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Differential geometry of $\mathfrak{g}$-manifolds

D.V. Alekseevsky[1]
Erwin Schrödinger International Institute of Mathematical Physics, Wien, Austria

Peter W. Michor
Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

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Abstract: An action of a Lie algebra $\mathfrak{g}$ on a manifold M is just a Lie algebra homomorphism $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$. We define orbits for such an action. In general the space of orbits $M/\mathfrak{g}$ is not a manifold and even has a bad topology. Nevertheless for a $\mathfrak{g}$-manifold with equidimensional orbits we treat such notions as connection, curvature, covariant differentiation, Bianchi identity, parallel transport, basic differential forms, basic cohomology, and characteristic classes, which generalize the corresponding notions for principal G-bundles. As one of the applications, we derive a sufficient condition for the projection $M \to M/\mathfrak{g}$ to be a bundle associated to a principal bundle.

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1. Introduction

Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let M be a smooth manifold. We say that $\mathfrak{g}$ acts on M or that M is a $\mathfrak{g}$-manifold if there is a Lie algebra homomorphism $\zeta = \zeta^M : \mathfrak{g} \to \mathfrak{X}(M)$ from $\mathfrak{g}$ into the Lie algebra of all vector fields on M. Many notions and results of the theory of G-manifolds and of the theory of principal bundles may be extended to the category of $\mathfrak{g}$-manifolds. This is the guideline for our approach to $\mathfrak{g}$-manifolds.

Now we describe the structure of the paper and we state some principal results.

In Section 2 notations are fixed and different properties of an action of a Lie algebra $\mathfrak{g}$ on a manifold M are defined. The pseudogroup $\Gamma(\mathfrak{g})$ of local transformations generated by an action of a Lie algebra $\mathfrak{g}$ is considered, and its graph is defined. We consider also the groupoid $P(\mathfrak{g})$ of germs of elements from $\Gamma(\mathfrak{g})$ and under some conditions we may define the adjoint representation of P into the adjoint group $\mathrm{Ad}(\mathfrak{g})$ associated with the Lie algebra $\mathfrak{g}$. Some technical lemmas are proved which will be used in Section 5.

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E-mail: Peter.Michor@esi.ac.at

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In Section 3 the main definition of a principal connection on a $\mathfrak{g}$-manifold M is given as such a $\mathfrak{g}$-invariant field Φ of endomorphisms of TM, whose value in a point $x \in M$ is a projection $\Phi_x : T_xM \to \mathfrak{g}(x)$ of the tangent space onto the 'vertical subspace' $\mathfrak{g}(x) := \{\zeta_X(x) : X \in \mathfrak{g}\}$. We say, that a principal connection Φ admits a principal connection form if it may be represented as $\Phi = \zeta_\omega$, where ω is a $\mathfrak{g}$-valued $\mathfrak{g}$-equivariant 1-form on M such that

(1) $T_xM = \mathfrak{g}(x) \oplus \ker(\omega_x)$ for each $x \in M$,

(2) $\mathfrak{g} = \mathfrak{g}_x \oplus \omega(T_xM)$, where $\mathfrak{g}_x = \{X \in \mathfrak{g} : \zeta_X(x) = 0\}$ is the isotropy subalgebra.

Any such form defines a principal connection $\Phi = \zeta_\omega$. On the other hand, a simple example shows that not every principal connection admits a principal connection form.

Principal connections may exist only if the action of $\mathfrak{g}$ on M has constant rank, see Proposition 3.2 which also gives some sufficient conditions for the existence of principal connections.

In order to define the curvature of a principal connection Φ we recall in Section 4 the definition of the *algebraic* bracket $[\varphi,\psi]^\wedge$ of $\mathfrak{g}$-valued differential forms on a manifold M which turns the space $\Omega(M;\mathfrak{g})$ of such forms into a graded Lie algebra. We also recall the definition of the *differential* Frölicher-Nijenhuis bracket which extends the Lie bracket of vector fields to a graded bracket on the space $\Omega(M;TM)$ of tangent bundle valued differential forms on M. An action of a Lie algebra $\mathfrak{g}$ on M (i.e., a homomorphism $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$) induces a linear mapping

$$\zeta : \Omega(M;\mathfrak{g}) \to \Omega(M;TM).$$

It is not a homomorphism of graded Lie algebras, but becomes an antihomomorphism when it is restricted to the subalgebra $\Omega^p_{\text{hor}}(M;\mathfrak{g})^{\mathfrak{g}}$ of $\mathfrak{g}$-equivariant horizontal forms. In general the Frölicher–Nijenhuis bracket $[\zeta_\varphi,\zeta_\psi]$ of two $\mathfrak{g}$-equivariant forms $\varphi,\psi \in \Omega(M;\mathfrak{g})$ may be expressed in terms of $[\varphi,\psi]^\wedge$ and exterior differentials. See Proposition 4.4 for the relevant formulas.

In Section 6 we give a local description of a principal connection Φ and of its curvature on a locally trivial $\mathfrak{g}$-manifold with standard fiber S. We show that locally a connection is described by a 1-form on the base with values in the centralizer $Z_{\mathfrak{X}(S)}(\mathfrak{g})$, which may be considered as the Lie algebra of infinitesimal automorphisms of the $\mathfrak{g}$-manifold S. We prove that it is isomorphic to the normalizer $N_{\mathfrak{g}}(\mathfrak{g}_x)$ of the isotropy subalgebra $\mathfrak{g}_x$ of a point $x \in S$. As a corollary we obtain the existence of a unique principal connection, which is moreover flat, under the assumption that $N_{\mathfrak{g}}(\mathfrak{g}_x) = 0$.

We treat the case of a homogeneous $\mathfrak{g}$-manifold M in Section 5. First we consider a $\mathfrak{g}$-manifold with a free transitive action ζ of $\mathfrak{g}$ and we remark that the inverse mapping $\kappa = \zeta^{-1}$ is a Maurer–Cartan form (i.e., a 1-form that satisfies the Maurer–Cartan equation). For a transitive free action of $\mathfrak{g}$ on a simply connected manifold M, we define by means of the graph of the pseudogroup a $\mathfrak{g}$-equivariant mapping $M \to G$, the 'Cartan development' of M into the simply connected Lie group with Lie algebra $\mathfrak{g}$. It is a local diffeomorphism but in general it is neither surjective nor injective. As an immediate application, we obtain a well defined mapping from any locally flat simply connected G-structure of finite type into the standard maximally homogeneous G-structure. It generalizes the developing of a locally flat conformal manifold into the conformal sphere. We prove that on a simply connected $\mathfrak{g}$-manifold M with free transitive $\mathfrak{g}$-action ζ the centralizer of $\zeta(\mathfrak{g})$ in the Lie algebra $\mathfrak{X}(M)$ of all vector fields on M is isomorphic to $\mathfrak{g}$. The corresponding free transitive action $\hat{\zeta} : \mathfrak{g} \to \mathfrak{X}(M)$, a Lie algebra antihomomorphism, is called the dual action of $\mathfrak{g}$. This is not true in general, if M is not simply connected.

Let $H\backslash G$ be a homogeneous G-manifold for a Lie group G with isotropy group $H \subset G$ of a point o. Then $p : G \to H\backslash G$ is a principal H-bundle with left principal H-action, and G acts from the right by automorphisms of principal bundles. Let $\kappa \in \Omega^1(G, \mathfrak{g})$ be the right invariant Maurer–Cartan form on G, associated with the left action of G on itself. Then any reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathrm{Ad}(H)\mathfrak{m} = \mathfrak{m}$ defines a G-invariant principal connection $\omega := \mathrm{pr}_{\mathfrak{h}} \circ \kappa$ for the principal bundle $p : G \to H\backslash G$, and a G-invariant displacement form $\theta := \mathrm{pr}_{\mathfrak{m}} \circ \kappa$. Any G-invariant principal connection of $p : G \to H\backslash G$ has this form.

We generalize these classical results in Section 5 to the case of a homogeneous $\mathfrak{g}$-manifold M. The role of the principal bundle $p : G \to H\backslash G$ is taken by the manifold P of germs of transformations of the pseudogroup $\Gamma(\mathfrak{g})$, at a fixed point. We prove that the principal connection forms on the $\mathfrak{g}$-manifold M correspond bijectively to the invariant principal connections of the principal bundle $P \to M$.

For a locally trivial $\mathfrak{g}$-manifold M with a principal connection Φ we define the horizontal lift of vector fields on the orbit space N and the parallel transport along a smooth curve on N. The parallel transport however is only locally defined. If the parallel transport is defined on the whole fiber along any smooth curve, then the connection is called complete. We show that any principal connection is complete if all vector fields in the centralizer $Z_{\mathfrak{X}(S)}(\mathfrak{g})$ are complete.

As final result in this section we prove the following: If a locally trivial $\mathfrak{g}$-manifold M with standard fiber S admits a complete principal connection Φ, whose holonomy Lie algebra consists of complete vector fields on S, then the bundle $M \to N = M/\mathfrak{g}$ is isomorphic to the bundle $P[S] = P \times_H S$ associated to a principal H-bundle $P \to N$, where H is the holonomy group. Moreover, the connection Φ is induced by a principal connection on P.

In the last Section 7 we assume that the $\mathfrak{g}$-manifold M admits not only a principal connection Φ, but also a principal connection form $\omega \in \Omega^1(M; \mathfrak{g})$ with curvature form Ω. We define the Chern–Weil homomorphism γ from the algebra $S(\mathfrak{g}^*)^{\mathfrak{g}}$ of $\mathrm{ad}(\mathfrak{g})$-invariant polynomials on $\mathfrak{g}$ into the algebra $\Omega^{\mathrm{closed}}(M)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant closed forms on M. We prove that for any $f \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ the cohomology class $[\gamma(f)]$ depends only on f and the $\mathfrak{g}$-action. If the action of $\mathfrak{g}$ is free the image of γ consists of horizontal forms. The associated cohomology classes are basic and may be considered as characteristic classes of the $\mathfrak{g}$-manifold, or of the 'bundle below M', even if the action of $\mathfrak{g}$ is not locally trivial. If on the other hand M is a homogeneous $\mathfrak{g}$-manifold, our cohomology classes are characteristic classes of the 'bundle above M', the principal bundle $P \to M$ consisting of germs of pseudogroup transformations constructed in Theorem 5.8.

2. Lie algebra actions alias $\mathfrak{g}$-manifolds

2.1. Actions of Lie algebras on a manifold. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let M be a smooth manifold. We say that $\mathfrak{g}$ *acts on* M or that M is a $\mathfrak{g}$-*manifold* if there is a Lie algebra homomorphism $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$, from $\mathfrak{g}$ into the Lie algebra of all vector fields on M.

If we have a right action of a Lie group on M, then the fundamental vector field mapping is an action of the corresponding Lie algebra on M.

Lemma. *If a Lie algebra $\mathfrak{g}$ acts on a manifold M, then it spans an integrable distribution on M, which need not be of constant rank. So through each point of M there is a unique maximal leaf of that distribution; we also call it the $\mathfrak{g}$-orbit through that point. It is an initial submanifold of M in the sense that a mapping from a manifold into the orbit is smooth if and only if it is smooth into M, see* [9, 2.14*ff*].

Proof. See [19] or [20] for integrable distributions of non-constant rank, or [9, 3.25]. Let Fl^{ξ}_t denote the flow of a vector field ξ. One may check easily that for $X, Y \in \mathfrak{g}$ we have

$$\frac{d}{dt}(\mathrm{Fl}^{\zeta_X}_{-t})\,\zeta(e^{t\,\mathrm{ad}(X)}Y) = 0,$$

which implies $(\mathrm{Fl}^{\zeta_X}_t)^*\zeta_Y = \zeta(e^{t\,\mathrm{ad}(X)}Y)$. So condition (2) of [9, Theorem 3.25] is satisfied and all assertions follow. □

An action of a Lie algebra $\mathfrak{g}$ on a manifold M may have the following properties:

(1) It is called *effective* if $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ is injective. So for each $X \in \mathfrak{g}$ there is some $x \in M$ such that $\zeta_X(x) \neq 0$.

(2) The action is called *free* if for each $x \in M$ the mapping $X \mapsto \zeta_X(x)$ is injective. Then the distribution spanned by $\zeta(\mathfrak{g})$ is of constant rank.

(3) The action is called *transitive* if for each $x \in M$ the mapping $X \mapsto \zeta_X(x)$ is surjective onto T_xM. If M is connected then it is the only orbit and we call M a *homogeneous* $\mathfrak{g}$-*space*.

(4) The action is called *complete* if each fundamental vector field ζ_X is complete, i.e., it generates a global flow. In this case the action can be integrated to a right action of a connected Lie group G with Lie algebra $\mathfrak{g}$, by a result of Palais, [15].

(5) The action is said to be of *constant rank k* if all orbits have the same dimension k.

(6) We call it an *isostabilizer action* if all the isotropy algebras $\mathfrak{g}_x := \ker(\zeta_x : \mathfrak{g} \to T_xM)$ are conjugate in $\mathfrak{g}$ under the connected adjoint group. An isostabilizer action is of constant rank.

(7) The action is called *locally trivial* if there exists a connected manifold S with a transitive action of $\mathfrak{g}$ on S, a submersion $p : M \to N$ onto a smooth manifold with trivial $\mathfrak{g}$-action, such that for each point $x \in N$ there exists an open neighbourhood U and a $\mathfrak{g}$-equivariant diffeomorphism $\varphi : p^{-1}(U) \to U \times S$ with $p \circ \varphi = p$. By '$\mathfrak{g}$-equivariant' we mean that for each $X \in \mathfrak{g}$ the fundamental vector fields $\zeta^M_X|_{p^{-1}(U)}$ and $0 \times \zeta^S_X$ are φ-related: $T\varphi \circ \zeta^M_X|_{p^{-1}(U)} = (0 \times \zeta^S_X) \circ \varphi$. A pair like (U, φ) is called a *bundle chart.*

Note that N is canonically isomorphic to the space of orbits $M/\mathfrak{g}$, and that a locally trivial action is isostabilizer and of constant rank.

(8) A free and locally trivial $\mathfrak{g}$-action is called a *principal action.* A complete principal $\mathfrak{g}$-action can be integrated to an almost free action of a Lie group G (i.e., with discrete isotropy groups). If its action is free, it defines a principal bundle $p : M \to M/G$, and the action of $\mathfrak{g}$ on M is the associated action of the Lie algebra of G. This explains the name.

In the general case, we will consider a locally trivial $\mathfrak{g}$-manifold M as some generalization of the notion of a principal G-bundle, and we will extend to this case some of the main differential geometric constructions of the geometry of principal bundles.

A smooth mapping $f : M \to N$ between $\mathfrak{g}$-manifolds M and N is called $\mathfrak{g}$-*equivariant* if for each $X \in \mathfrak{g}$ the fundamental vector fields ζ_X^M and ζ_X^N are f-related: $Tf \circ \zeta_X^M = \zeta_X^N \circ f$. In view of [9, Section 47] we may also say, that the *generalized Lie derivative* of f is zero:

$$\mathcal{L}_X f = \mathcal{L}_{\zeta_X^M, \zeta_X^N} f := \zeta_X^N \circ f - Tf \circ \zeta_X^M = 0.$$

Note that the integrable distribution of a $\mathfrak{g}$-manifold M of constant rank is a special case (in a certain sense the simplest case) of a foliation. To make this statement more precise we define the *degree of cohomogeneity* of a foliation (integrable distribution) $\mathcal{D}$ on M as the minimum of the difference between the rank of $\mathcal{D}$ and the rank of a $\mathfrak{g}$-manifold structure on M where $\mathfrak{g}$ runs through all finite dimensional subalgebras of constant rank in the Lie algebra $\mathfrak{X}(\mathcal{D})$ of global vector fields on M which are tangent to $\mathcal{D}$:

$$\min\{\operatorname{rank}(\mathcal{D}) - \operatorname{rank}(\mathfrak{g}) : \mathfrak{g} \subset \mathfrak{X}(\mathcal{D}), \dim(\mathfrak{g}) < \infty\}$$

Then we may say that the foliation associated with a $\mathfrak{g}$-manifold of constant rank has degree of cohomogeneity 0, or that it is a 'homogeneous foliation'.

2.2. The pseudogroup of a $\mathfrak{g}$-manifold. Let M be a $\mathfrak{g}$-manifold which we assume to be effective and connected. Local flows of fundamental vector fields, restricted to open subsets, and their compositions, form the *pseudogroup* $\Gamma(\mathfrak{g})$ of the $\mathfrak{g}$-action.

Let us first recall the following definition: A pseudogroup of diffeomorphisms of the manifold M is a set Γ consisting of diffeomorphisms $\varphi : U \to V$ between connected open subsets of M, subject to the following conditions:

(1) If $\varphi : U \to V$ is an element of Γ then also $\varphi^{-1} : V \to U$.

(2) If $\varphi : U \to V$ and $\psi : V \to W$ are elements of Γ then also the composition $\psi \circ \varphi : U \to W$ is in Γ.

(3) If $\varphi : U \to V$ is an element of Γ then also its restriction to any connected open subset $U_1 \subset U$ is an element of Γ.

(4) If $\varphi : U \to V$ is a diffeomorphism between connected open subsets of M which coincides on an open neighbourhood of each of its points with an element of Γ then also φ is in Γ. Now in more details $\Gamma(\mathfrak{g})$ consists of diffeomorphisms of the following form:

$$(5) \qquad \mathrm{Fl}_{t_n}^{\zeta_{X_n}} \circ \ldots \circ \mathrm{Fl}_{t_2}^{\zeta_{X_2}} \circ \mathrm{Fl}_{t_1}^{\zeta_{X_1}} |_U,$$

where $X_i \in \mathfrak{g}$, $t_i \in \mathbb{R}$, and $U \subset M$ are such that $\mathrm{Fl}^{\zeta_{X_1}}_{t_1}$ is defined on U, $\mathrm{Fl}^{\zeta_{X_2}}_{t_2}$ is defined on $\mathrm{Fl}^{\zeta_{X_1}}_{t_1}(U)$, and so on.

2.3. The graph of the pseudogroup of a $\mathfrak{g}$-manifold. Let M be a connected $\mathfrak{g}$-manifold. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We consider the distribution of rank equal to $\dim \mathfrak{g}$ on $G \times M$ which is given by

$$\{(L_X(g), \zeta^M_X(x)) : (g,x) \in G \times M, X \in \mathfrak{g}\} \subset TG \times TM, \tag{1}$$

where L_X is the left invariant vector field on G generated by $X \in \mathfrak{g}$. Obviously this distribution is integrable and thus we may consider the foliation induced by it, which we will call the *graph of the pseudogroup* $\Gamma(\mathfrak{g})$. Note that the flow of the vector field (L_X, ζ^M_X) on $G \times M$ is given by

$$\mathrm{Fl}^{(L_X,\zeta^M_X)}_t(x,g) = (g \cdot \exp^G(tX), \mathrm{Fl}^{\zeta_X}_t(x)).$$

In the sense of [9, Section 9], this foliation is the horizontal foliation for a flat connection of the trivial fiber bundle $G \times M \to G$. The first projection $\mathrm{pr}_1 : G \times M \to G$, when restricted to a leaf, is locally a diffeomorphism. For $x \in M$ we consider the leaf $L(x)$ through $(e,x) \in G \times M$. Then $W_x := \mathrm{pr}_1(L(x))$ is a connected open set in G.

In particular we may use the theory of parallel transport [9, 9.8]: Let $c : (a,b) \to G$ be a piecewise smooth curve with $0 \in (a,b)$ and $c(0) = g$. Then there is an open subset V of $\{g\} \times M \times \{0\}$ in $\{g\} \times M \times \mathbb{R}$ and a smooth mapping $\mathrm{Pt}_c : V \to G \times M$ such that:

(1) $\mathrm{pr}_1(\mathrm{Pt}(c,(g,x),t)) = c(t)$ if defined, and $\mathrm{Pt}(c,(g,x),0) = (g,x)$.

(2) $d/dt\, \mathrm{Pt}(c,(g,x),t)$ is tangent to the graph foliation.

(3) Reparametrization invariance: If $f : (a',b') \to (a,b)$ is piecewise smooth with $0 \in (a',b')$, then $\mathrm{Pt}(c,(g,x),f(t)) = \mathrm{Pt}(c \circ f, \mathrm{Pt}(c,u_x,f(0)),t)$ if defined.

(4) V is maximal for properties (1) and (2).

(5) If the curve c depends smoothly on further parameters then $\mathrm{Pt}(c,(g,x),t)$ depends also smoothly on those parameters. Now let $c : [0,1] \to G$ be piecewise smooth with $c(0) = e$, and let us assume that for some $x \in M$ the parallel transport $\mathrm{Pt}(c,(e,x),t)$ is defined for all $t \in [0,1]$. Then in particular $c([0,1]) \subset W_x$. Since $\{(e,x)\} \times [0,1] \subset V$ the parallel transport $\mathrm{Pt}(c,1)$ is defined on an open subset $\{e\} \times U$ of (e,x), and by (3) it is a diffeomorphism onto its image $\{c(1)\} \times U'$. We may choose U maximal with respect to this property. Since the connection is flat the parallel transport depends on the curve c only up to small (liftable) homotopies fixing end points, since $\mathrm{Pt}(c,(e,x))$ is just the unique lift over the local diffeomorphism $\mathrm{pr}_1 : L(x) \to W_x$. We put

$$\gamma_x(c) := \mathrm{pr}_2 \circ \mathrm{Pt}(c,1) \circ \mathrm{ins}_e : U \to \{e\} \times U \to \{c(1)\} \times U' \to U',$$

so $\gamma_x(c)$ is the parallel transport along c, from the fiber over e to the fiber over $c(1)$, viewed as a local diffeomorphism in M. Since c is homotopic within W_x to a finite sequence of left translates of 1-parameter subgroups, this parallel transport is a composition of a sequence of flows of fundamental vector fields, so $\gamma_x(c)$ is an element of the pseudogroup $\Gamma(\mathfrak{g})$ on M. So γ_x is a mapping from the set of homotopy classes fixing end points of curves starting at e in W_x into the pseudogroup $\Gamma(\mathfrak{g})$.

Conversely each element of $\Gamma(\mathfrak{g})$ of the form 2.2(5) applied to $x \in U$ is the parallel transport of (e,x) along the corresponding polygonial arc consisting of left translates of 1-parameter subgroups: first $[0,t_1] \ni t \mapsto \exp(tX_1)$, then $[t_1,t_2] \ni t \mapsto \exp(t_1X_1)\exp((t-t_1)X_2)$, and so on. Thus we have proved:

Lemma. *Let M be a connected $\mathfrak{g}$-manifold. Then any element $\varphi : U \to V$ of the pseudogroup $\Gamma(\mathfrak{g})$ is of the form $\varphi = \gamma_x(c)$ for $x \in U$ and a smooth curve $c : [0,1] \to W_x$.*

2.4. Lemma. *Let M be a $\mathfrak{g}$-manifold. Assume that for a point $x \in M$ the Lie algebra homomorphism $\operatorname{germ}_x \circ \zeta : \mathfrak{g} \to \mathfrak{X}(M)_{\text{germs at } x}$ is injective. Then for each $\varphi \in \Gamma(\mathfrak{g})$ which is defined near x there is a unique automorphism $\operatorname{Ad}(\varphi^{-1}) : \mathfrak{g} \to \mathfrak{g}$ satisfying $\varphi^*\zeta_X = \zeta_{\operatorname{Ad}(\varphi^{-1})X}$ for all $X \in \mathfrak{g}$.*

This mapping Ad generalizes the adjoint representation of a Lie group.

Proof. One may check easily that for $X, Y \in \mathfrak{g}$ we have

$$\frac{d}{dt}(\mathrm{Fl}^{\zeta_X}_{-t})^*\zeta(e^{t\,\mathrm{ad}(X)}Y) = 0$$

for all t for which the flow is defined. This implies

$$(\mathrm{Fl}^{\zeta_X}_t)^*\zeta_Y = \zeta(e^{t\,\mathrm{ad}(X)}Y). \tag{1}$$

We may apply (1) iteratively to elements of $\Gamma(\mathfrak{g})$ of the form 2.2(5) and thus we get

$$\gamma_x(c)^*\zeta_Y = \zeta_{\operatorname{Ad}(c(1))Y} \tag{2}$$

for each smooth curve starting from e in W_x which is liftable to $L(x)$ in the setting of 2.3. By the assumption, equation (2) now implies that $\operatorname{Ad}(c(1))$ depends only on $\gamma_x(c) \in \Gamma(\mathfrak{g})$ and we call it $\operatorname{Ad}(\gamma_x(c)^{-1})$. We use the inverse so that Ad becomes a 'homomorphism' in 2.5 below. □

2.5. Adjoint representation. Let M be a $\mathfrak{g}$-manifold with pseudogroup $\Gamma(\mathfrak{g})$ such that for each $x \in M$ the homomorphism $\operatorname{germ}_x \circ \zeta : \mathfrak{g} \to \mathfrak{X}(M)_{\text{germs at } x}$ is injective. We denote by $P_x(\mathfrak{g})$ the set of all germs at $x \in M$ of transformations in $\Gamma(\mathfrak{g})$ which are defined at x. Then the set $P(\mathfrak{g}) := \bigcup_{x \in M} P_x(\mathfrak{g})$ with the obvious partial composition is a groupoid. By Lemma 2.4 we have a well defined representation

$$\operatorname{Ad} : P \to \operatorname{Ad}(\mathfrak{g})$$

with values in the adjoint group, and we call it the *adjoint representation* of the groupoid P.

2.7. Lemma. *Let M be a $\mathfrak{g}$-manifold. Let $c : [0,1] \to M$ be a smooth curve in M with values in one $\mathfrak{g}$-orbit. Then there exists a smooth mapping $\varphi : [0,1] \times M \supset U \to M$ such that $\varphi_t \in \Gamma(\mathfrak{g})$ for each t, $[0,1] \times \{c(0)\} \subset U$, and $c(t) = \varphi(t, c(0))$ for all t.*

Since each $\mathfrak{g}$-orbit is an initial submanifold we may equivalently assume that c is a smooth curve in a $\mathfrak{g}$-orbit.

Proof. Let us call $c(0) = x$. Since $c'(t) \in \zeta_{c(t)}(\mathfrak{g})$ it is easy to get a smooth curve $b : [0,1] \to \mathfrak{g}$ such that $c'(t) = \zeta_{b(t)}(c(t))$. We may choose for example $b(t) = (\zeta_{c(t)}|^{-1}_{\ker(\zeta_{c(t)})^\perp})(c'(t))$ with respect to any inner product on $\mathfrak{g}$. Let G be a Lie group with Lie algebra $\mathfrak{g}$, let $g(t)$ be the integral curve of the time dependent vector field $(t,g) \mapsto L_{b(t)}(g)$ with $g(0) = e$. Then $(g(t), c(t))$ is a smooth curve in $G \times M$ which is tangent to the graph foliation of the pseudogroup $\Gamma(\mathfrak{g})$ and thus it lies in the leaf through $(e, x = c(0))$. From 2.3 we see that $\gamma_x(g|_{[0,t]}) = \varphi_t \in \Gamma(\mathfrak{g})$, where φ is the evolution operator of the time dependent vector field $(t,x) \to \zeta_{(b(t))}(x)$ on M. □

2.6. Isotropy groups. Let M be a connected $\mathfrak{g}$-manifold and let $x \in M$. Let us denote by $\Gamma(\mathfrak{g})_x$ the group of all germs at x of elements of the pseudogroup $\Gamma(\mathfrak{g})$ fixing x. It is called the *isotropy group.* Its natural representation into the space $J^k_x(M,\mathbb{R})_0$ of k-jets at x of functions vanishing at x is called the *isotropy representation of order* k. In general the isotropy representation of any order may have a nontrivial kernel. The simplest example is provided by the Lie algebra action defined by one vector field which is flat (vanishes together with all derivatives) at x. We remark that this cannot happen if $\Gamma(\mathfrak{g})$ is a 'Lie pseudogroup' (defined by a system of differential equations); in particular if $\zeta(\mathfrak{g})$ is the algebra of infinitesimal automorphisms of some geometrical structure.

Consider the following diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\zeta} & \mathfrak{X}(M) \\ {\scriptstyle\zeta_x}\downarrow & & \downarrow{\scriptstyle\mathrm{germ}_x} \\ T_xM & \xleftarrow[\mathrm{ev}_x]{} & \mathfrak{X}(M)_{\text{germs at } x}. \end{array}$$

The kernel of the linear mapping $\zeta_x : \mathfrak{g} \to T_xM$ is denoted by $\mathfrak{g}_x$ and it is called the *isotropy algebra* at x. The kernel of the Lie algebra homomorphism $\mathrm{germ}_x \circ \zeta : \mathfrak{g} \to \mathfrak{X}(M)_{\text{germs at } x_0}$ is denoted by $\mathfrak{g}_{\mathrm{germ}_x=0}$; it is an ideal of $\mathfrak{g}$ contained in the isotropy algebra $\mathfrak{g}_x$.

Lemma. *In this setting,* $\Gamma(\mathfrak{g})_x$ *is a Lie group (not necessarily second countable) whose Lie algebra is antiisomorphic to the quotient* $\mathfrak{g}_x/\mathfrak{g}_{\mathrm{germ}_x=0}$ *of the isotropy algebra* $\mathfrak{g}_x$.

If the Lie algebra homomorphism $\mathrm{germ}_x \circ \zeta : \mathfrak{g} \to \mathfrak{X}(M)_{\text{germs at } x}$ *is injective, then there is a canonical representation* $\mathrm{Ad} : \Gamma(\mathfrak{g})_x \to \mathrm{Aut}(\mathfrak{g})$ *which leaves invariant the isotropy subalgebra* $\mathfrak{g}_x$ *and coincides on* $\mathfrak{g}_x$ *with the adjoint representation of* $\Gamma(\mathfrak{g})_x$.

Proof. As in 2.3 we consider again the graph foliation of the $\mathfrak{g}$-manifold M on $G \times M$, where G is a connected Lie group with Lie algebra $\mathfrak{g}$, the leaf $L(x)$ through (e,x) of it, and the open set $W_x = \mathrm{pr}_1(L(x)) \subset G$. Let G_x be the connected subgroup of G corresponding to the isotropy algebra $\mathfrak{g}_x$. Then G_x is contained in W_x since for a smooth curve $c : [0,1] \to G_x$ the curve $(c(t), x)$ in $G \times M$ is tangent to the graph foliation; each curve in G_x and even each homotopy in G_x is liftable to $L(x)$. The universal cover of G_x may be viewed as the space of homotopy classes with fixed ends, of smooth curves in G_x starting from e. So by assigning the germ at x of $\gamma_x(c) \in \Gamma(\mathfrak{g})_x$ to the homotopy class of a curve c in G_x starting from e, we get a group homomorphism from the universal cover of G_x into $\Gamma(\mathfrak{g})_x$. Its tangent mapping at the identity is $-\operatorname{germ}_x \circ \zeta|_{\mathfrak{g}_x}$. Let us denote by $\Gamma(\mathfrak{g})_x^0$ the image of this group homomorphism.

Let φ_t be a smooth curve in the group of germs $\Gamma(\mathfrak{g})_x$, with $\varphi_0 = \mathrm{Id}$ in the sense that $(t,x) \to \varphi_t(x)$ is a smooth germ. Then $(d/dt\,\varphi_t) \circ \varphi_t^{-1}$ is the germ at x of a time dependent vector field with values in the distribution $\mathfrak{g}(M)$ spanned by $\mathfrak{g}$, which vanishes at x. So it has values in the set of germs at x of $\zeta_{\mathfrak{g}_x}$, and thus φ_t is in $\Gamma(\mathfrak{g})_x^0$, see the proof of Lemma 2.7. So the normal subgroup of $\Gamma(\mathfrak{g})_x$ of those elements which may be connected with the identity by a smooth curve in $\Gamma(\mathfrak{g})_x$, coincides with $\Gamma(\mathfrak{g})_x^0$, and the latter is a normal subgroup.

If we declare the Lie group $\Gamma(\mathfrak{g})_x^0$ to be open in $\Gamma(\mathfrak{g})_x$ we get a Lie group structure on $\Gamma(\mathfrak{g})_x$.

The statement about the adjoint representation follows immediately from Lemma 2.4. □

The antiisomorphism in this lemma comes because $\Gamma(\mathfrak{g})_x$ acts from the left on M, so the fundamental vector field mapping of this action should be a Lie algebra antiisomorphism, see [9, 5.12]. Since we started from a Lie algebra homomorphism $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$, the pseudogroup should really act from the right; so it should be viewed as an abstract pseudogroup and not one of transformations. We decided not to do this, but this will cause complicated sign conventions, especially in Theorem 5.8 below.

3. Principal connections for Lie algebra actions

3.1. Principal connections. Let M be a $\mathfrak{g}$-manifold. A vector valued 1-form $\Phi \in \Omega^1(M;TM)$, i.e., a vector bundle homomorphism $\Phi : TM \to TM$, is called a *connection* for the $\mathfrak{g}$-action if for each $x \in M$ the mapping $\Phi_x : T_xM \to T_xM$ is a projection onto $\mathfrak{g}(x) = \zeta(\mathfrak{g})(x) \subset T_xM$. The connection is called *principal* if it is $\mathfrak{g}$-equivariant, i.e., if for each $X \in \mathfrak{g}$ the Lie derivative vanishes: $\mathcal{L}_{\zeta_X}\Phi = [\zeta_X, \Phi] = 0$, where $[\cdot,\cdot]$ is the Frölicher–Nijenhuis bracket, see 4.4 below. The distribution $\ker(\Phi)$ is called the *horizontal distribution* of the connection Φ.

A Lie algebra valued 1-form $\omega \in \Omega^1(M;\mathfrak{g})$ is called a *principal connection form* if the following conditions are satisfied:

(1) ω is $\mathfrak{g}$-equivariant, i.e., for all $X \in \mathfrak{g}$ we have $\mathcal{L}_{\zeta_X}\omega = -\operatorname{ad}(X)\circ\omega$.

(2) For any $x \in M$ we have $\zeta_x = \zeta_x \circ \omega_x \circ \zeta_x : \mathfrak{g} \to T_xM \to \mathfrak{g} \to T_xM$. Thus for any $x \in M$ the kernel $\ker(\omega_x)$ is a complementary subspace to the vertical space $\mathfrak{g}(x)$ and the mapping $\omega_x : \mathfrak{g}(x) \cong \mathfrak{g}/\mathfrak{g}_x \to \mathfrak{g}$ is a right inverse to the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_x$.

Any principal connection form ω defines a principal connection $\Phi := \zeta_\omega$. The converse statement is not true in general as Example 5.7 shows.

3.2. Proposition. *Let M be a $\mathfrak{g}$-manifold.*

1. *If M admits a principal connection then the action of $\mathfrak{g}$ on M has constant rank.*

2. *Let us assume conversely that the* $\mathfrak{g}$*-action is of constant rank. Then* M *admits a principal connection if any of the following conditions is satisfied*:

(1) *The action is locally trivial.*

(2) *There exists a* $\mathfrak{g}$*-invariant Riemannian metric on* M.

(3) *The* $\mathfrak{g}$*-action is induced by a proper action of a Lie group* G *with Lie algebra* $\mathfrak{g}$.

3. *Let the* $\mathfrak{g}$*-action be locally trivial with standard fiber* S, *a homogeneous* $\mathfrak{g}$*-space which admits a principal connection form. Then* M *admits even a principal connection form* ω.

For assertion 3, see Example 5.7 for conditions assuring the existence of principal connection forms on the standard fiber S: the isotropy subalgebra $\mathfrak{g}_x$ of some point $x \in S$ admits an $\mathrm{ad}(g_x)$-invariant complement $\mathfrak{m}$ in $\mathfrak{g}$ which is also invariant under the isotropy representation of the pseudogroup $\Gamma(\mathfrak{g})$ generated by $\mathfrak{g}$.

Proof. 1. If a principal connection Φ exists, it is a projection onto the distribution spanned by $\mathfrak{g}$ (which we will call the vertical distribution sometimes), and its rank cannot fall locally. But the rank of the complementary projection $\chi := \mathrm{Id}_{TM} - \Phi$ onto the kernel of Φ also cannot fall locally, so the vertical distribution $\mathfrak{g}(M)$ is locally of constant rank.

2. First of all, we have the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1). The first implication is a theorem of Palais [16]. The implication (2) $\Rightarrow$ (1) may be proved as for an action of a Lie group that preserves a Riemannian metric, using a slice. Hence, we may assume that M is a connected locally trivial $\mathfrak{g}$-manifold.

Let $(U_\alpha, \varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times S)$ be a family of principal charts such that (U_α) is an open cover of M. Put $\Phi_\alpha(T\varphi_\alpha^{-1}(\xi_x, \eta_s)) = T\varphi_\alpha^{-1}(0_x, \eta_s)$ for $\xi_x \in T_xU_\alpha$ and $\eta_s \in T_sS$. Obviously that Φ_α is a principal connection on $p^{-1}(U_\alpha)$. Now let f_α be a smooth partition of unity on N which is subordinated to the open cover (U_α). Then $\Phi := \sum_\alpha (f_\alpha \circ p)\Phi_\alpha$ is a principal connection on M.

3. This is proved similarly as 2, starting from a principal connection form on the standard fiber S. □

3.3. Let M be a $\mathfrak{g}$-manifold. If a principal connection Φ exists then the distribution $\mathfrak{g}(M)$ spanned by $\mathfrak{g}$ is of constant rank and thus a vector bundle over M, and Φ factors to a $\mathfrak{g}$-equivariant right inverse of the vector bundle epimorphims $TM \to TM/\mathfrak{g}(M)$. Let us consider the following sequence of families of vector bundles over M, where $\mathrm{iso}(M) := \bigcup_{x\in M}\{x\} \times \mathfrak{g}_x = \ker(\zeta^M)$ is the *isotropy algebra bundle* over M:

$$\mathrm{iso}(M) \to M \times \mathfrak{g} \xrightarrow{\zeta^M} TM \to TM/\mathfrak{g}(M).$$

Then a principal connection form ω induces a $\mathfrak{g}$-equivariant right inverse on its image of the vector bundle homomorphism $\zeta^M : M \times \mathfrak{g} \to TM$, so it satisfies $\zeta^M \circ \omega \circ \zeta^M = \zeta^M$.

4. Frölicher–Nijenhuis bracket and curvature

4.1. Products of differential forms. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a Lie algebra $\mathfrak{g}$ in a finite dimensional vector space V and let M be a smooth manifold.

For $\varphi \in \Omega^p(M;\mathfrak{g})$ and $\Psi \in \Omega^q(M;V)$ we define the form $\rho^\wedge(\varphi)\Psi \in \Omega^{p+q}(M;V)$ by

$$\begin{aligned}(\rho^\wedge(\varphi)\Psi)&(X_1,\dots,X_{p+q})\\ &= \frac{1}{p!\,q!}\sum_\sigma \operatorname{sign}(\sigma)\rho(\varphi(X_{\sigma 1},\dots,X_{\sigma p}))\Psi(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)}).\end{aligned}$$

Then $\rho^\wedge(\varphi) : \Omega^*(M;V) \to \Omega^{*+p}(M;V)$ is a graded $\Omega(M)$-module homomorphism of degree p.

Recall also that $\Omega(M;\mathfrak{g})$ is a graded Lie algebra with the bracket $[\cdot,\cdot]^\wedge = [\cdot,\cdot]^\wedge_\mathfrak{g}$ given by

$$\begin{aligned}[\varphi,\psi]^\wedge&(X_1,\dots,X_{p+q})\\ &= \frac{1}{p!\,q!}\sum_\sigma \operatorname{sign}\sigma\, [\varphi(X_{\sigma 1},\dots,X_{\sigma p}),\psi(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})]_\mathfrak{g},\end{aligned}$$

where $[\cdot,\cdot]_\mathfrak{g}$ is the bracket in $\mathfrak{g}$. One may easily check that for the graded commutator in $\operatorname{End}(\Omega(M;V))$ we have

$$\rho^\wedge([\varphi,\psi]^\wedge) = [\rho^\wedge(\varphi),\rho^\wedge(\psi)] = \rho^\wedge(\varphi)\circ\rho^\wedge(\psi) - (-1)^{pq}\rho^\wedge(\psi)\circ\rho^\wedge(\varphi)$$

so that $\rho^\wedge : \Omega^*(M;\mathfrak{g}) \to \operatorname{End}^*(\Omega(M;V))$ is a homomorphism of graded Lie algebras.

For any vector space V let $\bigotimes V$ be the tensor algebra generated by V. For $\Phi,\Psi \in \Omega(M;\bigotimes V)$ we will use the associative bigraded product

$$\begin{aligned}(\Phi\otimes_\wedge\Psi)&(X_1,\dots,X_{p+q})\\ &= \frac{1}{p!\,q!}\sum_\sigma \operatorname{sign}(\sigma)\Phi(X_{\sigma 1},\dots,X_{\sigma p})\otimes\Psi(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)}).\end{aligned}$$

4.2. Basic differential forms. Let M be a $\mathfrak{g}$-manifold. A differential form $\varphi \in \Omega^p(M;V)$ with values in a vector space V (or even in a vector bundle over M) is called *horizontal* if it kills all fundamental vector fields ζ_X, i.e., if $i_{\zeta_X}\varphi = 0$ for each $X \in \mathfrak{g}$.

If moreover $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of the Lie algebra $\mathfrak{g}$ in V, then differential form $\varphi \in \Omega^p(M;V)$ is called *$\mathfrak{g}$-equivariant* if for the Lie derivative along fundamental vector fields we have: $\mathcal{L}_{\zeta_X}\varphi = -\rho(X)\circ\varphi$ for all $X \in \mathfrak{g}$.

Let us denote by $\Omega^p_{\mathrm{hor}}(M;V)^\mathfrak{g}$ the space of all V-valued differential forms on M which are horizontal and $\mathfrak{g}$-equivariant. It is called the space of *basic V-valued differential forms* on the $\mathfrak{g}$-manifold M. If the $\mathfrak{g}$-manifold M has constant rank and the action of $\mathfrak{g}$ defines a foliation, scalar valued basic forms are the usual basic differential forms of the foliation, see e.g. [13].

Note that the graded Lie module structure $\rho^\wedge$ from 4.1 restricts to a graded Lie module structure $\rho^\wedge : \Omega(M;\mathfrak{g})^\mathfrak{g} \times \Omega(M;V)^\mathfrak{g} \to \Omega(M;V)^\mathfrak{g}$. It is also compatible with the requirement of horizontality.

The exterior differential d acts on $(\Omega(M;\mathfrak{g}),[\cdot,\cdot]^\wedge)$ as a graded derivation of degree 1. It preserves the subalgebra $\Omega(M;\mathfrak{g})^\mathfrak{g}$ of $\mathfrak{g}$-invariant forms, but it does not preserve the subalgebra $\Omega^p_{\mathrm{hor}}(M;\mathfrak{g})^\mathfrak{g}$ of $\mathfrak{g}$-valued basic forms.

For $\varphi \in \Omega^p(M;\mathfrak{g})$ we consider the tangent bundle valued differential form $\zeta_\varphi \in \Omega^p(M;TM)$ which is given for $\xi_i \in T_xM$ by

$$(\zeta_\varphi)_x(\xi_1,\dots,\xi_p) := \zeta_{\varphi_x(\xi_1,\dots,\xi_p)}(x).$$

4.3. Frölicher–Nijenhuis bracket. Let M be a smooth manifold. We shall use now the Frölicher–Nijenhuis bracket

$$[\cdot,\cdot] : \Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^{p+q}(M;TM)$$

as guiding line for the further developments, since it is a natural and convenient way towards connections, curvature, and Bianchi identity, in many settings. See [9, Sections 8–11], as a convenient reference for this. We repeat here the global formula for the Frölicher–Nijenhuis bracket from [9, 8.9]: For $K \in \Omega^k(M;TM)$ and $L \in \Omega^\ell(M;TM)$ we have for the Frölicher–Nijenhuis bracket $[K,L]$ the following formula, where the ξ_i are vector fields on M.

$$\begin{aligned}
&[K,L](\xi_1,\dots,\xi_{k+\ell})\\
&= \frac{1}{k!\,\ell!}\sum_\sigma \operatorname{sign}\sigma\, [K(\xi_{\sigma 1},\dots,\xi_{\sigma k}), L(\xi_{\sigma(k+1)},\dots,\xi_{\sigma(k+\ell)})]\\
&+ \frac{-1}{k!\,(\ell-1)!}\sum_\sigma \operatorname{sign}\sigma\, L([K(\xi_{\sigma 1},\dots,\xi_{\sigma k}),\xi_{\sigma(k+1)}],\xi_{\sigma(k+2)},\dots)\\
&+ \frac{(-1)^{k\ell}}{(k-1)!\,\ell!}\sum_\sigma \operatorname{sign}\sigma\, K([L(\xi_{\sigma 1},\dots,\xi_{\sigma \ell}),\xi_{\sigma(\ell+1)}],\xi_{\sigma(\ell+2)},\dots) \qquad (1)\\
&+ \frac{(-1)^{k-1}}{(k-1)!\,(\ell-1)!\,2!}\sum_\sigma \operatorname{sign}\sigma\, L(K([\xi_{\sigma 1},\xi_{\sigma 2}],\xi_{\sigma 3},\dots),\xi_{\sigma(k+2)},\dots)\\
&+ \frac{(-1)^{(k-1)\ell}}{(k-1)!\,(\ell-1)!\,2!}\sum_\sigma \operatorname{sign}\sigma\, K(L([\xi_{\sigma 1},\xi_{\sigma 2}],\xi_{\sigma 3},\dots),\xi_{\sigma(\ell+2)},\dots).
\end{aligned}$$

For decomposable tangent bundle valued forms we have the following formula for the Frölicher–Nijenhuis bracket in terms of the usual operations with vector fields and differential forms, see [5], or [9, 8.7]. Let $\varphi \in \Omega^k(M)$, $\psi \in \Omega^\ell(M)$, and $X, Y \in \mathfrak{X}(M)$. Then

$$\begin{aligned}
[\varphi\otimes X,\psi\otimes Y] &= \varphi\wedge\psi\otimes[X,Y] + \varphi\wedge\mathcal{L}_X\psi\otimes Y - \mathcal{L}_Y\varphi\wedge\psi\otimes X\\
&\quad + (-1)^k\,(d\varphi\wedge i_X\psi\otimes Y + i_Y\varphi\wedge d\psi\otimes X).
\end{aligned} \qquad (2)$$

4.4. Proposition. *Let M be a $\mathfrak{g}$-manifold. Let $\varphi \in \Omega^p_{\text{hor}}(M;\mathfrak{g})^{\mathfrak{g}}$ and let $\psi \in \Omega^q_{\text{hor}}(M;\mathfrak{g})^{\mathfrak{g}}$. Then we have:*

(1) *ζ_ψ is horizontal and $\mathfrak{g}$-equivariant in the sense that the Lie derivative along any fundamental vector field ζ_X for $X \in \mathfrak{g}$ vanishes: $\mathcal{L}_{\zeta_X}(\zeta_\psi) = [\zeta_X,\zeta_\psi] = 0$.*

(2) *$[\zeta_\varphi,\zeta_\psi] = -\zeta_{[\varphi,\psi]^\wedge}$, so $\zeta : \Omega_{\text{hor}}(M;\mathfrak{g})^{\mathfrak{g}} \to \Omega(M;TM)$ is an antihomomorphism of graded Lie algebras. If $\omega \in \Omega^1(M;\mathfrak{g})^{\mathfrak{g}}$ is a principal connection form with principal connection $\Phi = \zeta_\omega$ and horizontal projection $\chi := \operatorname{Id}_{TM} - \Phi$ then we have furthermore:*

(3) *$[\Phi,\zeta_\psi] = -\zeta_{d\psi+[\omega,\psi]^\wedge}$.*

(4) $\frac{1}{2}[\Phi,\Phi] = -\zeta_{d\omega+\frac{1}{2}[\omega,\omega]^\wedge}$.

Compare this results with [9, 11.5], which, however, contains a sign mistake in (10). We give here a shorter proof of a stronger statement.

Proof. Let $X_1,\dots,X_k$ be a linear basis of the Lie algebra $\mathfrak{g}$. Then any form $\varphi \in \Omega^p(M;\mathfrak{g})$ can be uniquely written in the form $\varphi = \sum_{i=1}^k \varphi^i \otimes X_i$ for $\varphi^i \in \Omega^p(M)$. Then $\zeta_\varphi = \sum_{i=1}^k \varphi^i \otimes \zeta_{X_i}$.

Note that φ is horizontal if and only if all φ^i are horizontal. Also φ is $\mathfrak{g}$-equivariant, $\varphi \in \Omega^p(M;\mathfrak{g})^\mathfrak{g}$, i.e., $\mathcal{L}_{\zeta_X}\varphi = -\operatorname{ad}(X)\circ\varphi$ for all $X \in \mathfrak{g}$, if and only if for all $X \in \mathfrak{g}$ we have:

$$\sum_{i=1}^k \mathcal{L}_{\zeta_X}\varphi^i \otimes X_i = -\sum_{i=1}^k \varphi^i \otimes [X,X_i]. \tag{5}$$

Assertion (1) now follows from (5) and

$$\mathcal{L}_{\zeta_X}\zeta_\varphi = \sum_{i=1}^k \mathcal{L}_{\zeta_X}\varphi^i \otimes \zeta_{X_i} + \sum_{i=1}^k \varphi^i \otimes [\zeta_X,\zeta_{X_i}].$$

Using 4.3(2) we have for general $\varphi,\psi \in \Omega(M;\mathfrak{g})$

$$\begin{aligned}[\zeta_\varphi,\zeta_\psi] &= \sum_{i,j}[\varphi^i\otimes\zeta_{X_i},\psi^j\otimes\zeta_{X_j}] = \sum_{i,j}\varphi^i\wedge\psi^j\otimes[\zeta_{X_i},\zeta_{X_j}]\\ &\quad+\sum_{i,j}\varphi^i\wedge\mathcal{L}_{\zeta_{X_i}}\psi^j\otimes\zeta_{X_j} - \sum_{i,j}\mathcal{L}_{\zeta_{X_j}}\varphi^i\wedge\psi^j\otimes\zeta_{X_i}\\ &\quad+(-1)^p\sum_{i,j}\left(d\varphi^i\wedge i_{\zeta_{X_i}}\psi^j\otimes\zeta_{X_j} + i_{\zeta_{X_j}}\varphi^i\wedge d\psi^j\otimes\zeta_{X_i}\right).\end{aligned} \tag{6}$$

If ψ is $\mathfrak{g}$-equivariant then from (5) we have

$$\sum_{i,j}\varphi^i\wedge\mathcal{L}_{\zeta_{X_i}}\psi^j\otimes\zeta_{X_j} = -\sum_{i,j}\varphi^i\wedge\psi^j\otimes\zeta_{[X_i,X_j]} = -\zeta_{[\varphi,\psi]^\wedge}.$$

So for φ and ψ both horizontal and $\mathfrak{g}$-equivariant (6) reduces to assertion (2).

If $\varphi = \omega$, the connection form, then we have $(-1)^p\sum_{i,j} i_{\zeta_{X_j}}\varphi^i\wedge d\psi^j\otimes\zeta_{X_i} = \sum_j d\psi^j\otimes\zeta_{X_j} = -\zeta_{d\psi}$, so that (6) reduces to assertion (3). Similarly, for $\varphi=\psi=\omega$ formula (6) reduces to $[\Phi,\Phi] = -\zeta_{[\omega,\omega]^\wedge+2d\omega}$, so also (4) holds. □

4.5. Covariant exterior derivative. Let M be a $\mathfrak{g}$-manifold of constant rank, let $\Phi \in \Omega^1(M;TM)^\mathfrak{g}$ be a principal connection with associated horizontal projection $\chi := \mathrm{Id}_{TM} - \Phi$. Let V be any vector space of finite dimension. Then we define the *covariant exterior derivative*

$$d_\Phi := \chi^*\circ d : \Omega^p(M;V) \to \Omega^{p+1}_{\text{hor}}(M;V).$$

We also consider the following mapping as a form of the covariant exterior derivative:

$$\operatorname{ad}(\Phi) := [\Phi,\cdot\,] : \Omega^p(M;TM) \to \Omega^{p+1}(M;TM).$$

If a principal connection form $\omega : TM \to \mathfrak{g}$ exists and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ we also consider the following covariant exterior derivative:

$$d_\omega : \Omega^p(M;V) \to \Omega^{p+1}(M;V),$$
$$d_\omega \Psi := d\Psi + \rho^\wedge(\omega)\Psi.$$

Lemma. *In this situation we have:*

(1) $\operatorname{ad}(\Phi)$ *restricts to a mapping* $[\Phi, \cdot] : \Omega^p_{\text{hor}}(M;\mathfrak{g}(M))^{\mathfrak{g}} \to \Omega^{p+1}_{\text{hor}}(M;\mathfrak{g}(M))^{\mathfrak{g}}$, *where* $\mathfrak{g}(M) \subset TM$ *is the vertical bundle.*

(2) *Let* $\rho : \mathfrak{g} \to GL(V)$ *be a representation. Then* d_Φ *restricts to a mapping* $d_\Phi : \Omega^p_{\text{hor}}(M;V)^{\mathfrak{g}} \to \Omega^{p+1}_{\text{hor}}(M;V)^{\mathfrak{g}}$.

(3) *For* $\psi \in \Omega^p_{\text{hor}}(M;\mathfrak{g})^{\mathfrak{g}}$ *the two covariant derivatives correspond to each other up to a sign:* $\zeta_{d_\Phi\psi} = -[\Phi, \zeta_\psi]$.

(4) *Let* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ *be a representation. Then* d_ω *restricts to a mapping* $d_\omega : \Omega^p(M;V)^{\mathfrak{g}} \to \Omega^{p+1}(M;V)^{\mathfrak{g}}$. *For* $\Psi \in \Omega^p_{\text{hor}}(M;V)^{\mathfrak{g}}$ *and* $X \in \mathfrak{g}$ *we have* $i(\zeta_X)d_\omega\Psi = \rho(\omega(\zeta_X) - X)\Psi$. *If* M *is a free* $\mathfrak{g}$*-manifold, then* d_ω *also respects horizontality and we have* $d_\Phi = d_\omega : \Omega^p_{\text{hor}}(M;V)^{\mathfrak{g}} \to \Omega^{p+1}_{\text{hor}}(M;V)^{\mathfrak{g}}$, *where* $\Phi = \zeta_\omega$.

Proof. (1) Let $\Psi \in \Omega^p_{\text{hor}}(M;\mathfrak{g}(M))^{\mathfrak{g}}$ and $X \in \mathfrak{g}$. Then formulas [9, 8.11(2)] give us here

$$i_{\zeta_X}[\Phi,\Psi] = [i_{\zeta_X}\Phi,\Psi] + [\Phi, i_{\zeta_X}\Psi] - (i([\Phi,\zeta_X])\Psi - (-1)^p i([\Psi,\zeta_X])\Phi) = 0,$$

so that $[\Phi,\Psi]$ is again horizontal. It is also $\mathfrak{g}$-equivariant since $[\zeta_X,[\Phi,\Psi]] = [[\zeta_X,\Phi],\Psi] + [\Phi,[\zeta_X,\Psi]] = 0$ for all $X \in \mathfrak{g}$ by the graded Jacobi identity. That it has vertical values can be seen by contemplating one of the formulas in 4.3.

(2) Let $\Psi \in \Omega^p_{\text{hor}}(M;V)^{\mathfrak{g}}$. For $X \in \mathfrak{g}$ we have $\mathcal{L}_{\zeta_X}\Psi = -\rho(X)\circ\Psi$, then $d_\Phi\Psi$ is again $\mathfrak{g}$-equivariant, since we have

$$\begin{aligned}\mathcal{L}_{\zeta_X}\chi^* d\Psi &= \mathcal{L}_{\zeta_X}(d\Psi \circ \Lambda^{p+1}\chi) = \mathcal{L}_{\zeta_X}(d\Psi)\circ\Lambda^{p+1}\chi + d\Psi\circ\mathcal{L}_{\zeta_X}(\Lambda^{p+1}\chi)\\ &= (d\mathcal{L}_{\zeta_X}\Psi)\circ\Lambda^{p+1}\chi + 0 = \chi^* d(-\rho(X)\circ\Psi) = -\rho(X)\circ(\chi^* d\Psi),\end{aligned}$$

and clearly horizontal.

(3) Let again $X_1, \dots, X_k$ be a linear basis of the Lie algebra $\mathfrak{g}$ and consider $\psi = \sum_{i=1}^k \psi^i \otimes X_i \in \Omega^p_{\text{hor}}(M;\mathfrak{g})^{\mathfrak{g}}$ for $\psi^i \in \Omega^p_{\text{hor}}(M)$. Then we use [9, 8.7(5)] to get

$$\begin{aligned}(-1)^{p+1}[\Phi,\zeta_\psi] &= [\zeta_\psi,\Phi] = \sum_i [\psi^i \otimes \zeta_{X_i}, \Phi]\\ &= \sum_i \left(\psi^i \wedge [\zeta_{X_i},\Phi] - (-1)^p \mathcal{L}_\Phi\psi^i \otimes \zeta_{X_i} + (-1)^p d\psi^i \wedge i(\zeta_{X_i})\Phi\right).\end{aligned}$$

Since Φ is $\mathfrak{g}$-equivariant we have $[\zeta_{X_i},\Phi] = 0$. Moreover we have $\mathcal{L}_\Phi\psi^i = i_\Phi d\psi^i - d i_\Phi\psi^i = i_\Phi d\psi^i - 0$ and $i(\zeta_{X_i})\Phi = \zeta_{X_i}$. Thus we get

$$\begin{aligned}(-1)^{p+1}[\Phi,\zeta_\psi] &= (-1)^p \sum_i (d\psi^i - i_\Phi d\psi^i)\otimes\zeta_{X_i}\\ &= (-1)^p \sum_i \chi^* d\psi^i \otimes \zeta_{X_i} = (-1)^p \zeta_{d_\Phi\psi}.\end{aligned}$$

(4) Let $\Psi \in \Omega^p(M;V)^{\mathfrak{g}}$. For $X \in \mathfrak{g}$ we have $\mathcal{L}_{\zeta_X}\Psi = -\rho(X)\circ\Psi$, then $d_\omega\Psi$ is again $\mathfrak{g}$-equivariant, since we have

$$\begin{aligned}\mathcal{L}_{\zeta_X}(d\Psi + \rho^\wedge(\omega)\Psi) &= d\mathcal{L}_{\zeta_X}\Psi + \rho^\wedge(\mathcal{L}_{\zeta_X}\omega)\Psi + \rho^\wedge(\omega)\mathcal{L}_{\zeta_X}\Psi \\ &= -d\rho(X)\Psi - \rho^\wedge([X,\omega]^\wedge)\Psi - \rho^\wedge(\omega)\rho(X)\Psi \\ &= -\rho(X)(d\Psi + \rho^\wedge(\omega)\Psi).\end{aligned}$$

For $\Psi \in \Omega^p_{\text{hor}}(M;V)^{\mathfrak{g}}$ and $X \in \mathfrak{g}$ we use $i_{\zeta_X}\Psi = 0$ and $\mathcal{L}_{\zeta_X}\Psi = -\operatorname{ad}(X)\Psi$ to get

$$\begin{aligned}i_{\zeta_X}(d\Psi + \rho^\wedge(\omega)\Psi) &= i_{\zeta_X}d\Psi + di_{\zeta_X}\Psi + \rho(i_{\zeta_X}\omega)\Psi - \rho^\wedge(\omega)i_{\zeta_X}\Psi \\ &= \mathcal{L}_{\zeta_X}\Psi + \rho(\omega(\zeta_X))\Psi = \rho(\omega(\zeta_X) - X)\Psi.\end{aligned}$$

Let now M be a free $\mathfrak{g}$-manifold then $\omega(\zeta_X) - X = 0$ and $d_\omega\Psi$ is again horizontal. We use the principal connection Φ to split each vector field into the sum of a horizontal one and a vertical one. If we insert one vertical vector field, say ζ_X for $X \in \mathfrak{g}$, into $d_\Phi\Psi - d_\omega\Psi$, we get 0. Let now all vector fields ξ_i be horizontal, then we get

$$\begin{aligned}&(d_\Phi\Psi)(\xi_0,\dots,\xi_k) = (\chi^* d\Psi)(\xi_0,\dots,\xi_k) = d\Psi(\xi_0,\dots,\xi_k),\\ &(d\Psi + [\omega,\Psi]^\wedge)(\xi_0,\dots,\xi_k) = d\Psi(\xi_0,\dots,\xi_k). \quad \square\end{aligned}$$

4.6. Curvature. Let M be a $\mathfrak{g}$-manifold. If there exists a principal connection Φ then this is a projection onto the integrable vertical distribution induced by $\mathfrak{g}$, and the formula 4.3(1) for the Frölicher–Nijenhuis bracket reduces to

$$R(\xi,\eta) = \tfrac{1}{2}[\Phi,\Phi](\xi,\eta) = \Phi[\xi - \Phi\xi, \eta - \Phi\eta].$$

$R \in \Omega^2_{\text{hor}}(M;TM)^{\mathfrak{g}}$ is called the *curvature* of the connection Φ. From the graded Jacobi identity of the Frölicher–Nijenhuis bracket we get immediately the *Bianchi identity*

$$[\Phi, R] = \tfrac{1}{2}[\Phi,[\Phi,\Phi]] = 0.$$

Note that the kernel of $\operatorname{ad}(R)$ is invariant under $\operatorname{ad}(\Phi)$, and $\operatorname{ad}(\Phi)^2 = 0$ on it. It gives rise to a cohomology, depending on Φ.

If $\omega \in \Omega^1(M,\mathfrak{g})^{\mathfrak{g}}$ is a principal connection form, then formula (4) in Proposition 4.4 suggests to define

$$\Omega := d\omega + \tfrac{1}{2}[\omega,\omega]^\wedge$$

as the *curvature form* of ω; so we have $R = -\zeta_\Omega$. Then 4.4(3) suggests that the *Bianchi identity* should have the form $d_\omega\Omega = d\Omega + [\omega,\Omega]^\wedge = 0$. Indeed this is true and it follows directly from the graded Jacobi identity in $(\Omega(M,\mathfrak{g}), [\cdot,\cdot]^\wedge)$.

4.7. Proposition. *Let M be a $\mathfrak{g}$-manifold with principal connection Φ and horizontal projection $\chi := \operatorname{Id}_{TM} - \Phi$. Then we have:*

(1) $d_\Phi \circ \chi^* - d_\Phi = \chi^*[d,\chi^*] = \chi^* \circ i_R : \Omega^p(M;V) \to \Omega^{p+1}_{\text{hor}}(M;V)$, *where R is the curvature and V is any vector space and i_R is the insertion operator.*

(2) $d_\Phi \circ d_\Phi = \chi^* \circ i(R) \circ d : \Omega^p(M;V) \to \Omega^{p+2}_{\text{hor}}(M;V)$.

(3) *If $\omega \in \Omega^1(M;\mathfrak{g})^{\mathfrak{g}}$ is a principal connection form then the curvature form $\Omega = d\omega + \frac{1}{2}[\omega,\omega]^\wedge$ satisfies $i(\zeta_X)\Omega = [\omega(\zeta_X) - X, \omega]^\wedge - d(\omega(\zeta_X))$. If M is a free $\mathfrak{g}$-manifold then Ω is horizontal and $\Omega = d_\Phi\omega = d_\omega\omega \in \Omega^2_{\text{hor}}(M;\mathfrak{g})^{\mathfrak{g}}$.*

Note that by (2) the kernel of $\chi^* \circ i(R) \circ d$ is invariant under d_Φ, which gives rise to a cohomology associated to it.

Proof. (1) For $\Psi \in \Omega(P;V)$ we have

$$\begin{aligned}
(d_\Phi \chi^*\Psi)(\xi_0,\dots,\xi_k) &= (d\chi^*\Psi)(\chi(\xi_0),\dots,\chi(\xi_k)) \\
&= \sum_{0\leqslant i\leqslant k} (-1)^i \chi(\xi_i)((\chi^*\Psi)(\chi(\xi_0),\dots,\widehat{\chi(\xi_i)},\dots,\chi(\xi_k))) \\
&\quad + \sum_{i<j} (-1)^{i+j}(\chi^*\Psi)([\chi(\xi_i),\chi(\xi_j)],\chi(\xi_0),\dots,\widehat{\chi(\xi_i)},\dots,\widehat{\chi(\xi_j)},\dots) \\
&= \sum_{0\leqslant i\leqslant k} (-1)^i \chi(\xi_i)(\Psi(\chi(\xi_0),\dots,\widehat{\chi(\xi_i)},\dots,\chi(\xi_k))) \\
&\quad + \sum_{i<j} (-1)^{i+j}\Psi([\chi(\xi_i),\chi(\xi_j)] - \Phi[\chi(\xi_i),\chi(\xi_j)],\chi(\xi_0),\dots \\
&\qquad \dots,\widehat{\chi(\xi_i)},\dots,\widehat{\chi(\xi_j)},\dots) \\
&= (d\Psi)(\chi(\xi_0),\dots,\chi(\xi_k)) + (i_R\Psi)(\chi(\xi_0),\dots,\chi(\xi_k)) \\
&= (d_\Phi + \chi^* i_R)(\Psi)(\xi_0,\dots,\xi_k).
\end{aligned}$$

(2) $d_\Phi d_\Phi = \chi^* d\chi^* d = (\chi^* i_R + \chi^* d)d = \chi^* i_R d$ holds by (1).

(3) For $X \in \mathfrak{g}$ we have

$$\begin{aligned}
i_{\zeta_X}(d\omega + \tfrac{1}{2}[\omega,\omega]^\wedge) &= i_{\zeta_X} d\omega + \tfrac{1}{2}[i_{\zeta_X}\omega,\omega]^\wedge - \tfrac{1}{2}[\omega, i_{\zeta_X}\omega]^\wedge \\
&= \mathcal{L}_{\zeta_X}\omega - d(\omega(\zeta_X)) + [\omega(\zeta_X),\omega] = [\omega(\zeta_X) - X,\omega] - d\omega(\zeta_X).
\end{aligned}$$

If M is a free $\mathfrak{g}$-manifold then this is zero, and on horizontal vectors $d_\Phi\omega$ and $d_\omega\omega$ coincide. □

5. Homogeneous $\mathfrak{g}$-manifolds

5.1. Homogeneous free $\mathfrak{g}$-manifolds and Maurer–Cartan forms. Recall that a $\mathfrak{g}$-valued 1-form κ on a manifold M is called *Maurer–Cartan form* if $\kappa_x : T_xM \to \mathfrak{g}$ is a linear isomorphism for each $x \in M$ and if κ satisfies the *Maurer–Cartan equation* $d\kappa + \frac{1}{2}[\kappa,\kappa] = 0$. This concept is also sometimes called a *flat Cartan connection*, and a manifold with a flat Cartan connection is sometimes called a *principal homogeneous space*. See [7] for Maurer–Cartan forms.

Lemma. *To each free transitive $\mathfrak{g}$-action $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ there corresponds a unique Maurer–Cartan form $\kappa : TM \to \mathfrak{g}$, given by $\kappa_x = \zeta_x^{-1}$, and conversely. Then κ is*

$\mathfrak{g}$-equivariant with respect to the $\mathfrak{g}$-action $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$, and κ is the unique principal connection form on the g-manifold M.

Note also that an action of a Lie algebra $\mathfrak{g}$ is free if and only if the associated pseudogroup has discrete isotropy groups.

Proof. If $\kappa \in \Omega^1(M;\mathfrak{g})$ and $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ are inverse to each other then for $X, Y \in \mathfrak{g}$ we have

$$\begin{aligned}
&(d\kappa + \tfrac{1}{2}[\kappa,\kappa]^\wedge)(\zeta_X,\zeta_Y)\\
&\quad = \zeta_X(\kappa(\zeta_Y)) - \zeta_Y(\kappa(\zeta_X)) - \kappa([\zeta_X,\zeta_Y]) + [\kappa(\zeta_X),\kappa(\zeta_Y)]\\
&\quad = -\kappa([\zeta_X,\zeta_Y]) + [X,Y]\\
&\quad = -\kappa([\zeta_X,\zeta_Y] - \zeta_{[X,Y]}),
\end{aligned}$$

so that ζ is a Lie algebra homomorphism if and only if κ fulfills the Maurer–Cartan equation. For fixed $\mathfrak{g}$-action ζ the form κ is $\mathfrak{g}$-equivariant, since we have $\mathcal{L}_{\zeta_X}\kappa = i_{\zeta_X}d\kappa + di_{\zeta_X}\kappa = -i_{\zeta_X}(\frac{1}{2}[\kappa,\kappa]^\wedge) + dX = -[i_{\zeta_X}\kappa,\kappa] + 0 = -\operatorname{ad}(X)\kappa$. Thus κ is a principal connection form for this $\mathfrak{g}$-action, and it is the unique one by Proposition 5.7 below. □

5.2. Cartan's developing. It is well known that a free homogeneous G-manifold may be identified with the Lie group G by fixing a point. For $\mathfrak{g}$-manifolds the situation is more complicate. The following result may also be found in [7].

Proposition. *Let M be a free transitive $\mathfrak{g}$-manifold which is simply connected. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then there exist $\mathfrak{g}$-equivariant local diffeomorphisms $M \to G$. Namely for each $x \in M$ there is a unique $\mathfrak{g}$-equivariant smooth mapping $C_x : M \to W_x$ with $C_x(x) = e$ which is locally a diffeomorphism, where $W_x \subset G$ is defined in* 2.3.

If the $\mathfrak{g}$-action on M integrates to a G-action on M, then this mapping is automatically a global diffeomorphism.

The embedding $M \to G$ is called *Cartan's developing.* Its origins lie in Cartan's developing of a locally Euclidean space into the standard Euclidean space. If Cartan's developing is injective then the $\mathfrak{g}$-manifold M admits an extension to a $\mathfrak{g}$-manifold which is isomorphic to the Lie group G with the left action of $\mathfrak{g}$.

Proof. We consider again $G \times M$ with the graph foliation as in 2.3. Then $\mathrm{pr}_2 : G \times M \to M$ is a principal G-bundle with left multiplication as principal action, and since M is a free $\mathfrak{g}$-manifold the graph foliation is transversal to the fibers of pr_2 and is the horizontal foliation of a principal connection on the G-bundle. For $x \in M$ the restriction of pr_2 to the leaf $L(x)$ through (e,x) is a $\mathfrak{g}$-equivariant covering mapping which is a diffeomorphism since M is simply connected. Then

$$\varphi : M \xrightarrow{\mathrm{pr}_2^{-1}} L(x) \xrightarrow{\mathrm{pr}_1} W_x \subset G$$

is the looked for $\mathfrak{g}$-equivariant mapping which locally is a diffeomorphism since pr_1 also is one.

It remains to show that φ is a diffeomorphism if the $\mathfrak{g}$-action is complete. We consider the $\mathfrak{g}$-action on $G \times M$ on the factor M alone in this case. Then the graph foliation gives us a flat principal connection for this action, see Section 3, and by Proposition 6.6 below this connection is complete. Thus $\mathrm{pr}_1 : L(x) \to G$ is also a covering map, and since G is simply connected it is a diffeomorphism also and we are done. □

5.3. Example. The result of Proposition 5.2 is the best possible in general, as the following example shows. Let G be a simply connected Lie group, let W be a not simply connected open subset of G, and let M be a simply connected subset of the universal cover of W such that the projection $\varphi : M \to W$ is still surjective. We have an action of the Lie algebra $\mathfrak{g}$ of G on M by pulling back all left invariant vector fields on G to M via φ. Then φ is as constructed in 5.2, but it is only locally a diffeomorphism.

For example, let W be an annulus in $\mathbb{R}^2$, and let M be a piece of finite length of the spiral covering the annulus. Other examples can be found in [10, 11].

5.4. As an immediate application of the Cartan developing, we have the following proposition:

Proposition. *Let H be a connected linear Lie group of finite type, let G be the simply connected full prolongation of H such that G/H is the standard maximally homogeneous H-structure (see [1]).*

Then for any simply connected manifold N with a locally flat H-structure $p : M \to N$ there exists a map $\psi : N \to G/H$, which is a local isomorphism of H-structures.

In the case of a flat conformal structure we obtain the well known developing of a locally flat conformal manifold into the conformal sphere.

Proof. The mapping ψ is the unique one making the following diagram commutative:

$$\begin{array}{ccc} M^\infty & \xrightarrow{\varphi} & G \\ {\scriptstyle p_\infty}\downarrow & & \downarrow{\scriptstyle p_0} \\ N & \xrightarrow{\psi} & G/H. \end{array}$$

Here M^∞ is the full prolongation of the H-structure p with the natural free transitive action of the Lie algebra $\mathfrak{g}$ of G, see [1], and φ is the Cartan developing of the $\mathfrak{g}$-manifold M^∞ into G. □

5.5. The dual $\mathfrak{g}$-action for simply connected homogeneous free $\mathfrak{g}$-manifolds.
As motivation we recall that on a Lie group G (viewed as a homogeneous free right G-manifold) the fundamental vector fields correspond to the left invariant ones; they generate right translations, and correspond to the left Maurer–Cartan form κ on G. The diffeomorphisms which commute with all right translations are exactly the

left translations; the vector fields commuting with all left invariant ones are exactly the right invariant ones; they generate left translations, and correspond to the right Maurer–Cartan form $\hat{\kappa} = \mathrm{Ad} \cdot \kappa$.

Now let M be a free homogeneous $\mathfrak{g}$-manifold with action $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ and the corresponding principal connection form $\kappa = \zeta^{-1}$, see 5.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Choose a point $x_0 \in M$. Assume that M is simply connected and consider the Cartan developing $C_{x_0} : M \to G$. Then for $X \in \mathfrak{g}$ the fundamental vector field $\zeta_X \in \mathfrak{X}(M)$ is C_{x_0}-related to the left invariant vector field L_X on G. Let now $\hat{\zeta}_X \in \mathfrak{X}(M)$ denote the unique vector field on M which is C_{x_0}-related to the right invariant vector field $R_X \in \mathfrak{X}(G)$. Since $[L_X, R_Y] = 0$ we get $[\zeta_X, \hat{\zeta}_Y] = 0$, and even each local vector field $\xi \in \mathfrak{X}(U)$ for connected open $U \subset M$ with $[\xi, \zeta_Y] = 0$ for all $Y \in \mathfrak{g}$ extends to one of the form $\hat{\zeta}_X$. So we get a Lie algebra antihomomorphism

$$\hat{\zeta} : \mathfrak{g} \to \mathfrak{X}(M)$$

whose image is the centralizer algebra

$$Z_{\mathfrak{X}(M)}(\mathfrak{g}) := \{\eta \in \mathfrak{X}(M) : [\eta, \zeta_X] = 0 \text{ for all } X \in \mathfrak{g}\}.$$

This ('right') action $\hat{\zeta}$ of $\mathfrak{g}$ on M which commutes with the original action ζ is called the *dual action*. We have also the dual principal connection form $\hat{\kappa}$, inverse to $\hat{\zeta}$, see 5.1.

Note that for a free homogeneous $\mathfrak{g}$-manifold M which is not simply connected, the dual action of $\mathfrak{g}$ does not exist in general and the centralizer algebra $Z_{\mathfrak{X}(M)}(\mathfrak{g})$ is smaller than $\mathfrak{g}$: As an example we consider a Lie group G with the right action on $H\backslash G$ for a discrete subgroup H which is not central. Then the associated action of the Lie algebra $\mathfrak{g}$ is free, but its centralizer $Z_{\mathfrak{X}(H\backslash G)}(\mathfrak{g})$ is isomorphic to the subalgebra $\mathfrak{g}^H := \{X \in \mathfrak{g} : \mathrm{Ad}(h)X = X \text{ for all } h \in H\}$.

5.6. Homogeneous G-manifolds. As a motivation for what follows we consider here homogeneous G-manifolds. So let G be a connected Lie group with Lie algebra $\mathfrak{g}$, multiplication $\mu : G \times G \to G$, and for $g \in G$ let $\mu_g, \mu^g : G \to G$ denote the left and right translation, $\mu(g,h) = g \cdot h = \mu_g(h) = \mu^h(g)$. Let $H \subset G$ be a closed subgroup with Lie algebra $\mathfrak{h}$.

We consider the right coset space $M = H\backslash G$, the canonical projection $p : G \to H\backslash G$, the initial point $o = p(e) \in H\backslash G$ and the canonical right action of G on the right coset space $H\backslash G$, denoted by $\bar{\mu}^g : H\backslash G \to H\backslash G$. Then for $X \in \mathfrak{g}$ the left invariant vector field $L_X \in \mathfrak{X}(G)$ is p-related to the fundamental vector field $\zeta_X \in \mathfrak{X}(H\backslash G)$ of $\bar{\mu}^g$.

Suppose now that we are given a principal connection form $\omega \in \Omega^1(H\backslash G; \mathfrak{g})^{\mathfrak{g}}$. Then $\mathcal{L}_{\zeta_X}\omega = -\mathrm{ad}(X)\omega$ implies in turn

$$\begin{aligned} &(\bar{\mu}^{\exp(tX)})^*\omega = (\mathrm{Fl}_t^{\zeta_X})^*\omega = e^{-\mathrm{ad}(tX)}\omega = \mathrm{Ad}(\exp(-tX))\omega, \\ &(\bar{\mu}^g)^*\omega = \mathrm{Ad}(g^{-1})\omega \quad \text{ for all } g \in G, \end{aligned} \tag{1}$$

$$\omega_{o.g} = \mathrm{Ad}(g^{-1}) \circ \omega_o \circ T_o(\bar{\mu}^{g^{-1}}) \tag{2}$$

We also get a reductive decomposition of the Lie algebra $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \omega_o(T_o(H\backslash G)) =: \mathfrak{h} \oplus \mathfrak{m}_o, \tag{3}$$

where $\mathfrak{m}_o$ is a linear complement to $\mathfrak{h}$ which is invariant under $\mathrm{Ad}(H)$.

Conversely any $\mathrm{Ad}(H)$-invariant linear complement $\mathfrak{m}_o$ of $\mathfrak{h}$ in $\mathfrak{g}$ defines a principal connection form on the $\mathfrak{g}$-manifold $H\backslash G$ as follows: we consider the H-equivariant linear mapping

$$\omega_o : T_o(H\backslash G) \xleftarrow[\cong]{\zeta_o} \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}_o \subset \mathfrak{g}$$

and extend it to a principal connection form ω by (2).

There is a bijective correspondence between principal connection forms $\omega \in \Omega^1(H\backslash G, \mathfrak{g})^{\mathfrak{g}}$ and principal connections $\tilde{\omega} \in \Omega^1(G;\mathfrak{h})^H$ on the principal fiber bundle $p : G \to H\backslash G$ with *left* principal action of H, which is given by

$$\tilde{\omega}_g := T_g(\mu^{g^{-1}}) - \mathrm{Ad}(g) \circ \omega_{p(g)} \circ T_g p : T_g G \to \mathfrak{h}, \tag{4}$$

$$\tilde{\omega} := \kappa - \mathrm{Ad} \cdot p^*\omega, \tag{5}$$

where κ denotes the right Maurer–Cartan form. It is easily checked that $\tilde{\omega}$ is a principal connection for $p : G \to H\backslash G$: since the principal H-action is the left action on G we have $(\mu^h)^*\tilde{\omega} = \mathrm{Ad}(h)\omega$, and ω reproduces the generators in $\mathfrak{h}$ of right invariant vector fields on G. The principal curvature of $\tilde{\omega}$ is given by $d\tilde{\omega} - \frac{1}{2}[\tilde{\omega},\tilde{\omega}]^\wedge_{\mathfrak{h}}$: see [9, Proof of 11.2(3)]; compare with [8, I, Chap. X]. The curvature form of $\tilde{\omega}$ is related to the curvature form $\Omega = d\omega + \frac{1}{2}[\omega,\omega]^\wedge_{\mathfrak{g}}$ of ω by

$$\begin{aligned} d\tilde{\omega} - \tfrac{1}{2}[\tilde{\omega},\tilde{\omega}]^\wedge_{\mathfrak{h}} &= d\kappa - \tfrac{1}{2}[\kappa,\kappa]^\wedge_{\mathfrak{g}} - d\,\mathrm{Ad} \wedge p^*\omega - \mathrm{Ad} \cdot p^* d\omega \\ &\quad + [\kappa, \mathrm{Ad} \cdot p^*\omega]^\wedge_{\mathfrak{g}} - \tfrac{1}{2}[\mathrm{Ad} \cdot p^*\omega, \mathrm{Ad} \cdot p^*\omega]^\wedge_{\mathfrak{g}} \\ &= -\mathrm{Ad} \cdot p^*(d\omega + \tfrac{1}{2}[\omega,\omega]^\wedge_{\mathfrak{g}}) = -\mathrm{Ad} \cdot p^*\Omega, \end{aligned}$$

since for the right Maurer–Cartan form κ the Maurer–Cartan equation is given by $d\kappa - \frac{1}{2}[\kappa,\kappa]^\wedge_{\mathfrak{g}} = 0$, and since for $X \in \mathfrak{g}$ we have:

$$\begin{aligned} d\,\mathrm{Ad}(T(\mu^g)X) &= \frac{\partial}{\partial t}\Big|_0 \mathrm{Ad}(\exp(tX) \cdot g) = \mathrm{ad}(X)\,\mathrm{Ad}(g) \\ &= \mathrm{ad}(\kappa(T(\mu^g)X))\,\mathrm{Ad}(g), \\ d\,\mathrm{Ad} &= (\mathrm{ad} \circ \kappa)\,\mathrm{Ad}. \end{aligned}$$

5.7. Proposition. *Let M be a homogeneous $\mathfrak{g}$-space. Then there exists a unique principal connection $\Phi = \mathrm{Id}$ on M.*

On the other hand let M be an effective homogeneous $\mathfrak{g}$-space. Then principal connection forms ω correspond to reductive decompositions $\mathfrak{g} = \mathfrak{g}_x + \mathfrak{m}_x$, where $\mathfrak{g}_x$ is the isotropy subalgebra of a point $x \in M$, and where $\mathfrak{m}_x$ is an $\Gamma(\mathfrak{g})_x$-invariant complementary subspace.

Proof. The first statement is obvious.

We first check that for an effective homogeneous $\mathfrak{g}$-manifold M the homomorphism $\mathrm{germ}_x \circ \zeta^M : \mathfrak{g} \to \mathfrak{X}(M)_{\text{germs at } x}$ is injective for each $x \in M$. Let $\mathfrak{k}_x$ denote its kernel. Since $\mathfrak{g}$ is finite dimensional, we have $\mathfrak{k}_y = \mathfrak{k}_x$ for y near x, and since M is connected, this holds even for all $y \in M$. So $\mathfrak{k}_x$ is in the kernel of $\zeta^M : \mathfrak{g} \to \mathfrak{X}(M)$ which is zero since the $\mathfrak{g}$-action on M is supposed to be effective.

As in 2.3 we consider the graph foliation of the $\mathfrak{g}$-manifold M on $G \times M$, where G is a connected Lie group with Lie algebra $\mathfrak{g}$, the leaf $L(x)$ through (e, x) of it, and the open set $W_x = \mathrm{pr}_1(L(x)) \subset G$.

Consider a principal connection form $\omega \in \Omega^1(M; \mathfrak{g})^{\mathfrak{g}}$. Then $\mathcal{L}_{\zeta_X}\omega = -\operatorname{ad}(X)\omega$ implies $(\mathrm{Fl}_t^{\zeta_X})^*\omega = e^{-\operatorname{ad}(tX)}\omega = \operatorname{Ad}(\exp(-tX))\omega$, this holds then for all elements of the pseudogroup $\Gamma(\mathfrak{g})$ of the form 2.2(5) and finally we get for each smooth curve $c : [0,1] \to W_x$ with $c(0) = e$ which is liftable to $L(x)$:

$$(\gamma_x(c))^*\omega = \operatorname{Ad}(c(1)^{-1})\omega. \tag{1}$$

Thus we get a reductive decomposition of the Lie algebra $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{g}_x \oplus \omega_x(T_xM) =: \mathfrak{g}_x \oplus \mathfrak{m}_x, \tag{2}$$

where $\mathfrak{m}_x$ is a linear complement to $\mathfrak{g}_x$ which is invariant under $\operatorname{Ad}(\Gamma(\mathfrak{g})_x)$, see also 2.6.

If conversely we are given a reductive decomposition as in (2) which is invariant under $\operatorname{Ad}(\Gamma(\mathfrak{g})_x)$, then we consider the $\Gamma(\mathfrak{g})_x$-equivariant linear mapping

$$\omega_x : T_xM \underset{\cong}{\overset{\zeta_x}{\longleftarrow}} \mathfrak{g}/\mathfrak{g}_x \cong \mathfrak{m}_x \subset \mathfrak{g}$$

and we use (1) to define $\omega \in \Omega^1(M; \mathfrak{g})^{\mathfrak{g}}$ by

$$\omega_{\gamma_x(c)(x)} = \operatorname{Ad}(c(1)^{-1}) \circ \omega \circ T_x(\gamma_x(c)^{-1}), \tag{3}$$

for each smooth curve $c : [0,1] \to W_x$ with $c(0) = e$ which is liftable to $L(x)$. Since any element of the pseudogroup $\Gamma(\mathfrak{g})$ is of this form (see 2.3) we get a well defined principal connection form on M. $\square$

5.8. Theorem. *Let M be a homogeneous effective $\mathfrak{g}$-space. Let $x_0 \in M$ and let $P_{x_0}(\mathfrak{g})$ be the set of all germs at x_0 of transformations in $\Gamma(\mathfrak{g})$.*

Then $\mathrm{ev}_{x_0} : P_{x_0}(\mathfrak{g}) \to M$ is the projection of a smooth principal fiber bundle with structure group the isotropy group $\Gamma(\mathfrak{g})_{x_0}$ (see 2.6) and with principal right action just composition from the right, and its smooth structure is the unique one for which the smooth curves $[0,1] \to P_{x_0}(\mathfrak{g})$ correspond exactly to the germs along $[0,1] \times \{x_0\}$ of smooth mappings $\varphi : [0,1] \times U \to M$ with $\varphi_t \in \Gamma(\mathfrak{g})$ for all t. The total space $P_{x_0}(\mathfrak{g})$ is connected.

We have a canonical free transitive $\mathfrak{g}$-action $\zeta^P : \mathfrak{g} \to \mathfrak{X}(P_{x_0}(\mathfrak{g}))$ which is given by

$$\zeta^P_X(\varphi) := \zeta^M_X \circ \varphi = T\varphi \circ \zeta^M_{\operatorname{Ad}(\varphi^{-1})X},$$

and whose corresponding pseudogroup is generated by the local flows $\varphi \mapsto \mathrm{Fl}_t^{\zeta^M_X} \circ \varphi$. Each vector field ζ^P_X is invariant under the pullback by the principal right action of $\Gamma(\mathfrak{g})_{x_0}$. The projection $\mathrm{ev}_{x_0} : P_{x_0}(\mathfrak{g}) \to M$ is $\mathfrak{g}$-equivariant, so the vector fields ζ^P_X and ζ^M_X are ev_{x_0}-related. Its associated Maurer–Cartan form is called κ.

There exists also the dual free $\mathfrak{g}$-action (a Lie algebra antihomomorphism) $\hat{\zeta}^P : \mathfrak{g} \to \mathfrak{X}(P_{x_0}(\mathfrak{g}))$, given by

$$\hat{\zeta}^P_X := T\varphi \circ \zeta^M_X = \zeta^M_{\mathrm{Ad}(\varphi)X} \circ \varphi = \zeta^P_{\mathrm{Ad}(\varphi)X}(\varphi).$$

Its associated Maurer–Cartan form is called $\hat{\kappa}$, and its corresponding pseudogroup is generated by

$$\varphi \mapsto \varphi \circ \mathrm{Fl}_t^{\zeta^M_X}.$$

The pullback of $\hat{\zeta}^P_X$ by the principal right action of $\psi \in \Gamma(\mathfrak{g})_{x_0}$ is given by

$$(\psi^*)^* \hat{\zeta}^P_X = \hat{\zeta}^P_{\mathrm{Ad}(\psi^{-1})X}.$$

The principal connections forms $\omega \in \Omega^1(M;\mathfrak{g})^{\mathfrak{g}}$ correspond bijectively to principal connections forms $\tilde{\omega} \in \Omega^1(P_{x_0}(\mathfrak{g}), \mathfrak{g}_{x_0})^{\Gamma(\mathfrak{g})_{x_0}}$ on the principal $\Gamma(\mathfrak{g})_{x_0}$-bundle $P_{x_0}(\mathfrak{g}) \to M$ via

$$\tilde{\omega}_\varphi = -\hat{\kappa}_\varphi - \mathrm{Ad}(\varphi^{-1}) \circ \omega_{\varphi(x_0)} \circ T_\varphi(\mathrm{ev}_{x_0}) : T_\varphi(P_{x_0}(\mathfrak{g})) \to \mathfrak{g}_{x_0}.$$

The principal curvature forms are then related by

$$\begin{aligned}\tilde{\Omega} &:= d\tilde{\omega} + \tfrac{1}{2}[\tilde{\omega},\tilde{\omega}]^\wedge_{\mathfrak{g}_{x_0}} = -(\mathrm{Ad}\circ\mathrm{inv})\cdot \mathrm{ev}^*_{x_0}(d\omega + \tfrac{1}{2}[\omega,\omega]^\wedge_{\mathfrak{g}}) \\ &= -(\mathrm{Ad}\circ\mathrm{inv})\cdot \mathrm{ev}^*_{x_0}\,\Omega.\end{aligned}$$

The manifold $P_{x_0}(M)$ is not simply connected in general (e.g., a Lie group); nevertheless the dual action is defined.

Proof. Recall first from the Proof of 5.7 that for an effective homogeneous $\mathfrak{g}$-manifold M the homomorphism $\mathrm{germ}_x \circ \zeta^M : \mathfrak{g} \to \mathfrak{X}(M)_{\text{germs at } x}$ is injective for each $x \in M$.

As in 2.3 we consider the graph foliation of the $\mathfrak{g}$-manifold M on $G \times M$, where G is a connected Lie group with Lie algebia $\mathfrak{g}$, the leaf $L(x_0)$ through (e, x_0) of it, and the open set $W_{x_0} = \mathrm{pr}_1(L(x_0)) \subset G$.

Now we choose a splitting $\mathfrak{g} = \mathfrak{g}_{x_0} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is a linear complement to the isotropy algebra $\mathfrak{g}_{x_0}$. Let us now consider a small open ball B with center 0 in $\mathfrak{g}$, its diffeomorphic image $\exp(B) =: W \subset W_{x_0} \subset G$, and $\tilde{W} \subset L(x_0)$, an open neighbourhood of (e, x_0) in $L(x_0)$ such that $\mathrm{pr}_1|_{\tilde{W}} : \tilde{W} \to W$ is a diffeomorphism. Let $B^{\mathfrak{m}} := B \cap \mathfrak{m}$, $W^{\mathfrak{m}} := \exp(B^{\mathfrak{m}})$, and $\tilde{W}^{\mathfrak{m}} := (\mathrm{pr}_1|_{\tilde{W}})^{-1}(W^{\mathfrak{m}})$, and choose now B so small that $\mathrm{pr}_2 : \tilde{W}^{\mathfrak{m}} \to M$ is a diffeomorphism onto an open neighbourhood U of x_0. We consider the composed diffeomorphism

$$\lambda : U \xrightarrow{\mathrm{pr}_2^{-1}} \tilde{W}^{\mathfrak{m}} \xrightarrow{\mathrm{pr}_1} W^{\mathfrak{m}} \xrightarrow{\exp^{-1}} B^{\mathfrak{m}}.$$

Now for $f \in P_{x_0}(\mathfrak{g})$ with $\mathrm{ev}_{x_0}(f) = f(x_0) \in U$ we define

$$\varphi(f) = \gamma_{x_0}(c_{f(x_0)})^{-1} \circ f \in \Gamma(\mathfrak{g})_{x_0},$$

where $c_{f(x_0)} : [0,1] \to W^{\mathfrak{m}}$ is the curve $c_{f(x_0)}(t) = \exp(t\lambda(f(x_0)))$.

Next we choose an open cover (U_α) of M with transformations $f_\alpha : V_\alpha \to U_\alpha$ in the pseudogroup $\Gamma(\mathfrak{g})$, where V_α is a connected open neighbourhood of x_0 in U, and we define

$$\begin{aligned}&\varphi_\alpha : \mathrm{ev}_{x_0}^{-1}(U_\alpha) = P_{x_0}(\mathfrak{g})|U_\alpha \to U_\alpha \times \Gamma(\mathfrak{g}, x_0),\\ &\varphi_\alpha(f) := (f(x_0), \varphi(f_\alpha^{-1} \circ f)) = (f(x_0), \gamma_{x_0}(c_{f_\alpha^{-1}f(x_0)})^{-1} \circ f_\alpha^{-1} \circ f).\end{aligned}$$

These give a smooth principal fiber bundle atlas for $P_{x_0}(\mathfrak{g})$ since for $(x,h) \in (U_\alpha \cap U_\beta) \times \Gamma(\mathfrak{g})_{x_0}$ we have

$$\varphi_\alpha\varphi_\beta^{-1}(x,h) = \big(x, \gamma_{x_0}(c_{f_\alpha^{-1}(x)})^{-1} \circ f_\alpha^{-1} \circ f_\beta \circ \gamma_{x_0}(c_{f_\beta^{-1}(x)}) \circ h\big).$$

The smooth structure on $P_{x_0}(\mathfrak{g})$ induced by this atlas is the unique one where the smooth curves are exactly as described in the theorem, since this is visibly the case in each chart. Thus by the Lemma in 2.3 the total space $P_{x_0}(\mathfrak{g})$ is connected.

For $X \in \mathfrak{g}$ and $\varphi \in P_{x_0}(\mathfrak{g})$ we have

$$\frac{\partial}{\partial t} \mathrm{Fl}_t^{\zeta_X^M} \circ \varphi = \zeta_X^M \circ \mathrm{Fl}_t^{\zeta_X^M} \circ \varphi,$$

so a smooth vector field on $P_{x_0}(\mathfrak{g})$ is defined by

$$\zeta_X^P(\varphi) := \zeta_X^M \circ \varphi = T\varphi \circ T\varphi^{-1} \circ \zeta_X^M \circ \varphi = T\varphi \circ (\varphi^*\zeta_X^M) = T\varphi \circ \zeta^M_{\mathrm{Ad}(\varphi^{-1})X},$$

where we used 2.4, and its local flow is given by $\mathrm{Fl}_t^{\zeta_X^P}(\varphi) = \mathrm{Fl}_t^{\zeta_X^M} \circ \varphi$. Clearly $\zeta^P : \mathfrak{g} \to \mathfrak{X}(P_{x_0}(\mathfrak{g}))$ is a $\mathfrak{g}$-action, which is free, since for each $x \in M$ the homomorphism $\mathrm{germ}_x \circ \zeta^M : \mathfrak{g} \to \mathfrak{X}(M)_{\text{germs at } x}$ is injective. Consider now $\psi \in \Gamma(\mathfrak{g})_{x_0}$ and its principal right action ψ^* on $P_{x_0}(\mathfrak{g})$; it acts trivially by pullback on each vector field ζ_X^P since we have:

$$\begin{aligned}((\psi^*)^*\zeta_X^P)(\varphi) &= (T(\psi^*)^{-1} \circ \zeta_X^P \circ \psi^*)(\varphi) = (\psi^{-1})^*(\zeta_X^P(\varphi \circ \psi))\\ &= (\psi^{-1})^*(\zeta_X^M \circ \varphi \circ \psi) = \zeta_X^M \circ \varphi = \zeta_X^P(\varphi).\end{aligned}$$

The bundle projection $\mathrm{ev}_{x_0} : P_{x_0}(\mathfrak{g}) \to M$ is visibly $\mathfrak{g}$-equivariant. Now we describe the associated unique principal connection form (Maurer–Cartan form) $\kappa \in \Omega^1(P_{x_0}(\mathfrak{g}); \mathfrak{g})^{\Gamma(\mathfrak{g})_{x_0}}$: Consider a smooth curve f_t in $P_{x_0}(\mathfrak{g})$. Then $\partial/\partial t|_0 f_t$ is a tangent vector with foot point f_0 and we have

$$\kappa\Big(\frac{\partial}{\partial t}\Big|_0 f_t\Big) = (\mathrm{germ}_{x_0} \circ \zeta^M)^{-1}\Big(\Big(\frac{\partial}{\partial t}\Big|_0 f_t\Big) \circ f_0^{-1}\Big) \in \mathfrak{g}.$$

The dual action $\hat\zeta^P : \mathfrak{g} \to \mathfrak{X}(P_{x_0}(\mathfrak{g}))$ is given by

$$\begin{aligned}\hat\zeta_X^P(\varphi) &:= T\varphi \circ \zeta_X^M = T\varphi \circ \zeta_X^M \circ \varphi^{-1} \circ \varphi = ((\varphi^{-1})^*\zeta_X^M) \circ \varphi\\ &= \zeta^M_{\mathrm{Ad}(\varphi)X} \circ \varphi = \zeta^P_{\mathrm{Ad}(\varphi)X}(\varphi),\end{aligned}$$

by 2.4 again, and its local flow is given by $\mathrm{Fl}_t^{\hat\zeta_X^P}(\varphi) = \mathrm{germ}_{x_0}(\bar\varphi \circ \mathrm{Fl}_t^{\zeta_X^M})$, where $\bar\varphi$ is a representative of the germ φ. Then $\hat\zeta^P : \mathfrak{g} \to \mathfrak{X}(P_{x_0}(\mathfrak{g}))$. It is a Lie algebra antihomomorphism, since we have (using [9, 3.16])

$$\begin{aligned}
[\hat\zeta^P_X,\hat\zeta^P_Y](\varphi) &= \frac{1}{2}\frac{\partial^2}{\partial t^2}\Big|_0 \mathrm{Fl}^{\hat\zeta^P_Y}_{-t}\,\mathrm{Fl}^{\hat\zeta^P_X}_{-t}\,\mathrm{Fl}^{\hat\zeta^P_Y}_{t}\,\mathrm{Fl}^{\hat\zeta^P_X}_{t}(\varphi)\\
&= \frac{1}{2}\frac{\partial^2}{\partial t^2}\Big|_0 \varphi\circ \mathrm{Fl}^{\zeta^M_X}_{t}\circ \mathrm{Fl}^{\zeta^M_Y}_{t}\circ \mathrm{Fl}^{\zeta^M_X}_{-t}\circ \mathrm{Fl}^{\zeta^M_Y}_{-t}\\
&= T\varphi\circ[\zeta^M_Y,\zeta^M_X] = -T\varphi\circ\zeta^M_{[X,Y]} = -\hat\zeta^P_{[X,Y]}(\varphi).
\end{aligned}$$

Consider now $\psi\in\Gamma(\mathfrak g)_{x_0}$ and its principal right action ψ^* on $P_{x_0}(\mathfrak g)$; it acts by pullback on each vector field $\hat\zeta^P_X$ as follows:

$$\begin{aligned}
((\psi^*)^*\hat\zeta^P_X)(\varphi) &= (T(\psi^*)^{-1}\circ\hat\zeta^P_X\circ\psi^*)(\varphi) = (\psi^{-1})^*(\hat\zeta^P_X(\varphi\circ\psi))\\
&= (\psi^{-1})^*(T\varphi\circ T\psi\circ\zeta^M_X) = T\varphi\circ T\psi\circ\zeta^M_X\circ\psi^{-1}\\
&= T\varphi\circ((\psi^{-1})^*\zeta^M_X) = T\varphi\circ\zeta^M_{\mathrm{Ad}(\psi)X}\\
&= \hat\zeta^P_{\mathrm{Ad}(\psi)X}(\varphi).
\end{aligned}$$

Note that the vector fields $\hat\zeta^P_{-X}\in\mathfrak X(P_{x_0}(\mathfrak g))$ for $X\in\mathfrak g_{x_0}$ are the fundamental vector fields of the principal right action, and recall from 2.6 that the Lie algebra of the structure group $\Gamma(\mathfrak g)_{x_0}$ is antiisomorphic to the isotropy Lie algebra $\mathfrak g_{x_0}$. The associated unique principal connection form (dual Maurer–Cartan form) $\hat\kappa\in\Omega^1(P_{x_0}(\mathfrak g);\mathfrak g)$ is given by

$$\begin{aligned}
&\hat\kappa_\varphi = (\hat\zeta^P_\varphi)^{-1} : T_\varphi(P_{x_0}(\mathfrak g))\to\mathfrak g,\\
&\hat\kappa\Big(\frac{\partial}{\partial t}\Big|_0 f_t\Big) = (\mathrm{germ}_{x_0}\circ\zeta^M)^{-1}\Big(Tf_0^{-1}\circ\frac{\partial}{\partial t}\Big|_0 f_t\Big)\in\mathfrak g
\end{aligned}$$

for each smooth curve f_t in $P_{x_0}(\mathfrak g)$. Since $\hat\zeta^P:\mathfrak g\to\mathfrak X(P_{x_0}(\mathfrak g))$ is a Lie algebra antihomomorphism, $\hat\kappa$ satisfies the Maurer–Cartan equation in the form $d\hat\kappa-\frac12[\hat\kappa,\hat\kappa]^\wedge=0$ and is $\Gamma(\mathfrak g)_{x_0}$-equivariant in the form $(\psi^*)^*\hat\kappa=\mathrm{Ad}(\psi^{-1})\hat\kappa$.

Finally let $\omega\in\Omega^1(M;\mathfrak g)^{\mathfrak g}$ be a principal connection form on the $\mathfrak g$-manifold M. Then from 5.7(1) and from 2.4 for any $\varphi\in\Gamma(\mathfrak g)$ we have

$$\varphi^*\omega = \mathrm{Ad}(\varphi^{-1})\omega.$$

We consider the 1-form

$$\begin{aligned}
&\tilde\omega_\varphi := -\hat\kappa_\varphi - \mathrm{Ad}(\varphi^{-1})\circ\omega_{\varphi(x_0)}\circ T_\varphi(\mathrm{ev}_{x_0}) : T_\varphi(P_{x_0}(\mathfrak g))\to\mathfrak g_{x_0},\\
&\tilde\omega = -\hat\kappa - (\mathrm{Ad}\circ\mathrm{inv})\cdot\mathrm{ev}^*_{x_0}\,\omega.
\end{aligned}$$

Then $\tilde\omega$ is $\mathfrak g_{x_0}$-valued by property 5.7(1), and it is a principal connection form $\tilde\omega\in\Omega^1(P_{x_0}(\mathfrak g);\mathfrak g_{x_0})^{\Gamma(\mathfrak g)_{x_0}}$ on the principal $\Gamma(\mathfrak g)_{x_0}$-bundle $\mathrm{ev}_{x_0}:P_{x_0}(\mathfrak g)\to M$, with right principal action now, $\psi^*(\varphi)=\varphi\circ\psi$ for $\psi\in\Gamma(\mathfrak g)_{x_0}$ and $\varphi\in P_{x_0}(\mathfrak g)$, because we have in turn for $X \in \mathfrak{g}_{x_0}$:

$$\tilde\omega_\varphi(\zeta^P_{-X}(\varphi)) = -\hat\kappa_\varphi(\zeta^P_{-X}(\varphi)) - 0 = X \in \mathfrak{g}_{x_0},$$
$$\begin{aligned}(\psi^*\tilde\omega)_\varphi &= \tilde\omega_{\varphi\circ\psi}\circ T(\psi^*)\\
&= -(\psi^*\hat\kappa)_\varphi - \mathrm{Ad}(\psi^{-1}\circ\varphi^{-1})\circ\omega_{\varphi(x_0)}\circ T(\mathrm{ev}_{x_0})\circ T(\psi^*)\\
&= \mathrm{Ad}(\psi^{-1})(-\hat\kappa_\varphi - \mathrm{Ad}(\varphi^{-1})\circ\omega_{\varphi(x_0)}\circ T(\mathrm{ev}_{x_0}))\\
&= \mathrm{Ad}(\psi^{-1})\circ\tilde\omega_\varphi.\end{aligned}$$

On the other hand, for any principal connection form $\tilde\omega \in \Omega^1(P_{x_0}(\mathfrak{g});\mathfrak{g}_{x_0})^{\Gamma(\mathfrak{g})_{x_0}}$ on the principal $\Gamma(\mathfrak{g})_{x_0}$-bundle $\mathrm{ev}_{x_0}: P_{x_0}(\mathfrak{g}) \to M$ the $\mathfrak{g}$-valued 1-form $\bar\omega_\varphi := \mathrm{Ad}(\varphi)(-\hat\kappa_\varphi - \tilde\omega_\varphi)$ is horizontal and $\Gamma(\mathfrak{g})_{x_0}$-invariant, thus it is the pullback of a unique form $\omega \in \Omega^1(M;\mathfrak{g})$ which is easily seen to be a principal connection form on M. For the curvature we may compute as follows (compare 5.6)

$$\begin{aligned}\tilde\Omega := d\tilde\omega + \tfrac12[\tilde\omega,\tilde\omega]^\wedge_{\mathfrak{g}_{x_0}} &= d(-\hat\kappa - (\mathrm{Ad}\circ\mathrm{inv})\cdot\mathrm{ev}^*_{x_0}\omega)\\
&\quad + \tfrac12[-\hat\kappa - (\mathrm{Ad}\circ\mathrm{inv})\cdot\mathrm{ev}^*_{x_0}\omega, -\hat\kappa - (\mathrm{Ad}\circ\mathrm{inv})\cdot\mathrm{ev}^*_{x_0}\omega]^\wedge_{\mathfrak{g}_{x_0}}\\
&= -d\hat\kappa + \tfrac12[\hat\kappa,\hat\kappa]^\wedge_{\mathfrak{g}} - d(\mathrm{Ad}\circ\mathrm{inv})\wedge\mathrm{ev}^*_{x_0}\omega - (\mathrm{Ad}\circ\mathrm{inv})\cdot\mathrm{ev}^*_{x_0}d\omega\\
&\quad - [\hat\kappa,(\mathrm{Ad}\circ\mathrm{inv})\cdot\mathrm{ev}^*_{x_0}\omega]^\wedge_{\mathfrak{g}} - \tfrac12[(\mathrm{Ad}\circ\mathrm{inv})\cdot\mathrm{ev}^*_{x_0}\omega,(\mathrm{Ad}\circ\mathrm{inv})\cdot\mathrm{ev}^*_{x_0}\omega]^\wedge_{\mathfrak{g}}\\
&= -(\mathrm{Ad}\circ\mathrm{inv})\cdot\mathrm{ev}^*_{x_0}(d\omega + \tfrac12[\omega,\omega]^\wedge_{\mathfrak{g}}) = -(\mathrm{Ad}\circ\mathrm{inv})\cdot\mathrm{ev}^*_{x_0}\Omega,\end{aligned}$$

where we used the Maurer–Cartan equation for $\hat\kappa$ and

$$\begin{aligned}d(\mathrm{Ad}\circ\mathrm{inv})(\hat\zeta_X(\varphi)) &= \frac{\partial}{\partial t}\Big|_0(\mathrm{Ad}\circ\mathrm{inv})(\varphi\circ\mathrm{Fl}^{\zeta^M_X}_t) = \frac{\partial}{\partial t}\Big|_0\mathrm{Ad}(\mathrm{Fl}^{\zeta^M_X}_{-t}\circ\varphi^{-1})\\
&= -\mathrm{ad}(X)\circ\mathrm{Ad}(\varphi^{-1}) = -\mathrm{ad}(\hat\kappa(\hat\zeta^P_X(\varphi)))\circ\mathrm{Ad}(\varphi^{-1}),\\
d(\mathrm{Ad}\circ\mathrm{inv}) &= -\mathrm{ad}(\hat\kappa)(\mathrm{Ad}\circ\mathrm{inv}). \quad \square\end{aligned}$$

5.9. The Lie algebra $Z_{\mathfrak{X}}(M)(\mathfrak{g})$ of infinitesimal automorphisms of a homogeneous $\mathfrak{g}$-manifold. Let M be an effective homogeneous $\mathfrak{g}$-manifold. We will describe now the centralizer

$$Z_{\mathfrak{X}(M)}(\mathfrak{g}) := \{\eta \in \mathfrak{X}(M) : [\eta,\zeta_X] = 0 \text{ for all } X \in \mathfrak{g}\}$$

of $\zeta(\mathfrak{g})$ in the Lie algebra $\mathfrak{X}(M)$ of all vector fields on M.

Let $x_0 \in M$ be a fixed point with isotropy subalgebra $\mathfrak{g}_{x_0} = \ker(\zeta_{x_0}: \mathfrak{g} \to T_{x_0}M)$ and isotropy group $\Gamma(\mathfrak{g})_{x_0}$. We consider the *normalizer* $N_{\mathfrak{g}}(\Gamma(\mathfrak{g})_{x_0})$ of the isotropy group $\Gamma(\mathfrak{g})_{x_0}$ in $\mathfrak{g}$, and the 'Weyl algebra' $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}(x_0)$, which are given by

$$\begin{aligned}N_{\mathfrak{g}}(\Gamma(\mathfrak{g})_{x_0}) &:= \{X \in \mathfrak{g} : \mathrm{Ad}(\psi)X - X \in \mathfrak{g}_{x_0} \text{ for all } \psi \in \Gamma(\mathfrak{g})_{x_0}\},\\
\hat{\mathfrak{g}}(x_0) &:= N_{\mathfrak{g}}(\Gamma(\mathfrak{g})_{x_0})/\mathfrak{g}_{x_0}.\end{aligned}$$

It is clear that $\mathfrak{g}_{x_0}$ is an ideal in $N_{\mathfrak{g}}(\Gamma(\mathfrak{g})_{x_0})$, thus $\hat{\mathfrak{g}}(x_0)$ is a Lie algebra. Clearly, if $\varphi \in \Gamma(\mathfrak{g})$, then $\mathrm{Ad}(\varphi): \mathfrak{g} \to \mathfrak{g}$ induces an isomorphism $\hat{\mathfrak{g}}(\varphi(x_0)) \to \hat{\mathfrak{g}}(x_0)$. We can define a Lie algebra antihomomorphism

$$\hat\zeta^M : \hat{\mathfrak{g}}(x_0) = N_{\mathfrak{g}}(\Gamma(\mathfrak{g})_{x_0})/\mathfrak{g}_{x_0} \to \mathfrak{X}(M),$$

as follows: Any point $x \in M$ is of the form $x = \varphi(x_0)$ for some element of the pseudogroup $\Gamma(\mathfrak{g})$, and for $X \in N_{\mathfrak{g}}(\Gamma(\mathfrak{g})_{x_0})$ we have a well defined vector field

$$\hat\zeta^M_X(x = \varphi(x_0)) := T\varphi \cdot \zeta^M_X(x_0) = ((\varphi^{-1})^*\zeta_X)(\varphi(x_0)) = \zeta_{\mathrm{Ad}(\varphi)X}(x).$$

For all $X \in \hat{\mathfrak{g}}(x_0)$ and all $Y \in \mathfrak{g}$ we have

$$\begin{aligned}
((\mathrm{Fl}^{\zeta_Y}_t)^*\hat\zeta_X)(x = \varphi(x_0)) &:= T(\mathrm{Fl}^{\zeta_Y}_{-t}) \cdot \hat\zeta_X(\mathrm{Fl}^{\zeta_Y}_t(\varphi(x_0)))\\
&= T(\mathrm{Fl}^{\zeta_Y}_{-t}) \cdot T(\mathrm{Fl}^{\zeta_Y}_t \circ \varphi) \cdot \zeta_X(x_0)\\
&= T(\varphi) \cdot \zeta_X(x_0) = \hat\zeta_X(x = \varphi(x_0)),\\
[\hat\zeta_X, \zeta_Y] &= 0,
\end{aligned}$$

so that $\hat\zeta(\hat{\mathfrak{g}}(x_0)) \subset \mathfrak{X}(M)$ is contained in the centralizer $Z_{\mathfrak{X}(M)}(\mathfrak{g})$. On the other hand we have:

Lemma. *In this situation, $Z_{\mathfrak{X}(M)}(\mathfrak{g}) = \hat\zeta(\hat{\mathfrak{g}})$, and these are exactly the vector fields on M which are projections from all projectable vector fields in $\zeta^P(\mathfrak{g}) \subset \mathfrak{X}(P_{x_0}(\mathfrak{g}))$ for the principal fiber bundle projection $\mathrm{ev}_{x_0} : P_{x_0}(\mathfrak{g}) \to M$. The flow of $\hat\zeta_X$ for $X \in \hat{\mathfrak{g}}(x_0)$ is given by*

$$\mathrm{Fl}^{\hat\zeta_X}_t(x = \varphi(x_0)) = \varphi(\mathrm{Fl}^{\zeta_X}_t(x_0)).$$

Proof. Let $\xi \in \mathfrak{X}(M)$ be a vector field that commutes with the action of $\mathfrak{g}$. Then for any $\varphi \in \Gamma(\mathfrak{g})$ we have $\varphi^*\xi = T\varphi^{-1} \circ \xi \circ \varphi = \xi$. Then for $\psi \in \Gamma(\mathfrak{g})_{x_0}$ we have $T_{x_0}\psi \cdot \xi(x_0) = \xi_{x_0}$. If we choose any $X \in \mathfrak{g}$ with $\zeta_X(x_0) = \xi(x_0)$ we get by 2.4 for all $\psi \in \Gamma(\mathfrak{g})_{x_0}$

$$\zeta_{\mathrm{Ad}(\psi^{-1})X}(x_0) = (\psi^*\zeta_X)(x_0) = T_{x_0}\psi.\xi(x_0) = \xi(x_0) = \zeta_X(x_0).$$

so that $X - \mathrm{Ad}(\psi^{-1})X \in \mathfrak{g}_{x_0}$ and $X \in N_{\mathfrak{g}}(\Gamma(\mathfrak{g})_{x_0})$. Moreover for $x \in M$ and $\varphi \in \Gamma(\mathfrak{g})$ with $x = \varphi(x_0)$ we have

$$\hat\zeta_X(x) = T_{x_0}\varphi.\zeta_X(x_0) = T_{x_0}\varphi \cdot \xi(x_0) = \xi(\varphi(x_0)) = \xi(x).$$

The statement about the projectable vector fields on $P_{x_0(\mathfrak{g})}$ is easily checked, and the formula for the flow of $\hat\zeta_X$ also follows by projecting it from $P_{x_0}(\mathfrak{g})$. □

6. Parallel transport

6.1. Local description of principal connections. Let M be a locally trivial $\mathfrak{g}$-manifold with projection $p : M \to N := M/\mathfrak{g}$ and with standard fiber S.

Let $(U_\alpha, \varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times S)$ be an atlas of bundle charts as specified in 2.1(7). Then we have $(\varphi_\beta \circ \varphi_\alpha^{-1})(x, s) = (x, \varphi_{\beta\alpha}(x, s))$ for $(x, s) \in (U_\alpha \cap U_\beta) \times S$, where $\varphi_{\alpha\beta}(x, \cdot)$ is a $\mathfrak{g}$-equivariant diffeomorphism of S for each $x \in M$. See also 5.9.

Let $\Phi \in \Omega^1(M; TM)^{\mathfrak{g}}$ be a principal connection. Then we have

$$((\varphi_\alpha)^{-1})^*\Phi)(\xi_x, \eta_y) =: -\Gamma^\alpha(\xi_x, y) + \eta_y \text{ for } \xi_x \in T_xU_\alpha \text{ and } \eta_y \in T_yS,$$

since it reproduces vertical vectors. The Γ^α are given by

$$(0_x, \Gamma^\alpha(\xi_x, y)) := -T(\varphi_\alpha) \cdot \Phi \cdot T(\varphi_\alpha)^{-1} \cdot (\xi_x, 0_y).$$

We may consider Γ^α as an element of the space $\Omega^1(U_\alpha; Z_{\mathfrak{X}(S)}(\mathfrak{g}))$, i.e., as a 1-form on U^α with values in the centralizer $Z_{\mathfrak{X}(S)}(\mathfrak{g})$ of $\zeta^S(\mathfrak{g})$ in the Lie algebra $\mathfrak{X}(S)$ of all vector fields on the standard fiber. This space is finite dimensional by Lemma 5.9. This follows from the naturality of the Frölicher–Nijenhuis bracket [9, 8.15] via the following computation, with some abuse of notation:

$$\begin{aligned} 0 = [\zeta_X^M, \Phi] &= \varphi_\alpha^*[0 \times \zeta_X^S, \mathrm{pr}_2 - \Gamma^\alpha] = \varphi_\alpha^*(0 \times [\zeta_X^S, \mathrm{Id}_{TS} - \Gamma^\alpha]) \\ &= -\varphi_\alpha^*(0 \times [\zeta_X^S, \Gamma^\alpha]), \end{aligned}$$

since Id_{TS} is in the center of the Frölicher–Nijenhuis algebra.

The Γ^α are called the *Christoffel forms* of the connection Φ with respect to the bundle atlas $(U_\alpha, \varphi_\alpha)$.

From [9, 9.7] we get that the transformation law for the Christoffel forms is

$$T_y(\varphi_{\alpha\beta}(x, \cdot)) \cdot \Gamma^\beta(\xi_x, y) = \Gamma^\alpha(\xi_x, \varphi_{\alpha\beta}(x, y)) - T_x(\varphi_{\alpha\beta}(\cdot, y)) \cdot \xi_x.$$

The curvature R of Φ satisfies

$$(\varphi_\alpha^{-1})^* R = d\Gamma^\alpha + \tfrac{1}{2}[\Gamma^\alpha, \Gamma^\alpha]^\wedge_{\mathfrak{X}(S)}.$$

Here $d\Gamma^\alpha$ is the exterior derivative of the 1-form $\Gamma^\alpha \in \Omega^1(U_\alpha; Z_{\mathfrak{X}(S)}(\mathfrak{g}))$ with values in the finite dimensional Lie algebra $Z_{\mathfrak{X}(S)}(\mathfrak{g})$.

6.2. Asystatic locally trivial $\mathfrak{g}$-manifolds. A locally trivial $\mathfrak{g}$-manifold M is called *asystatic* if the normalizer $N_{\mathfrak{g}}(\mathfrak{g}_x) = \mathfrak{g}_x$ for any isotropy subalgebra $\mathfrak{g}_x$ of M. From 6.1 and 5.9 we have immediately:

Proposition. *An asystatic locally trivial $\mathfrak{g}$-manifold admits a unique principal connection. This principal connection is flat. Its horizontal space at $x \in M$ is the subspace of T_xM on which the isotropy representation of $\mathfrak{g}_x$ vanishes.*

6.3. Horizontal lifts on locally trivial $\mathfrak{g}$-manifolds. Let Φ be a connection on the locally trivial $\mathfrak{g}$-manifold $(M, p, N = M/\mathfrak{g}, S)$. Then the projection $(\pi_M, Tp) : TM \to M \times_N TN$ onto the fibered product restricts to an isomorphism $\ker(\Phi) \to M \times_N TN$ whose inverse will be denoted by $C : M \times_N TN \to TM$ and will be called the *horizontal lift*. If $\xi \in \mathfrak{X}(N)$ is a vector field on the base then its horizontal lift $C(\xi)$ is given by $C(\xi)(y) = C(y, \xi(p(y)))$. In a bundle chart we have $T(\varphi_\alpha) \cdot C(\xi)(\varphi_\alpha^{-1}(x, s)) = (\xi(x), \Gamma^\alpha(\xi(x))(s))$. Thus we see from 6.1 that $C(\xi)$ commutes with all fundamental vector fields: $[\zeta_X^M, C(\xi)] = 0$ for all $X \in \mathfrak{g}$ and $\xi \in \mathfrak{X}(N)$.

Note that the $\mathfrak{g}$-equivariant vector fields on M which are horizontal in the sense that they take values in the kernel of the connection Φ are exactly the horizontal lifts of vector fields on the base manifold N.

6.4. Theorem. (Parallel transport). Let Φ be a connection on a locally trivial $\mathfrak{g}$-manifold $(M, p, N = M/\mathfrak{g}, S)$ and let $c : (a,b) \to N$ be a smooth curve with $0 \in (a,b)$, $c(0) = x$.

Then there is a neighbourhood U of $M_x \times \{0\}$ in $M_x \times (a,b)$ and a smooth mapping $\mathrm{Pt}_c : U \to M$ such that:

(1) $p(\mathrm{Pt}(c, u_x, t)) = c(t)$ if defined, and $\mathrm{Pt}(c, u_x, 0) = u_x$.

(2) $\Phi(d/dt\, \mathrm{Pt}(c, u_x, t)) = 0$ if defined.

(3) Reparametrization invariance: If $f : (a', b') \to (a,b)$ is smooth with $0 \in (a', b')$, then $\mathrm{Pt}(c, u_x, f(t)) = \mathrm{Pt}(c \circ f, \mathrm{Pt}(c, u_x, f(0)), t)$ if defined.

(4) U is maximal for properties (1) and (2).

(5) If the curve c depends smoothly on further parameters then $\mathrm{Pt}(c, u_x, t)$ depends also smoothly on those parameters.

(6) If $\xi \in \mathfrak{X}(N)$ is a vector field on the base and $C(\xi) \in \mathfrak{X}(M)$ is its horizontal lift, then $\mathrm{Pt}(\mathrm{Fl}^{\xi}(x), u_x, t) = \mathrm{Fl}_t^{C(\xi)}(u_x)$.

(7) For each $X \in \mathfrak{g}$ the restrictions of the fundamental field ζ_X to $M_x = p^{-1}(x)$ and to $M_{c(t)}$ are $\mathrm{Pt}(c,t)$-related: $T(\mathrm{Pt}(c,t)) \circ \zeta_X|_{M_x} = (\zeta_X|_{M_{c(t)}}) \circ \mathrm{Pt}(c,t)$.

Proof. All assertions but the last two of this theorem follow from the general result [9, 9.8]. The assertion (6) is obvious and for (7) we first note that it suffices to show it for curves of the form $c(t) = \mathrm{Fl}_t^{\xi}(x)$. But then by (6) and by 6.3 we have

$$\begin{aligned}\frac{d}{dt}\mathrm{Pt}(c,t)^*(\zeta_X|_{M_{c(t)}}) &= \frac{d}{dt}(\mathrm{Fl}_t^{C(\xi)})^*(\zeta_X)|_{M_x} = (\mathrm{Fl}_t^{C(\xi)})^*([C(\xi), \zeta_X])|_{M_x} \\ &= 0\end{aligned}$$

so that $\mathrm{Pt}(c,t)^*(\zeta_X|M_{c(t)})$ is constant in t and thus equals $\zeta_X|_{M_x}$. $\square$

6.5. Parallel transport. Now we consider a $\mathfrak{g}$-manifold M which admits a principal connection Φ. Guided by the last remark in 6.3 we call *parallel transport* each local flow Fl_t^{ξ} along any horizontal $\mathfrak{g}$-equivariant vector field on M.

6.6. Complete connections. Let M be a locally trivial $\mathfrak{g}$-manifold with projection $p : M \to N := M/\mathfrak{g}$ and with standard fiber S. Following [9, 9.9] we call a principal connection Φ *complete* if for each curve $c : (a,b) \to N$ the parallel transport $\mathrm{Pt}(c, \cdot)$ is defined on the whole of $p^{-1}(c(0)) \times (a,b)$.

Proposition. *In this situation, if each vector field in the centralizer $Z_{\mathfrak{X}(S)}(\mathfrak{g})$ of the $\mathfrak{g}$-action on S is complete, then each principal connection on M is complete.*

Proof. It suffices to show that for each curve $c : (a,b) \to U$ the parallel transport $\mathrm{Pt}(c,t)$ is defined on the whole of $M_{c(0)}$ for each $t \in (a,b)$, where $(U, \varphi : M|U \to U \times S)$ is a bundle chart, since we may piece together such local solutions. So we may assume that $M = N \times S$ is a trivial $\mathfrak{g}$-manifold. Then by 6.1 any principal connection is of the form $\Phi(\xi_x, \eta_s) = \eta_s - \Gamma(\xi_x)(s)$, where $\Gamma \in \Omega^1(N; \mathfrak{h})$ is the Christoffel form with values in the centralizer algebra $\mathfrak{h} := Z_{\mathfrak{X}(S)}(\mathfrak{g})$, which is finite dimensional by 5.9. Since all vector fields in this Lie algebra are complete we may integrate its action on S to a right action $r : S \times H \to S$ of a connected Lie group H with Lie algebra $\mathfrak{h}$. Then $t \mapsto \Gamma(c'(t))$ is a smooth curve in $\mathfrak{h}$ which we may integrate to a smooth curve $b(t) \in H$ with $b(0) = e$ and $b'(t) = L_{\Gamma(c'(t))}(b(t))$ where L_X is the left invariant vector field on H generated by $X \in \mathfrak{h}$. It is an integral curve of a time dependent vector field on H which is, locally in time, bounded with respect to a left invariant Riemannian metric on H. So indeed $b : (a, b) \to H$. But then $Pt(c, t, u) = (c(t), r(u, b(t)))$ for each $u \in M_{c(0)}$. □

6.7. Theorem. *Let $p : M \to N := M/\mathfrak{g}$ be a locally trivial $\mathfrak{g}$-manifold with standard fiber S. Let Φ be a complete principal connection on M. Let us assume that the holonomy Lie algebra of Φ in the sense explained in the proof consists of complete vector fields on S.*

Then there exists a finite dimensional Lie group H with Lie algebra $\mathfrak{h}$, a principal H-bundle $P \to N$, an irreducible principal connection form ω on P, and a left action of H on the standard fiber S such that:

(1) *The fundamental vector field mapping of the H-action on S is an injective Lie algebra antihomomorphism $\mathfrak{h} \to Z_{\mathfrak{X}(S)}(\mathfrak{g})$. The H-action on S commutes pointwise with the $\mathfrak{g}$-action.*

(2) *The associated bundle $P[S] = P \times_H S$ is isomorphic to the bundle $M \to N$.*

(3) *The $\mathfrak{g}$-principal connection Φ on M is induced by the principal connection form ω on P.*

Proof. We suppose first that the base N is connected. Let x_0 be a fixed point in N, and let us identify the standard fiber S with the fiber M_{x_0} of M over x_0. Since Φ is a complete connection on the bundle $M \to N$ we may consider the holonomy group $\mathrm{Hol}(\Phi, x_0)$ consisting of all parallel transports with respect to Φ along closed loops in N through x_0, and the holonomy Lie algebra $\mathrm{hol}(\Phi, x_0)$, which is defined as follows (see [9, 9.10]):

Let $C : TN \times_N M \to TM$ be the horizontal lift and let R be the curvature of the connection Φ. For any $x \in N$ and $X_x \in T_xN$ the horizontal lift $C(X_x) := C(X_x, \cdot) : M_x \to TM$ is a vector field along M_x. For X_x and $Y_x \in T_xN$ we consider $R(CX_x, CY_x) \in \mathfrak{X}(M_x)$. Now we choose any piecewise smooth curve c from x_0 to x and consider the diffeomorphism $\mathrm{Pt}(c, t) : S = M_{x_0} \to M_x$ and the pullback $\mathrm{Pt}(c, 1)^* R(CX_x, CY_x) \in \mathfrak{X}(S)$. Then $\mathrm{hol}(\Phi, x_0)$ is the closed linear subspace, generated by all these vector fields (for all $x \in N$, X_x, $Y_x \in T_xN$ and curves c from x_0 to x) in $\mathfrak{X}(S)$ with respect to the compact C^∞-topology.

In each local chart $(U_\alpha, \varphi_\alpha : M|U \to U \times S)$ the curvature is expressed by the Christoffel form via $(\varphi_\alpha^{-1})^* R = d\Gamma^\alpha + \frac{1}{2}[\Gamma^\alpha, \Gamma^\alpha]^\wedge_{\mathfrak{X}(S)}$, see 6.1, and since Γ^α takes values in $Z_{\mathfrak{X}(S)}(\mathfrak{g})$, the local expression of the curvature $(\varphi_\alpha^{-1})^* R$ does it also. The parallel transport $\mathrm{Pt}^\Phi(c, t)$ along any curves relates $\mathfrak{g}$-fundamental vector fields to itself by 6.4(7). Thus the holonomy Lie algebra $\mathrm{hol}(\Phi, x_0)$ is contained in the centralizer algebra $Z_{\mathfrak{X}(S)}(\mathfrak{g})$, so it is finite dimensional.

By assumption $\mathrm{hol}(\Phi, x_0) \subset \mathfrak{X}(S)$ consists of complete vector fields. Thus all conditions of Theorem [9, 9.11] are satisfied and all conclusions follow from it. □

6.8. Remark. In the situation of Theorem 6.7 let us suppose that the centralizer algebra $Z_{\mathfrak{X}(S)}(\mathfrak{g})$ consists of complete vector fields. Then the each principal connection Φ is complete by 6.6 and the holonomy Lie algebra $\operatorname{hol}(\Phi, x_0) \subset Z_{\mathfrak{X}(S)}(\mathfrak{g})$ is also complete, see the Proof of 6.7. Thus the conclusions of Theorem 6.7 hold.

7. Characteristic classes for $\mathfrak{g}$-manifolds

7.1. Basic cohomology. Let M be a $\mathfrak{g}$-manifold. Following 4.2, by $\Omega^p_{\text{hor}}(M)^{\mathfrak{g}}$ we denote the space of all real valued horizontal forms on M which are $\mathfrak{g}$-invariant: $\mathcal{L}_{\zeta_X}\varphi = 0$ for all $X \in \mathfrak{g}$. These forms are called *basic forms* of the $\mathfrak{g}$-manifold M.

Lemma. *In this situation the exterior derivative restricts to a mapping*

$$d : \Omega^p_{\text{hor}}(M)^{\mathfrak{g}} \to \Omega^{p+1}_{\text{hor}}(M)^{\mathfrak{g}}$$

Proof. Let $\varphi \in \Omega^p_{\text{hor}}(M)^{\mathfrak{g}}$ then for $X \in \mathfrak{g}$ we have

$$i_{\zeta_X} d\varphi = i_{\zeta_X} d\varphi + d i_{\zeta_X}\varphi = \mathcal{L}_{\zeta_X}\varphi = 0$$
$$\mathcal{L}_{\zeta_X} d\varphi = d\mathcal{L}_{\zeta_X}\varphi = 0. \quad \square$$

The cohomology of the resulting differential complex will be called the *basic cohomology* of the $\mathfrak{g}$-manifold M:

$$H^p_{\mathfrak{g}}(M) := \frac{\ker(d : \Omega^p_{\text{hor}}(M)^{\mathfrak{g}} \to \Omega^{p+1}_{\text{hor}}(M)^{\mathfrak{g}})}{\operatorname{im}(d : \Omega^{p-1}_{\text{hor}}(M)^{\mathfrak{g}} \to \Omega^p_{\text{hor}}(M)^{\mathfrak{g}})}$$

In the case of a $\mathfrak{g}$-manifold M of constant rank this cohomology is exactly the basic cohomology of the orbit foliation of M, defined by Reinhard [17] and intensively studied in the theory of foliations, see [13], appendix B by V. Sergiescu. Note that this cohomology may be of infinite dimension, see [18] and [6].

If $f : M \to N$ is a smooth $\mathfrak{g}$-equivariant mapping between $\mathfrak{g}$-manifolds M and N, then the pullback operator induces a mapping $f^* : \Omega^p_{\text{hor}}(N)^{\mathfrak{g}} \to \Omega^p_{\text{hor}}(M)^{\mathfrak{g}}$ which in turn induces a linear mapping in basic cohomology $f^* : H^p_{\mathfrak{g}}(N) \to H^p_{\mathfrak{g}}(M)$. If $f, g : M \to N$ are smoothly homotopic through $\mathfrak{g}$-equivariant mappings then they induce the same mapping in basic cohomology.

7.2. Chern–Weil forms. If $f \in L^k(\mathfrak{g}) := (\bigotimes^k \mathfrak{g}^*)$ is a k-linear function on $\mathfrak{g}$ and if $\psi_i \in \Omega^{p_i}(M;\mathfrak{g})$ we can construct the following differential forms (see 4.1):

$$\psi_1 \otimes_\wedge \cdots \otimes_\wedge \psi_k \in \Omega^{p_1+\cdots+p_k}(M; \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}),$$
$$f^{\psi_1,\ldots,\psi_k} := f \circ (\psi_1 \otimes_\wedge \cdots \otimes_\wedge \psi_k) \in \Omega^{p_1+\cdots+p_k}(M).$$

The exterior derivative of the latter one is clearly given by

$$d(f \circ (\psi_1 \otimes_\wedge \cdots \otimes_\wedge \psi_k)) f \circ d(\psi_1 \otimes_\wedge \cdots \otimes_\wedge \psi_k)$$
$$= f \circ \left(\sum_{i=1}^k (-1)^{p_1+\cdots+p_{i-1}} \psi_1 \otimes_\wedge \cdots \otimes_\wedge d\psi_i \otimes_\wedge \cdots \otimes_\wedge \psi_k \right).$$

Note that the form $f^{\psi_1,\dots,\psi_k}$ is basic, i.e., $\mathfrak{g}$-invariant and horizontal, if all $\psi_i \in \Omega^{p_i}_{\mathrm{hor}}(M;\mathfrak{g})^{\mathfrak{g}}$ and f is invariant under the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ ($f \in L^k(\mathfrak{g})^{\mathfrak{g}}$) in the following sense:

7.3. Definition. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. $f \in L^k(V)$ is called $\mathfrak{g}$-*invariant* if $\sum_{i=1}^k f(v_1,\dots,\rho(X)v_i,\dots,v_k) = 0$ for each $X \in \mathfrak{g}$. If f is $\mathfrak{g}$-invariant then we have for $\psi_i \in \Omega^{p_i}(M;V)$ and any $\varphi \in \Omega^p(M;\mathfrak{g})$, by applying alternation:

$$f \circ \left(\sum_{i=1}^k (-1)^{(p_1+\cdots+p_{i-1})p}\psi_1 \otimes_\wedge \cdots \otimes_\wedge \rho^\wedge(\varphi)\psi_i \otimes_\wedge \cdots \otimes_\wedge \psi_k\right) = 0.$$

7.4. Lemma. *Let M be a $\mathfrak{g}$-manifold with a principal connection form ω and let Ω be its curvature form. Let $f \in L^k(\mathfrak{g})^{\mathfrak{g}}$ be $\mathfrak{g}$-invariant under the adjoint action then the differential form $f^\Omega := f^{\Omega,\dots,\Omega} \in \Omega^{2k}(M)^{\mathfrak{g}}$ is a closed $\mathfrak{g}$-invariant form.*

If moreover M is a free $\mathfrak{g}$-manifold, then Ω and consequently f^Ω are horizontal, so $f^\Omega \in \Omega^{2p}_{\mathrm{hor}}(M)^{\mathfrak{g}}$ is a closed basic form.

Proof. We have in turn by 7.2 and the Bianchi identity 4.6

$$\begin{aligned}
df^\Omega &= d(f \circ (\Omega \otimes_\wedge \cdots \otimes_\wedge \Omega))\\
&= f \circ \left(\sum_{i=1}^k \Omega \otimes_\wedge \cdots \otimes_\wedge d\Omega \otimes_\wedge \cdots \otimes_\wedge \Omega\right)\\
&= -f \circ \left(\sum_{i=1}^k \Omega \otimes_\wedge \cdots \otimes_\wedge [\omega,\Omega]^\wedge \otimes_\wedge \cdots \otimes_\wedge \Omega\right)\\
&= -f \circ \left(\sum_{i=1}^k \Omega \otimes_\wedge \cdots \otimes_\wedge \mathrm{ad}^\wedge(\omega)\Omega \otimes_\wedge \cdots \otimes_\wedge \Omega\right)
\end{aligned}$$

which is 0 by 7.3. The second statement is obvious. □

7.5. Proposition. *Let ω_0 and ω_1 be two principal connection forms on the $\mathfrak{g}$-manifold M with curvature forms $\Omega_0, \Omega_1 \in \Omega^2(M;\mathfrak{g})^{\mathfrak{g}}$, and let $f \in L^k(\mathfrak{g})^{\mathfrak{g}}$. Then the cohomology classes of the two closed forms f^{Ω_0} and f^{Ω_1} in $H^{2k}(M)$ coincide.*

If M is a free $\mathfrak{g}$-manifold then the curvature forms Ω_0, Ω_1 are horizontal and define the same basic cohomology classes in $H^{2k}_{\mathfrak{g}}(M) = H^{2k}(\Omega^*_{\mathrm{hor}}(M)^{\mathfrak{g}})$.

Thus for $f \in L^k(\mathfrak{g})^{\mathfrak{g}}$ the cohomology class $[f^\Omega] \in H^{2p}(M)$ depends only on f and the $\mathfrak{g}$-action and we call it a *characteristic class* for the $\mathfrak{g}$-action.

If $M \to M/G$ is a principal G-bundle, thus M a free $\mathfrak{g}$-manifold, we have just reconstructed the usual Chern–Weil characteristic classes.

If M is a homogeneous $\mathfrak{g}$-manifold (e.g., a homogeneous G-manifold $H\backslash G$), by Theorem 5.8 these characteristic classes in $H^{2m}(M)$ are usual characteristic classes of the principal $\Gamma(\mathfrak{g})_{x_0}$-bundle $P_{x_0}(\mathfrak{g}) \to M$, but possibly not all of them: only those arising from invariant polynomials on $\mathfrak{g}_{x_0}$ which are restrictions of invariant polynomials on $\mathfrak{g}$ appear.

Proof. For each $t \in \mathbb{R}$ we have a principal connection form $\omega_t := (1-t)\omega_0 + t\omega_1$, and also consider its curvature $\Omega_t := d\omega_t + \frac{1}{2}[\omega_t, \omega_t]^\wedge$. Since $\partial_t \omega_t = \omega_1 - \omega_0$ we get

$$\begin{aligned}
\partial_t \Omega_t &= d\partial_t\omega_t + [\omega_t, \partial_t\omega_t]^\wedge \\
&= d(\omega_1 - \omega_0) + [\omega_t, \omega_1 - \omega_0]^\wedge = d_{\omega_t}(\omega_1 - \omega_0).
\end{aligned}$$

Note that $d_{\omega_t}(\omega_1 - \omega_0)$ makes sense since $\omega_1 - \omega_0 \in \Omega^p_{\text{hor}}(M;\mathfrak{g})^{\mathfrak{g}}$. We will also need the Bianchi identity $d_{\omega_t}\Omega_t = d\Omega_t + [\omega_t, \Omega_t]^\wedge = 0$, see 4.6. Since Ω_t is a 2-form we may assume that f is symmetric. Then we have in turn:

$$\begin{aligned}
\partial_t f^{\Omega_t} &= \partial_t f(\Omega_t, \dots, \Omega_t) = p \cdot f(\partial_t\Omega_t, \Omega_t, \dots, \Omega_t) \\
&= p \cdot f(d_{\omega_t}(\omega_1 - \omega_0), \Omega_t, \dots, \Omega_t) \\
&= p \cdot f(d_{\omega_t}(\omega_1 - \omega_0), \Omega_t, \dots, \Omega_t) - p\sum_{i=2}^{p} f(\omega_1 - \omega_0, \Omega_t, \dots, d_{\omega_t}\Omega_t, \dots, \Omega_t) \\
&= p \cdot f(d(\omega_1 - \omega_0), \Omega_t, \dots, \Omega_t) - p\sum_{i=2}^{p} f(\omega_1 - \omega_0, \Omega_t, \dots, d\Omega_t, \dots, \Omega_t) \\
&\quad + p \cdot f([\omega_t, \omega_1 - \omega_0]^\wedge, \Omega_t, \dots, \Omega_t) \\
&\quad - p\sum_{i=2}^{p} f(\omega_1 - \omega_0, \Omega_t, \dots, [\omega_t, \Omega_t]^\wedge, \dots, \Omega_t) \\
&= p \cdot df(\omega_1 - \omega_0, \Omega_t, \dots, \Omega_t),
\end{aligned}$$

where we again used 7.3 in the form

$$0 = f([\omega_t, \omega_1 - \omega_0]^\wedge, \Omega_t, \dots, \Omega_t) - \sum_{i+2}^{p} f(\omega_1 - \omega_0, \Omega_t, \dots, [\omega_t, \Omega_t]^\wedge, \dots, \Omega_t).$$

Since $T(f, \Omega_t) := f(\omega_1 - \omega_0, \Omega_t, \dots, \Omega_t) \in \Omega^{2p-1}_{\text{hor}}(M)^{\mathfrak{g}}$, the following form is exact in $(\Omega^*_{\text{hor}}(M)^{\mathfrak{g}}, d)$:

$$f^{\Omega_1} - f^{\Omega_0} = \int_0^1 \partial_t f^{\Omega_t} dt = \int_0^1 p dT(f, \Omega_t) dt = d\left(\int_0^1 p \cdot T(f, \Omega_t)\, dt\right). \quad \square$$

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