \notin \mathcal{R}^*(M)$. Therefore $m \geqslant 1$. As $T_i^* \cap (H \cup e) = \{e, b_i\}$, for each $i \in \{1, \ldots, m\}$, it follows that $C = \{e, b_1, \ldots, b_m\}$ is contained in a series class of $M \mid (H \cup e)$ since e is not a coloop of $M \mid (H \cup e)$. But $M \mid (H \cup e)$ is not connected and so C is the ground set of a connected component of $M \mid (H \cup e)$ (the other connected component of this matroid is $[M \mid H]\backslash\{b_1, \ldots, b_m\}$). In particular, C is a circuit of M. By orthogonality, $C \cap T_i^* \neq \{e\}$, for every $i \in \{1, \ldots, n\}$, and so $n = m$. We have (i)(b).

Suppose that H does not span e in M. In this case, H is a hyperplane of M. Thus $C^* \cup e$ is a cocircuit of M. If $H = H' \cup \{b_{m+1}, \ldots, b_n\}$, then (ii)(a) occurs. We may assume that $m \geqslant 1$. Therefore $\{b_1, \ldots, b_m\}$ is the set of coloops of $M \mid H$. For $i \in \{1, \ldots, m\}$, $H_i = H \triangle \{a_i, b_i\}$ is also a hyperplane of $M\backslash e$. Note that $\{b_1, \ldots, b_m\} \triangle \{a_i, b_i\}$ is the set of coloops of $M \mid H_i$ and $M \mid (H' \cup \{b_{m+1}, \ldots, b_n\})$ is the other connected component of this matroid. But $(C^* \cup e) \triangle T_i^*$ is a cocircuit of M. So $H_i \cup e$ is a hyperplane of M. That is, H_i spans e in M. There is a circuit C_i of M such that $e \in C_i \subseteq H_i \cup e$. By orthogonality with T_j^*, for $j \in \{1, \ldots, m\}$, $\{b_1, \ldots, b_m\} \triangle \{a_i, b_i\} \subseteq C_i$. We have two subcases.

- If $C_i \cap (H' \cup \{b_{m+1}, \ldots, b_n\}) \neq \emptyset$, then $M \mid H_i$ is connected and $C^* \triangle \{a_i, b_i\} \in \mathcal{R}^*(M)$.
- If $C_i \cap (H' \cup \{b_{m+1}, \ldots, b_n\}) = \emptyset$, then $M \mid H_i$ has two connected components, namely: $M \mid C_i$ and $M \mid (H' \cup \{b_{m+1}, \ldots, b_n\})$. By orthogonality, $C_i \cap T_j^* \neq \{e\}$, for $j \in \{m+1, \ldots, n\}$, and so $n = m$. That is, $C_i = \{e, b_1, \ldots, b_n\} \triangle \{a_i, b_i\}$.

We have (ii)(b) or (ii)(c).

Now, assume that at least two of the sets $C, C_1, \ldots, C_n$ is a circuit of M. We have two cases to consider. First, suppose that C and C_i are circuits of M. Hence $C \triangle C_i = \{a_i, b_i\}$ is a disjoint union of circuits of M. So M has a circuit having at most 2 elements; a contradiction because M is 3-connected. Suppose that C_i and C_j are circuits of M, for $i \neq j$. As $C_i \triangle C_j = \{a_i, b_i, a_j, b_j\}$ is a disjoint union of circuits of M and each circuit of M has at least 3 elements, it follows that $\{a_i, b_i, a_j, b_j\}$ is a circuit of M. Therefore $\{a_i, a_j\}$ is a circuit of $\mathrm{co}(M\backslash e)$. Note that $\mathrm{co}(M\backslash e) \cong U_{1,2}$ or $\mathrm{co}(M\backslash e) \cong U_{1,3}$ because $\mathrm{co}(M\backslash e)$ is 3-connected; a contradiction since $r^*(\mathrm{co}(M\backslash e)) \geqslant 3$. $\square$

Now, we choose the elements of the cosimplification of $M\backslash e$ to narrow the options given by the previous lemma.

Lemma 2.2. *Suppose that e is an element of a 3-connected binary matroid M such that the cosimplification of $M\backslash e$ is 3-connected. If $r^*(M) \geqslant 4$, then it is possible to choose the ground set of $\mathrm{co}(M\backslash e)$ so that, for each $C^* \in \mathcal{R}^*(\mathrm{co}(M\backslash e))$, $C^* \triangle X \in \mathcal{R}^*(M)$, where $X = \emptyset$ or $X = \{e\}$ or $X = T^* - e$, for some triad T^* of M meeting both $\{e\}$ and C^*.*

Proof. Let $T_1^*, \ldots, T_n^*$ be the triads of M that contains e, for a non-negative integer n. For $i \in \{1, \ldots, n\}$, we set $T_i^* = \{e, a_i, b_i\}$. Define $C = \{e, b_1, \ldots, b_n\}$ and, for $i \in \{1, \ldots, n\}$, $C_i = C \triangle \{a_i, b_i\}$. By the second part of Lemma 2.1, at most one of the sets $C, C_1, \ldots, C_n$ is a circuit of M. If one of these sets is a circuit of M, then $n \geqslant 2$ because M is 3-connected. When this happens, we may relabel the elements in the triads and the triads in such a way that C_n is a circuit of M. In particular Lemma 2.1(i)(b) does not occur. If (i)(a) or (ii)(a) or (ii)(b) of Lemma 2.1 happens, then the result follows. We may assume that Lemma 2.1(ii)(c) holds. By our choice of labels, (ii)(c)(1) occurs for $i = 1$ and the result also follows in this case. □

3. An auxiliary function

Let M be a 3-connected binary matroid M. For $A \subseteq E(M)$, we define

$$\operatorname{dep}_A(M) = \left|\mathcal{R}^*_A(M)\right| - \dim_A(M).$$

(When $|A| = 1$, say $A = \{a\}$, we use $\operatorname{dep}_a(M)$ instead of $\operatorname{dep}_A(M)$.) A subset Υ of $\mathcal{R}^*_A(M)$ is said to be *inessential* provided both $\mathcal{R}^*_A(M) - \Upsilon$ and $\mathcal{R}^*_A(M)$ span the same linear subspace of the cocycle space of M. Observe that $\operatorname{dep}_A(M)$ is the cardinality of any maximal inessential subset Υ of $\mathcal{R}^*_A(M)$. To prove Proposition 1.1, we need to show only that $\operatorname{dep}_f(M) > 0$, for some element f of M, when M is not graphic.

Lemma 3.1. *Suppose that e is an element of a 3-connected binary matroid M such that the cosimplification of $M\backslash e$ is 3-connected. If $r^*(M) \geqslant 4$, then it is possible to choose the ground set of $\operatorname{co}(M\backslash e)$ so that, for each $A \subseteq E(\operatorname{co}(M\backslash e))$,*

$$\operatorname{dep}_A(M) \geqslant \operatorname{dep}_{A'}(M) \geqslant \operatorname{dep}_A\bigl(\operatorname{co}(M\backslash e)\bigr),$$

where A' is the minimal subset of $E(M)$ satisfying $A \subseteq A'$ and, for each triad T^* of M that meets both e and A, $T^* - e \subseteq A'$.

Proof. By Lemma 2.2, it is possible to choose the ground set of $N := \operatorname{co}(M\backslash e)$ so that, for each $C^* \in \mathcal{R}^*(N)$, $C^* \triangle X \in \mathcal{R}^*(M)$, where $X = \emptyset$ or $X = \{e\}$ or $X = T^* - e$, for some triad T^* of M meeting both $\{e\}$ and C^*.

Let $T^*_1, \dots, T^*_r$ be the triads of M containing e. Note that $T^*_1 - e, \dots, T^*_r - e$ are pairwise disjoint sets. For $i \in \{1, \dots, r\}$, we set $T^*_i = \{e, a_i, b_i\}$, where $a_i \in E(N)$. In particular, $N = M\backslash e/\{b_1, \dots, b_r\}$. We may assume that $A \cap \{a_1, \dots, a_r\} = \{a_{l+1}, \dots, a_r\}$, for a non-negative integer l. (That is, T^*_i avoids A if and only if $i \leqslant l$.) Therefore

$$A' = A \cup \{b_{l+1}, \dots, b_r\} \quad \text{and} \quad A' \cap \bigl(T^*_1 \cup \dots \cup T^*_l\bigr) = \emptyset. \tag{3.1}$$

Suppose that $\mathcal{R}^*_A(N) = \{C^*_1, C^*_2, \dots, C^*_n\}$, where $n = |\mathcal{R}^*_A(N)|$. By the first paragraph, for each $i \in \{1, 2, \dots, n\}$, we can define $D^*_i := C^*_i \triangle X_i$ in such a way that $D^*_i \in \mathcal{R}^*(M)$, where $X_i = \emptyset$ or $X_i = \{e\}$ or $X_i = T^* - e$, for some $T^* \in \{T^*_1, \dots, T^*_r\}$ satisfying $T^* \cap C^* \neq \emptyset$.

Now, we establish that

$$\bigl\{D^*_1, D^*_2, \dots, D^*_n\bigr\} \subseteq \mathcal{R}^*_{A'}(M). \tag{3.2}$$

If $f \in D^*_i \cap A'$, then, by (3.1), $f \in X_i$ and $X_i = T^* - e$, for some triad T^* meeting both C^*_i and e, say $T^* = \{e, f, g\}$. Therefore $g \in C^*_i$. By definition of A', $\{f, g\} \subseteq A'$ and so $g \in A$; a contradiction because $g \in C^*_i$. Thus (3.2) follows.

Observe that $D^*_1, D^*_2, \dots, D^*_n$ are pairwise different because $\{e\}, T^*_1 - e, \dots, T^*_r - e$ are pairwise disjoint sets and none of them is contained in $E(N)$. (If $D^*_i = D^*_j$, for $i \neq j$, then $C^*_i \triangle X_i = C^*_j \triangle X_j$ and so

$$\emptyset \neq C^*_i \triangle C^*_j = X_i \triangle X_j = X_i \cup X_j \nsubseteq E(N).)$$

We may relabel the cocircuits of $\mathcal{R}_A^*(N)$ such that $\Upsilon = \{C_i^*\colon\ m+1 \leqslant i \leqslant n\}$ is a maximal inessential subset of $\mathcal{R}_A^*(N)$, for some positive integer m. The result follows provided we can prove that $\Upsilon' = \{D_i^*\colon\ m+1 \leqslant i \leqslant n\}$ is an inessential subset of $\mathcal{R}_{A'}^*(M)$.

For each integer i such that $m+1 \leqslant i \leqslant n$, we need to prove only that D_i^* is spanned in the cocycle space of M by $\mathcal{R}_{A'}^*(M) - \Upsilon'$. We achieve this goal by establishing that

$$D_i^* \text{ is spanned in the cocycle space of } M \text{ by } D_1^*, D_2^*, \ldots, D_m^*, T_1^*, \ldots, T_l^*. \tag{3.3}$$

(Remember that $T_1^*, \ldots, T_l^*$ are the triads of M containing e and avoiding A and so A', by (3.1). Note that each triad of M containing e is non-separating because $\mathrm{co}(M\backslash e)$ is a 3-connected binary matroid.) By hypothesis, when $m+1 \leqslant i \leqslant n$, there are integers $i_1, i_2, \ldots, i_k$ satisfying $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m$ such that

$$C_i^* = C_{i_1}^* \triangle C_{i_2}^* \triangle \cdots \triangle C_{i_k}^*,$$

say $(i_1, i_2, \ldots, i_k) = (1, 2, \ldots, k)$. That is,

$$\left[C_1^* \triangle C_2^* \triangle \cdots \triangle C_k^*\right] \triangle C_i^* = \emptyset. \tag{3.4}$$

We set

$$X = \left[D_1^* \triangle D_2^* \triangle \cdots \triangle D_k^*\right] \triangle D_i^*. \tag{3.5}$$

In particular, X belongs to the cocycle space of M. By (3.4),

$$X = [X_1 \triangle X_2 \triangle \cdots \triangle X_k] \triangle X_i.$$

For each $j \in \{1, \ldots, l\}$, we have that $X \cap (T_j^* - e) = \emptyset$ or $T_j^* - e \subseteq X$ because

$$\{X_1, X_2, \ldots, X_n\} \subseteq \left\{\emptyset, \{e\}, T_1^* - e, \ldots, T_l^* - e\right\}.$$

(The sets $\emptyset, \{e\}, T_1^* - e, \ldots, T_l^* - e$ are pairwise disjoint.) Suppose that X contains $T_1^* - e, \ldots, T_s^* - e$, for a non-negative integer s, and so X is disjoint from $T_{s+1}^* - e, \ldots, T_l^* - e$. Therefore

$$X \triangle \left[T_1^* \triangle \cdots \triangle T_s^*\right]$$

is a cocycle of M contained in $\{e\}$. Thus

$$X = T_1^* \triangle \cdots \triangle T_s^*. \tag{3.6}$$

Replacing (3.6) into (3.5), we obtain:

$$T_1^* \triangle \cdots \triangle T_s^* = \left[D_1^* \triangle D_2^* \triangle \cdots \triangle D_k^*\right] \triangle D_i^*.$$

This identity can be rewritten as

$$D_i^* = T_1^* \triangle \cdots \triangle T_s^* \triangle D_1^* \triangle D_2^* \triangle \cdots \triangle D_k^*.$$

Therefore (3.3) holds and the result follows. $\square$

4. Proof of Proposition 1.1

We argue by contradiction. Choose a counter-example such that $|E(M)|$ is minimum. In particular, $\mathrm{dep}_e(M) = 0$, for every element e of M. We divide the proof into some lemmas.

Lemma 4.1. *$r^*(N) = r^*(M)$, for every non-graphic 3-connected minor N of M.*

Proof. Suppose that N is a non-graphic 3-connected minor of M such that $r^*(N) < r^*(M)$. As $r^*(N) \geqslant 3$, it follows that $r^*(M) \geqslant 4$. By the dual of Lemma 3.4 of [13], there is an element e of M such that $\mathrm{co}(M\backslash e)$ is a 3-connected matroid having an N-minor. By Lemma 3.1, we can choose the elements of $\mathrm{co}(M\backslash e)$ so that

$$\mathrm{dep}_f(M) \geqslant \mathrm{dep}_f\big(\mathrm{co}(M\backslash e)\big), \tag{4.1}$$

for each $f \in E(\mathrm{co}(M\backslash e))$. By the choice of M, there is an element f of $\mathrm{co}(M\backslash e)$ such that $\mathrm{dep}_f(\mathrm{co}(M\backslash e)) > 0$; a contradiction to (4.1). Therefore $r^*(N) = r^*(M)$, for every non-graphic 3-connected minor of M. □

Lemma 4.2. *M is regular.*

Proof. Suppose that M is non-regular. By Tutte's characterization of regular matroids, M has F_7 or F_7^* as a minor (see [12]). By the choice of M, M cannot be isomorphic to:

(i) F_7 because each 4-element cocircuit of F_7 is non-separating and, for each element f of F_7, the sum in the cocycle space of F_7 of the 3 4-element cocircuits avoiding f is zero; or
(ii) F_7^* because each 3-element cocircuit of F_7^* is non-separating and, for each element f of F_7^*, the sum in the cocycle space of F_7^* of the 4 3-element cocircuits avoiding f is zero.

But Seymour proved that F_7 is a splitter for the class of binary matroids without minor isomorphic to F_7^* ((7.6) of [10]). Hence M has a minor isomorphic to F_7^*. By Lemma 4.1, $r^*(M) = r^*(F_7^*) = 3$. Thus M is isomorphic to F_7^*; a contradiction. Therefore M is regular. □

Lemma 4.3. *M is isomorphic to $M^*(G)$, where G is a graph such that $|V(G)| = 6$ and $G\backslash X$ is isomorphic to $K_{3,3}$, for some non-empty subset X of $E(G)$.*

Proof. By Tutte's characterization of graphic matroids, M has $M^*(K_{3,3})$ or $M^*(K_5)$ as a minor (see [12]). By the choice of M, M cannot be isomorphic to

(i) $M^*(K_{3,3})$ because the edge-set of each 4-edge circuit of $K_{3,3}$ is a non-separating cocircuit of $M^*(K_{3,3})$ and, for each vertex v of $K_{3,3}$, the sum in the cocycle space of $M^*(K_{3,3})$ of the edge-set of the 3 4-edge circuits of $K_{3,3}$ avoiding v is zero; or
(ii) $M^*(K_5)$ because the edge-set of each triangle of K_5 is a non-separating cocircuit of $M^*(K_5)$ and, for each vertex v of K_5, the sum in the cocycle space of $M^*(K_5)$ of the edge-set of the 4 triangles of K_5 avoiding v is zero.

But Seymour proved that $M^*(K_5)$ is a splitter for the class of regular matroids without minor isomorphic to $M^*(K_{3,3})$ ((7.5) of [10]). Hence M has a minor isomorphic to $M^*(K_{3,3})$. By Lemma 4.1, $r^*(M) = r^*(M^*(K_{3,3})) = 5$. In particular M does not have R_{12} as a minor. By Theorem 14.2 of Seymour [10], M is cographic or M is isomorphic to R_{10}.

By the choice of M, M is not isomorphic to R_{10} because each 4-element cocircuit of R_{10} is non-separating and R_{10}/f is isomorphic to $M^*(K_{3,3})$, for every element f, and so the non-separating cocircuits of R_{10} avoiding f are linearly dependent in the cocycle space of R_{10}. Hence M is cographic.

There is a simple graph G having 6 vertices such that $M = M^*(G)$ (remember that $r^*(M) = 5$) and $G\backslash X = K_{3,3}$, for some set of edges X. $\square$

Let $\{V_1, V_2\}$ be a partition of $V(G)$ such that V_1 and V_2 are independent sets of vertices of $G\backslash X$. If T is a triangle of G such that $V(T)$ meets both V_1 and V_2, then $G/E(T)$ is a block. That is $E(T)$ is a non-separating cocircuit of M.

Now, we prove that V_1 or V_2 is an independent set of vertices of G. If V_1 and V_2 are not independent, then, for $i \in \{1, 2\}$, there is an edge e_i joining two vertices belonging to V_i, say u_i and v_i. Observe that $G' = G[\{u_1, u_2, v_1, v_2\}]$ is isomorphic to K_4. If T_1, T_2, T_3, T_4 are the triangles of G', then $E(T_1)$, $E(T_2)$, $E(T_3)$, $E(T_4)$ are non-separating cocircuits of M because $V(T_i)$ meets both V_1 and V_2, for every $i \in \{1, 2, 3, 4\}$. But

$$E(T_1) \triangle E(T_2) \triangle E(T_3) \triangle E(T_4) = \emptyset;$$

a contradiction because $E(G) - E(G') \neq \emptyset$. Hence V_1 or V_2 is an independent set of G, say V_2.

Let v be a fixed vertex of G belonging to V_2. If $G[V_1]$ is isomorphic to K_3, then $\triangle_T E(T) = \emptyset$, where T runs over the 6 triangles of $G - v$ such that $V(T)$ meets both V_1 and V_2; a contradiction. Thus $|X| \in \{1, 2\}$. If $|X| = 2$ and T_1, T_2, T_3, T_4 are the triangles of $G - v$, then

$$E(T_1) \triangle E(T_2) \triangle E(T_3) \triangle E(T_4)$$

is a 4-element non-separating cocircuit of M; a contradiction and so $|X| = 1$. If u is adjacent to the edge belonging to X, then $E(C_1), E(C_2), E(C_3)$ are linearly dependent cocircuits of M, where C_1, C_2, C_2 are the 4-element circuits of $G - u$; a contradiction. Therefore Proposition 1.1 follows.

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