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ON THE TRANSFORMATION LAW FOR THETA-CONSTANTS

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0. Introduction

In the 1960's, J.-I. Igusa conducted a series of deep investigations. He began by considering Siegel modular forms of genus two [3]. A key tool in his consideration was his theta-constants, and next he turned to a study of their properties [4]. He then returned to genus two, and constructed non-singular compactifications of Siegel modular varieties [5].

We have become very interested in these latter spaces [7, 8]. In the course of our work, we have found that our study of these spaces provides an interpretation of the way theta-constants transform, and it is this geometric interpretation of the transformation law that we offer here.

Before we explain what is going on, we must indicate some differences in language and notation between Igusa and ourselves: We use 'degree' instead of 'genus'. For us, the integral symplectic group of $2g\times 2g$ matrices is $\mathrm{Sp}_{2g}(\mathbb{Z})$, while for Igusa it is $\mathrm{Sp}_g(\mathbb{Z})$. Finally, and most importantly, our formulas are written so as to be correct with all groups acting on the right (so that if a group G acts on a space X we may properly write X/G), whereas, Igusa's groups act on the left. Accordingly, for us 'characteristics' are row vectors, whereas for Igusa they are column vectors. (Sometimes in the literature one sees characteristics, intended to be column vectors, written as row vectors for typographical convenience. For us the usage is precise, rather than just expedient.)

Now let us define our objects of interest.

Let $\boldsymbol{m}=(m_1,\dots,m_{2g})$ be an integer (row) vector.

Set $\boldsymbol{m}'=(m_1,\dots,m_g)$, $\boldsymbol{m}''=(m_{g+1},\dots,m_{2g})$. The theta-function

$$\theta_{\boldsymbol{m}}(\tau,z)=\sum_p e(\tfrac{1}{2}(p+m'/2)\,\tau\,{}^{\mathrm{t}}(p+m'/2)+(p+m'/2)^{\mathrm{t}}(z+m''/2)) \tag{0.1}$$

is a function on $\mathbb{S}_g \times \mathbb{C}^g$, where $\mathbb{S}_g$ is the Siegel space of degree g,

$$\mathbb{S}_g = \{\tau \in M_g(\mathbb{C}) \mid \tau = {}^{t}\tau,\ \operatorname{Im}\tau > 0\},$$

the summation is over $p \in \mathbb{Z}^{2g}$, and $e(\cdot) = \exp(2\pi i \cdot)$.

We call $\boldsymbol{m}$ even (odd) accordingly as $\boldsymbol{m}' \cdot \boldsymbol{m}'' = m_1 m_{g+1} + m_2 m_{g+2} + \cdots + m_g m_{2g}$ is even or odd, and we call the value of $\boldsymbol{m}' \cdot \boldsymbol{m}'' \bmod 2$ the parity of $\boldsymbol{m}$.

The theta-constant $\theta_{\boldsymbol{m}}(\tau)$ is defined to be $\theta_{\boldsymbol{m}}(\tau, 0)$.

Igusa [4, I] derives several transformation laws for theta functions. In particular, we have

$$\theta_{\boldsymbol{m}}(\tau, -z) = (-1)^{\boldsymbol{m}' \cdot \boldsymbol{m}''}\, \theta_{\boldsymbol{m}}(\tau, z), \tag{0.2}$$

$$\theta_{\boldsymbol{m}+2\boldsymbol{n}}(\tau, z) = (-1)^{\boldsymbol{m}' \cdot \boldsymbol{n}''}\, \theta_{\boldsymbol{m}}(\tau, z). \tag{0.3}$$

Equation (0.2) clearly implies the theta constant $\theta_{\boldsymbol{m}}(\tau)$ is identically zero for $\boldsymbol{m}$ odd, while (0.3) implies that it suffices to know the theta functions as $\boldsymbol{m}$ ranges over a set of representatives for $\mathbb{Z}^{2g}/(2\mathbb{Z})^{2g}$. Thus we restrict our attention to $\boldsymbol{m} = (m_1, \ldots, m_{2g})$ with $m_i = 0$ or 1, and call such an $\boldsymbol{m}$ a characteristic.

The principal interest of this paper is the following formula of [4, I]:

$$\theta_{T \cdot \boldsymbol{m}}(T \cdot \tau) = \varepsilon(T, \boldsymbol{m}) \det(c\tau + d)^{1/2}\, \theta_{\boldsymbol{m}}(\tau). \tag{0.4}$$

Here $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z})$, and $\varepsilon(T, \boldsymbol{m})$ is a certain eighth root of unity. The determination of $\varepsilon(T, \boldsymbol{m})$ is a subtle matter (and of course also depends on the choice of square root in (0.4)) which we do not address here. For this we refer the reader to [4, I]. Much work has been done on this point recently.

We are interested in the left-hand side of (0.4). The expression $T \cdot \tau$ is just the usual left action of the symplectic group on the Siegel space

$$T \cdot \tau = (a\tau + b)(c\tau + d)^{-1}. \tag{0.5}$$

However, the left action of T on $\boldsymbol{m}$ is given by

$$T \cdot \boldsymbol{m} = \boldsymbol{m} T^{-1} + ((d\,{}^{t}c)_0 \ \ (b\,{}^{t}a)_0) \tag{0.6}$$

where X_0 is the row vector determined by the diagonal entries of X. (The right-hand side is the transpose of that in [4, I, p. 226] to account for the fact that for us the characteristics are row vectors.)

It is obvious from (0.6) that if T is congruent to the identity matrix modulo two, T acts trivially on characteristics. In addition, it is easy to check that for a symplectic matrix T, $T \cdot \boldsymbol{m}$ has the same parity as $\boldsymbol{m}$, and it is not hard to check that there are $2^{g-1}(2^g + 1)$ even characteristics and $2^{g-1}(2^g - 1)$ odd characteristics.

Now let us specialize to the case of degree 2. Setting $g = 2$, we find 10 even characteristics, given by the first column of Table 1, and 6 odd characteristics, given by the first column of Table 2.

Let $\Gamma = \Gamma(2)$ be the principal congruence subgroup of level 2 in $\mathrm{Sp}_4(\mathbb{Z})$, i.e.

$$\Gamma = \{T \in \mathrm{Sp}_4(\mathbb{Z}) \mid T \equiv I \pmod 2\}.$$

Table 1

even characteristic	triadic syntheme	pair of anisotropic planes
1 1 1 1	123 456	(0100)∧(0001), (1000)∧(0010)
0 1 0 0	124 356	(1110)∧(0001), (1001)∧(0011)
0 1 1 0	125 346	(1100)∧(0001), (1000)∧(0011)
1 1 0 0	126 345	(0110)∧(0001), (1001)∧(0010)
0 0 0 1	134 256	(0100)∧(1011), (1100)∧(0110)
0 0 1 1	135 246	(0100)∧(1001), (1000)∧(0110)
1 0 0 1	136 245	(0100)∧(0011), (1100)∧(0010)
0 0 0 0	234 156	(1110)∧(1011), (1101)∧(0111)
0 0 1 0	235 146	(1100)∧(1001), (1000)∧(0111)
1 0 0 0	236 145	(0110)∧(0011), (1101)∧(0010)

Table 2

odd characteristic	monad	spread of lines
1 0 1 0	1	(0001),(0100),(0111),(1101),(1111)
1 0 1 1	2	(0001),(0101),(0110),(1100),(1110)
1 1 1 0	3	(0011),(0100),(0101),(1001),(1011)
0 1 0 1	4	(0010),(1000),(1011),(1110),(1111)
0 1 1 1	5	(0010),(1001),(1010),(1100),(1101)
1 1 0 1	6	(0011),(0110),(0111),(1000),(1010)

By the above remarks, the quotient $Sp_4(\mathbb{Z})/\Gamma = Sp_4(\mathbb{Z}/2)$ acts on the characteristics, preserving parity. Igusa [3, II] observes further:

The action of $Sp_4(\mathbb{Z}/2)$ on the six odd characteristics, given by (0.6), gives an isomorphism of $Sp_4(\mathbb{Z}/2)$ with the symmetric group on 6 elements Σ_6. (0.7)

It is the aim of this paper to show that formula (0.6) in fact has a natural geometric interpretation, which yields as a by-product the isomorphism of (0.7). In fact, we will relate the geometry of the vector space $(\mathbb{Z}/2)^4$, with the canonical action of $Sp_4(\mathbb{Z}/2)$ thereon, given by right multiplication by matrices, the natural action of Σ_6 on subsets of $\{1,\ldots,6\}$, induced by permutation of the symbols $1,\ldots,6$, and Igusa's formula (0.6) for the actions of $Sp_4(\mathbb{Z}/2)$ on both even and odd characteristics.

The connection between these is via our study of the space M^* of [7], which we introduce in Section 1. In Section 2, we further describe M^* as a moduli space, and in Section 3 bring the theta-constants into the picture. Our main results are proved in Section 4.

1. The Igusa space

In this section we summarize the results of [7, 8]. We refer the reader to those two papers for details.

As Γ is a subgroup of $\mathrm{Sp}_4(\mathbb{Z})$, which acts on $\mathbb{S}_2$, we may form the quotient $M=\mathbb{S}_2/\Gamma$. This space is a branched cover of $\mathbb{S}_2/\mathrm{Sp}_4(\mathbb{Z})$, and so the quotient group $G=\mathrm{Sp}_4(\mathbb{Z})/\Gamma=\mathrm{Sp}_4(\mathbb{Z}/2)$ acts on M. We call M the Siegel space of degree 2 and level 2.

M is a quasi-projective algebraic variety. It was first compactified by Satake [9], and we call the resulting space $\bar{M}$ the Satake space. This compactification was desingularized by Igusa [5], and we call the resulting space M^* the Igusa space. Thus we have

$$M \longrightarrow \bar{M} \stackrel{\pi}{\longleftarrow} M^* \tag{1.1}$$

and π is a 'blow-down' map. M^* is a non-singular projective three-fold.

The action of G on M extends to an action on $\bar{M}$, and also to an action on M^*.

The modern way to view the compactification procedure is via the use of the Tits building. We define a symplectic form on $(\mathbb{Z})^4$ by $\lambda(x,y)=xJ\,{}^t y$, where J is the matrix

$$J=\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

and take $\mathrm{Sp}_4(\mathbb{Z})$ to be the group of integral matrices preserving this form. The group G acts on $V=(\mathbb{Z}/2)^4$ by $v\to vg$, preserving the mod 2 reduction of the form, which we again denote λ. A subspace $V^1\subset V$ is called isotropic if $\lambda(v_1,v_2)=0$ for all $v_1, v_2\in V^1$. An isotropic subspace must have dimension 1 or 2, i.e., must be a line or a plane, and all lines are isotropic, as λ is a skew form. We shall call a plane $V^1\subset V$ anisotropic if $\lambda(v_1,v_2)\neq 0$ for some $v_1, v_2\in V^1$, in which case the restriction of λ to V is a non-singular form. We shall denote a line by l, an isotropic plane by h, and anisotropic plane by δ. Note that δ determines its orthogonal complement $\delta^\perp$, another anisotropic plane, and we denote by Δ the un-ordered pair $\Delta=\{\delta,\delta^\perp\}$.

We then have

$$\bar{M}=M\coprod\bigcup_l B(l),\qquad M^*=M\coprod\bigcup_l D(l) \tag{1.2}$$

where the spaces $B(l)$ and $D(l)$ are described in [7, 8]. The Igusa space M^* is a non-singular variety. By abuse of language, we call $\partial=M^*-M$ the 'boundary' of M^*, and each $D(l)$ a boundary component. Each $D(l)$ is a complex surface, and the different copies (they are mutually isomorphic) intersect transversely. What is important for our purposes here is the following: These components are indexed by the lines $l\subset V$.

$$B(l_1)\cap B(l_2)\neq\emptyset \;\Leftrightarrow\; D(l_1)\cap D(l_2)\neq\emptyset \;\Leftrightarrow\; l_1, l_2\subset h$$

for some isotropic plane h. In fact, if l_1, l_2, l_3 are the three lines in h, then $B(l_i)\cap B(l_j)=B(l_1)\cap B(l_2)\cap B(l_3)$ is a single triple point, and $D(l_1)\cap D(l_2)\cap D(l_3)$ is a single point as well.

Finally, this indexing is G-equivariant. That is, if $l \subset V$, $g \in G$,

$$B(l) \cdot g = B(lg), \qquad D(l) \cdot g = D(lg). \tag{1.3}$$

In addition to the boundary components, there is another important collection of subvarieties of M^*. The Siegel space $\mathbb{S}_2$ contains as a subspace

$$\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_1 = \left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix} \middle| \operatorname{Im} \tau_1 > 0,\ \operatorname{Im} \tau_3 > 0 \right\}, \tag{1.4}$$

as well as the union of all its translates under the action of $\mathrm{Sp}_4(\mathbb{Z})$. Thus M contains the quotient of this union

$$M = \mathbb{S}_2/\Gamma \supset (\mathbb{S}_1 \times \mathbb{S}_1)\mathrm{Sp}_4(\mathbb{Z})/\Gamma.$$

This quotient is a disjoint union of 10 subvarieties. They each extend to non-singular subvarieties of M^*, and remain disjoint. Following [2], we shall call their union, Θ, the Humbert surface in M^*, and each component $H(\Delta)$ of Θ a Humbert component. Then

$$\Theta = \coprod H(\Delta), \tag{1.5}$$

Again, this indexing is G-equivariant, i.e., for $\Delta = \{\delta, \delta^\perp\}$, $g \in G$,

$$H(\Delta)g = H(\Delta g). \tag{1.6}$$

Indeed, we can say more about the relationship between $H(\Delta)$ and $D(l)$:

$$H(\Delta) \cap D(l) \neq \emptyset \text{ exactly when } l \in \delta \text{ or } l \in \delta^\perp, \text{ where } \Delta = \{\delta, \delta^\perp\}. \tag{1.7}$$

(Thus each $H(\Delta)$ intersects 6 $D(l)$'s and each $D(l)$ intersects 4 $H(\Delta)$'s.) The blow-down map π of (1.1) restricts to a blow-down map π: $D(l) \to B(l)$, giving $D(l)$ the structure of a singular fibration over $B(l)$, and $H(\Delta) \cap D(l)$, when non-empty, is a section of this fibration. This is all carefully described in [8, Section 2, especially Figs. 2.3.1, 2.3.3].

If we consider the situation in $\bar{M}$, we have correspondingly

$$H(\Delta) \supset B(l) \text{ exactly when } l \in \delta, \text{ or } l \in \delta^\perp, \text{ where } D\{\delta, \delta^\perp\}. \tag{1.8}$$

(Thus here the different copies of $H(\Delta)$ are not disjoint, and if $H(\Delta) \supset B(l_1)$, and l_1, l_2, and l_3 are the three lines in an isotropic plane h, then it will be the case that $H(\Delta) \supset B(l_2)$, while $H(\Delta) \cap B(l_3)$ is a single point, for proper choice of l_2 and l_3.)

2. M^* as a moduli space

In the previous section we discussed the structure of M^* from the symplectic point of view. Now we discuss M^* as a moduli space, which shows its relationship to the symmetric group. Again, we refer the reader to [7, 8] for details and proofs.

The space M^* is the moduli space of stable curves of genus 2 with level 2 structure.

Stable curves include non-singular ones, and ones with particular kinds of singularities. In particular, $M^\circ = M^* - \partial \cup \Theta$ is the moduli space of non-singular curves with level 2 structure.

In general, a level n structure is a choice of symplectic basis for the points of level n in the Jacobian of a curve. In our case, it has a much simpler interpretation. A curve R of genus 2 is hyperelliptic, i.e., given by an equation $y^2 = f(x)$, branched at six points of $\mathbb{P}^1$. The inverse images $\{p_i\}$, $i = 1, \ldots, 6$, of these points on R are the six Weierstrass points of R. Then a level 2 structure on R is precisely an ordering of $p_1, \ldots, p_6$. The correspondence is as follows: Let dz_1, dz_2 be a basis for the space of holomorphic differentials on R. Then the symplectic basis of the points of order 2 on the Jacobian of R, i.e. of the half-periods, is given by the columns of the matrix

$$\begin{bmatrix} \int_{p_1}^{p_2} dz_1 & \int_{p_4}^{p_5} dz_1 & \int_{p_2}^{p_3} dz_1 & \int_{p_5}^{p_6} dz_1 \\ \int_{p_1}^{p_2} dz_2 & \int_{p_4}^{p_5} dz_2 & \int_{p_2}^{p_3} dz_2 & \int_{p_5}^{p_6} dz_2 \end{bmatrix}.$$

As points in $\mathbb{S}_2$ are equivalent under the action of $\mathrm{Sp}_4(\mathbb{Z})$ iff they parameterize the same curve, the group $\mathrm{Sp}_4(\mathbb{Z})$ is the automorphism group of level 2 structures on a fixed curve, so by the above we obtain an isomorphism

$$\varrho : \Sigma_6 \to \mathrm{Sp}_4(\mathbb{Z}/2) \tag{2.1}$$

Now the points of $\partial \cup \Theta$ parameterize curves with singularities.

The components $D(l)$ of ∂ parameterize curves with nodes, i.e., where two of the Weierstrass points coalesce, $p_i = p_j$ for some i, j. Thus these components are naturally parameterized by $\{i, j\} \subset \{1, \ldots, 6\}$. The components $H(\Delta)$ of Θ parameterize curves with double points which decompose into the one-point union of two elliptic curves, each of which contains three of the Weierstrass points and the singular point as its four points of order two. Thus these components are naturally parameterized by unordered partitions $\{\{i, j, k\}, \{i', j', k'\}\}$ of $\{1, \ldots, 6\}$. We are thus led to the following:

Definition 2.2. Let $G = \mathrm{Sp}_4(\mathbb{Z}/2)$ act on $V = (\mathbb{Z}/2)^4$ by right multiplication, $(v, g) \to vg^{-1}$. This action induces actions on $\{l\}$ of lines in V, on $\{h\}$ of isotropic planes in V, and $\Delta = \{\delta, \delta^\perp\}$ of pairs of anisotropic planes in V. These latter actions will be called *natural*.

Definition 2.3. Let Σ_6 act on $\{1, \ldots, 6\}$ on the left by permuting monads, i.e., $\{i\} \subset \{1, \ldots, 6\}$. This action induces actions on duads, i.e., $\{i, j\} \subset \{1, \ldots, 6\}$, duadic synthemes, i.e., $\{\{i, j\}, \{i', j'\}, \{i'', j''\}\}$ unordered partitions of $\{1, \ldots, 6\}$, and triadic synthemes, i.e., $\{\{i, j, k\}, \{i', j', k'\}\}$ unordered partitions of $\{1, \ldots, 6\}$. These latter actions will be called *natural*.

Table 3

duad	line
12	0 0 0 1
13	0 1 0 0
14	1 1 1 1
15	1 1 0 1
16	0 1 1 1
23	0 1 0 1
24	1 1 1 0
25	1 1 0 0
26	0 1 1 0
34	1 0 1 1
35	1 0 0 1
36	0 0 1 1
45	0 0 1 0
46	1 0 0 0
56	1 0 1 0

Table 4

duadic syntheme	isotropic plane
12 34 56	(0001)∧(1010)
12 35 46	(0001)∧(1000)
12 36 45	(0001)∧(0010)
13 24 56	(0100)∧(1010)
13 25 46	(0100)∧(1000)
13 26 45	(0010)∧(0100)
14 23 56	(0101)∧(1010)
14 25 36	(0011)∧(1100)
14 26 35	(0110)∧(1001)
15 23 46	(0101)∧(1000)
15 24 36	(0011)∧(1101)
15 26 34	(0110)∧(1011)
16 23 45	(0010)∧(0101)
16 24 35	(0111)∧(1001)
16 25 34	(0111)∧(1011)

Historical remark 2.4. The terms monad, duad, duadic syntheme, and triadic syntheme are due to Sylvester [10].

Remark 2.5. There are fifteen duads, fifteen duadic synthemes, and ten triadic synthemes, to be found in Tables 3, 4, and 1 respectively.

Theorem 2.6. *The isomorphism ϱ of* (2.1) *induces isomorphisms between the natural actions of* $\mathrm{Sp}_4(\mathbb{Z}/2)$ *on lines, isotropic planes, and pairs of anisotropic planes and the natural actions of Σ_6 on duads, duadic synthemes, and triadic synthemes respectively.*

(Note that we write a plane as the wedge product $l_1 \wedge l_2$ of two lines therein; the third line l_3 is then $l_1 + l_2$.)

In order to prove 2.6, we shall have to investigate ϱ further. This we defer until Section 4, when 2.6 will be included in a more comprehensive result.

3. Humbert surfaces and theta-constants

Let m be an even weight. For $T \in \Gamma(2)$, the eighth root of unity in (0.4) is actually a fourth root of unity. Thus we see that the fourth power of any theta-constant $\theta_m^4(\tau)$ is a modular form of weight 2 with respect to the subgroup Γ, hence descends to a modular form on M.

One can check directly that the theta-constant $\theta_m(\tau)$, for $m = (1, 1, 1, 1)$, vanishes on the subspace $\mathbb{S}$ of $\mathbb{S}_2$. More precisely, we have for $\tau \in \mathbb{S}$ (see [2, p. 335])

$$\theta_m^4(\tau) = \begin{cases} \theta^4_{(m_1, m_3)}(\tau_1)\, \theta^4_{(m_2, m_4)}(\tau_3), & m \neq (1, 1, 1, 1) \\ 0, & m = (1, 1, 1, 1), \end{cases} \tag{3.1}$$

and also

$$\lim_{\tau_3 \to \infty} \theta_m^4(\tau) = \begin{cases} \theta^4_{(m_1, m_3)}(\tau_1), & m_2 = 0, \\ 0, & m_2 = 1. \end{cases} \tag{3.2}$$

The θ_m^4 all extend to modular forms on $\bar{M}$, with their behavior on the boundary given by (3.2), and hence to forms on the Igusa space M^*. By considering the action of $G = \mathrm{Sp}_4(\mathbb{Z}/2)$ on M^*, it then follows from (3.1) that

Each Humbert component $H(\Delta)$ is the closure (in M^*) of the zero-set of θ_m^4 in M for exactly one even characteristic $\boldsymbol{m}$. (3.3)

Since there are four even characteristics $\boldsymbol{m}$ with $m_2 = 1$, it follows from (3.2) that

Each boundary component $D(l)$ intersects exactly four Humbert components $H(\Delta)$. (3.4)

Then dually, it follows that

Each Humbert component $H(\Delta)$ intersects exactly six boundary components $D(l)$. (3.5)

Now (3.3) clearly establishes a correspondence between pairs of anisotropic planes and even characteristics. There is the natural action of $\mathrm{Sp}_4(\mathbb{Z}/2)$ on the latter, giving us an action on the former, and we shall see it is precisely Igusa's action (0.6).

4. The main theorem

It is an ancient fact that there is a set of five lines in $V = (\mathbb{Z}/2)^4$ no two of which are contained in the same isotropic plane, and in fact there are six such sets. We will call each of them a spread of lines.

There are 15 lines and 15 isotropic planes. Each plane contains 3 lines and each line is contained in three planes. It then follows readily that a spread of lines may also be described as a set of five lines, such that every isotropic plane contains one of them. Since there are 15 lines and $30 = 6 \cdot 5$ lines in the six spreads of lines, it follows that each line is contained in exactly two spreads. It is also apparent that the property defining a spread is invariant under the action of $\mathrm{Sp}_4(\mathbb{Z}/2)$, so the set of spreads is acted on by this group.

The spreads are listed in the third column of Table 2. We denote a spread by σ.

Now we can begin to establish our isomorphism

Definition 4.1. If $(\upsilon_1, \ldots, \upsilon_{2g}) \in (\mathbb{Z}/2)^{2g}$, $\upsilon' \cdot \upsilon'' = \upsilon_1 \upsilon_{g+1} + \cdots + \upsilon_g \upsilon_{2g}$ will be called the parity of υ.

Lemma 4.2. *For* $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z}/2)$, *and* $\upsilon \in V = (\mathbb{Z}/2)^{2g}$, *let*

$$\phi(T, \upsilon) = \upsilon T^{-1} + ((d\,{}^{\mathrm{t}}c)_0 \ (b\,{}^{\mathrm{t}}a)_0)$$

and

$$\psi(T, v) = vT^{-1}.$$

Then $\phi(T_1 T_2, v) = \phi(T_1, \phi(T_2, v))$, *and* $\psi(T_1 T_2, v) = \psi(T_1, \psi(T_2, v))$.

Furthermore, for any $v_1, v_2 \in V$, *the parity of* $v_1 + v_2$ *is the same as the parity of* $\phi(T, v_1) + \psi(T, v_2)$.

Proof. The first assertion is that $T \to \phi(T, \ )$ is a left action of $\mathrm{Sp}_{2g}(\mathbb{Z}/2)$ on $\{v \in V\}$, which is an easy computation, and that $T \to \psi(T, \ )$ is a left action of $\mathrm{Sp}_{2g}(\mathbb{Z}/2)$ on $\{v \in V\}$, which is immediate.

In view of this, it suffices to check the second assertion for matrices of the form

$$T_1 = \begin{pmatrix} A & \\ 0 & {}^{t}A^{-1} \end{pmatrix}, \quad T_2 = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \text{with } B = {}^{t}B, \quad \text{and} \quad T_3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

as these generate $\mathrm{Sp}_{2g}(\mathbb{Z}/2)$.

Write $v = (v', v'')$. Of course, $v' \cdot v'' = v'\,{}^{t}v''$.

In case $T = T_1$, $\phi(T, (v', v'')) = \psi(T, (v', v'')) = (v' A^{-1}, v''\,{}^{t}A)$, so

$$(v_1' + v_2') A^{-1}\,{}^{t}((v_1'' + v_2'')\,{}^{t}A) = (v_1' + v_2')\,{}^{t}(v_1'' + v_2'').$$

In case $T = T_2$, $\phi(T, (v', v'')) = (v', v'B + v'' + B_0)$, $\phi(T, (v', v'')) = (v', v'B + v'')$, so

$$\begin{aligned}(v_1' + v_2')\,{}^{t}(v_1'' + v_2'' + (v_1' + v_2') B + B_0) &= (v_1' + v_2')\,{}^{t}(v_1'' + v_2'') + (v_1' + v_2')\,{}^{t}B\,{}^{t}(v_1' + v_2') \\ &\quad + (v_1' + v_2')\,{}^{t}B_0 \\ &= (v_1' + v_2')\,{}^{t}(v_1'' + v_2'') \quad \text{as } B \text{ is symmetric.}\end{aligned}$$

In case $T = T_3$, $\phi(T, (v', v'')) = \psi(T, (v', v'')) = \psi(T, (v', v'')) = (v'', v')$ and the result is clear.

Henceforth we specialize to the case of degree $g = 2$.

Lemma 4.3. *Let* m *be a characteristic. The equation*

$$(m + v)' \cdot (m + v)'' = 1, \quad v \in V \tag{4.4}$$

has 6 solutions.

(a) *In case* m *is even, upon proper ordering these are lines* $l_1, \ldots, l_6$ *with* l_1, l_2, l_3 *lying in an anisotropic plane* δ *and* l_4, l_5, l_6 *in its orthogonal complement* $\delta^{\perp}$.

(b) *In case* m *is odd, they consist of the zero element and the five lines in some spread of lines.*

(c) *Equation* (4.4) *establishes a* 1–1 *correspondence* η *between even characteristics and pairs* $\Delta = \{\delta, \delta^{\perp}\}$ *of anisotropic planes, and between odd characteristics and spreads* σ *of lines.*

Proof. Equation (4.4) says $\boldsymbol{m}+\upsilon$ is odd, and we have already seen that there are six such.

The group $\mathrm{Sp}_4(\mathbb{Z}/2)$ clearly acts transitively on anisotropic pairs Δ and on spreads σ. Thus to verify (a) and (b) it suffices to check (4.4) on one representative of each type.

If $\Delta=\{(0100)\wedge(0001),\ (1000)\wedge(0010)\}$, (4.4) has the solution $\boldsymbol{m}=(1111)$.

If $\sigma=\{(0001), (0100), (0111), (1101), (1111)\}$, (4.4) has the solution $\boldsymbol{m}=(1010)$.

Part (c) amounts to saying that (4.4) establishes a bijection in each of the two cases, but that is immediate.

We now extend our correspondence η. We label each spread by an integer between 1 and 6, i.e., a monad, as in Table 2. Thus η is now a correspondence between odd characteristics, monads, and spreads of lines.

Each line is contained in two spreads. Thus we label it by a duad, consisting of the monads corresponding to each of the spreads containing it, as in Table 3, and extend η to the correspondence between duads and lines.

Each isotropic plane contains three lines which together form a duadic syntheme, so we label it thereby as in Table 4, and extend η to this correspondence between duadic synthemes and isotropic planes.

Finally, each anisotropic plane contains three lines which together form a triad, and an anisotropic pair thus corresponds to a triadic syntheme, as in Table 1. Thus we extend the correspondence η between even characteristics and anisotropic pairs to include triadic synthemes as well.

Theorem 4.5. (i) *The correspondence* η *induces isomorphisms between the following:*

(a) $\{$*monads*$\} \leftrightarrow \{$*spreads of lines*$\}$,

(b) $\{$*duads*$\} \leftrightarrow \{$*lines*$\}$,

(c) $\{$*duadic synthemes*$\} \leftrightarrow \{$*isotropic planes*$\}$,

(d) $\{$*triadic synthemes*$\} \leftrightarrow \{$*pairs of anisotropic planes*$\}$.

These isomorphisms are as representation spaces of Σ_6 *and* $\mathrm{Sp}_4(\mathbb{Z}/2)$ *respectively, where in all cases each group acts by the natural action.*

(ii) *The isomorphism in* (a) *agrees with the map* ϱ *of* 2.1.

Proof. (i). The isomorphism in (a) is a tautology from the definition of monad. Then (b), (c), and (d) follow immediately once it is seen that a single line corresponds to a duad, an isotropic plane to a duadic syntheme, and a pair of anisotropic planes to a triadic syntheme. It is trivial to check that indeed $12\leftrightarrow(0001)$, $34\leftrightarrow(1011)$, $56\leftrightarrow(1010)$, so $(12)(34)(56)\leftrightarrow(0001)\wedge(1010)$, and $13\leftrightarrow(0100)$, $23\leftrightarrow(0101)$, so $(123)\leftrightarrow(0100)\wedge(0001)$, and similarly $(456)\leftrightarrow(1000)\wedge(0010)$.

(ii) The map ϱ^{-1} gives some action on monads; the question is which. One can check directly from the definition of ϱ^{-1} what its action is on boundary components and hence on spreads of lines (i.e., monads) but there is an easier way:

The action on Weierstrass points is a six-dimensional representation which contains a one-dimensional trivial representation, as Σ_6 leaves $p_1+p_2+\cdots+p_6$ invariant.

Thus we may write it as $\mathrm{tr}+F$, tr a trivial one-dimensional representation, F a five-dimensional representation yet to be determined. Whatever this representation is, it induces the natural representation on duads, whose description in terms of Young diagrams is $[6]+[51]+[42]$. However, the action on duads is $\Sigma^2(\mathrm{tr}+F)-(\mathrm{tr}+F)$, where Σ^2 denotes the second symmetric power, which contains F as a subrepresentation, so F must be $[51]$ and so $\mathrm{tr}+F=[6]+[51]$ is the natural action of Σ_6 on monads.

Now we can give our interpretation of Igusa's transformation law:

Corollary 4.6. *The correspondence η induces isomorphisms between the following:*
(a) $\{$*odd characteristics*$\} \leftrightarrow \{$*spreads of lines* $\sigma\}$,
(b) $\{$*even characteristics*$\} \leftrightarrow \{$*pairs of anistropic planes* $\Delta\}$.
These isomorphisms are as representation spaces of $\mathrm{Sp}_4(\mathbb{Z}/2)$, *where* $\mathrm{Sp}_4(\mathbb{Z}/2)$ *acts on characteristics on the left as in* (0.6), *and on* $\{\sigma\}$ *and* $\{\Delta\}$ *on the right in the natural way, i.e., if* $T\in \mathrm{Sp}_4(\mathbb{Z}/2)$, $\boldsymbol{m}$ *is a characteristic, and* ω *denotes either* σ *or* Δ, *then* $\boldsymbol{m}\leftrightarrow\omega \Leftrightarrow T\cdot \boldsymbol{m}\leftrightarrow \omega T^{-1}$.

Proof. This is immediate from the identification η of Theorem 4.5 (a) and (d) and the conclusion of Lemma 4.2.

Our viewpoint also applies to a further result of Igusa. In [3,II] he shows that the ring of modular forms with respect to $\Gamma(2)$ is generated over $\mathbb{C}$ by:

(a) $\{\theta_m^4\}$,

(b) $\{(\theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4})^2\}$, $\boldsymbol{m}_1,\ldots,\boldsymbol{m}_4$ compatible,

(c) $\{(\theta_{m_5}\theta_{m_6}\theta_{m_7}\theta_{m_8}\theta_{m_9}\theta_{m_{10}})^2\}$, $\boldsymbol{m}_5,\ldots,\boldsymbol{m}_{10}$ complementary to a compatible $\boldsymbol{m}_1,\ldots,\boldsymbol{m}_4$,

(d) the product $\theta_{m_1}\cdots\theta_{m_{10}}$

where the characteristics $\boldsymbol{m}$ are of course all even, and in (b) they satisfy the compatibility condition

$$\boldsymbol{m}_1+\boldsymbol{m}_2+\boldsymbol{m}_3+\boldsymbol{m}_4=(0000)\in V. \tag{4.7}$$

The compatibility condition is easy to interpret in terms of the geometry of Igusa space M^*.

As we observed following (1.2), each isotropic plane h contains 3 lines l_1, l_2, l_3 with $D(l_1)\cap D(l_2)\cap D(l_3)$ a single point. There are a total of six Humbert surfaces that pass through $D(l_1)\cap D(l_2)\cap D(l_3)$ (see [7, Section 2, Fig. 2.3.2]), and it is easy to check that these are precisely the Humbert surfaces corresponding to the characteristics in a complementary six-tuple as in (c), so a compatible four-tuple corresponds to the four Humbert surfaces *not* passing through $D(l_1)\cup D(l_2)\cup D(l_3)$. It then follows that we can associate to each compatible four-tuple or complementary six-tuple an isotropic plane, and indeed in an equivariant way. It is also clear that $(\theta_{m_1}\cdots\theta_{m_4})^2$ cannot be a cusp form, as it does not vanish on $D(l_1)\cup D(l_2)\cup D(l_3)$, while it vanishes to order one on the other 12 boundary components (recall θ_m has weight 1/2), nor can $(\theta_m)^4(\theta_{m_1}\cdots\theta_{m_4})^2$ be a cusp form. On the other hand, it is easy to check that $(\theta_{m_1}\cdots\theta_{m_6})^2$ is a cusp form, vanishing to order two on $D(l_1)\cup D(l_2)\cup D(l_3)$, and to order one on the other 12 boundary components. (Of course, $\theta_{m_1}\cdots\theta_{m_{10}}$ is a cusp form.)

There is one further isomorphism, whose proof we omit, as it is similar to that of 4.5. Dual to the spreads of lines are spreads of planes. A spread of planes is a set of five isotropic planes such that every line is contained in some member of the set. There are six spreads of planes, with each isotropic plane contained in exactly two spreads. There are also six totals (another term due to Sylvester), where a total is a set of five duadic synthemes such that every duad is contained in some member of the set. There is an obvious notion of natural action for each of these, and we have

Theorem 4.8. *There is a correspondence*

$$\{totals\} \leftrightarrow \{spreads\ of\ planes\}$$

which induces an isomorphism as representation spaces of Σ_6 and $\mathrm{Sp}_4(\mathbb{Z}/2)$, where each group acts by the natural action.

If Σ_6 is regarded as the symmetric group on monads, the automorphism taking an element to the (product of) cycle(s) determined by its action on totals gives the 'odd' automorphism of this group (see [11], where the totals are listed.)

Remark 4.9. The terms of Young diagrams, the representations of Σ_6 on monads, duads, triadic synthemes, totals and duadic synthemes are $[6]+[51]$, $[6]+[51]+[42]$, $[6]+[42]$, $[6]+[222]$, and $[6]+[42]+[222]$ respectively.

Remark 4.10. Although we do not need it for the results of this paper, it is interesting to observe that the space M^* arises from two different geometries. We have see in Section 1 that M^* is a compactification of $M=\mathbb{S}_2/\Gamma(2)$. On the other hand, by the work of Deligne and Mostow [1], M^* may also be obtained as a non-singular compactification of the quotient $Q=\mathbb{B}^3/\Gamma$ of the ball in $\mathbb{C}^3$ by a discrete subgroup Γ of PU(1,3). In this geometry, $M^*=Q\coprod H$, i.e., here the cusps are not the boundary components but rather the components of the Humbert surface. In fact, Q is the quotient of the stable points in $(\mathbb{P}^1)^6$ under a certain action of $\mathrm{PGL}_2(\mathbb{C})$, and

M^* the desingularized quotient of the semi-stable points in $(\mathbb{P}^1)^6$ under a certain equivalence relation. (Here stable and semi-stable are in the sense of geometric invariant theory.) We are being vague here, referring the reader to [6] for details, wherein Q is denoted $Q(1\ 1\ 1\ 1\ 1\ 1)$. Now there is an action of Σ_6 on $(\mathbb{P}^1)^6$, by permuting coordinates, which descends to an action on M^*. One can also recover some of the results of this section by studying this action of Σ_6.

References

[1] P. Deligne and G.D. Mostow, Monodromy of hypergeometric functions and non-integral lattice monodromy, Publ. Math. I.H.E.S., to appear.

[2] G. van der Geer, On the geometry of a Siegel modular threefold, Math. Ann. 260 (1982) 317–350.

[3] J.-I. Igusa, On Siegel modular forms of genus two. I, Amer. J. Math. 84 (1962) 175–200. II, Amer. J. Math. 86 (1964) 392–412.

[4] J.-I. Igusa, On the graded ring of theta constants. I, Amer. J. Math. 86 (1964) 219–246. II, Amer. J. Math. 88 (1966) 221–236.

[5] J.-I. Igusa, A desingularization problem in the theory of Siegel modular functions, Math. Ann. 168 (1967) 228–260.

[6] F.C. Kirwan, R. Lee and S.H. Weintraub, Quotients of the complex ball by discrete groups, to appear.

[7] R. Lee and S.H. Weintraub, Cohomology of a Siegel modular variety of degree two, in: R. Schultz, ed., Group actions on Manifolds (Amer. Math. Soc., Providence, RI, 1985).

[8] R. Lee and S.H. Weintraub, Cohomology of $Sp_4(\mathbb{Z})$ and related groups and spaces, Topology 24 (1985) 391–410.

[9] I. Satake, On the compactification of the Siegel space, J. Indian Math. Soc. 20 (1956) 259–281.

[10] J.J. Sylvester, Elementary researches in the analysis of combinatorial aggregation, Philosophical Magazine 24 (1844) 285–296. (Collected mathematical papers, Vol. I, pp. 91–102.)

[11] J.A. Todd, The 'odd' number six, Proc. Camb. Phil. Soc. 41 (1945) 66–68.