Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Advances in Mathematics 224 (2010) 1311–1348

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Combinatorial Hopf algebras, noncommutative Hall–Littlewood functions, and permutation tableaux

J.-C. Novelli [a], J.-Y. Thibon [a,*], L.K. Williams [b]

[a] *Institut Gaspard Monge, Université Paris-Est, Marne-la-Vallée, 5 Boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France*

[b] *Harvard University, Department of Mathematics, 432 Science Center, 1 Oxford Street, Cambridge, MA 02138, USA*

Received 4 July 2008; accepted 21 January 2010

Available online 1 February 2010

Communicated by the Managing Editors of AIM

Abstract

We introduce a new family of noncommutative analogues of the Hall–Littlewood symmetric functions. Our construction relies upon Tevlin's bases and simple q-deformations of the classical combinatorial Hopf algebras. We connect our new Hall–Littlewood functions to permutation tableaux, and also give an exact formula for the q-enumeration of permutation tableaux of a fixed shape. This gives an explicit formula for: the steady state probability of each state in the partially asymmetric exclusion process (PASEP); the polynomial enumerating permutations with a fixed set of *weak excedances* according to *crossings*; the polynomial enumerating permutations with a fixed set of *descent bottoms* according to occurrences of the *generalized pattern* 2–31.

Keywords: Noncommutative symmetric functions; Combinatorial Hopf algebras; Permutation tableaux; PASEP

Contents

* Corresponding author.
E-mail addresses: novelli@univ-mlv.fr (J.-C. Novelli), jyt@univ-mlv.fr (J.-Y. Thibon), lauren@math.harvard.edu (L.K. Williams).

doi:10.1016/j.aim.2010.01.006

1. Introduction

The combinatorics of Hall–Littlewood functions is one of the most interesting aspects of the modern theory of symmetric functions [27]. These are bases of symmetric functions, depending on a parameter q which was originally regarded as the cardinality of a finite field. They are named after Littlewood's explicit realization of the Hall algebra in terms of symmetric functions, which gave meaning to arbitrary complex values of q [23].

Combinatorics entered the scene with the observation by Foulkes [7] that the transition matrices between Schur functions and Hall–Littlewood P-functions seemed to be given by polynomials, which were nonnegative q-analogues of the well-known Kostka numbers, counting Young tableaux according to shape and weight. The conjecture of Foulkes was established by Lascoux and Schützenberger [20], who introduced the charge statistic on Young tableaux to explain the powers of q. Almost simultaneously, Lusztig [25] obtained an interpretation in terms of the intersection homology of nilpotent orbits, and it is now known that these Kostka–Foulkes polynomials are particular Kazhdan–Lusztig polynomials associated with the affine Weyl groups of type A [26]. But this was not the end of the story. Some ten years later, Kirillov and Reshetikhin [16] discovered an interpretation of Kostka–Foulkes polynomials in statistical physics, as generating functions of Bethe ansatz configurations for some generalizations of Heisenberg's *XXX*-magnet model, and obtained a closed expression in the form of a sum of products of q-binomial coefficients.

All of these results have been generalized in many directions. Generalized Hall algebras (associated with quivers) have been introduced [36]. Ribbon tableaux [21] and k-Schur functions [19] give rise to generalizations of the charge polynomials, sometimes interpretable as Kazhdan–Lusztig polynomials [22]. Intersection homology has been computed for other varieties. The Kirillov–Reshetikhin formula is now included in a vast corpus of fermionic formulas, available for a large number of models [11]. However, the relations – if any – between these theories are generally unknown.

The present article is devoted to a different kind of generalization of the Hall–Littlewood theory. It is by now well known that many aspects of the theory of symmetric functions can be lifted to noncommutative symmetric functions, or quasi-symmetric functions, and that those points which do not have a good analogue at this level can sometimes be explained by lifting them to more complicated combinatorial Hopf algebras.[1]

The paradigm here is the Littlewood–Richardson rule, which becomes trivial in the algebra of free symmetric functions, all the difficulty having been diluted in the definition of the algebra [5].

A theory of noncommutative and quasi-symmetric Hall–Littlewood functions has been worked out by Hivert [12]. Since there is no Hall algebra to use as a starting point, Hivert's choice was to imitate Littlewood's definition, which can be reformulated in terms of an action of the affine Hecke algebra on polynomials. By replacing the usual action by a quasi-symmetrizing one, Hivert obtained interesting bases, behaving in much the same way as the original ones, and were easily deformable with a second parameter, so that analogues of Macdonald's functions could also be defined [13,3].

However, Hivert's analogues of the Kostka–Foulkes polynomials are just Kostka–Foulkes *monomials*, i.e., powers of q, given moreover by a simple explicit formula. So the combinatorial connections to tableaux, geometry and statistical physics do not show up in this theory.

[1] There is no general agreement on the precise definition of a combinatorial Hopf algebra, see [1] and [24] for attempts at making this concept precise, and [29,31,33] for more examples.

More recently, new possibilities arose with Tevlin's [39] discovery of a plausible analogue of monomial symmetric functions on the noncommutative side. Tevlin's constructions are incompatible with the Hopf structure (his monomial functions are not dual to products of complete functions in any reasonable sense), so it seemed unlikely that they could lead to interesting combinatorics. Nevertheless, Tevlin computed analogues of the Kostka matrices in his setting, and conjectured that they had nonnegative integer coefficients. This conjecture was proved in [14], and turned out to be more interesting than expected. The proof required the use of larger combinatorial Hopf algebras, and led to a vast generalization of the Genocchi numbers.

In this paper we give a new generalization of Hall–Littlewood functions, starting from Tevlin's bases. We define q-analogues $S^I(q)$ of the products of complete homogeneous functions by embedding **Sym** in an associative deformation of **WQSym** and projecting back to **Sym** by the map introduced in [14]. This defines a nonassociative q-product $\star_q$ on **Sym**, and our Hall–Littlewood functions are equal to the products (see Section 3.8)

$$S^I(q) = S^{i_1} \star_q \left(S^{i_2} \star_q \left(\cdots \left(S^{i_{r-1}} \star_q S^{i_r}\right)\right)\right). \tag{1}$$

These functions can be regarded as interpolating between the S^I (at $q=1$) and a new kind of noncommutative Schur functions (at $q=0$), have nonnegative coefficients, which can be expressed in closed form as products of q-binomial coefficients, and have a transparent combinatorial interpretation. As a consequence, the basis $R_I(q)$, defined by Moebius inversion on the composition lattice, is also nonnegative on the same basis.

The really interesting phenomenon occurs with Tevlin's second basis (denoted here and in [14] by L_I), an analogue of Gessel's fundamental basis F_I. One can observe that the last column of the matrix $M(S,L)$ (which expresses S^{1^n} in terms of the L_I's) gives the enumeration of *permutation tableaux* [38] by shape. This observation can be easily shown, and one may wonder whether the expansion of $S^{1^n}(q)$ on L_I gives rise to interesting q-analogues. This is clearly not the case (there are negative coefficients), but it turns out that introducing a simple q-analogue $L_I(q)$ of L_I, we obtain again nonnegative polynomials in the matrix $M(S(q),L(q))$. Finally, the matrix $M(R(q),L(q))$ gives the q-enumeration of permutation tableaux according to shape and rank, and another (yet unknown) statistic. Other (conjectural) combinatorial interpretations in terms of permutations or packed words are also proposed.[2]

As permutation tableaux occur in geometry (they are a distinguished subset of Postnikov's Ⅎ-*diagrams*, which parameterize cells in the totally nonnegative part of the Grassmannian [35]) and the q-enumeration of permutation tableaux is (up to a shift) counting cells according to dimension. Additionally, permutation tableaux occur in physics – Corteel and Williams [4] found a close connection to a well-known model from statistical physics called the partially asymmetric exclusion process, which in turn is related to the Hamiltonian of the XXZ quantum spin chain [6]. Therefore we may say that our new Hall–Littlewood functions have some of the features which were absent from Hivert's theory. However, we do not have the algebraic side coming from affine Hecke algebras, and it is an open question whether both points of view can be unified.

We conclude this paper with exact formulas for the q-enumeration of permutation tableaux of types A and B, according to shape. In the type A case, by the result of Corteel and Williams [4],

[2] Since many of our results and conjectures are stated in terms of permutations, one might be tempted to work only with **FQSym**, bypassing **WQSym** entirely. However, as was already clear in [14], one cannot make sense of the definition of Tevlin's monomial basis Ψ using **FQSym**.

this gives an exact formula for the steady state probability of each state of the partially asymmetric exclusion process (with arbitrary q, and $\alpha = \beta = 1$). Applying results of [38], this also gives an exact formula for the number of permutations with a fixed *weak excedance set* enumerated according to *crossings*, and for the number of permutations with a fixed set of *descent bottoms*, enumerated according to occurrences of the pattern 2–31.

2. Notations and background

2.1. Words, permutations, and compositions

We assume that the reader is familiar with the standard notations of the theory of noncommutative symmetric functions [8,5]. We shall need an infinite totally ordered alphabet $A = \{a_1 < a_2 < \cdots < a_n < \cdots\}$, generally assumed to be the set of positive integers. We denote by $\mathbb{K}$ a field of characteristic 0, and by $\mathbb{K}\langle A\rangle$ the free associative algebra over A when A is finite, and the projective limit $\operatorname{proj\,lim}_B \mathbb{K}\langle B\rangle$, where B runs over finite subsets of A, when A is infinite. The *evaluation* $\mathrm{ev}(w)$ of a word w is the sequence whose ith term is the number of times the letter a_i occurs in w. The *standardized word* $\mathrm{Std}(w)$ of a word $w \in A^*$ is the permutation obtained by iteratively scanning w from left to right, and labelling $1, 2, \ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. For example, $\mathrm{Std}(bbacab) = 341625$. For a word w on the alphabet $\{1, 2, \ldots\}$, we denote by $w[k]$ the word obtained by replacing each letter i by the integer $i + k$. If u and v are two words, with u of length k, one defines the *shifted concatenation* $u \bullet v = u \cdot (v[k])$ and the *shifted shuffle* $u \Cup v = u \sqcup\!\sqcup (v[k])$, where $\sqcup\!\sqcup$ is the usual shuffle product.

Recall that a permutation σ admits a *descent* at position i if $\sigma(i) > \sigma(i+1)$. Symmetrically, σ admits a *recoil* at i if $\sigma^{-1}(i) > \sigma^{-1}(i+1)$. The descent and recoil sets of σ are the positions of the descents and recoils, respectively.

A *composition* of an integer n is a sequence $I = (i_1, \ldots, i_r)$ of positive integers of sum n. In this case we write $I \models n$. The integer r is called the *length* of the composition. The *descent set* of I is $\mathrm{Des}(I) = \{i_1,\ i_1 + i_2, \ldots, i_1 + \cdots + i_{r-1}\}$. The *reverse refinement order*, denoted by $\succeq$, on compositions is such that $I = (i_1, \ldots, i_k) \succeq J = (j_1, \ldots, j_l)$ iff $\mathrm{Des}(I) \supseteq \mathrm{Des}(J)$, or equivalently, $\{i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_k\}$ contains $\{j_1, j_1 + j_2, \ldots, j_1 + \cdots + j_l\}$. In this case, we say that I is finer than J. For example, $(2, 1, 2, 3, 1, 2) \succeq (3, 2, 6)$. The *descent composition* $\mathrm{DC}(\sigma)$ of a permutation $\sigma \in \mathfrak{S}_n$ is the composition I of n whose descent set is the descent set of σ. Similarly we can define recoil compositions.

If $I = (i_1, \ldots, i_q)$ and $J = (j_1, \ldots, j_p)$ are two compositions, then $I \cdot J$ refers to their concatenation $(i_1, \ldots, i_q, j_1, \ldots, j_p)$, and $I \triangleright J$ is equal to $(i_1, \ldots, i_q + j_1, j_2, \ldots, j_p)$.

The *major index* $\mathrm{maj}(K)$ of a composition $K = (k_1, \ldots, k_r)$ is equal to the dot product of $(k_1, \ldots, k_r)$ with $(r-1, r-2, \ldots, 2, 1, 0)$, i.e. $\sum_{i=1}^{r} (r-i) k_i$.

2.2. Word quasi-symmetric functions: **WQSym**

Let $w \in A^*$. The *packed word* $u = \mathrm{pack}(w)$ associated with w is obtained by the following process. If $b_1 < b_2 < \cdots < b_r$ are the letters occurring in w, u is the image of w by the homomorphism $b_i \mapsto a_i$. A word u is said to be *packed* if $\mathrm{pack}(u) = u$. We denote by PW the set of packed words. With such a word, we associate the polynomial

$$\mathbf{M}_u := \sum_{\mathrm{pack}(w)=u} w. \tag{2}$$

For example, restricting A to the first five integers,

$$\begin{aligned}\mathbf{M}_{13132} = {} & 13132 + 14142 + 14143 + 24243 + 15152 + 15153 + 25253 + 15154 \\ & + 25254 + 35354.\end{aligned} \tag{3}$$

Under the abelianization $\chi : \mathbb{K}\langle A\rangle \to \mathbb{K}[X]$, the $\mathbf{M}_u$ are mapped to the monomial quasi-symmetric functions M_I ($I = (|u|_a)_{a\in A}$ being the evaluation vector of u).

These polynomials span a subalgebra of $\mathbb{K}\langle A\rangle$, called **WQSym** for Word Quasi-Symmetric functions [12]. These are the invariants of the noncommutative version of Hivert's quasi-symmetrizing action [12], which is defined by $\sigma \cdot w = w'$ where w' is such that $\mathrm{Std}(w') = \mathrm{Std}(w)$ and $\chi(w') = \sigma \cdot \chi(w)$. Thus, two words are in the same $\mathfrak{S}(A)$-orbit iff they have the same packed word.

The graded dimension of **WQSym** is the sequence of ordered Bell numbers ([37, A000670]) $1, 1, 3, 13, 75, 541, 4683, 47293, 545835, \ldots$. Hence, **WQSym** is much larger than **Sym**, which can be embedded in it in various ways [30,31].

The product of the $\mathbf{M}_u$ of **WQSym** is given by

$$\mathbf{M}_{u'}\mathbf{M}_{u''} = \sum_{u\in u' *_W u''} \mathbf{M}_u, \tag{4}$$

where the *convolution* $u' *_W u''$ of two packed words is defined as

$$u' *_W u'' = \sum_{v,w;\ u=v\cdot w\in \mathrm{PW},\ \mathrm{pack}(v)=u',\ \mathrm{pack}(w)=u''} u. \tag{5}$$

For example,

$$\mathbf{M}_{11}\mathbf{M}_{21} = \mathbf{M}_{1121} + \mathbf{M}_{1132} + \mathbf{M}_{2221} + \mathbf{M}_{2231} + \mathbf{M}_{3321}, \tag{6}$$

$$\begin{aligned}\mathbf{M}_{21}\mathbf{M}_{121} = {} & \mathbf{M}_{12121} + \mathbf{M}_{12131} + \mathbf{M}_{12232} + \mathbf{M}_{12343} + \mathbf{M}_{13121} + \mathbf{M}_{13232} + \mathbf{M}_{13242} \\ & + \mathbf{M}_{14232} + \mathbf{M}_{23121} + \mathbf{M}_{23131} + \mathbf{M}_{23141} + \mathbf{M}_{24131} + \mathbf{M}_{34121}.\end{aligned} \tag{7}$$

2.3. *Matrix quasi-symmetric functions:* MQSym

This algebra is introduced in [12,5]. We start from a totally ordered set of commutative variables $X = \{x_1 < \cdots < x_n\}$ and consider the ideal $\mathbb{K}[X]^+$ of polynomials without constant term. We denote by $\mathbb{K}\{X\} = T(\mathbb{K}[X]^+)$ its tensor algebra. We will also consider tensor products of elements of this algebra. To avoid confusion, we denote by "$\cdot$" the product of the tensor algebra and call it the dot product. We reserve the notation $\otimes$ for the external tensor product.

A natural basis of $\mathbb{K}\{X\}$ is formed by dot products of nonconstant monomials (called *multiwords* in the sequel), which can be represented by nonnegative integer matrices $M = (m_{ij})$, where m_{ij} is the exponent of the variable x_i in the jth factor of the tensor product. Since constant monomials are not allowed, such matrices have no zero column. We say that they are *horizontally packed*. A multiword $\mathbf{m}$ can be encoded in the following way. Let V be the *support* of $\mathbf{m}$, that is, the set of those variables x_i such that the ith row of M is nonzero, and let P be the matrix obtained from M by removing the null rows. We set $\mathbf{m} = V^P$. A matrix such as P, without zero rows or columns, is said to be *packed*.

For example, the multiword $\mathbf{m} = a \cdot ab^3e^5 \cdot a^2d$ is encoded by $\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array}\begin{bmatrix} 1&1&2\\0&3&0\\0&0&0\\0&0&1\\0&5&0 \end{bmatrix}$. Its support is the set $\{a, b, d, e\}$, and the associated packed matrix is $\begin{bmatrix} 1&1&2\\0&3&0\\0&0&1\\0&5&0 \end{bmatrix}$.

Let $\mathbf{MQSym}(X)$ be the linear subspace of $\mathbb{K}\{X\}$ spanned by the elements

$$\mathbf{MS}_M = \sum_{V \in \mathcal{P}_k(X)} V^M \tag{8}$$

where $\mathcal{P}_k(X)$ is the set of k-element subsets of X, and M runs over packed matrices of height $h(m) < n$.

For example, on the alphabet $\{a < b < c < d\}$

$$\mathbf{MS}_{\begin{bmatrix} 1&1&2\\0&3&0\\0&0&1 \end{bmatrix}} = \begin{array}{c} a \\ b \\ c \\ d \end{array}\begin{bmatrix} 1&1&2\\0&3&0\\0&0&1\\0&0&0 \end{bmatrix} + \begin{array}{c} a \\ b \\ c \\ d \end{array}\begin{bmatrix} 1&1&2\\0&3&0\\0&0&0\\0&0&1 \end{bmatrix} + \begin{array}{c} a \\ b \\ c \\ d \end{array}\begin{bmatrix} 1&1&2\\0&0&0\\0&3&0\\0&0&1 \end{bmatrix} + \begin{array}{c} a \\ b \\ c \\ d \end{array}\begin{bmatrix} 0&0&0\\1&1&2\\0&3&0\\0&0&1 \end{bmatrix}.$$

One can show that $\mathbf{MQSym}$ is a subalgebra of $\mathbb{K}\{X\}$. Actually,

$$\mathbf{MS}_P \mathbf{MS}_Q = \sum_{R \in \sqcup\!\sqcup(P,Q)} \mathbf{MS}_R$$

where the *augmented shuffle* of P and Q, $\sqcup\!\sqcup(P, Q)$ is defined as follows: let r be an integer between $\max(p, q)$ and $p + q$, where $p = h(P)$ and $q = h(Q)$. Insert null rows in the matrices P and Q so as to form matrices $\tilde{P}$ and $\tilde{Q}$ of height r. Let R be the matrix $(\tilde{P}, \tilde{Q})$. The set $\sqcup\!\sqcup(P, Q)$ is formed by all the matrices without null rows obtained in this way.

For example:

$$\mathbf{MS}\begin{bmatrix} 2&1\\1&0 \end{bmatrix}\mathbf{MS}_{[3\ 1]} = \mathbf{MS}\begin{bmatrix} 2&1&0&0\\1&0&0&0\\0&0&3&1 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2&1&0&0\\1&0&3&1 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 2&1&0&0\\0&0&3&1\\1&0&0&0 \end{bmatrix}$$
$$+ \mathbf{MS}\begin{bmatrix} 2&1&3&1\\1&0&0&0 \end{bmatrix} + \mathbf{MS}\begin{bmatrix} 0&0&3&1\\2&1&0&0\\1&0&0&0 \end{bmatrix}.$$

2.4. *Free quasi-symmetric functions:* FQSym

The Hopf algebra $\mathbf{FQSym}$ is the subalgebra of $\mathbf{WQSym}$ spanned by the polynomials [5]

$$\mathbf{G}_\sigma := \sum_{\mathrm{Std}(u)=\sigma} \mathbf{M}_u = \sum_{\mathrm{Std}(w)=\sigma} w. \tag{9}$$

The multiplication rule is, for $\alpha \in \mathfrak{S}_k$ and $\beta \in \mathfrak{S}_l$,

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \sum_{\substack{\gamma \in \mathfrak{S}_{k+l};\ \gamma = u \cdot v \\ \mathrm{Std}(u)=\alpha, \mathrm{Std}(v)=\beta}} \mathbf{G}_\gamma. \tag{10}$$

As a Hopf algebra, **FQSym** is self-dual. The scalar product materializing this duality is the one for which $(\mathbf{G}_\sigma, \mathbf{G}_\tau) = \delta_{\sigma,\tau^{-1}}$ (Kronecker symbol). Hence, $\mathbf{F}_\sigma := \mathbf{G}_{\sigma^{-1}}$ is the dual basis of $\mathbf{G}$. Their product is given by

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{\gamma \in \alpha \Cup \beta} \mathbf{F}_\gamma. \tag{11}$$

2.5. *Embeddings*

2.5.1. **Sym** *into* **MQSym**

Recall that the algebra of *noncommutative symmetric functions* is the free associative algebra $\mathbf{Sym} = \mathbb{C}\langle S_1, S_2, \ldots \rangle$ generated by an infinite sequence of noncommutative indeterminates S_k, called *complete* symmetric functions. For a composition $I = (i_1, \ldots, i_r)$, one sets $S^I = S_{i_1} \cdots S_{i_r}$. The family (S^I) is a linear basis of **Sym**. A useful realization, denoted by $\mathbf{Sym}(A)$, can be obtained by taking an infinite alphabet $A = \{a_1, a_2, \ldots\}$ and defining its complete homogeneous symmetric functions by the generating function

$$\sum_{n \geqslant 0} t^n S_n(A) = (1 - ta_1)^{-1}(1 - ta_2)^{-1} \cdots.$$

Given a packed matrix P, the vector of its *column sums* will be denoted by $\mathrm{Col}(P)$. The algebra morphism defined on generators by

$$\beta : S_n \mapsto \sum_{\mathrm{Col}(P)=(n)} \mathbf{MS}_P \tag{12}$$

is an embedding of Hopf algebras [5]. By definition of **MQSym**, for an arbitrary composition, we have

$$\beta(S^I) = \sum_{\mathrm{Col}(P)=I} \mathbf{MS}_P. \tag{13}$$

2.5.2. **Sym** *into* **WQSym**

The algebra morphism defined on generators by

$$\alpha : S_n \mapsto \sum_{\mathrm{Std}(u)=12\cdots n} \mathbf{M}_u \tag{14}$$

is also an embedding of Hopf algebras. For an arbitrary composition,

$$\alpha\big(S^I\big) = \sum_{\mathrm{DC}(u)\preceq I} \mathbf{M}_u. \tag{15}$$

Indeed, when **Sym** is realized as **Sym**(A), the latter sum is equal to $S^I(A)$.

2.6. *Epimorphisms*

We shall also need to project back from the algebras **MQSym** and **WQSym** to **Sym**. The crucial projection is the one associated with the (non-Hopf) quotient of **WQSym** introduced in [14].

2.6.1. **MQSym** *to* **WQSym**

To a packed matrix M, one associates a packed word $w(M)$ as follows. Read the entries of M columnwise, from top to bottom and left to right. The word $w(M)$ is obtained by repeating m_{ij} times each row index i.

Let $\mathcal{J}$ be the ideal of **MQSym** generated by the differences

$$\big\{\mathbf{MS}_P - \mathbf{MS}_Q \mid w(P) = w(Q)\big\}. \tag{16}$$

Then the quotient $\mathbf{MQSym}/\mathcal{J}$ is isomorphic as an algebra to **WQSym**, via the identification $\overline{\mathbf{MS}}_M = \mathbf{M}_{w(M)}$. More precisely, $\eta : \overline{\mathbf{MS}}_M \mapsto \mathbf{M}_{w(M)}$ is a morphism of algebras.

2.6.2. **WQSym** *to* **Sym**

Let w be a packed word. The *Word composition* (W-composition) of w is the composition whose descent set is given by the positions of the last occurrences of each letter in w.

For example,

$$\mathrm{WC}(1543421323) = (2, 3, 2, 2, 1). \tag{17}$$

Indeed, the descent set is $\{2, 5, 7, 9, 10\}$ since the last 5 is in position 2, the last 4 is in position 5, the last 1 is in position 7, the last 2 is in position 9, and the last 3 is in position 10.

The following tables group the packed words in PW_2 and PW_3 according to their W-composition.

2	11
11	12
	21

3	21	12	111
111	112	122	123
	121	211	132
	212		213
	221		231
			312
			321

(18)

Let $\sim$ be the equivalence relation on packed words defined by $u \sim v$ iff $\mathrm{WC}(u) = \mathrm{WC}(v)$. Let $\mathcal{J}'$ be the subspace of **WQSym** spanned by the differences

$$\{\mathbf{M}_u - \mathbf{M}_v \mid u \sim v\}. \tag{19}$$

Then, it has been shown [14] that $\mathcal{J}'$ is a two-sided ideal of **WQSym**, and that the quotient $\mathbf{T}'$ defined by $\mathbf{T}' = \mathbf{WQSym}/\mathcal{J}'$ is isomorphic to **Sym** as an algebra.

More precisely, recall that Ψ_n is a *noncommutative power sum of the first kind* [8]. Tevlin defined the *noncommutative monomial symmetric functions* Ψ_I [39] as quasideterminants in the Ψ_n's. We do not need the precise definition of Ψ_I here, only the following result.

Proposition 2.1. *(See [14].)* $\zeta : \overline{\mathbf{M}}_u \mapsto \Psi_{\mathrm{WC}(u)}$ *is a morphism of algebras.*

3. Quantizations and noncommutative Hall–Littlewood functions

In this section, we introduce a new q-analogue $S^I(q)$ of the basis S^I of **Sym**, giving two different but equivalent definitions. When we examine the transition matrices between this new basis and other bases, we will see a connection to permutation tableaux and hence to the asymmetric exclusion process. The new basis elements $S^I(q)$ play the role of the classical Hall–Littlewood Q'_μ [27, Ex. 7.(a), p. 234], and of Hivert's $H_I(q)$.

3.1. The special inversion statistic

Let $u = u_1 \cdots u_n$ be a packed word. We say that an inversion $u_i = b > u_j = a$ (where $i < j$ and $a < b$) is *special* if u_j is the *rightmost* occurrence of a in u. Let $\mathrm{sinv}(u)$ denote the number of special inversions in u. Note that if u is a permutation, this coincides with its ordinary inversion number.

3.2. Quantizing **WQSym**

Let $\mathbf{M}'_u = q^{\mathrm{sinv}(u)}\mathbf{M}_u$ and define a linear map ϕ_q by $\phi_q(\mathbf{M}_u) = \mathbf{M}'_u$. We define a new associative product $\star_q$ on **WQSym** by requiring that

$$\mathbf{M}'_u \star_q \mathbf{M}'_v = \phi_q(\mathbf{M}_u\mathbf{M}_v). \tag{20}$$

For example, by (6), one has

$$\begin{aligned}\mathbf{M}'_{11} \star_q \mathbf{M}'_{21} &= \mathbf{M}'_{1121} + \mathbf{M}'_{1132} + \mathbf{M}'_{2221} + \mathbf{M}'_{2231} + \mathbf{M}'_{3321} \\ &= q\mathbf{M}_{1121} + q\mathbf{M}_{1132} + q^3\mathbf{M}_{2221} + q^3\mathbf{M}_{2231} + q^5\mathbf{M}_{3321}.\end{aligned} \tag{21}$$

This algebra structure on the vector space **WQSym** will be denoted by $\mathbf{WQSym}_q$.

3.3. Quantizing **MQSym**

Similarly, the q-product $\star_q$ can be defined on **MQSym**, by requiring that the $\mathbf{MS}'_M = q^{\mathrm{sinv}(w(M))}\mathbf{MS}_M$ multiply as the $\mathbf{MS}_M$.

3.4. Two equivalent definitions of $S^I(q)$

Embedding **Sym** into $\mathbf{MQSym}_q$ and projecting back to **Sym**, we define q-analogues of the products S^I by

$$S^I(q) = \zeta \circ \eta\big(\beta(S_{i_1}) \star_q \cdots \star_q \beta(S_{i_r})\big). \tag{22}$$

Equivalently, since under the above embeddings, the image of **Sym** in **MQSym** is contained in the image of **WQSym**, one can embed **Sym** into $\mathbf{WQSym}_q$ and project back to **Sym**, which yields

$$S^I(q) = \zeta\big(\alpha(S_{i_1}) \star_q \cdots \star_q \alpha(S_{i_r})\big). \tag{23}$$

3.5. The transition matrix $M(S(q), \Psi)$

For any two bases F, G of **Sym**, we denote by $M_n(F, G)$ the matrix indexed by compositions of n, whose entry in row I and column J is the coefficient of G_I in the G-expansion of F_J. We will give two combinatorial formulas (Propositions 3.1 and 3.2) and one recursive formula (Theorem 3.6) for the elements of the transition matrix $M(S(q), \Psi)$, where the Ψ_I's are Tevlin's noncommutative monomial symmetric functions.

3.5.1. First examples

Let $[n]$ denote the q-analogue $1 + q + \cdots + q^{n-1}$ of n. The first transition matrices $SP_n = M_n(S(q), \Psi)$ are

$$SP_3 = M_3(S(q), \Psi) = \begin{pmatrix} [1] & [1] & [1] & [1] \\ [1] & [3] & [2] & [2][2] \\ [1] & [1] & [2] & [2] \\ [1] & [3] & [3] & [2][3] \end{pmatrix},$$

$$SP_4 = \begin{pmatrix} [1] & [1] & [1] & [1] & [1] & [1] & [1] & [1] \\ [1] & [4] & [3] & [2][3] & [2] & [2][3] & [2][2] & [2][2][2] \\ [1] & [1] & [3] & [3] & [2] & [2] & [2][2] & [2][2] \\ [1] & [4] & [4][3]/[2] & [3][4] & [3] & [3][3] & [3][3] & [2][3][3] \\ [1] & [1] & [1] & [1] & [2] & [2] & [2] & [2] \\ [1] & [4] & [3] & [2][3] & [3] & [3][3] & [2][3] & [2][2][3] \\ [1] & [1] & [3] & [3] & [3] & [3] & [2][3] & [2][3] \\ [1] & [4] & [4][3]/[2] & [3][4] & [4] & [3][4] & [3][4] & [2][3][4] \end{pmatrix}.$$

The coefficient of Ψ_I in $S^J(q)$ will be denoted by $C_I^J(q)$.

3.5.2. Combinatorial interpretations

Recall that by Proposition 2.1, $\zeta \circ \eta$ is a morphism of algebras sending $\mathbf{MS}_M$ to $\Psi_{\mathrm{WC}(w(M))}$. Hence, our first definition of $S^J(q)$ gives the following:

Proposition 3.1. *Let $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$ be two compositions. Let $M(I, J)$ be the set of integer matrices $M = (m_{p,q})_{1 \leqslant p \leqslant l; 1 \leqslant q \leqslant k}$ without null rows such that*

$$\mathrm{WC}\big(w(M)\big) = I \quad \textit{and} \quad \mathrm{Col}(M) = J. \tag{24}$$

Then

$$C_I^J(q) = \sum_{M \in M(I,J)} q^{\mathrm{sinv}(w(M))}. \tag{25}$$

For example, the six matrices corresponding to the coefficient $[4][3]/[2]$ of M_4 in row $(2,1,1)$ and column $(2,2)$ are

$$\begin{bmatrix} 2 & . \\ . & 1 \\ . & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & . \\ . & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ . & 1 \\ 1 & . \end{bmatrix} \begin{bmatrix} . & 1 \\ 2 & . \\ . & 1 \end{bmatrix} \begin{bmatrix} . & 1 \\ 1 & 1 \\ 1 & . \end{bmatrix} \begin{bmatrix} . & 1 \\ . & 1 \\ 2 & . \end{bmatrix}. \tag{26}$$

The corresponding statistics are

$$\{0,\ 1,\ 2,\ 2,\ 3,\ 4\}. \tag{27}$$

The second definition of $S^J(q)$ yields a different combinatorial description:

Proposition 3.2. *Let I and J be two compositions and let $W(I,J)$ be the set of packed words w such that*

$$\mathrm{WC}(w) = I \quad \textit{and} \quad \mathrm{DC}(w) \preceq J. \tag{28}$$

Then

$$C_I^J(q) = \sum_{w \in W(I,J)} q^{\mathrm{sinv}(w)}. \tag{29}$$

For example, the six packed words corresponding to the coefficient $[4][3]/[2]$ of M_4 in row $I = (2,1,1)$ and column $J = (2,2)$ are

$$1123,\ 1213,\ 1312,\ 2213,\ 2312,\ 3312. \tag{30}$$

These words are the column readings of the six matrices from (26). The first word has descent composition (4) and the others have descent composition $(2,2)$.

3.5.3. The q-product on **Sym**

To explain the factorization of the coefficients of the transition matrix $M(S(q), \Psi)$, we need a recursive formula for $S^J(q)$. This will be given in Theorem 3.6.

In **WQSym**, let

$$\tilde{S}_n = \alpha(S_n) = \sum_{u;\, u \uparrow n} \mathbf{M}_u \tag{31}$$

where $u \uparrow n$ means that u is a nondecreasing packed word of length n, and define

$$\tilde{S}^J = \tilde{S}_{j_1} \star_q \cdots \star_q \tilde{S}_{j_r} \tag{32}$$

so that $S^J(q) = \zeta(\tilde{S}^J)$. Let $J = (j_1, \ldots, j_r)$ and set $J' = (j_2, \ldots, j_r)$. Since $\star_q$ is associative in $\mathbf{WQSym}_q$, we have

$$S^{j_1,J'}(q)=\zeta\big(\tilde{S}^{j_1,J'}\big)=\sum_{\substack{u,v;u\uparrow j_1\\ \mathrm{DC}(v)\preceq J'}} q^{\mathrm{sinv}(v)}\zeta(\mathbf{M}_u\star_q\mathbf{M}_v). \tag{33}$$

This expression can be simplified by means of the following lemma.

Lemma 3.3. *Let u be a nondecreasing packed word. Then*

$$\zeta(\mathbf{M}_u\star_q\mathbf{M}_v)=\zeta(\mathbf{M}_u\star_q\mathbf{M}_{v'}) \tag{34}$$

for all v' such that $\mathrm{WC}(v')=\mathrm{WC}(v)$.

Proof. Since u is nondecreasing, each packed word $z=x\cdot y$ appearing in the expansion of $\mathbf{M}_u\star_q\mathbf{M}_v$ is completely determined by the letters used in x and the letters used in y. Looking at the packed words z and z' occurring in $\mathbf{M}_u\star_q\mathbf{M}_v$ and in $\mathbf{M}_u\star_q\mathbf{M}_{v'}$ with given letters used for their prefixes and suffixes, we have $\mathrm{sinv}(z')=\mathrm{sinv}(z)+\mathrm{sinv}(v')-\mathrm{sinv}(v)$, whence the result. □

For example,

$$\mathbf{M}_{11}\star_q\mathbf{M}_{12}=\mathbf{M}_{1112}+\mathbf{M}_{1123}+q^2\mathbf{M}_{2212}+q^2\mathbf{M}_{2213}+q^4\mathbf{M}_{3312}, \tag{35}$$

$$\mathbf{M}_{11}\star_q q\mathbf{M}_{21}=q\mathbf{M}_{1121}+q\mathbf{M}_{1132}+q^3\mathbf{M}_{2221}+q^3\mathbf{M}_{2231}+q^5\mathbf{M}_{3321}. \tag{36}$$

Let now $\sigma:\mathbf{Sym}\to\mathbf{WQSym}$ be the section of the projection ζ defined by

$$\sigma(\Psi_I)=\mathbf{M}_{1^{i_1}2^{i_2}\cdots r^{i_r}}. \tag{37}$$

We can define a (non-associative!) q-product on **Sym** by

$$f\star_q g=\zeta\big(\sigma(f)\star_q\sigma(g)\big). \tag{38}$$

Then Lemma 3.3 implies that

$$S^I(q)=S^{i_1}\star_q\big(S^{i_2}\star_q\big(\cdots\big(S^{i_{r-1}}\star_q S^{i_r}\big)\big)\big). \tag{39}$$

3.5.4. Closed form for the coefficients

From Lemma 3.3, we now have

$$S^{j_1,J'}(q)=\zeta\big(\tilde{S}^{j_1,J'}\big)=\sum_{\substack{u,v;u\uparrow j_1\\ v\uparrow j_2+\cdots+j_r}} C^{J'}_{\mathrm{WC}(v)}(q)\zeta(\mathbf{M}_u\star_q\mathbf{M}_v). \tag{40}$$

Note that $\zeta(\mathbf{M}_u\star_q\mathbf{M}_v)$ and $\zeta(\mathbf{M}_{u'}\star_q\mathbf{M}_{v'})$ are linear combinations of disjoint sets of Ψ_K as soon as the nondecreasing words v and v' are different. So the computation of the coefficient C^I_J boils down to the evaluation of

$$\sum_{u;u\uparrow j_1}\zeta(\mathbf{M}_u\star_q\mathbf{M}_v)=\zeta\big(\tilde{S}^{j_1}\star_q\mathbf{M}_v\big), \tag{41}$$

where v is a nondecreasing word. Let us first characterize the terms of the product yielding a given Ψ_I.

Lemma 3.4. *Let u be a nondecreasing word of length k over $[1,r]$. Given a composition $I=(i_1,\dots,i_r)$ of length r, there exists at most one nondecreasing word v over $[1,r]$ such that uv is packed and $\mathrm{WC}(uv)=I$. Such a v exists precisely when $u=u_1\cdots u_k$ satisfies $u_i<u_{i+1}$ for $i\in\mathrm{Des}(I)$.*

In this case, let $y=1^{i_1}2^{i_2}\cdots r^{i_r}$. Then $\mathrm{sinv}(uv)$ is equal to

$$\sum_{1\leqslant i\leqslant k}(u_i-y_i). \tag{42}$$

This sum is also equal to

$$\sum_{1\leqslant i\leqslant k}u_i-\bigl(k+\mathrm{maj}(\overline{K})\bigr), \tag{43}$$

where K is the composition of k such that $\mathrm{Des}(K)=\mathrm{Des}(I)\cap[1,k-1]$.

Proof. The construction of v was already given in the proof of Theorem 6.1 of [14]. It comes essentially from the facts that the letters which should be used in v are determined by the letters used in u, and that a word is uniquely determined by its packed word and its alphabet.

Now, for each letter x of u, its contribution to $\mathrm{sinv}(uv)$ is given by the number of different letters strictly smaller than x appearing in v. This is equal to u_i-y_i. The sum of the y_i is $k+\mathrm{maj}(\overline{K})$. $\square$

For example, given $I=1221$, there are 10 nondecreasing words u of $[1,4]$ of length 3 satisfying the conditions of the lemma. The following table gives the corresponding v and the sinv statistics of the products uv.

u	v	sinv
122	334	0
123	224	1
124	223	2
133	224	2
134	223	3
144	223	4
233	114	3
234	113	4
244	113	5
344	112	6

(44)

We are now in a position to compute

$$\zeta\bigl(\tilde{S}^{j_1}\star_q\mathbf{M}_v\bigr) \tag{45}$$

when v is a nondecreasing word.

Lemma 3.5. *Let I be a composition of $k+n$ and let I' be the composition of n such that* $\mathrm{Des}(I') = \{a_1, \dots, a_s\}$ *satisfies* $\{k+a_1, \dots, k+a_s\} = \mathrm{Des}(I) \cap [k+1, k+n]$.

Let v be the nondecreasing word of evaluation I'. The coefficient of Ψ_I in $\zeta(\tilde{S}_k \star_q \mathbf{M}_v)$ is the q-binomial coefficient

$$\begin{bmatrix} k+r-s \\ r-s \end{bmatrix}_q \tag{46}$$

where $r = l(I)$, K is the composition of k such that $\mathrm{Des}(K) = \mathrm{Des}(I) \cap [1, k-1]$, *and $s = l(K)$.*

Proof. To start with, write

$$\tilde{S}_k \star_q \mathbf{M}_v = \sum_w q^{\mathrm{sinv}(w)} \mathbf{M}_w, \tag{47}$$

where w runs over packed words of the form $w = u'v'$, with u' nondecreasing and $\mathrm{pack}(v') = v$. From Lemma 3.4, we see that in order to have $\mathrm{WC}(u'v') = I$, $u' = x_1 \cdots x_k$ must be a word over the interval $[1, r]$ with equalities $x_i = x_j$ allowed precisely when cells i and j are in the same row of the diagram of K. The commutative image of the formal sum of such words, which are the nondecreasing reorderings of the quasi-ribbons of shape K [17], is the quasi-symmetric *quasi-ribbon* polynomial $F_K(t_1, \dots, t_r)$, introduced in [9]. Hence, the coefficient of Ψ_I is

$$q^{-\mathrm{maj}(\bar{K})} \begin{bmatrix} k+r-s \\ r-s \end{bmatrix}_q \tag{48}$$

given the generating function [10]

$$\sum_{m \geqslant 0} t^m F_K\bigl(1, q, \dots, q^{m-1}\bigr) = \frac{t^{l(K)} q^{\mathrm{maj}(\bar{K})}}{(t;q)_{k+1}}. \qquad \square \tag{49}$$

The example presented in (44) corresponds to the case $I = (1, 2, 2, 1)$ and $i = 3$, so that $K = (1, 2)$. We then find $\begin{bmatrix} 3+4-2 \\ 4-2 \end{bmatrix}_q = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$, which indeed corresponds to the statistic in the last column of (44).

Summarizing the above discussion, we can now state the main result of this section:

Theorem 3.6. *Let $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$ be compositions of n. Then the coefficient $C_I^J(q)$ of Ψ_I in $S^J(q)$ is given by the following rule:*

(i) *if $i_1 < j_1$, then $C_I^J(q) = C^J_{(i_1+i_2, i_3, \dots, i_k)}(q)$,*
(ii) *otherwise,*

$$C_I^J(q) = \begin{bmatrix} k+j_1-1 \\ j_1 \end{bmatrix}_q C_{I'}^{(j_2, \dots, j_l)}(q) \tag{50}$$

where the diagram of I' is obtained by removing the first j_1 cells of the diagram of I.

3.6. The transition matrix $M(R(q),\Psi)$

3.6.1. q-deformed ribbons

We now define a q-ribbon basis $R_I(q)$ in terms of the $S^J(q)$'s by analogy to the relationship between the ordinary R_I's and S^J's:

$$R_I(q) := \sum_{J\preceq I} (-1)^{l(J)-l(I)} S^J(q). \tag{51}$$

The coefficient of Ψ_I in the expansion of $R^J(q)$ will be denoted by $D_I^J(q)$.

3.6.2. First examples

We get the following transition matrices between $R(q)$ and Ψ for $n=3,4$:

$$RP_3 = M_3(R,\Psi) = \begin{pmatrix} 1 & . & . & . \\ 1 & q+q^2 & q & . \\ 1 & . & q & . \\ 1 & q+q^2 & q+q^2 & q^3 \end{pmatrix},$$

$$RP_4 = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ 1 & q[3] & q+q^2 & . & q & q^2 & . & . \\ 1 & . & q+q^2 & . & q & . & . & . \\ 1 & q[3] & q+2q^2+q^3+q^4 & q^3[3] & q+q^2 & q^2+q^3+q^4 & q^3 & . \\ 1 & . & . & . & q & . & . & . \\ 1 & q[3] & q+q^2 & . & q+q^2 & q^2+q^3+q^4 & q^3 & . \\ 1 & . & q+q^2 & . & q+q^2 & . & q^3 & . \\ 1 & q[3] & q+2q^2+q^3+q^4 & q^3[3] & q[3] & q^2+q^3+2q^4+q^5 & q^3[3] & q^6 \end{pmatrix}.$$

3.6.3. Combinatorial interpretations

By definition of the transition matrix from $S(q)$ to $R(q)$, the matrices $M(R(q),\Psi)$ can be described as follows:

Proposition 3.7. *Let I and J be compositions of n, and let $W'(I,J)$ be the set of packed words w such that*

$$\mathrm{WC}(w) = I \quad \textit{and} \quad \mathrm{DC}(w) = J. \tag{52}$$

Then

$$D_I^J(q) = \sum_{w\in W'(I,J)} q^{\mathrm{sinv}(w)}. \tag{53}$$

Proof. This follows directly from the combinatorial interpretation of C_I^J in terms of packed words (see Proposition 3.2). □

In terms of **MQSym**, this can be rewritten as follows:

Corollary 3.8. *Let I and J be compositions of n. Then $D_I^J(q)$ is given by the statistic $\operatorname{sinv}(w(M))$ applied to the elements M of the subset of $M(I,J)$ where in each pair of consecutive columns, the bottom-most nonzero entry of the left one is strictly below the top-most nonzero entry of the right one.*

3.7. The transition matrix $M(L(q),\Psi)$

3.7.1. A new q-analogue of the L basis of **Sym**

Let $\operatorname{st}(I,J)$ be the statistic on pairs of compositions of the same weight defined by

$$\operatorname{st}(I,J) := \begin{cases} \#\{(i,j)\in \operatorname{Des}(I)\times\operatorname{Des}(J) \mid i \geqslant j\} & \text{if } I \succeq J, \\ -\infty & \text{otherwise.} \end{cases} \tag{54}$$

We define a new basis $L(q)$ by

$$L_J(q) := \sum_{I \models |J|} q^{st(I,J)}\Psi_I = \sum_{I\succeq J} q^{st(I,J)}\Psi_I. \tag{55}$$

For $q=1$, this reduces to Tevlin's basis L_I (in the notation of [14]). Since $M(L_J(q),\Psi)$ is unitriangular, $L_I(q)$ is a basis of **Sym**.

3.7.2. First examples

Here are the first transition matrices from $L(q)$ to Ψ:

$$MLP_3 = M_3(L(q),\Psi) = \begin{pmatrix} 1 & . & . & . \\ 1 & q & . & . \\ 1 & . & q & . \\ 1 & q & q^2 & q^3 \end{pmatrix}, \tag{56}$$

$$MLP_4 = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ 1 & q & . & . & . & . & . & . \\ 1 & . & q & . & . & . & . & . \\ 1 & q & q^2 & q^3 & . & . & . & . \\ 1 & . & . & . & q & . & . & . \\ 1 & q & . & . & q^2 & q^3 & . & . \\ 1 & . & q & . & q^2 & . & q^3 & . \\ 1 & q & q^2 & q^3 & q^3 & q^4 & q^5 & q^6 \end{pmatrix}. \tag{57}$$

Note that up to some minor changes (conjugation w.r.t. mirror image of compositions), these are the matrices expressing Hivert's Hall–Littlewood $\tilde{H}_J$ on the basis R_I [12]. This allows us to derive the expression of their inverse, that is, transition matrices from Ψ to $L(q)$ (see [12], Theorem 6.6):

$$\Psi_J = \sum_{I\succeq J} (-1/q)^{l(I)-l(J)} q^{-\operatorname{st}'(I,J)} L_I(q), \tag{58}$$

where $\operatorname{st}'(I,J)$ is

$$\operatorname{st}'(I,J) := \begin{cases} \#\{(i,j)\in \operatorname{Des}(I)\times\operatorname{Des}(J) \mid i \leqslant j\} & \text{if } I \succeq J, \\ -\infty & \text{otherwise.} \end{cases} \tag{59}$$

3.8. The transition matrix $M(S(q), L(q))$

The coefficient of $L_I(q)$ in $S^J(q)$ will be denoted by $E_I^J(q)$. In this section, we will see a connection to permutation tableaux and hence to the asymmetric exclusion process.

3.8.1. First examples

Here are the first transition matrices from $S(q)$ to $L(q)$:

$$SL_3 = M_3\big(S(q), L(q)\big) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ . & 1+q & 1 & 2+q \\ . & . & 1 & 1 \\ . & . & . & 1 \end{pmatrix},$$

$$SL_4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1+q+q^2 & 1+q & 2+2q+q^2 & 1 & 2+2q+q^2 & 2+q & 3+3q+q^2 \\ . & . & 1+q & 1+q & 1 & 1 & 2+q & 2+q \\ . & . & q & 1+2q+q^2 & . & 1+q & 1+q & 3+3q+q^2 \\ . & . & . & . & 1 & 1 & 1 & 1 \\ . & . & . & . & . & 1+q & 1 & 2+q \\ . & . & . & . & . & . & 1 & 1 \\ . & . & . & . & . & . & . & 1 \end{pmatrix}.$$

In fact the right-hand column of each of these matrices contains the (un-normalized) steady-state probabilities of each state of the partially asymmetric exclusion process (PASEP). More specifically, the steady-state probabilities of the states ••, •∘, ∘•, and ∘∘ (the states of the PASEP on 2 sites) are $\frac{1}{q+5}$, $\frac{q+2}{q+5}$, $\frac{1}{q+5}$, and $\frac{1}{q+5}$, respectively; compare this with the right-hand column of SL_3. The steady-state probabilities of • • •, • • ∘, • ∘ •, • ∘ ∘, ∘ • •, ∘ • ∘, ∘ ∘ •, ∘ ∘ ∘ are given by the right-hand column of SL_4. This will be proved in Section 4, building on work of [4].

Note that since all coefficients of the matrix SP_n are explicit (and products of q-binomials) and since SL_n comes from SP_n by adding and substracting rows, we have a simple expression of $E_I^J(q)$ as an alternating sum of $C_I^J(q)$, hence of products of q-binomials.

Note that these matrices are invertible for generic values of q, in particular for $q = 0$. Hence, although this will not be used in the sequel, it is interesting to observe that at this point, we can define Hall–Littlewood type functions by

$$\tilde{H}_J(q) = \sum_I E_I^J(q) L_I \tag{60}$$

which interpolate between S^I (at $q = 1$) and a new kind of noncommutative Schur functions Σ_I at $q = 0$.

We will see in the next section that the last column gives the q-enumeration of permutation tableaux according to shape. Let us write down the precise statement in that case.

Proposition 3.9. *Let $c_I(q)$ be the coefficient of Ψ_I in $S^{1^n}(q)$. Let $e_I(q)$ be the coefficient of $L_I(q)$ in $S^{1^n}(q)$.*

Then,

$$e_I(q) = \sum_{J \preceq I} (-1/q)^{l(I)-l(J)} q^{-\mathrm{st}'(I,J)} c_J(q), \quad \textit{and} \tag{61}$$

$$c_{j_1,\dots,j_r}(q) = [r]_q^{j_1} [r-1]_q^{j_2} \cdots [2]_q^{j_{r-1}} [1]_q^{j_r}. \tag{62}$$

Proof. Straightforward from Theorem 3.6 and Eq. (58). □

We shall use the notation $\mathrm{QFact}_A(J) := c_J(q)$ in the sequel, regarding it as a generalized q-factorial defined for all compositions (the classical one coming from $J = (1^n)$).

3.8.2. A combinatorial lemma

We need to describe the q-product in the $L(q)$ basis. Our first objective will be to understand how $\mathrm{st}(I, K)$ can be related to $\mathrm{st}(I, J)$ and $\mathrm{st}(J, K)$ for all $K \succeq J \succeq I$.

Lemma 3.10. *Let $X = \{x_1 < \cdots < x_r\} \subseteq Z = \{z_1 < \cdots < z_m\}$ be two sets of positive integers. For an integer y, let*

$$\nu(y) = \#\{z \in Z \mid z \geqslant y\} + \#\{x \in X \mid x \leqslant y\}, \tag{63}$$

and for a set Y,

$$\nu(Y) = \sum_{y \in Y} \nu(y). \tag{64}$$

For $r \leqslant s \leqslant m$, let

$$\Sigma_s(X, Z) = \sum_{\substack{X \subseteq Y \subseteq Z \\ |Y| = s}} q^{\nu(Y)}. \tag{65}$$

Then,

$$\Sigma_s(X, Z) = q^{\nu(X) + (r+1)(s-r) + \binom{s-r}{2}} \begin{bmatrix} m-r \\ s-r \end{bmatrix}_q. \tag{66}$$

Proof. Let $Z/X = U = \{u_1 < \cdots < u_{m-r}\}$ and $\nu_i = \nu(u_i)$. Then $\nu_i = m - i + 1$, so that all ν_j are consecutive integers. By definition,

$$\begin{aligned} \Sigma_s(X, Z) &= \sum_{X \subseteq \{y_1 < \cdots < y_s\} \subseteq Z} q^{\nu(y_1) + \cdots + \nu(y_s)} \\ &= q^{\nu(X)} \sum_{k_1 < \cdots < k_{s-r}} q^{\nu_{k_1} + \cdots + \nu_{k_{s-r}}} \\ &= q^{\nu(X)} e_{s-r}\big(q^{\nu_1}, \dots, q^{\nu_{m-r}}\big), \end{aligned} \tag{67}$$

where $e_n(X)$ is the usual elementary symmetric function on the alphabet X. Thus,

$$\begin{aligned}\Sigma_s(X,Z) &= q^{\nu(X)} e_{s-r}\big(q^{r+1},\dots,q^m\big)\\ &= q^{\nu(X)} q^{(r+1)(s-r)} e_{s-r}\big(1,q,\dots,q^{m-r-1}\big)\\ &= q^{\nu(X)+(r+1)(s-r)+\binom{s-r}{2}} \begin{bmatrix} m-r\\ s-r\end{bmatrix}_q. \qquad \square \end{aligned} \tag{68}$$

For example, with $X=\{3,7\}$ and $Z=\{1,\dots,10\}$, one has

$$\Sigma_3(X,Z) = q^{\nu(3)+\nu(7)} \sum_{y\in Z/X} q^{\nu(y)} = q^{15}\big(q^{10}+q^9+\dots+q^4+q^3\big) = q^{18}\begin{bmatrix}8\\1\end{bmatrix}_q. \tag{69}$$

$$\Sigma_4(X,Z) = q^{15} e_2\big(q^3,\dots,q^{10}\big) = q^{21} e_2\big(1,\dots,q^7\big) = q^{22}\begin{bmatrix}8\\2\end{bmatrix}_q. \tag{70}$$

3.8.3. The q-product on the basis $L(q)$

Recall that the q-product on **Sym** is a nonassociative product but that $S^I(q)$ is the q-product on the parts of I, multiplied from right to left (see (38)).

Lemma 3.11. *For an integer p and a composition I,*

$$L_p(q)\star_q L_I(q) = \sum_{K\succeq p\triangleright I} q^{st(K,p\triangleright I)} \begin{bmatrix} m_K+p\\ p\end{bmatrix}_q \Psi_K, \tag{71}$$

where $m_K = \#\{k\in \mathrm{Des}(K) \mid k\geqslant p\}$.

Proof. Note that $L_p(q)=S_p$. The q-products $S_p\star_q \Psi_L$ are easily computed by means of Lemma 3.5. $\square$

We are now in a position to expand such q-products on the $L(q)$ basis:

Lemma 3.12. *For an integer p and a composition I, we have*

$$L_p(q)\star_q L_I(q) = \sum_{\substack{J\succeq p\triangleright I\\ j_1\geqslant p}} q^{st(J,p\triangleright I)+\binom{l(J)-l(I)}{2}-\binom{l(I)}{2}} \begin{bmatrix} l(I)+p-1\\ l(J)-1\end{bmatrix}_q L_J(q). \tag{72}$$

Proof. From Lemma 3.11, with $r=l(I)-1$, we have

$$\begin{aligned} L_p(q)\star_q L_I(q) &= \sum_{K\succeq p\triangleright I} q^{st(K,p\triangleright I)} \sum_{s\geqslant r} q^{sr} \begin{bmatrix} p-r\\ p-s\end{bmatrix}_q \begin{bmatrix} p+r\\ s\end{bmatrix}_q \Psi_K\\ &= \sum_{K\succeq p\triangleright I}\sum_{s\geqslant r} \begin{bmatrix} p+r\\ s\end{bmatrix}_q \left(q^{st(K,p\triangleright I)+rs} \begin{bmatrix} p-r\\ p-s\end{bmatrix}_q \Psi_K\right). \end{aligned} \tag{73}$$

Thanks to Lemma 3.10, noting that $\mathrm{st}(K,J)+\mathrm{st}(J,p\triangleright I) = \nu(\mathrm{Des}(J))$ with $X=\mathrm{Des}(I)$ and $Z=\mathrm{Des}(K)\cap[p,\infty]$, this is equal to

$$= \sum_{s \geqslant r} \begin{bmatrix} p+r \\ s \end{bmatrix}_q \sum_{\substack{J \succeq p \triangleright I \\ l(J)=s+1,\ j_1 \geqslant p}} q^{\mathrm{st}(J,p\triangleright I)+\binom{s-r}{2}-\binom{r+1}{2}} \sum_{K \succeq J} q^{\mathrm{st}(K,J)} \Psi_K$$

$$= \sum_{\substack{J \succeq p \triangleright I \\ j_1 \geqslant p}} q^{st(J,p\triangleright I)+\binom{l(J)-l(I)}{2}-\binom{l(I)}{2}} \begin{bmatrix} l(I)+p-1 \\ l(J)-1 \end{bmatrix}_q L_J(q). \qquad \square \tag{74}$$

For example,

$$L_2(q) \star_q L_{21}(q) = [3]L_{41}(q) + [3]L_{311}(q) + [3]L_{221}(q) + qL_{2111}(q), \tag{75}$$

$$L_2(q) \star_q L_{12}(q) = [3]L_{32}(q) + q[3]L_{311}(q) + [3]L_{212}(q) + q^2 L_{2111}(q), \tag{76}$$

$$\begin{aligned} L_3(q) \star_q L_{22}(q) = [4]L_{52} + q\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q L_{511} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q L_{412} + q^2[4]L_{4111} \\ + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q L_{322}(q) + q^2[4]L_{3211} + q[4]L_{3112} + q^4 L_{31111}. \end{aligned} \tag{77}$$

Since $S^J(q) = L_{j_1}(q) \star_q (L_{j_2}(q) \star_q (\cdots (L_{j_{r-1}}(q) \star_q L_{j_r}(q)) \cdots))$, formula (72) implies the following:

Corollary 3.13. *The coefficient $E_I^J(q)$ is in $\mathbb{N}[q]$.*

Corollary 3.14. *Recall that $e_I(q)$ is the coefficient of $L_I(q)$ in $S^{1^n}(q)$. Then, for any composition $I = (i_1, \ldots, i_r)$,*

$$\begin{aligned} &e_{1+i_1,i_2,\ldots,i_r}(q) = [r]_q e_I + \sum_{k=1}^{n} q^{k-1} e_{i_1,\ldots,i_k+i_{k+1},\ldots,i_r}(q), \\ &e_{1,i_1,i_2,\ldots,i_r}(q) = e_I(q). \end{aligned} \tag{78}$$

Conversely, this property and the trivial initial conditions determine completely the e_I.

Proof. This follows from the fact that $S^{1^n}(q) = L_1(q) \star_q (\cdots (L_1(q) \star_q L_1(q)) \cdots)$, by putting $p = 1$ into (72). $\square$

3.8.4. Towards a combinatorial interpretation of $E_I^J(q)$

In Theorem 3.17, we will give a combinatorial interpretation of the coefficients $E_I^J(q)$ expressing $S^J(q)$ in terms of the $L_I(q)$'s. But first we need a new combinatorial algorithm sending a permutation to a composition.

Let σ be a permutation in $\mathfrak{S}_n$. We compute a composition $\mathrm{LC}(\sigma)$ of n as follows.

- Consider the Lehmer code of its inverse $\mathrm{Lh}(\sigma)$, that is, the word whose ith letter is the number of letters of σ to the left of i and greater than i.
- Fix $S = \emptyset$ and read $\mathrm{Lh}(\sigma)$ from right to left. At each step, if the entry k is strictly greater than the size of S, add the (k-#(S))-th element of the sequence $[1, n]$ with the elements of S removed.

- The set S is the descent set of a composition C, and $\mathrm{LC}(\sigma)$ is the mirror image $\bar{C}$ of C.

For example, with $\sigma = (637124985)$, the Lehmer code of its inverse is $\mathrm{Lh}(\sigma) = (331240010)$. Then S is $\emptyset$ at first, then the set $\{1\}$ (second step), then the set $\{1,4\}$ (fifth step), then the set $\{1,4,2\}$ (eighth step). Hence C is $(1,1,2,5)$, so that $\mathrm{LC}(\sigma) = (5,2,1,1)$.

One can find in Appendix B the permutations of $\mathfrak{S}_3$ and $\mathfrak{S}_4$ arranged by rows according to their LC statistics and by columns according to their recoil compositions.

3.8.5. A left $\mathbf{Sym}_q$*-module*

Let $\sim$ be the equivalence relation on $\mathfrak{S}_n$ defined by $\sigma \sim \tau$ whenever $\mathrm{LC}(\sigma) = \mathrm{LC}(\tau)$. Let $\mathcal{M}$ be the quotient of **FQSym** [5] by the subspace

$$\mathcal{V} = \{\mathbf{F}_\sigma - \mathbf{F}_\tau \mid \sigma \sim \tau\}. \tag{79}$$

For a composition I, set $\kappa(I) = \mathrm{maj}(\bar{I})$, and for a permutation, $\kappa(\sigma) = \kappa(\mathrm{LC}(\sigma))$. Let $\mathcal{F}_I$ denote the equivalence class of $q^{\kappa(\sigma)}\mathbf{F}_\sigma$. Denote by $\circ_q$ be the q-product of **FQSym** inherited from **WQSym**. More precisely, if $\mathbf{F}'_\sigma = q^{\mathrm{inv}(\sigma)}\mathbf{F}_\sigma$ and $\phi_q(\mathbf{F}_\sigma) = \mathbf{F}'_\sigma$, then

$$\mathbf{F}'_\sigma \circ_q \mathbf{F}'_\tau = \phi_q(\mathbf{F}_\sigma \mathbf{F}_\tau). \tag{80}$$

This is the same structure as the one considered in [5]. In particular, in the basis $\mathbf{G}_\sigma = \mathbf{F}_{\sigma^{-1}}$, the product is given by the q-convolution

$$\mathbf{G}_\alpha \circ_q \mathbf{G}_\beta = \sum_{\substack{\gamma = u\cdots v \\ \mathrm{Std}(u)=\alpha,\ \mathrm{Std}(v)=\beta}} q^{\mathrm{inv}(\gamma)-\mathrm{inv}(\alpha)-\mathrm{inv}(\beta)}\mathbf{G}_\gamma. \tag{81}$$

Lemma 3.15. *The quotient vector space $\mathcal{M}$ is a left* **Sym***-module for the q-product of* **FQSym***, that is,*

$$F \equiv G \bmod \mathcal{V} \quad \Longrightarrow \quad S_p \circ_q F \equiv S_p \circ_q G \bmod \mathcal{V}. \tag{82}$$

Proof. Let $\sigma^{-1} \sim \tau^{-1} \in \mathfrak{S}_l$ and $n = p + l$. We need to compare the codes of the permutations appearing in the q-convolutions

$$U = \mathbf{G}_{12\cdots p} \circ_q \mathbf{G}_\sigma \quad \text{and} \quad V = \mathbf{G}_{12\cdots p} \circ_q \mathbf{G}_\tau. \tag{83}$$

For a subset $S = \{s_1 < s_2 < \cdots < s_p\}$ of $[n]$, let σ_S and τ_S be the elements of U and V whose prefix of length p is $s_1 s_2 \cdots s_p$. Then, the codes of σ_S and τ_S coincide on the first p positions, and are equivalent on the last l ones, so that $\sigma_S \sim \tau_S$. Moreover, σ_S and τ_S arise with the same power of q, so we have a module for the q-structure as well. $\square$

For example,

$$\begin{aligned}\mathbf{F}_{12} \circ_q q\mathbf{F}_{132} = {}& q\mathbf{F}_{12354} + q^2\mathbf{F}_{13254} + q^3\mathbf{F}_{13524} + q^4\mathbf{F}_{13542} + q^3\mathbf{F}_{31254} \\ &+ q^4\mathbf{F}_{31524} + q^5\mathbf{F}_{31542} + q^5\mathbf{F}_{35124} + q^6\mathbf{F}_{35142} + q^7\mathbf{F}_{35412}\end{aligned}$$

$$\begin{aligned}
&= \mathcal{F}_{41} + q\mathcal{F}_{41} + \mathcal{F}_{311} + \mathcal{F}_{221} + q^2\mathcal{F}_{41} \\
&\quad + q\mathcal{F}_{311} + q\mathcal{F}_{221} + q^2\mathcal{F}_{311} + q^2\mathcal{F}_{221} + q\mathcal{F}_{2111}. \qquad (84)\\
\mathbf{F}_{12} \circ_q q\mathbf{F}_{312} &= q\mathbf{F}_{12534} + q^2\mathbf{F}_{15234} + q^3\mathbf{F}_{15324} + q^4\mathbf{F}_{15342} + q^3\mathbf{F}_{51234} \\
&\quad + q^4\mathbf{F}_{51324} + q^5\mathbf{F}_{51342} + q^5\mathbf{F}_{53124} + q^6\mathbf{F}_{53142} + q^7\mathbf{F}_{53412} \\
&= \mathcal{F}_{41} + q\mathcal{F}_{41} + \mathcal{F}_{311} + \mathcal{F}_{221} + q^2\mathcal{F}_{41} \\
&\quad + q\mathcal{F}_{311} + q\mathcal{F}_{221} + q^2\mathcal{F}_{311} + q^2\mathcal{F}_{221} + q\mathcal{F}_{2111}. \qquad (85)\\
\mathbf{F}_{12} \circ_q q\mathbf{F}_{213} &= q\mathbf{F}_{12435} + q^2\mathbf{F}_{14235} + q^3\mathbf{F}_{14325} + q^4\mathbf{F}_{14352} + q^3\mathbf{F}_{41235} \\
&\quad + q^4\mathbf{F}_{41325} + q^5\mathbf{F}_{41352} + q^5\mathbf{F}_{43125} + q^6\mathbf{F}_{43152} + q^7\mathbf{F}_{43512} \\
&= \mathcal{F}_{41} + q\mathcal{F}_{41} + \mathcal{F}_{311} + \mathcal{F}_{221} + q^2\mathcal{F}_{41} \\
&\quad + q\mathcal{F}_{311} + q\mathcal{F}_{221} + q^2\mathcal{F}_{311} + q^2\mathcal{F}_{221} + q\mathcal{F}_{2111}. \qquad (86)
\end{aligned}$$

We have now:

Lemma 3.16. *The left q-product of a $\mathcal{F}_I$ by a complete function is given by* (72):

$$S_p \circ_q \mathcal{F}_I = \sum_{\substack{J \succeq p \triangleright I \\ j_1 \geqslant p}} q^{st(J, p\triangleright I) + \binom{l(J)-l(I)}{2} - \binom{l(I)}{2}} \begin{bmatrix} l(I)+p-1 \\ l(J)-1 \end{bmatrix}_q \mathcal{F}_J. \tag{87}$$

Proof. Let us first show that this is true at $q = 1$. Let σ be such that $\mathrm{LC}(\sigma^{-1}) = I$. By definition of LC, the permutations τ occurring in $\mathbf{G}_{12\cdots p} \circ_q \mathbf{G}_\sigma$ satisfy $\overline{\mathrm{LC}(\tau^{-1})} \succeq \overline{\mathrm{LC}(\sigma^{-1})}$, and the codes of those permutations have the form

$$s_1 s_2 \cdots s_p t_1 t_2 \cdots t_l, \tag{88}$$

where $t = t_1 t_2 \cdots t_l$ is the code of σ and $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_p$. The compositions J such that $l(J) - l(I)$ has a fixed value m will all be obtained by fixing the last m values $s_p, \ldots, s_{p-m+1}$ in a way depending on the code t, the first $p - m$ being allowed to be any weakly increasing sequence

$$s_1 \leqslant s_2 \leqslant \cdots \leqslant s_p \leqslant l(J) - 1, \quad \text{which leaves } \binom{p + l(I) - 1}{l(J) - 1} \text{ choices.} \tag{89}$$

Now, in the q-convolution $\mathbf{G}_{12\cdots p} \circ_q \mathbf{G}_\sigma$, these permutations τ occur with a coefficient $q^{\mathrm{inv}(\tau)-\mathrm{inv}(\sigma)}$, so that the coefficient of $\mathcal{F}_J$ is, up to a power of q, the q-binomial coefficient $\begin{bmatrix} p+l(I)-1 \\ l(J)-1 \end{bmatrix}_q$. By our choice of the normalization $\mathcal{F}_I = q^{\kappa(\sigma)}\mathbf{F}_\sigma$, this power of q is the same as in (72). □

As one can check on the previous examples, we have indeed

$$\mathcal{F}_2 \circ_q \mathcal{F}_{21} = (1 + q + q^2)\mathcal{F}_{41} + (1 + q + q^2)\mathcal{F}_{311} + (1 + q + q^2)\mathcal{F}_{221} + q\mathcal{F}_{2111}. \tag{90}$$

By Lemmas 3.16 and (72), the two bases $\mathcal{F}$ and $L(q)$ have the same multiplication formula, so that $E_I^J(q)$ is also the coefficient of $\mathcal{F}_I$ in the expansion of $S^J(q)$. Hence

Theorem 3.17. *Let I and J be two compositions of n. Let* $\mathrm{PP}(I,J)$ *be the set of permutations whose* LC *statistic is I and whose recoil composition is finer than J. Then,*

$$E_I^J(q) = q^{-\mathrm{maj}(\overline{\mathrm{LC}(\sigma)})} \sum_{\sigma \in \mathrm{PP}(I,J)} q^{\mathrm{inv}(\sigma)}. \tag{91}$$

3.9. The transition matrix $M(R(q), L(q))$

The last transition matrix which remains to be computed is the one from $R(q)$ to $L(q)$.

3.9.1. First examples

We have the following matrices for $n = 3, 4$:

$$RL_3 = M_3\big(R(q), L(q)\big) = \begin{pmatrix} 1 & . & . & . \\ . & 1+q & 1 & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix},$$

$$RL_4 = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ . & 1+q+q^2 & 1+q & . & 1 & q & . & . \\ . & . & 1+q & . & 1 & . & . & . \\ . & . & q & 1+q+q^2 & . & 1+q & 1 & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & 1+q & 1 & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{pmatrix}.$$

3.9.2. Combinatorial interpretation

The coefficient of $L_I(q)$ in $R_J(q)$ will be denoted by $F_I^J(q)$.

From the characterization in Theorem 3.17 of $M(S(q), L(q))$ in terms of permutations we obtain:

Theorem 3.18. *Let I and J be two compositions. Let* $\mathrm{PP}'(I,J)$ *be the set of permutations whose* LC *statistic is I and whose recoil composition is J. The coefficient F_I^J of $L_I(q)$ in the expansion of $R_J(q)$ is given by*

$$q^{-\mathrm{maj}(\overline{\mathrm{LC}(\sigma)})} \sum_{\sigma \in \mathrm{PP}'(I,J)} q^{\mathrm{inv}(\sigma)}. \tag{92}$$

4. The PASEP and type A permutation tableaux

Permutation tableaux (of type A) are certain fillings of Young diagrams with 0's and 1's which are in bijection with permutations (see [38] for two bijections). They are a distinguished subset of Postnikov's (type A) ⅃-diagrams [35], which index cells of the totally nonnegative part of the

Grassmannian. Additionally, permutation tableaux are of interest as they are closely connected to a model from statistical physics called the partially asymmetric exclusion process (PASEP) [4].

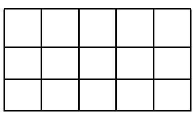
The PASEP with n sites can be defined as a Markov chain on 2^n states – all words of length n in 0 and 1 – where a 1 in the ith position represents a particle in the ith position of a one-dimensional lattice of n sites. The particles may hop left and right and in and out of the lattice, subject to the condition that no two particles may occupy the same site. Provided that this is satisfied, a new particle may enter the lattice at the left with probability $\frac{\alpha}{n+1}$, a particle may exit the lattice to the right with probability $\frac{\beta}{n+1}$, and a particle may hop to an empty site to its right or left with probabilities $\frac{1}{n+1}$ and $\frac{q}{n+1}$ respectively. See [4] for more details.

When $\alpha = \beta = 1$, the connection between the PASEP and permutation tableaux can be described as follows. Given a state τ of the PASEP with n sites, let $\lambda(\tau)$ denote the Young diagram with semiperimeter $n+1$, whose southeast border (read from northeast to southwest) is the sequence of steps $V - f(\tau_1) - f(\tau_2) - \cdots - f(\tau_n)$, where $f(1)$ is a vertical step V, and $f(0)$ is a horizontal step H. Let Z_n denote the q-enumeration of all permutation tableaux of semiperimeter $n+1$. By results of [4], the steady state probability that the PASEP with n sites is in configuration τ is equal to the q-enumeration of permutation tableaux of shape $\lambda(\tau)$ divided by Z_n. Moreover, Z_n is the *partition function* of the PASEP.

In this section we will give an explicit formula for the q-enumeration of permutation tableaux of a given shape. So in particular this is an explicit formula for the steady state probability of each state of the PASEP. Additionally, by results of [38], this formula counts permutations with a given set of weak excedances according to *crossings*; it also counts permutations with a given set of *descent bottoms* according to occurrences of the pattern 2–31. A *weak excedance* of a permutation π is a value $\pi(i)$ such that $\pi(i) \geqslant i$. For a descent i of π, that is, $\pi_i > \pi_{i+1}$, we say that π_i is a descent top, and π_{i+1} a descent bottom [38].

4.1. *Permutation tableaux*

Regard the following $(k, n-k)$ rectangle (here $k = 3$ and $n = 8$)



(93)

as a poset $Q^A_{k,n}$: the elements of the poset are the boxes, and box b is less than b' if b is southwest of b'. We then define a *type A Young diagram* contained in a $(k, n-k)$ rectangle to be an order ideal in the poset $Q^A_{k,n}$. This corresponds to the French notation for representing Young diagrams. We will sometimes refer to such a Young diagram by the partition λ given by the lengths of the rows of the order ideal. Note that we allow partitions to have parts of size 0.

As in [38], we define a type A *permutation tableau* $\mathcal{T}$ to be a type A Young diagram Y_λ together with a filling of the boxes with 0's and 1's such that the following properties hold:

(1) Each column of the diagram contains at least one 1.
(2) There is no 0 which has a 1 below it in the same column *and* a 1 to its left in the same row.

We call such a filling a *valid* filling of Y_λ. Here is an example of a type A permutation tableau.

$$\begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 1 & & \\ \hline 1 & 0 & 1 & 1 & \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline \end{array} \tag{94}$$

Note that if we forget the requirement (1) in the definition of type A permutation tableaux then we recover the description of a (type A) Ⅎ-diagram [35], an object which represents a cell in the totally nonnegative part of a Grassmannian. In that case, the total number of 1's corresponds to the dimension of the cell.

We define the *rank* $\operatorname{rank}(\mathcal{T})$ of a permutation tableau (of type A) $\mathcal{T}$ with k columns to be the total number of 1's in the filling minus k. (We subtract k since there must be at least k 1's in a valid filling of a tableau with k columns.)

4.2. Enumeration of permutation tableaux by shape

Starting from a partition with k rows and $n-k$ columns, one encodes it as a composition $I=(i_1,\dots,i_k)$ of n as follows: i_1-1 is the number of columns of length k, i_2-1 is the number of columns of length $k-1$, and $\dots$, i_k-1 is the number of columns of length 1.

Let $\ell(I)$ denote the number of parts of I. Then the number PT_I^A of permutation tableaux of shape corresponding to I is given by a simple formula coming from combinatorics of noncommutative symmetric functions. Indeed, according to [39, Proposition 9.2],

$$L_1^n=\sum_{I\vDash n} g_I\Psi_I, \tag{95}$$

where

$$g_I=\prod_{k=1}^{l(I)}\bigl(l(I)-k+1\bigr)^{i_k}. \tag{96}$$

Hence, the coefficient e_J of

$$L_1^n=\sum_{J\vDash n} e_J L_J \tag{97}$$

is given by

$$e_J=\sum_{I\succeq J}(-1)^{l(I)-l(J)}\prod_{k=1}^{l(I)}\bigl(l(I)-k+1\bigr)^{i_k}. \tag{98}$$

Moreover, since

$$L_1^n=\sum_{I\vDash n} R_I, \tag{99}$$

from [14], Theorem 5.1, we see that e_I is the number of permutations such that $\mathrm{GC}(\sigma)=I$ (the definition of GC is recalled at the beginning of Appendix A). Finally, permutation tableaux of a given shape are in bijection with permutations with given descent bottoms [38]. It is easy to check that the set of descent bottoms of σ coincides with $\mathrm{Des}(\mathrm{GC}(\bar{\sigma}))$, where $\bar{\sigma} = (n - \sigma_n + 1)\cdots(n - \sigma_1 + 1)$ is the reverse complement of σ. Hence, the number of permutations such that $\mathrm{GC}(\sigma) = I$ is also the number of permutation tableaux of shape I.

Theorem 4.1.

$$\mathrm{PT}_I^A = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)} \mathrm{Fact}(J), \tag{100}$$

where the sum is over the compositions J coarser than I and where Fact *is defined by*

$$\mathrm{Fact}(j_1, \dots, j_p) := p^{j_1}(p-1)^{j_2} \cdots 2^{j_{p-1}} 1^{j_p}. \tag{101}$$

For example, with $I = (3, 4, 1)$, we get

$$\mathrm{PT}_{341}^A = 3^3 2^4 1^1 - 2^7 1^1 - 2^3 1^5 + 1^8 = 297. \tag{102}$$

4.2.1. q-enumeration of permutation tableaux according to their shape

In this section, we make the connection between the coefficients $e_I(q)$ previously seen, and the q-enumeration of permutation tableaux. Recall that $e_I(q)$ is the coefficient of $L_I(q)$ in $S^{1^n}(q)$. We saw in Corollary 3.14 that for all compositions $I = (i_1, \dots, i_r)$, the following hold:

- $e_{(1,i_1,i_2,\dots,i_r)}(q) = e_I(q)$.
- $e_{(1+i_1,i_2,\dots,i_r)}(q) = [r]_q e_I + \sum_{k=1}^{r-1} q^{k-1} e_{(i_1,\dots,i_k+i_{k+1},\dots,i_r)}(q)$.

It is possible to transform this result into a q-enumeration of permutation tableaux by their rank. Let

$$\mathrm{PT}_I^A(q) := \sum_T q^{\mathrm{rank}(T)}, \tag{103}$$

where the sum is over all permutation tableaux whose shape corresponds to I.

The following result generalizes Theorem 4.1. Its proof follows directly from Proposition 3.9, Corollary 3.14, and Lemma 4.5 below.

Theorem 4.2. *Let I be a composition. Then,*

$$\mathrm{PT}_I^A(q) = e_I(q) = \sum_{J \preceq I} (-1/q)^{l(I)-l(J)} q^{-\mathrm{st}'(I,J)} \mathrm{QFact}_A(J), \tag{104}$$

where QFact_A *is recalled to be*

$$\mathrm{QFact}_A(j_1, \dots, j_p) := [p]_q^{j_1} [p-1]_q^{j_2} \cdots [2]_q^{j_{p-1}} [1]_q^{j_p}. \tag{105}$$

By [4], Theorem 4.2 gives an explicit formula for the steady state probabilities in the PASEP with n sites.

Corollary 4.3. *Recall the notation of Theorem* 4.2. *Let I be a composition of $n+1$, and let Z_n denote the q-enumeration of all permutation tableaux of semiperimeter $n+1$. Let τ denote the state of the PASEP in which all sites of* $\mathrm{Des}(I)$ *are occupied by a particle and all sites of* $[n-1]\setminus\mathrm{Des}(I)$ *are empty. Then the probability that in the steady state, the PASEP is in state τ, is*

$$\frac{\sum_{J\preceq I}(-1/q)^{l(I)-l(J)}q^{-\mathrm{st}'(I,J)}\,\mathrm{QFact}_A(J)}{Z_n}.$$

By [38], this is also an explicit formula enumerating permutations with a fixed set of *weak excedances* according to the number of *crossings*; equivalently, an explicit formula enumerating permutations with a fixed set of *descent bottoms* according to the number of occurrences of the *generalized pattern* 2–31.

More specifically, let I be a composition of $n+1$, let $DB(I)$ be the descent set of the reverse composition of I, and let $W(I)=\{1\}\cup\{1+DB(I)\}$. Here $1+DB(I)$ denotes the set obtained by adding 1 to each element of $DB(I)$. If σ is a permutation, let $(2\text{–}31)\sigma$ denote the number of occurrences of the pattern 2–31 in σ, and let $cr(\sigma)$ denote the number of *crossings* of σ. Let $T_I(q)=\sum_\sigma q^{(2-31)\sigma}$ be the sum over all permutations in S_{n+1} whose set of descent bottoms is $DB(I)$. And let $T'_I(q)=\sum_\sigma q^{cr(\sigma)}$ be the sum over all permutations in S_{n+1} whose set of weak excedances is $W(I)$.

Corollary 4.4.

$$T_I(q)=T'_I(q)=\sum_{J\preceq I}(-1/q)^{l(I)-l(J)}q^{-\mathrm{st}'(I,J)}\,\mathrm{QFact}_A(J).$$

For example, with $I=(3,4,1)$, the compositions coarser than I are $(3,4,1)$, $(7,1)$, $(3,5)$, and (8), so we get

$$\begin{aligned}\mathrm{PT}^A_{341}(q)&=\frac{1}{q^2}\left(\frac{[3]_q^3[2]_q^4}{q}-\frac{[2]_q^7}{q}-\frac{[2]_q^3}{1}+\frac{1}{1}\right)\\&=q^7+7q^6+24q^5+52q^4+76q^3+75q^2+47q+15.\end{aligned}\tag{106}$$

The descent set $\mathrm{Des}(I)$ of I is $\{3,7\}$, which corresponds to the following state of the PASEP: $\tau=\bullet\circ\circ\circ\bullet\circ\circ$. Therefore the probability that in the steady state, the PASEP is in state τ, is $\frac{q^7+7q^6+24q^5+52q^4+76q^3+75q^2+47q+15}{Z_7}$.

The polynomial $q^7+7q^6+24q^5+52q^4+76q^3+75q^2+47q+15$ also enumerates the permutations in S_8 with set of descent bottoms $\{1,5\}$ according to occurrences of the pattern 2–31. And it enumerates permutations in S_8 with weak excedances in positions $\{1,2,6\}$ according to crossings.

The reader might want to compare (104) with (61).

Lemma 4.5. *Let $I=(i_1,\dots,i_r)$ be a composition. Then*

$$\mathrm{PT}^A_{(1,i_1,i_2,\dots,i_r)}(q)=\mathrm{PT}^A_I(q),\tag{107}$$

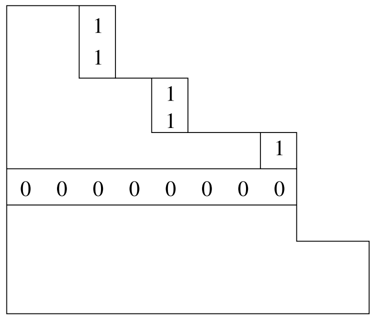


Fig. 1.

$$\mathrm{PT}^A_{(1+i_1,i_2,\dots,i_r)}(q) = [r]_q\mathrm{PT}^A_I + \sum_{k=1}^{n} q^{k-1}\mathrm{PT}^A_{(i_1,\dots,i_k+i_{k+1},\dots,i_r)}(q). \tag{108}$$

Proof. First note that $PT^A_{(1,i_1,i_2,\dots,i_r)}(q) = PT^A_I(q)$: this just says that the q-enumeration of permutation tableaux of shape λ is the same as the q-enumeration of permutation tableaux of shape λ', where λ' is obtained from λ by adding a row of length 0.

Therefore we just need to prove the second equality. Let $\lambda = (\lambda_1,\dots,\lambda_r)$ be a partition. Then, in terms of partitions, the statement translates as:

$$\mathrm{PT}^A_\lambda(q) = [r]PT^A_{(\lambda_1-1,\lambda_2-1,\dots,\lambda_r-1)}(q) + \sum_{k=1}^{r} q^{k-1}PT^A_{(\lambda_1,\dots,\lambda_{r-k},\widehat{\lambda_{r-k+1}},\lambda_{r-k+2}-1,\dots,\lambda_r-1)}(q), \tag{109}$$

where $\widehat{\lambda_{r-k+1}}$ means that this part has been removed.

To this aim, we need to introduce the notion of a *restricted* zero. We say that a zero in a tableau is *restricted* if there is a 1 below it in the same column. Note that every entry to the left of and in the same row as the restricted zero must also be zero.

We will prove the recurrence by examining the various possibilities for the set S of r boxes of the Young diagram λ which are rightmost in their row. We will partition (most of) the permutation tableaux with shape λ based on the position of the highest restricted zero among S.

We will label rows of the Young diagram from top to bottom, from 1 to r. Consider the set of tableaux obtained via the following procedure: choose a row k for $1 \leqslant k \leqslant r-1$, and fill it entirely with 0's. Also fill each box of S in row ℓ for any $\ell > k$ with a 1. Now ignore row k and the filled boxes of S, and fill the remaining boxes (which can be thought of as boxes of a partition of shape $\lambda' := (\lambda_1,\dots,\lambda_{k-1},\widehat{\lambda_k},\lambda_{k+1}-1,\dots,\lambda_r-1))$ in any way which gives a legitimate permutation tableau of shape λ' (see Fig. 1). Note that if we add back the ignored boxes, we will increase the rank of the first tableau by $k-1$. So the q-enumeration of the tableaux under consideration is exactly $\sum_{k=1}^r q^{k-1}PT_{(\lambda_1,\dots,\lambda_{r-k},\widehat{\lambda_{r-k+1}},\lambda_{r-k+2}-1,\dots,\lambda_r-1)}(q)$.

Let us denote the columns of λ which contain a north-east corner of the Young diagram λ as $c_1,\dots,c_h$; we will call them *corner columns*. Denote the lengths of those columns by $C_1,\dots,C_h$, so $C_1 > \dots > C_h$. And denote the differences of their lengths by $d_1 := C_1 - C_2,\dots,d_{h-1} := C_{h-1} - C_h, d_h := C_h$.

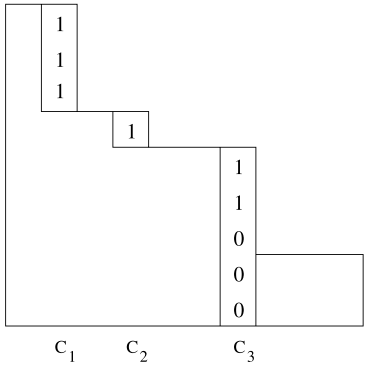


Fig. 2.

Clearly, our procedure constructs all permutation tableaux of shape λ with the following description: at least one box of S is a restricted zero. Furthermore, if we choose the restricted zero of S (say in box b) which is in the lowest row (say row k), then every box of S in a row above k is filled with a 1. Equivalently, each corner column c_j left of b has its top d_j boxes filled with 1's, and contains at least $d_j + 1$ ones total; and the corner column containing b contains at least $d + 1$ ones total, where d is the number of boxes above b in the same column.

The permutation tableaux of shape λ which this procedure has *not constructed* are those tableaux such that either no box of S is a restricted zero, or else there *is* a box of S which is a restricted zero. Let b denote the lowest such box. The condition that all boxes of S above b must be 1's is violated. Let W denote this set of tableaux.

The following construction gives rise to all permutation tableaux in W (see Fig. 2). Choose a corner column c_j and a number m such that $1 \leqslant m \leqslant d_j$. Fill the top m boxes of c_j with 1's and the remaining boxes with 0's. For each $i < j$, fill the top d_i boxes of column c_i with 1's. Now ignore the boxes that have been filled, and choose any filling of the remaining boxes – which form a partition of shape $\lambda'' := (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_r - 1)$ – which gives a legitimate permutation tableau of shape λ''. Note that adding back the boxes we had ignored will add $d_1 + \cdots + d_{h-1} + m$ to the rank of the tableau of shape λ'. Since the quantity $d_1 + \cdots + d_{h-1} + m$ can range between 0 and $r - 1$, the rank of the tableaux in W is $[r]PT_{(\lambda_1-1,\lambda_2-1,\ldots,\lambda_r-1)}(q)$. $\square$

Note that we could give an alternative (direct) proof of Theorem 4.2 by using the following recurrences for permutation tableaux (which had been observed in [38]). See Fig. 3 for an illustration of the second recurrence.

Lemma 4.6. *The following recurrences for type A permutation tableaux hold.*

- $PT^A_{(i_2,i_3,\ldots,i_n)}(q) = PT^A_{(1,i_2,i_3,\ldots,i_n)}(q)$.
- $PT^A_{(i_1,i_2,\ldots,i_n)}(q) = qPT^A_{(i_1-1,i_2+1,i_3,\ldots,i_n)}(q) + PT^A_{(i_1-1,i_2,\ldots,i_n)}(q) + PT^A_{(1,i_1+i_2-1,i_3,\ldots,i_n)}(q)$.

5. Permutation tableaux and enumeration formulas in type B

One can also define [18] type B ⅃-diagrams and permutation tableaux, where the Type B ⅃-diagrams index cells in the odd-orthogonal Grassmannian, and type B permutation tableaux are in bijection with signed permutations. In this section, we will enumerate permutation tableaux

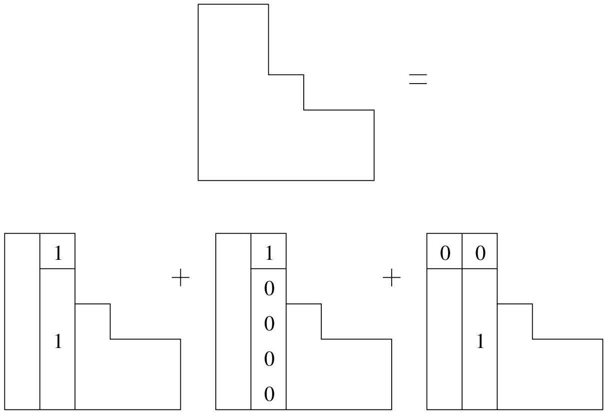
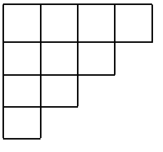


Fig. 3.

of type B of a fixed shape, according to rank. This formula can be given an interpretation in terms of signed permutations.

To define type B_n Young diagrams, regard the following shape



(110)

as representing a poset Q_n^B (here $n = 4$): the elements of the poset are the boxes, and box b is less than b' if b is southwest of b'. We then define a *type B_n Young diagram* to be an order ideal in the poset Q_n^B.

As in [18], we define a type B *permutation tableau* $\mathcal{T}$ to be a type B Young diagram Y_λ together with a filling of the boxes with 0's and 1's such that the following properties hold:

(1) Each column of the diagram contains at least one 1.
(2) There is no 0 which has a 1 below it in the same column *and* a 1 to its left in the same row.
(3) If a diagonal box contains a 0, every box in that row must contain a 0.

Here is an example of a type B permutation tableau.

$$\begin{array}{|c|c|c|} \cline{1-1} 1 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline 1 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array} \tag{111}$$

Note that if we forget requirement (1) in the definition of a type B permutation tableaux then we recover the description of a type B ⅃-diagram [18], an object which represents a cell in the totally nonnegative part of an odd orthogonal Grassmannian.

As before, we define the *rank* $\operatorname{rank}(\mathcal{T})$ of a permutation tableau $\mathcal{T}$ (of type B) with k columns to be the total number of 1's in the filling minus k.

Starting from a type B Young diagram Y_λ inside a staircase of height $n+1$, we encode it as a composition of n as follows. If k is the width of the widest row of Y_λ, then $I = (i_1, \ldots, i_{k+1})$ is defined by: $i_1 + 1$ is the number of rows of length k, i_2 is the number of rows of length $k-1, \ldots, i_{k+1}$ is the number of rows of length 0.

We now explain how to enumerate type B permutation tableaux of a fixed shape according to their rank.

Define QFact_B by

$$\begin{aligned}\mathrm{QFact}_B(j_1, \ldots, j_p) &:= \mathrm{QFact}_A(j_1, \ldots, j_p) \prod_{t=1}^{p-1} \left(1 + q^t\right) \\ &= [p]_q^{j_1} [p-1]_q^{j_2} \ldots [2]_q^{j_{p-1}} [1]_q^{j_p} \prod_{t=1}^{p-1} \left(1 + q^t\right). \qquad (112)\end{aligned}$$

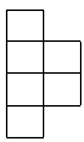
Theorem 5.1. *Let I be a composition.*

$$\mathrm{PT}_I^B(q) = \sum_{J \preceq I} (-1/q)^{l(I)-l(J)} q^{-\mathrm{st}'(I,J)}\, \mathrm{QFact}_B(J) \qquad (113)$$

where p is the length of J.

Note that the formula enumerating type B permutation tableaux is very similar to the formula enumerating type A permutation tableaux.

As an example, suppose we want to enumerate according to rank the type B permutation tableaux that have the following shape:



(114)

We take $n = 4$ (we would get the same answer for any $n > 4$), and $k = 2$ since the widest row has width 2. Then the corresponding composition is $I = (1, 2, 0)$. We then get

$$\begin{aligned}\mathrm{PT}_{(1,2,0)}^B(q) &= q^{-2}\left(q^{-1}[3]_q[2]_q^2(1+q)\left(1+q^2\right) - q^{-1}[2]_q^3(1+q) - [2]_q(1+q) + 1\right) \\ &= q^4 + 4q^3 + 8q^2 + 10q + 6.\end{aligned}$$

We will prove Theorem 5.1 directly: we first prove some recurrences for type B permutation tableaux, and then prove that the formula in Theorem 5.1 satisfies the same recurrences.

Lemma 5.2. *The following recurrences for type B permutation tableaux hold.*

$$\mathrm{PT}_{(0,i_2,i_3,\ldots,i_k)}^B(q) = \mathrm{PT}_{(i_2,i_3,\ldots,i_k)}^B(q), \qquad (115)$$

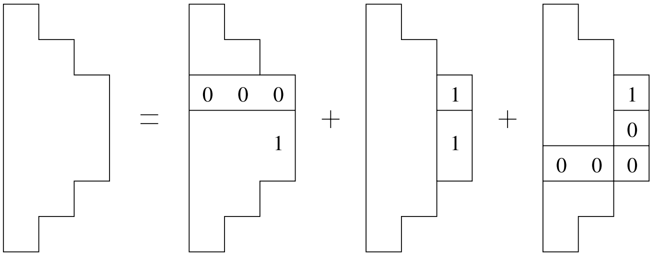


Fig. 4.

$$\mathrm{PT}^B_{(i_1,i_2,i_3,\ldots,i_k)}(q) = \mathrm{PT}^B_{(i_1-1,i_2,i_3,\ldots,i_k)}(q) + q\,PT^B_{(i_1-1,i_2+1,i_3,\ldots,i_k)}(q) + PT^B_{(0,i_1+i_2-1,i_3,\ldots,i_k)}(q). \tag{116}$$

Proof. The first recurrence says that enumerating permutation tableaux of a shape which has a unique row of maximal width is the same as enumerating permutation tableaux of the shape obtained from the first shape by deleting the rightmost column. This is clear, since the rightmost column will have only one box which must be filled with a 1.

To see that the second recurrence holds, see Fig. 4. Consider the topmost box b of the rightmost column of an arbitrary type B permutation tableau of shape corresponding to $(i_1,\ldots,i_k)$. Since $i_1 > 0$, the rightmost column has at least two boxes. If b contains a 0, then by definition of type B permutation tableaux, there is a 1 below it in the same column – which implies that the entire row containing b must be filled with 0's. We can delete that entire row and what remains will be a type B permutation tableau (of smaller shape).

If b contains a 1 and there is another 1 in the same column, then we can delete the box b and what remains will be a type B permutation tableau.

If b contains a 1 and there is no other 1 in the same column, then let b' denote the bottom box of that column. By the definition of type B permutation tableau, the entire row of b' is filled with 0's. If we delete the entire row of b' and every box below and in the same column as b, then what remains will be a type B permutation tableau. $\square$

Now we prove Theorem 5.1.

Proof of the theorem.

Let

$$g_I(q) := \sum_{J \preceq I} (-1/q)^{\ell(I)-\ell(J)} q^{-\mathrm{st}'(I,J)}\, \mathrm{QFact}_B(J). \tag{117}$$

We want to prove that $\mathrm{PT}^B_I(q) = g_I(q)$. We claim that it is enough to prove the following two facts:

(1) $g_{(0,i_2,i_3,\ldots,i_n)}(q) = g_{(i_2,i_3,\ldots,i_n)}(q)$,

(2) $g_{(i_1,i_2,\ldots,i_n)}(q) = q g_{(i_1-1,i_2+1,i_3,\ldots,i_n}(q) + g_{(i_1-1,i_2,i_3,\ldots,i_n)}(q) + g_{(0,i_1+i_2-1,i_3,\ldots,i_n)}(q)$ when $i_1 > 0$.

By Lemma 5.2, both of these recurrences are true for $PT_I^B(q)$. And the two recurrences together clearly determine $g_I(q)$ for any composition I, which is why it suffices to prove these recurrences.

Consider the first recurrence. To prove it, we will pair up the terms that occur in

$$g_{0,i_2,\dots,i_n}(q) := \sum_{J \preceq I} (-1/q)^{\ell(I)-\ell(J)} q^{-\mathrm{st}'(I,J)}\, \mathrm{QFact}_B(J), \tag{118}$$

pairing each composition of the form $J := (0, j_1, j_2, \dots, j_r)$ with the composition $J' := (j_1, j_2, \dots, j_r)$.

Note that $\ell(J) = \ell(J') + 1$ and $\mathrm{st}'(I, J) = \mathrm{st}'(I, J') + 1$. Also

$$\mathrm{QFact}_B(0, j_1, j_2, \dots, j_r) = \mathrm{QFact}_B(j_1, j_2, \dots, j_r)\big(1 + q^r\big) \tag{119}$$

so that

$$\mathrm{QFact}_B(0, j_1, j_2, \dots, j_r) - \mathrm{QFact}_B(j_1, j_2, \dots, j_r) = q^r\, \mathrm{QFact}_B(j_1, j_2, \dots, j_r). \tag{120}$$

And now it follows from the fact that

$$\mathrm{st}'\big((0, I), (0, J)\big) = q^{r-1}\mathrm{st}'(I, J), \tag{121}$$

that the contribution to $g_{0,i_2,\dots,i_n}(q)$ by the pair of compositions J and J' is exactly the contribution to $g_{i_2,\dots,i_n}(q)$ by the composition $(j_1, \dots, j_r)$. So $g_{(0,i_2,i_3,\dots,i_n)}(q) = g_{(i_2,i_3,\dots,i_n)}(q)$.

Now let us turn our attention to the second recurrence. We prove the second recurrence by showing that each term of $g_{(i_1,\dots,i_n)}(q)$ comes from either one term each from $qg_{(i_1-1,i_2+1,i_3,\dots,i_n)}(q)$ and $g_{(i_1-1,i_2,\dots,i_n)}(q)$, or one term each from $qg_{(i_1-1,i_2+1,i_3,\dots,i_n)}(q)$ and $g_{(i_1-1,i_2,\dots,i_n)}(q)$ and two terms from $g_{(0,i_1+i_2-1,i_3,\dots,i_n)}(q)$.

Let us denote the relevant compositions by $I := (i_1, \dots, i_n)$, $I' := (i_1 - 1, i_2 + 1, i_3, \dots, i_n)$, $I'' := (i_1 - 1, i_2, \dots, i_n)$ and $I''' := (0, i_1 + i_2 - 2, i_3, \dots, i_n)$.

First, consider the terms of $g_{(i_1,\dots,i_n)}(q)$ corresponding to compositions J such that the first part of J is i_1, i.e., J has the form $(i_1, j_2, j_3, \dots, j_r)$. Let us compare this term to the terms of $qg_{(i_1-1,i_2+1,i_3,\dots,i_n)}(q)$ and $g_{(i_1-1,i_2,\dots,i_n)}(q)$ corresponding to the partitions $J' := (i_1 - 1, j_2 + 1, j_3, \dots, j_r)$ and $J'' := (i_1 - 1, j_2, j_3, \dots, j_r)$, respectively. All three terms have the same sign and the same st': $\mathrm{st}'(I, J) = \mathrm{st}'(I', J') = \mathrm{st}'(I'', J'')$. And now it is easy to see that $q\, \mathrm{QFact}_B(J') + \mathrm{QFact}_B(J'') = \mathrm{QFact}_B(J)$:

$$\begin{aligned}
&q[r]^{i_1-1}[r-1]^{j_2+1}[r-2]^{j_3}\cdots + [r]^{i_1-1}[r-1]^{j_2}[r-2]^{j_3}\cdots \\
&\quad = \big(q[r-1]+1\big)\big([r]^{i_1-1}[r-1]^{j_2}[r-2]^{j_3}\cdots\big) \\
&\quad = [r]^{i_1}[r-1]^{j_2}[r-2]^{j_3}\cdots.
\end{aligned} \tag{122}$$

Note that all terms contain the extra factor $\prod_{t=1}^{r-1}(1 + q^t)$. Therefore the term corresponding to J is equal to the sum of the terms corresponding to J' and J''.

Now consider each term of $g_{(i_1,\dots,i_n)}(q)$ which corresponds to a composition J such that the first part of J is *not* i_1, i.e., J has the form $(j_1, j_2, j_3, \dots, j_r)$ where $j_1 = i_1 + i_2 + \cdots + i_k$ where $k \geqslant 2$. Let us compare this to the following four terms: the term of $qg_{(i_1-1,i_2+1,i_3,\dots,i_n}(q)$

corresponding to the composition $J' := J$; the term of $g_{(i_1-1,i_2,\dots,i_n)}(q)$ corresponding to the composition $J'' := (j_1 - 1, j_2, \dots, j_r)$; and the two terms of $g_{(0,i_1+i_2-1,i_3,\dots,i_n)}(q)$ corresponding to the compositions $J''' := (0, j_1 - 1, j_2, \dots, j_r)$ and $J^{(4)} := J''$. Note that the terms corresponding to J, J', J'', and $J^{(4)}$ have the same sign, while the term corresponding to J''' has the opposite sign. And all five terms have the same st' statistic. The quantity QFact_B is nearly the same for every term, and if we divide each term by $\mathrm{QFact}_B(J'')$, it remains to verify the equation: $[r+1]_q = q[r+1]_q + 1 - (1+q^{r+1}) + 1$. This is clearly true.

We have now accounted for all terms involved in the recurrence. This completes the proof of the theorem. □

It is very likely that colored Hopf algebra analogues of **WQSym**, **FQSym**, **Sym** already defined in [28,34,29,2,33,32] could be used to justify the q-enumeration of permutation tableaux of type B. Based on preliminary calculations, we believe that the type B analogue of the matrices from Section 3.8.1 are given by computing the transition matrix between the S^I and two new bases Ψ_I^B and $L^B(q)$. Here

$$\Psi_I^B := \frac{\Psi_I}{\prod_{i=2}^{r}(1+q^{i-1})} \tag{123}$$

and $L^B(q)$ is defined by having $M_{L(q),\Psi}$ as transition matrix from the Ψ^B. Note also that this interpretation would immediately generalize to colored algebras with any number of colors and not only to two colors.

Acknowledgments

This work has been partially supported by Agence Nationale de la Recherche, grant ANR-06-BLAN-0380. The authors would also like to thank the contributors of the MuPAD project, and especially those of the Combinat package, for providing the development environment for this research (see [15] for an introduction to MuPAD-Combinat).

Appendix A. Conjectures

We define the *descent tops* (also called the *Genocchi descent set*) of a permutation $\sigma \in S_n$ as $\mathrm{GDes}(\sigma) := \{i \in [2,n] \mid \sigma(j) = i \Rightarrow \sigma(j+1) < \sigma(j)\}$. In other words, $\mathrm{GDes}(\sigma)$ is the set of values of the descents of σ. We also define the *Genocchi composition of descents* $\mathrm{GC}(\sigma)$ as the integer composition I of n whose descent set is $\{d-1 \mid d \in \mathrm{GDes}(\sigma)\}$.

From Theorem 5.1 of [14], it is easy to see that $E_I^J(1)$ is equal to the number of packed words w such that

$$\mathrm{GC}\big(\mathrm{Std}(w)\big) = I \quad \text{and} \quad \mathrm{ev}(w) = J. \tag{124}$$

So $E_I^J(q)$ is the generating function of a statistic in q over this set of words. We propose the following conjecture:

Conjecture 6.1. *Let I and J be compositions of n and let $W''(I,J)$ be the set of packed words w such that*

$$\mathrm{GC}\big(\mathrm{Std}(w)\big) = I \quad \textit{and} \quad \mathrm{ev}(w) = J. \tag{125}$$

Then

$$E_I^J(q) = \sum_{w \in W''(I,J)} q^{\mathrm{totg}(w)}, \tag{126}$$

where totg *is the number of occurrences of the patterns* 21–1 *and* 31–2 *in* w.

For example, the coefficient $2 + 2q + q^2$ in row (3, 1) and column (2, 1, 1) comes from the fact that the five words 1132, 1231, 1312, 2311, and 3112 respectively have 0, 0, 1, 1, and 2 occurrences of the previous patterns.

There should exist a connection between the sinv statistic and the pattern counting on special packed words but we have not been able to find it.

Note that packed words w are in bijection with pairs (σ, J) where σ is a permutation and J a composition finer than the recoil composition of σ. Since the patterns 31–2 in $\mathrm{Std}(w)$ come from patterns 21–1 or 31–2 in w, Conjecture 6.1 is equivalent to

Conjecture 6.2. *Let I and J be compositions of n and let $P''(I, J)$ be the set of permutations σ such that*

$$\mathrm{GC}(\sigma) = I \quad \textit{and} \quad \mathrm{DC}\big(\sigma^{-1}\big) \preceq J. \tag{127}$$

Then

$$E_I^J(q) = \sum_{w \in P'(I,J)} q^{\mathrm{tot}(\sigma)}, \tag{128}$$

where $\mathrm{tot}(\sigma)$ *is the number of occurrences of the pattern* 31–2 *in* σ.

If we apply Schützenberger's involution to permutations, that is, $\sigma \mapsto \omega\sigma\omega$, where $\omega = n \cdots 21$ (also known as taking the *reverse complement*), the statistic descent tops is transformed into descent bottoms, and patterns 31–2 are transformed into patterns 2–31. In that case it follows from results of [38] that for $J = 1^n$, the sum in Eq. (128) gives the q-enumeration of permutation tableaux of a given shape.

Therefore if we assume Conjecture 6.2, Theorem 4.2 implies the following.

Conjecture 6.3. *When $K = 1^n$,*

$$E_I^K(q) = \mathrm{PT}_I^A(q) = \sum_{J \preceq I} (-1/q)^{l(I)-l(J)} q^{-\mathrm{st}'(I,J)} \, \mathrm{QFact}_A(J). \tag{129}$$

Going from $S(q)$ to $R(q)$ is simple, and allows us to reformulate Conjecture 6.3 as follows:

Conjecture 6.4. *Let I and J be two compositions of n. Let* $\mathrm{PP}'(I, J)$ *be the set of permutations σ such that* $\mathrm{GC}(\sigma) = I$ *and* $\mathrm{DC}(\sigma^{-1}) = J$.

Then

$$F_I^J(q) = \sum_{\sigma \in \mathrm{PP}(I,J)} q^{\mathrm{tot}(\sigma)}. \tag{130}$$

For example, the coefficient $1+q+q^2$ in row (3, 1) and column (3, 1) comes from the fact that the words 1243, 1423, 4123 respectively have 0, 1 and 2 occurrences of the pattern 31–2.

Appendix B. Tables

Here are the transition matrices from $R(q)$ to $L(q)$ (the matrices of the coefficients $F_I^J(q)$) for $n=3$ and $n=4$, where the numbers have been replaced by the corresponding list of permutations having given recoil composition and LC-composition.

To save space and for better readability, 0 has been omitted.

LC*Rec*	3	21	12	111
3	123			
21		132 312	213	
12			231	
111				321

(131)

LC*Rec*	4	31	22	211	13	121	112	1111
4	1234							
31		1243, 1423 4123	1324 3124		2134	2143		
22			1342 3142		2314			
211			3412	1432, 4132 4312		2413 4213	3214	
13					2341			
121						2431 4231	3241	
112							3421	
1111								4321

(132)

References

[1] M. Aguiar, N. Bergeron, F. Sottile, Combinatorial Hopf algebras and generalized Dehn–Sommerville relations, Compos. Math. 142 (2006) 1–30.

[2] P. Baumann, C. Hohlweg, A Solomon descent theory for the wreath products $G \wr S_n$, Trans. Amer. Math. Soc. 360 (2008) 1475–1538.

[3] N. Bergeron, M. Zabrocki, q and q,t-analogs of non-commutative symmetric functions, Discrete Math. 298 (2005) 79–103.

[4] S. Corteel, L. Williams, Tableaux combinatorics for the asymmetric exclusion process, Adv. in Appl. Math. 39 (2007) 293–310.

[5] G. Duchamp, F. Hivert, J.-Y. Thibon, Noncommutative symmetric functions VI: Free quasi-symmetric functions and related algebras, Internat. J. Algebra Comput. 12 (2002) 671–717.

[6] F. Essler, V. Rittenberg, Representations of the quadratic algebra and partially asymmetric diffusion with open boundaries, J. Phys. A 29 (1996) 3375–3407.

[7] H.O. Foulkes, A survey of some combinatorial aspects of symmetric functions, in: Permutations, Gauthier-Villars, Paris, 1974.
[8] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, J.-Y. Thibon, Noncommutative symmetric functions, Adv. Math. 112 (1995) 218–348.
[9] I. Gessel, Multipartite P-partitions and inner products of skew Schur functions, in: C. Greene (Ed.), Combinatorics and Algebra, in: Contemp. Math., vol. 34, 1984, pp. 289–301.
[10] I. Gessel, C. Reutenauer, Counting permutations with given cycle structure and descent set, J. Combin. Theory Ser. A 64 (2) (1993) 189–215.
[11] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Z. Tsuboi, Paths, crystals and fermionic formulae, Prog. Math. Phys. 23 (2002) 205–272.
[12] F. Hivert, Hecke algebras, difference operators, and quasi-symmetric functions, Adv. Math. 155 (2000) 181–238.
[13] F. Hivert, A. Lascoux, J.-Y. Thibon, Noncommutative symmetric functions and quasi-symmetric functions with two and more parameters, preprint arXiv:math/0106191.
[14] F. Hivert, J.-C. Novelli, L. Tevlin, J.-Y. Thibon, Permutation statistics related to a class of noncommutative symmetric functions and generalizations of the Genocchi numbers, preprint arXiv:0710.0447v1 [math.CO].
[15] F. Hivert, N. Thiéry, MuPAD-Combinat, an open-source package for research in algebraic combinatorics, Sém. Lothar. Combin. 51 (2004) 70 (electronic).
[16] A.N. Kirillov, N.Yu. Reshetikhin, Bethe ansatz and the combinatorics of Young tableaux, J. Sov. Math. 41 (1988) 925–955.
[17] D. Krob, J.-Y. Thibon, Noncommutative symmetric functions IV: Quantum linear groups and Hecke algebras at $q = 0$, J. Algebraic Comb. 6 (1997) 339–376.
[18] T. Lam, L. Williams, Total positivity for cominuscule Grassmannians, New York J. Math. 14 (2008) 53–99.
[19] L. Lapointe, A. Lascoux, J. Morse, Tableau atoms and a new Macdonald positivity conjecture, Duke Math. J. 116 (1) (2003) 103–146.
[20] A. Lascoux, M.P. Schützenberger, Sur une conjecture de H.O. Foulkes, C. R. Acad. Sci. Paris 286A (1978) 323–324.
[21] A. Lascoux, B. Leclerc, J.-Y. Thibon, Ribbon tableaux, Hall–Littlewood functions, quantum affine algebras, and unipotent varieties, J. Math. Phys. 38 (1997) 1041–1068.
[22] B. Leclerc, J.-Y. Thibon, Littlewood–Richardson coefficients and Kazhdan–Lusztig polynomials, Adv. Stud. Pure Math. 28 (2000) 155–220.
[23] D.E. Littlewood, On certain symmetric functions, Proc. London Math. Soc. 43 (1961) 485–498.
[24] J.-L. Loday, M. Ronco, Combinatorial Hopf algebras, preprint arXiv:0810.0435.
[25] G. Lusztig, Green polynomials and singularities of unipotent classes, Adv. Math. 42 (1981) 169–178.
[26] G. Lusztig, Singularities, character formulas, and a q-analog of weight multiplicities, Analyse et topologie sur les espaces singuliers (II–III), Astérisque 101–102 (1983) 208–227.
[27] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., Oxford University Press, 1995.
[28] R. Mantaci, C. Reutenauer, A generalization of Solomon's descent algebra for hyperoctahedral groups and wreath products, Comm. Algebra 23 (1995) 27–56.
[29] J.-C. Novelli, J.-Y. Thibon, Polynomial realizations of some trialgebras, FPSAC'06, San Diego.
[30] J.-C. Novelli, J.-Y. Thibon, Construction of dendriform trialgebras, C. R. Acad. Sci. Paris 342 (2006) 365–369.
[31] J.-C. Novelli, J.-Y. Thibon, Hopf algebras and dendriform structures arising from parking functions, Fund. Math. 193 (2007) 189–241.
[32] J.-C. Novelli, J.-Y. Thibon, Free quasi-symmetric functions of arbitrary level, arXiv:math/0405597.
[33] J.-C. Novelli, J.-Y. Thibon, Superization and (q, t)-specialization in combinatorial Hopf algebras, arXiv:0803.1816.
[34] S. Poirier, Cycle type and descent set in wreath products, Discrete Math. 180 (1998) 315–343.
[35] A. Postnikov, Total positivity, Grassmannians, and networks, preprint arXiv:math/0609764v1 [math.CO].
[36] C.M. Ringel, Hall algebras and quantum groups, Invent. Math. 101 (1990) 583–591.
[37] N.J.A. Sloane, The on-line encyclopedia of integer sequences, http://www.research.att.com/~njas/sequences/.
[38] E. Steingrimsson, L. Williams, Permutation tableaux and permutation patterns, J. Combin. Theory Ser. A 114 (2007) 211–234.
[39] L. Tevlin, Noncommutative monomial symmetric functions, Proc. FPSAC'07, Tianjin, China.