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On the maximum of a subcritical branching process in a random environment ☆

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Abstract

Let $\{\xi_n\}$ be a subcritical branching process in random environment with independent identically distributed generating functions $f_n(s)$. It is shown that if there exists a positive number æ such that $\boldsymbol{E}(f_0'(1))^{æ} = 1$ then, for $x \to +\infty$,

$$\boldsymbol{P}\left(\sup_n \xi_n > x\right) \sim Kx^{-æ},$$

where K is a positive constant.

MSC: 60J80; 60J15; 60F17

Keywords: Branching processes; Random environment

1. Introduction and main result

Let $\{\xi_n, n \in \mathbf{N}_0\}$ be a branching process in random environment $\{\pi_n, n \in \mathbf{N}_0\}$, where $\pi_n = \{\pi_n^{(0)}, \pi_n^{(1)}, \pi_n^{(2)}, \ldots\}$, $\pi_n^{(i)} \geqslant 0$, $\sum_{i=0}^{\infty} \pi_n^{(i)} = 1$, $n \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$, and besides the sequences π_n are identically distributed and independent for different n. By definition it means that ξ_n are non-negative, integer-valued random variables and

$$\xi_0 = 1, \quad \boldsymbol{E}(s^{\xi_{n+1}} | \xi_0, \xi_1, \ldots, \xi_n, \pi_0, \pi_1, \ldots, \pi_n) = (f_n(s))^{\xi_n},$$

where $f_n(s) = \sum_{i=0}^{\infty} \pi_n^{(i)} s^i$, $s \in [-1, 1]$, $n \in \mathbf{N}_0$.

In Afanasyev (1999) for the critical case (that is when $\boldsymbol{E} \ln f_0'(1) = 0$) it is shown that, as $x \to +\infty$,

$$\boldsymbol{P}\left(\sup_{n \in \mathbf{N}_0} \xi_n > x\right) \sim \frac{K_1}{\ln x},$$

where K_1 is a positive constant (note, that one of the assumptions in Afanasyev (1999) is that $f_n(s)$ are linear-fractional generating functions). The aim of the present paper is

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to find the asymptotics of $\boldsymbol{P}(\sup_{n\in\mathbf{N}_0}\xi_n > x)$ as $x\to+\infty$ in the subcritical case (i.e., when $\boldsymbol{E}\ln f_0'(1)<0$). We assume that there exists a positive number æ such that

$$\boldsymbol{E}(f_0'(1))^{æ}=1. \tag{1}$$

Let us analyse this assumption in more details. Set $X_n=\ln f_{n-1}'(1)$, $n\in\mathbf{N}$, and consider the function $\Theta(t)=\boldsymbol{E}\exp(tX_1)$, $t\in\mathbf{R}$. Assumption (1) is equivalent to the following condition: there exists a positive number æ such that

$$\Theta(æ)=1. \tag{2}$$

Since the sequences π_n, the components of environment, are identically distributed and independent for different n, X_n are also identically distributed and independent random variables for different n. We consider the random walk $S_0=0$, $S_n=\sum_{i=1}^n X_i$, $n\in\mathbf{N}$, which is closely connected with $\{\xi_n\}$. Since $\{\xi_n\}$ is a subcritical branching process in random environment, $\boldsymbol{E}X_1=\boldsymbol{E}\ln f_0'(1)<0$, i.e., $\{S_n\}$ is a random walk with negative drift. Condition (2) is well known for random walks with negative drift and it allows us to find the asymptotics of $\boldsymbol{P}(\sup_{n\in\mathbf{N}_0}S_n>x)$ (see [Feller, 1971, Chapter XII]) and of $\boldsymbol{P}(\sum_{n=1}^{\infty}\exp S_n>x)$ (see Kesten, 1973), as $x\to+\infty$. Given this condition it is convenient to pass from $\{S_n\}$ to the conjugate random walk $\tilde{S}_0=0$, $\tilde{S}_n=\sum_{i=1}^n\tilde{X}_i$. These random walks are connected by the relation $\boldsymbol{P}(\tilde{X}_1\leqslant x)=\int_{-\infty}^{x}\exp(æu)\,\mathrm{d}\boldsymbol{P}(X_1\leqslant u)$. This transformation allows us to reduce the respective problems for random walks with negative drift to the problems for random walks with positive drift. Let us show why the drift of $\{\tilde{S}_n\}$ is positive.

The function $\Theta(t)$ is convex on $(0,æ)$, since $\Theta''(t)=\boldsymbol{E}(X_1^2\exp(tX_1))>0$, $t\in(0,æ)$. Moreover, $\lim_{t\downarrow 0}\Theta'(t)=\lim_{t\downarrow 0}\boldsymbol{E}(X_1\exp(tX_1))=\boldsymbol{E}X_1<0$. Provided that

$$\boldsymbol{E}(X_1^+\exp(æX_1))<+\infty \tag{3}$$

the limit $\lim_{t\uparrow æ}\Theta'(t)=\boldsymbol{E}(X_1\exp(æX_1))$ exists. Since $\Theta(0)=\Theta(æ)=1$, it follows from the arguments above that $\boldsymbol{E}(X_1\exp(æX_1))>0$. Hence

$$\boldsymbol{E}\tilde{X}_1=\boldsymbol{E}(X_1\exp(æX_1))>0 \tag{4}$$

as required.

In the present paper we use a modification of this method – the passage to the "conjugate" random environment. The modified branching process in this environment turns out to be supercritical (it means that $\boldsymbol{E}\ln f_0'(1)>0$).

In connection with the problem under consideration, it is necessary to mention paper (Kesten et al., 1975), the main part of which is, in essence, devoted to finding asymptotics of $P(\sum_{n=0}^{+\infty}\xi_n>x)$ as $x\to+\infty$ for a subcritical branching process in random environment. It is established in Kesten et al. (1975) that if conditions (2) and (3) are valid and the distribution of X_1 is non-lattice then, as $x\to+\infty$,

$$\boldsymbol{P}\left(\sum_{n=0}^{\infty}\xi_n>x\right)\sim K_2x^{-æ},$$

where K_2 is a positive constant. One should note, however, that only the case when $æ\in(0,2]$ and $f_n(s)$ are linear-fractional generating functions was considered.

The next important step in the investigation of probability $\boldsymbol{P}(\sum_{n=0}^{\infty} \xi_n > x)$ was made in paper (Dembo et al., 1996). It is shown there that if æ > 1 and $0 < \gamma <$ æ, and $\boldsymbol{E}\xi_1^{\gamma} < +\infty$, then

$$\boldsymbol{P}\left(\sum_{n=0}^{\infty} \xi_n > x\right) \leqslant K(\gamma) x^{-\gamma},$$

where $K(\gamma)$ is independent of x and $K(\gamma) > 0$.

Problems of finding asymptotics $\boldsymbol{P}(\sup_{n \in \mathbf{N}_0} \xi_n > x)$ and $\boldsymbol{P}(\sum_{n=0}^{\infty} \xi_n > x)$ are closely connected. One can use the method from the present paper to find the asymptotic $\boldsymbol{P}(\sum_{n=0}^{\infty} \xi_n > x)$ taking into account the relation $\boldsymbol{P}(\sum_{n=0}^{\infty} \xi_n > x) = \boldsymbol{P}(\sup_{n \in \mathbf{N}_0} \zeta_n > x)$, where $\zeta_n = \sum_{i=0}^{n} \xi_i$.

In the present paper we prove the following result (observe that we do not assume that $f_n(s)$ are linear-fractional generating functions).

Theorem 1. *Let $\{\xi_n\}$ be a subcritical branching process in random environment. Let conditions* (2) *and* (3) *be satisfied for some positive* æ *and let the distribution of X_1 be non-lattice. In addition, let $\boldsymbol{E}(\xi_1 \ln^+ \xi_1 \exp((æ - 1)X_1)) < +\infty$ and, if æ $\geqslant 1$, there exists a number $p >$ æ such that $\boldsymbol{E}(\xi_1^p \exp((æ - p)X_1)) < +\infty$. Then, as $x \to +\infty$,*

$$\boldsymbol{P}\left(\sup_{n \in \mathbf{N}_0} \xi_n > x\right) \sim K x^{-æ}, \tag{5}$$

where K is a positive constant being independent of x.

2. Passage to a supercritical process

Consider the random sequence $\{X_1, \pi_0^{(0)}, \pi_0^{(1)}, \ldots\}$. Recall, that $X_1 = \ln f_0'(1) = \ln \sum_{i=0}^{\infty} i\pi_0^{(i)}$. Let $F_1(x_1)$, $F_2(x_1, x_2)$, $F_3(x_1, x_2, x_3)$, ... be the distribution functions of the random vectors X_1, $(X_1, \pi_0^{(0)}), (X_1, \pi_0^{(0)}, \pi_0^{(1)}), \ldots$ respectively. Set

$$\tilde{F}_n(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} e^{æ u_1}\, dF_n(u_1, \ldots, u_n), \quad n \in \mathbf{N}.$$

In other words, for $n \in \mathbf{N}$

$$\tilde{F}_n(x_1, x_2, \ldots, x_n) = \boldsymbol{E}(\exp(æ X_1) \chi_{\{X_1 \leqslant x_1, \pi_0^{(0)} \leqslant x_2, \ldots, \pi_0^{(n-2)} \leqslant x_n\}}),$$

where χ_A is the indicator function of a set A. Note, that these functions are indeed distribution functions. For example, by Lebesgue's theorem and (2) we deduce that

$$\lim_{\substack{x_1 \to +\infty, \ldots, \\ x_n \to +\infty}} \tilde{F}_n(x_1, \ldots, x_n) = \boldsymbol{E} \exp(æ X_1) = 1.$$

These distribution functions are consistent, because

$$\begin{aligned} \lim_{x_n \to +\infty} \tilde{F}_n(x_1, \ldots, x_n) &= \boldsymbol{E}(\exp(æ X_1) \chi_{\{X_1 \leqslant x_1, \pi_0^{(0)} \leqslant x_2, \ldots, \pi_0^{(n-3)} \leqslant x_{n-1}\}}) \\ &= \tilde{F}_{n-1}(x_1, \ldots, x_{n-1}). \end{aligned}$$

By Kolmogorov's theorem there exists a random sequence $\tilde{X}_1$, $\tilde{\pi}_0^{(0)}$, $\tilde{\pi}_0^{(1)}, \ldots$ (may be, on another probability space) such that

$$\tilde{F}_1(x_1) = \boldsymbol{P}(\tilde{X}_1 \leqslant x_1), \tilde{F}_2(x_1, x_2) = \boldsymbol{P}(\tilde{X}_1 \leqslant x_1, \tilde{\pi}_0^{(0)} \leqslant x_2), \ldots$$

The following statement is of importance in the subsequent arguments.

Lemma 1. *For any measurable function* $g(x_1,x_2,\dots,x_{n+2})$ *defined for all* $(x_1,x_2,\dots,x_{n+2})\in\mathbf{R}^{n+2}$, *the equalities*

$$\boldsymbol{E}g(X_1,\pi_0^{(0)},\dots,\pi_0^{(n)})=\boldsymbol{E}(\exp(-æ\tilde{X}_1)g(\tilde{X}_1,\tilde{\pi}_0^{(0)},\dots,\tilde{\pi}_0^{(n)})),$$

$$\boldsymbol{E}g(\tilde{X}_1,\tilde{\pi}_0^{(0)},\dots,\tilde{\pi}_0^{(n)})=\boldsymbol{E}(\exp(æX_1)g(X_1,\pi_0^{(0)},\dots,\pi_0^{(n)}))$$

are valid if at least one of the expectations exists in each equality.

Proof. Clearly,

$$\begin{aligned}&\boldsymbol{E}(\exp(-æ\tilde{X}_1)g(\tilde{X}_1,\tilde{\pi}_0^{(0)},\dots,\tilde{\pi}_0^{(n)}))\\&=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp(-æu_1)g(u_1,u_2,\dots,u_{n+2})\,\mathrm{d}\tilde{F}_{n+2}(u_1,u_2,\dots,u_{n+2})\\&=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp(-æu_1)g(u_1,u_2,\dots,u_{n+2})\exp(æu_1)\,\mathrm{d}F_{n+2}(u_1,u_2,\dots,u_{n+2})\\&=\boldsymbol{E}g(X_1,\pi_0^{(0)},\dots,\pi_0^{(n)}).\end{aligned}$$

The remaining case is considered in a similar way. Lemma 1 is proved. □

Put $\tilde{\Theta}(t)=\boldsymbol{E}\exp(t\tilde{X}_1),\ t\in\mathbf{R}$.

Corollary 1. $\tilde{\Theta}(-æ)=1$ *and* $\tilde{\Theta}(t)<1$ *for all* $t\in(-æ,0)$.

Proof. By Lemma 1

$$\tilde{\Theta}(t)=\boldsymbol{E}\exp(t\tilde{X}_1)=\boldsymbol{E}\exp((æ+t)X_1)=\Theta(æ+t).$$

It remains to recall that $\Theta(t)<1$ for $t\in(0,æ)$ and $\Theta(0)=1$. □

Now we study some properties of $\{\tilde{X}_1,\tilde{\pi}_0^{(0)},\tilde{\pi}_0^{(1)},\dots\}$. First it is clear that $\tilde{\pi}_0^{(n)}\geqslant 0$ a.s. for any $n\in\mathbf{N}_0$. Further, since $\sum_{i=0}^{n}\pi_0^{(i)}\leqslant 1$, it follows that $\sum_{i=0}^{n}\tilde{\pi}_0^{(i)}\leqslant 1$ a.s. for any $n\in\mathbf{N}_0$. Therefore, $\sum_{i=0}^{\infty}\tilde{\pi}_0^{(i)}\leqslant 1$ a.s. Observe now that by the monotone convergence theorem, Lemma 1, and condition (2)

$$\begin{aligned}\boldsymbol{E}\sum_{i=0}^{+\infty}\tilde{\pi}_0^{(i)}&=\lim_{n\to\infty}\boldsymbol{E}\sum_{i=0}^{n}\tilde{\pi}_0^{(i)}=\lim_{n\to\infty}\boldsymbol{E}\left(\exp(æX_1)\sum_{i=0}^{n}\pi_0^{(i)}\right)\\&=\boldsymbol{E}\left(\exp(æX_1)\sum_{i=0}^{\infty}\pi_0^{(i)}\right)=\boldsymbol{E}\exp(æX_1)=1.\end{aligned}$$

Therefore, $\sum_{i=0}^{\infty}\tilde{\pi}_0^{(i)}=1$ a.s.

Thus, we have established that the sequence $\tilde{\pi}_0=\{\tilde{\pi}_0^{(0)},\tilde{\pi}_0^{(1)},\dots\}$ may be viewed as the initial element of a random environment. Consider the generating function $\tilde{f}_0(s)=\sum_{i=0}^{\infty}\tilde{\pi}_0^{(i)}s^i,\ s\in[-1,1]$. We show that if, along with (2), condition (3) is valid then

$$\tilde{X}_1=\ln\tilde{f}_0'(1)\quad\text{a.s.}\tag{6}$$

Since $X_1 = \ln \sum_{i=0}^{\infty} i\pi_0^{(i)}$, it follows that $\sum_{i=0}^{n} i\pi_0^{(i)} \leqslant \exp X_1 \ \forall n \in \mathbf{N}_0 \Rightarrow \sum_{i=0}^{n} i\tilde{\pi}_0^{(i)} \leqslant \exp \tilde{X}_1$ a.s. $\forall n \in \mathbf{N}_0$. Therefore,

$$\ln \tilde{f}_0'(1) \leqslant \tilde{X}_1 \quad \text{a.s.} \tag{7}$$

Now, by the monotone convergence theorem and Lemma 1, we obtain

$$\begin{aligned} \boldsymbol{E} \ln^+ \tilde{f}_0'(1) &= \boldsymbol{E} \ln^+ \sum_{i=0}^{\infty} i\tilde{\pi}_0^{(i)} = \lim_{n\to\infty} \boldsymbol{E} \ln^+ \sum_{i=0}^{n} i\tilde{\pi}_0^{(i)} \\ &= \lim_{n\to\infty} \boldsymbol{E} \left(\mathrm{e}^{æX_1} \ln^+ \sum_{i=0}^{n} i\pi_0^{(i)} \right) = \boldsymbol{E} \left(\mathrm{e}^{æX_1} \ln^+ \sum_{i=0}^{\infty} i\pi_0^{(i)} \right) \\ &= \boldsymbol{E}(X_1^+ \mathrm{e}^{æX_1}). \end{aligned} \tag{8}$$

Here the last expectation is finite by condition (3). Analogously we demonstrate that

$$\boldsymbol{E} \ln^- \tilde{f}_0'(1) = \boldsymbol{E}(X_1^- \exp(æX_1)) < +\infty. \tag{9}$$

Conditions (7)–(9) imply (6).

Along with the initial element $\tilde{\pi}_0$ of the environment we consider elements $\tilde{\pi}_1, \tilde{\pi}_2, \ldots,$ each of which is distributed the same as $\tilde{\pi}_0$ and demand the independence of all elements of the sequence $\tilde{\pi}_0, \tilde{\pi}_1, \tilde{\pi}_2, \ldots$. It is this sequence we take as the environment conjugate to $\{\pi_n\}$. Now we consider a branching process in random environment $\{\tilde{\pi}_n\}$. We denote this process by $\{\tilde{\xi}_n\}$. Put $\tilde{f}_n(s) = \sum_{i=0}^{\infty} \tilde{\pi}_n^{(i)} s^i$ for $n \in \mathbf{N}$ and $\tilde{X}_n = \tilde{f}_{n-1}'(1)$ for $n = 2, 3, \ldots$. Taking into account (6) we conclude that $\tilde{S}_0 = 0$, $\tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_i$, $n \in \mathbf{N}$, is a random walk. Since $\boldsymbol{P}(\tilde{X}_1 \leqslant x) = \int_{-\infty}^{x} \exp(æu)\, \mathrm{d}\boldsymbol{P}(X_1 \leqslant u)$, it follows that $\{\tilde{S}_n\}$ is conjugate to $\{S_n\}$. In particular, relation (4) is valid. But this means that $\{\tilde{\xi}_n\}$ is a supercritical process.

Set $\pi_{m,n} = \{\pi_m^{(0)}, \pi_m^{(1)}, \ldots, \pi_m^{(n)}\}, m, n \in \mathbf{N}_0$. Lemma 1 admits an obvious generalization.

Lemma 1′. *For any measurable function g defined on $\mathbf{R}^{m(n+1)}$, the equalities*

$$\boldsymbol{E} g(\pi_{0,n}; \pi_{1,n}; \ldots; \pi_{m-1,n}) = \boldsymbol{E}(\exp(-æ\tilde{S}_m) g(\tilde{\pi}_{0,n}; \tilde{\pi}_{1,n}; \ldots; \tilde{\pi}_{m-1,n})),$$

$$\boldsymbol{E} g(\tilde{\pi}_{0,n}; \tilde{\pi}_{1,n}; \ldots; \tilde{\pi}_{m-1,n}) = \boldsymbol{E}(\exp(æS_m) g(\pi_{0,n}; \pi_{1,n}; \ldots; \pi_{m-1,n}))$$

are valid if at least one of the expectations exists in each equality.

As a corollary we have the following result. Let

$$T_x = \inf\{n:\ \xi_n > x\}, \quad \tilde{T}_x = \inf\{n:\ \tilde{\xi}_n > x\}.$$

Lemma 2. *For all $x \in (0, +\infty)$*

$$\boldsymbol{P}\left(\sup_{n \in \mathbf{N}_0} \xi_n > x \right) = \boldsymbol{E}(\exp(-æ\tilde{S}_{\tilde{T}_x});\ \tilde{T}_x < +\infty).$$

Proof. Since for $x \in (0,1]$ both sides of this equality are equal to 1, it is sufficient to consider the case $x \in (1,+\infty)$. Clearly,

$$\boldsymbol{P}\left(\sup_{n \in \mathbf{N}_0} \xi_n > x\right) = \boldsymbol{P}(T_x < +\infty) = \sum_{m=1}^{\infty} \boldsymbol{P}(T_x = m), \tag{10}$$

$$\boldsymbol{P}(T_x = m) = \boldsymbol{P}(\xi_1 \leqslant x, \ldots, \xi_{m-1} \leqslant x) - \boldsymbol{P}(\xi_1 \leqslant x, \ldots, \xi_{m-1} \leqslant x, \xi_m \leqslant x). \tag{11}$$

Denote by $\boldsymbol{E}_\pi$ and $\boldsymbol{P}_\pi$ the expectation and probability under fixed environment. Then, for all $m \in \mathbf{N}$,

$$\boldsymbol{P}(\xi_1 \leqslant x, \ldots, \xi_m \leqslant x) = \boldsymbol{E}\boldsymbol{P}_\pi(\xi_1 \leqslant x, \ldots, \xi_m \leqslant x).$$

Clearly, $\boldsymbol{P}_\pi(\xi_1 \leqslant x; \ldots; \xi_m \leqslant x)$ is a non-random function g of the random vector $\{\pi_{0,[x]}; \ldots; \pi_{m-1,[x]}\}$, where $[x]$ is the integer part of x. By Lemma 1′

$$\begin{aligned}\boldsymbol{P}(\xi_1 \leqslant x, \ldots, \xi_m \leqslant x) &= \boldsymbol{E}g(\pi_{0,[x]}; \ldots; \pi_{m-1,[x]})\\ &= \boldsymbol{E}(\mathrm{e}^{-æ\tilde{S}_m} g(\tilde{\pi}_{0,[x]}; \ldots; \tilde{\pi}_{m-1,[x]})).\end{aligned}$$

It is not difficult to see that $g(\tilde{\pi}_{0,[x]}; \ldots; \tilde{\pi}_{m-1,[x]})$ plays the same role for $\{\tilde{\pi}_n\}$ as $g(\pi_{0,[x]}; \ldots; \pi_{m-1,[x]})$ for $\{\pi_n\}$, i.e.

$$g(\tilde{\pi}_{0,[x]}; \ldots; \tilde{\pi}_{m-1,[x]}) = \boldsymbol{P}_{\tilde{\pi}}(\tilde{\xi}_1 \leqslant x, \ldots, \tilde{\xi}_m \leqslant x)).$$

Thus,

$$\boldsymbol{P}(\xi_1 \leqslant x, \ldots, \xi_m \leqslant x) = \boldsymbol{E}(\exp(-æ\tilde{S}_m)\boldsymbol{P}_{\tilde{\pi}}(\tilde{\xi}_1 \leqslant x, \ldots, \tilde{\xi}_m \leqslant x)). \tag{12}$$

Similarly,

$$\begin{aligned}\boldsymbol{P}(\xi_1 \leqslant x, \ldots, \xi_{m-1} \leqslant x) &= \boldsymbol{E}(\exp(-æ\tilde{S}_{m-1})\boldsymbol{P}_{\tilde{\pi}}(\tilde{\xi}_1 \leqslant x, \ldots, \tilde{\xi}_{m-1} \leqslant x))\\ &= \boldsymbol{E}(\exp(-æ\tilde{S}_m)\boldsymbol{P}_{\tilde{\pi}}(\tilde{\xi}_1 \leqslant x, \ldots, \tilde{\xi}_{m-1} \leqslant x)) \end{aligned}\tag{13}$$

since $\exp(-æ\tilde{X}_m)$ and $\boldsymbol{P}_{\tilde{\pi}}(\tilde{\xi}_1 \leqslant x, \ldots, \tilde{\xi}_{m-1} \leqslant x)$ are independent and $\boldsymbol{E}\exp(-æ\tilde{X}_m) = 1$ by Corollary 1. Relations (11)–(13) imply

$$\boldsymbol{P}(T_x = m) = \boldsymbol{E}(\exp(-æ\tilde{S}_m)\boldsymbol{P}_{\tilde{\pi}}(\tilde{T}_x = m)).$$

Whence, taking into account (10), we obtain

$$\begin{aligned}\boldsymbol{P}\left(\sup_{n \in N_0} \xi_n > x\right) &= \sum_{m=1}^{\infty} \boldsymbol{E}(\mathrm{e}^{-æ\tilde{S}_m}\boldsymbol{P}_{\tilde{\pi}}(\tilde{T}_x = m)) = \sum_{m=1}^{\infty} \boldsymbol{E}\boldsymbol{E}_{\tilde{\pi}}(\mathrm{e}^{-æ\tilde{S}_m}\chi_{\{\tilde{T}_x = m\}})\\ &= \boldsymbol{E}\sum_{m=1}^{\infty} \mathrm{e}^{-æ\tilde{S}_m}\chi_{\{\tilde{T}_x = m\}} = \boldsymbol{E}(\mathrm{e}^{-æ\tilde{S}_{\tilde{T}_x}}; \tilde{T}_x < +\infty).\end{aligned}$$

Lemma 2 is proved. □

To reformulate the hypotheses of Theorem 1 in terms of $\{\tilde{\xi}_n\}$ we need the following statement.

Lemma 3. *The following equalities*:

$$\boldsymbol{E}(\xi_1 \ln^+ \xi_1 \exp((æ - 1)X_1)) = \boldsymbol{E}(\tilde{\xi}_1 \ln^+ \tilde{\xi}_1 \exp(-\tilde{X}_1)),$$
$$\boldsymbol{E}(\xi_1^p \exp((æ - p)X_1)) = \boldsymbol{E}(\tilde{\xi}_1^p \exp(-p\tilde{X}_1))$$

hold if at least one of the expectations exists in each equality.

Proof. It is clear that, for $n \in \mathbf{N}$, $\boldsymbol{E}_\pi(\xi_1 \wedge n)^p = h(\pi_0^{(0)}, \pi_0^{(1)}, \ldots, \pi_0^{(n)})$, where h is a non-random function and, moreover, $\boldsymbol{E}_{\tilde\pi}(\tilde\xi_1 \wedge n)^p = h(\tilde\pi_0^{(0)}, \tilde\pi_0^{(1)}, \ldots, \tilde\pi_0^{(n)})$. Hence by Lemma 1

$$\begin{aligned}\boldsymbol{E}((\xi_1 \wedge n)^p \exp((æ - p)X_1)) &= \boldsymbol{E}(\exp((æ - p)X_1)\boldsymbol{E}_\pi(\xi_1 \wedge n)^p)\\ &= \boldsymbol{E}(\exp(((æ - p)X_1)h(\pi_0^{(0)}, \pi_0^{(1)}, \ldots, \pi_0^{(n)}))\\ &= \boldsymbol{E}(\mathrm{e}^{-p\tilde X_1} h(\tilde\pi_0^{(0)}, \tilde\pi_0^{(1)}, \ldots, \tilde\pi_0^{(n)}))\\ &= \boldsymbol{E}(\exp(-p\tilde X_1)\boldsymbol{E}_{\tilde\pi}(\tilde\xi_1 \wedge n)^p)\\ &= \boldsymbol{E}((\tilde\xi_1 \wedge n)^p \exp(-p\tilde X_1)).\end{aligned}$$

Using the monotone convergence theorem as $n \to \infty$, we obtain the second statement of Lemma 3. The first statement can be proved by similar arguments. Lemma 3 is proved. □

3. Properties of the natural martingale of the super-critical process

Below, to simplify presentation, we omit the symbol $\sim$ in all notations connected with the branching process in the conjugate random environment. By this reason from now on we consider $\{\xi_n\}$ as a supercritical branching process in random environment $\{\pi_n\}$. In view of Lemma 2 it is necessary to investigate the asymptotic behavior of

$$\boldsymbol{E}(\exp(-æS_{T_x}); T_x < +\infty) = x^{-æ}\boldsymbol{E}\left(\left(\frac{\xi_{T_x}}{\exp S_{T_x}}\right)^{æ}\left(\frac{x}{\xi_{T_x}}\right)^{æ}; T_x < +\infty\right) \tag{14}$$

as $x \to \infty$. It is convenient to rewrite this expectation as

$$\boldsymbol{E}\left(\left(\frac{\xi_{T_x}}{\exp S_{T_x}}\right)^{æ}\left(\frac{x}{\xi_{T_x}}\right)^{æ}; T_x < +\infty\right) = M_1(x,k) + M_2(x,k), \tag{15}$$

where $k \in \mathbf{N}_0$ is arbitrary and

$$M_1(x,k) = \boldsymbol{E}\left(\left(\frac{\xi_k}{\exp S_k}\right)^{æ}\left(\frac{x}{\xi_{T_x}}\right)^{æ}; T_x < +\infty\right),$$

$$M_2(x,k) = \boldsymbol{E}\left(\left(\left(\frac{\xi_{T_x}}{\exp S_{T_x}}\right)^{æ} - \left(\frac{\xi_k}{\exp S_k}\right)^{æ}\right)\left(\frac{x}{\xi_{T_x}}\right)^{æ}; T_x < +\infty\right).$$

In calculating the limits of $M_1(x,k)$ and $M_2(x,k)$ as $k \to \infty$, Lemma 7 below plays an important role, the proof of which is based on Lemmas 4–6.

Lemma 4. *Let α_n be the number of direct descendant of a particle from the $(n-1)$th generation and $\zeta_n^{(m)} = \boldsymbol{E}_\pi(\alpha_n \exp(-X_n))^m, n, m \in \mathbf{N}$. Then, for $n \in \mathbf{N}$, $m = 2, 3, \ldots,$ the following inequality holds*:

$$\boldsymbol{E}_\pi\left(\frac{\xi_n}{\exp S_n}\right)^m \leqslant \frac{(m!)^2}{2^m}\sigma_m(n),$$

where

$$\sigma_m(n) = \sum_{k_{m-1}=0}^{n-1} \zeta_{k_{m-1}+1}^{(m)}\left(\ldots\left(\sum_{k_2=0}^{k_3-1}\zeta_{k_2+1}^{(3)}\left(\sum_{k_1=0}^{k_2-1}\zeta_{k_1+1}^{(2)}\mathrm{e}^{-S_{k_1}}\right)\mathrm{e}^{-S_{k_2}}\right)\ldots\right)\mathrm{e}^{-S_{k_{m-1}}}.$$

Proof. Denote by $\alpha_n^{(1)},\alpha_n^{(2)},\dots,\alpha_n^{(\xi_{n-1})}$ the numbers of direct descendants of the 1th, 2th, $\dots,(\xi_{n-1})$th particle from the $(n-1)$th generation, respectively. Let $\boldsymbol{E}_{\pi,\xi_{n-1}}$ be the conditional expectation given that the random environment and the number of particles in the $(n-1)$th generation are fixed. Then, on account of

$$\boldsymbol{E}_{\pi,\xi_{n-1}}\left(\frac{\alpha_n^{(i)}}{\exp X_n}\frac{\alpha_n^{(j)}}{\exp X_n}\right)=1 \quad \text{for } i\neq j, \quad \boldsymbol{E}_{\pi}\frac{\xi_{n-1}}{\exp S_{n-1}}=1$$

we see that

$$\begin{aligned}\boldsymbol{E}_{\pi}\left(\frac{\xi_n}{\exp S_n}\right)^2&=\boldsymbol{E}_{\pi}\left(\boldsymbol{E}_{\pi,\xi_{n-1}}\left(\frac{\xi_n}{\exp S_n}\right)^2\right)\\&=\boldsymbol{E}_{\pi}\left(\exp(-2S_{n-1})\boldsymbol{E}_{\pi,\xi_{n-1}}\left(\frac{\alpha_n^{(1)}+\alpha_n^{(2)}+\cdots+\alpha_n^{(\xi_{n-1})}}{\exp X_n}\right)^2\right)\\&=\boldsymbol{E}_{\pi}\left(\exp(-2S_{n-1})\boldsymbol{E}_{\pi,\xi_{n-1}}\left(\sum_{i=1}^{\xi_{n-1}}\left(\frac{\alpha_n^{(i)}}{\exp X_n}\right)^2+\sum_{\substack{i,j=1\\(i\neq j)}}^{\xi_{n-1}}\frac{\alpha_n^{(i)}}{\exp X_n}\frac{\alpha_n^{(j)}}{\exp X_n}\right)\right)\\&\leqslant\boldsymbol{E}_{\pi}(\exp(-2S_{n-1})(\zeta_n^{(2)}\xi_{n-1}+\xi_{n-1}^2))=\zeta_n^{(2)}\mathrm{e}^{-S_{n-1}}+\boldsymbol{E}_{\pi}\left(\frac{\xi_{n-1}}{\exp S_{n-1}}\right)^2.\end{aligned}$$

From this it follows that:

$$\boldsymbol{E}_{\pi}\left(\frac{\xi_n}{\exp S_n}\right)^2\leqslant\sum_{i=0}^{n-1}\zeta_{i+1}^{(2)}\mathrm{e}^{-S_i}$$

proving the statement of the lemma for $m=2$.

Now we use the induction method. Suppose that the statement of the lemma is valid for some $m=2,3,\dots$ Let us prove its validity for $m+1$. We have

$$\begin{aligned}\boldsymbol{E}_{\pi}\left(\frac{\xi_n}{\exp S_n}\right)^{m+1}&=\boldsymbol{E}_{\pi}\left(\boldsymbol{E}_{\pi,\xi_{n-1}}\left(\frac{\xi_n}{\exp S_n}\right)^{m+1}\right)\\&=\boldsymbol{E}_{\pi}\left(\exp(-(m+1)S_{n-1})\boldsymbol{E}_{\pi,\xi_{n-1}}\right.\\&\qquad\left.\left(\frac{\alpha_n^{(1)}+\alpha_n^{(2)}+\cdots+\alpha_n^{(\xi_{n-1})}}{\exp X_n}\right)^{m+1}\right).\end{aligned}\tag{16}$$

By Lyapunov's inequality, for $k_1+\cdots+k_l=m+1$, $k_1,\dots,k_l\in\mathbf{N}$, $1\leqslant i_1<i_2<\cdots<i_l\leqslant n$, $i_1,i_2,\dots,i_l\in\mathbf{N}$, $l\in\mathbf{N}$

$$\boldsymbol{E}_{\pi}\left(\left(\frac{\alpha_n^{(i_1)}}{\exp X_n}\right)^{k_1}\cdot\dots\cdot\left(\frac{\alpha_n^{(i_l)}}{\exp X_n}\right)^{k_l}\right)$$

$$= \boldsymbol{E}_\pi \left(\frac{\alpha_n^{(i_1)}}{\exp X_n} \right)^{k_1} \cdot \ldots \cdot \boldsymbol{E}_\pi \left(\frac{\alpha_n^{(i_l)}}{\exp X_n} \right)^{k_l}$$

$$\leqslant \left(\boldsymbol{E}_\pi \left(\frac{\alpha_n^{(i_1)}}{\exp X_n} \right)^{m+1} \right)^{k_1/(m+1)} \cdot \ldots \cdot \left(\boldsymbol{E}_\pi \left(\frac{\alpha_n^{(i_l)}}{\exp X_n} \right)^{m+1} \right)^{k_l/(m+1)}$$

$$= \boldsymbol{E}_\pi \left(\frac{\alpha_n}{\exp X_n} \right)^{m+1} = \zeta_n^{(m+1)}.$$

For $k \in \mathbf{N}$ and $a_1, \ldots, a_k \in \mathbf{R}$ we write the following obvious identity:

$$(a_1 + \cdots + a_k)^{m+1} = \sum_{1 \leqslant i_1, i_2, \ldots, i_{m+1} \leqslant k} a_{i_1} a_{i_2} \ldots a_{i_{m+1}}.$$

First we consider the summands in the last sum, whose indices $i_1, i_2, \ldots, i_{m+1}$ are different. The number of such summands is $k(k-1)\ldots(k-m) \leqslant k^{m+1}$. Now we consider the remaining summands, the number of which is $k^{m+1} - k(k-1)\ldots(k-m)$ and, as it is not difficult to show, is not greater than $k^m m(m+1)/2$. Hence we conclude that

$$\boldsymbol{E}_{\pi, \xi_{n-1}} \left(\frac{\alpha_n^{(1)} + \alpha_n^{(2)} + \cdots + \alpha_n^{(\xi_{n-1})}}{\exp X_n} \right)^{m+1} \leqslant \zeta_n^{(m+1)} \xi_{n-1}^m \frac{m(m+1)}{2} + \xi_{n-1}^{m+1}.$$

Applying this equality to (16) we obtain

$$\boldsymbol{E}_\pi \left(\frac{\xi_n}{\exp S_n} \right)^{m+1} \leqslant \boldsymbol{E}_\pi \left(\mathrm{e}^{-(m+1)S_{n-1}} \left(\zeta_n^{(m+1)} \xi_{n-1}^m \frac{m(m+1)}{2} + \xi_{n-1}^{m+1} \right) \right)$$

$$= \frac{m(m+1)}{2} \zeta_n^{(m+1)} \boldsymbol{E}_\pi \left(\frac{\xi_{n-1}}{\exp S_{n-1}} \right)^m \mathrm{e}^{-S_{n-1}} + \boldsymbol{E}_\pi \left(\frac{\xi_{n-1}}{\exp S_{n-1}} \right)^{m+1}.$$

Hence it follows that:

$$\boldsymbol{E}_\pi \left(\frac{\xi_n}{\exp S_n} \right)^{m+1} \leqslant \frac{m(m+1)}{2} \sum_{k_m=0}^{n-1} \zeta_{k_m+1}^{(m+1)} \boldsymbol{E}_\pi \left(\frac{\xi_{k_m}}{\exp S_{k_m}} \right)^m \mathrm{e}^{-S_{k_m}}. \tag{17}$$

By the induction assumption

$$\boldsymbol{E}_\pi \left(\frac{\xi_{k_m}}{\exp S_{k_m}} \right)^m \leqslant \frac{(m!)^2}{2^m} \sigma_m(k_m). \tag{18}$$

Relations (17) and (18) imply

$$\boldsymbol{E}_\pi \left(\frac{\xi_n}{\exp S_n} \right)^{m+1} \leqslant \frac{m(m+1)}{2} \frac{(m!)^2}{2^m} \sum_{k_m=0}^{n-1} \zeta_{k_m+1}^{(m+1)} \sigma_m(k_m) \mathrm{e}^{-S_{k_m}}$$

$$\leqslant \frac{((m+1)!)^2}{2^{m+1}} \sigma_{m+1}(n).$$

Lemma 4 is proved. □

By the assumption of Theorem 1 and Lemma 3, for æ$\geqslant 1$ there exists $p >$ æ such that $\boldsymbol{E}((\xi_1 \exp(-X_1))^p < +\infty$. Clearly, for all $n \in \mathbf{N}$,

$$\boldsymbol{E} \left(\frac{\xi_1}{\exp X_1} \right)^p = \boldsymbol{E} \left(\frac{\alpha_1}{\exp X_1} \right)^p = \boldsymbol{E} \left(\frac{\alpha_n}{\exp X_n} \right)^p.$$

Thus, there exists $p > æ$ such that for all $n \in \mathbf{N}$

$$\boldsymbol{E}\left(\frac{\alpha_n}{\exp X_n}\right)^p = L_1 < +\infty, \tag{19}$$

where L_1 is a positive constant being independent of n. Without loss of generality we may assume that

$$æ < p < [æ] + 1, \tag{20}$$

where $[æ]$ is the integer part of æ. Hence by Corollary 1

$$q = \max(\boldsymbol{E}e^{-(p-1)X_1}, \boldsymbol{E}e^{-(p-2)X_1}, \ldots, \boldsymbol{E}e^{-(p-[æ])X_1}) < 1. \tag{21}$$

Lemma 5. *Let* $æ \geqslant 2$ *and conditions* (19), (20) *be valid. Then, for* $n \in \mathbf{N}$ *and* $m = 2, 3, \ldots, [æ]$ *the inequality*

$$\boldsymbol{E}(\sigma_m(n)e^{-(p-m)S_n}) \leqslant (L_1 n)^{m-1} q^{n-m+1}$$

holds.

Proof. First we consider the case $m = 2$. Obviously,

$$\left(\sum_{k_1=0}^{n-1} \zeta_{k_1+1}^{(2)} e^{-S_{k_1}}\right) e^{-(p-2)S_n} = \sum_{k_1=0}^{n-1} (\zeta_{k_1+1}^{(2)} e^{-(p-2)X_{k_1+1}}) e^{-(p-1)S_{k_1}} e^{-(p-2)(S_n - S_{k_1+1})}$$

and all three factors on the right-hand side of this equality are independent random variables. Hence, taking into account (21), we obtain

$$\boldsymbol{E}\left(\left(\sum_{k_1=0}^{n-1} \zeta_{k_1+1}^{(2)} e^{-S_{k_1}}\right) e^{-(p-2)S_n}\right)$$
$$\leqslant \sum_{k_1=0}^{n-1} \boldsymbol{E}(\zeta_{k_1+1}^{(2)} e^{-(p-2)X_{k_1+1}}) q^{n-1} \leqslant \sum_{k_1=0}^{n-1} L_1 q^{n-1} = L_1 n q^{n-1}.$$

The validity of last inequality can be explained in the following manner:

$$\boldsymbol{E}(\zeta_{k_1+1}^{(2)} e^{-(p-2)X_{k_1+1}}) = \boldsymbol{E}\left(\boldsymbol{E}_\pi \left(\frac{\alpha_{k_1+1}}{\exp X_{k_1+1}}\right)^2 e^{-(p-2)X_{k_1+1}}\right)$$
$$= \boldsymbol{E}\frac{\alpha_{k_1+1}^2}{\exp(pX_{k_1+1})} \leqslant \boldsymbol{E}\left(\frac{\alpha_{k_1+1}}{\exp X_{k_1+1}}\right)^p = L_1.$$

Thus, the statement of the lemma is proved for $m = 2$. Now we use induction. Assume that the statement of the lemma is true for a natural number $m - 1$, where $m \leqslant æ$, and prove its validity for m. It is clear that

$$\sigma_m(n)e^{-(p-m)S_n} = \sum_{k_{m-1}=0}^{n-1} \left(\frac{\zeta_{k_{m-1}+1}^{(m)}}{\exp((p-m)X_{k_{m-1}+1})}\right)$$
$$\times \left(\frac{\sigma_{m-1}(k_{m-1})}{\exp((p-(m-1))S_{k_{m-1}})}\right) e^{-(p-m)(S_n - S_{k_{m-1}+1})},$$

where all three factors on the right-hand side of this equality are independent random variables. Hence, taking into account the validity of the lemma for the second factor, we obtain

$$\begin{aligned} \boldsymbol{E}(\sigma_m(n)\mathrm{e}^{-(p-m)S_n}) &\leqslant \sum_{k_{m-1}=0}^{n-1} \boldsymbol{E}\left(\frac{\zeta^{(m)}_{k_{m-1}+1}}{\exp((p-m)X_{k_{m-1}+1})}\right)(L_1 k_{m-1})^{m-2} \\ &\quad \times q^{k_{m-1}-m+2} q^{n-k_{m-1}-1} \leqslant L_1^{m-1} \sum_{k_{m-1}=0}^{n-1} k_{m-1}^{m-2} q^{n-m+1} \\ &\leqslant (L_1 n)^{m-1} q^{n-m+1}. \end{aligned}$$

Here we make use of the relationship

$$\begin{aligned} \boldsymbol{E}\frac{\zeta^{(m)}_{k_{m-1}+1}}{\exp((p-m)X_{k_{m-1}+1})} &= \boldsymbol{E}\left(\boldsymbol{E}_\pi\left(\frac{\alpha_{k_{m-1}+1}}{\exp X_{k_{m-1}+1}}\right)^m \mathrm{e}^{-(p-m)X_{k_{m-1}+1}}\right) \\ &= \boldsymbol{E}\frac{\alpha^m_{k_{m-1}+1}}{\exp(pX_{k_{m-1}+1})} \leqslant \boldsymbol{E}\left(\frac{\alpha_{k_{m-1}+1}}{\exp X_{k_{m-1}+1}}\right)^p = L_1. \end{aligned}$$

Lemma 5 is proved. □

Lemma 6. *Let* æ $\geqslant 1$ *and conditions* (19), (20) *be satisfied. Then, for* $n \in \mathbf{N}_0$,

$$\boldsymbol{E}\left|\frac{\xi_{n+1}}{\exp S_{n+1}} - \frac{\xi_n}{\exp S_n}\right|^p \leqslant C_p\, n^{[æ]-1} q^n,$$

where C_p *is a positive constant which is independent of* n.

Proof. Using the notation of Lemma 4 we write

$$\begin{aligned} &\boldsymbol{E}_\pi\left|\frac{\xi_{n+1}}{\exp S_{n+1}} - \frac{\xi_n}{\exp S_n}\right|^p \\ &\quad = \boldsymbol{E}_\pi\left(\mathrm{e}^{-pS_n}\boldsymbol{E}_{\pi,\xi_n}\left|\frac{\alpha^{(1)}_{n+1}}{\exp X_{n+1}} + \cdots + \frac{\alpha^{(\xi_n)}_{n+1}}{\exp X_{n+1}} - \xi_n\right|^p\right) \\ &\quad = \boldsymbol{E}_\pi\left(\mathrm{e}^{-pS_n}\boldsymbol{E}_{\pi,\xi_n}\left|\left(\frac{\alpha^{(1)}_{n+1}}{\exp X_{n+1}} - 1\right) + \cdots + \left(\frac{\alpha^{(\xi_n)}_{n+1}}{\exp X_{n+1}} - 1\right)\right|^p\right). \end{aligned} \tag{22}$$

The summands under the module sign in the last expectations are independent identically distributed random variables with zero mean.

First we consider the case æ $\geqslant 2$. By the Dharmadhikary–Jogdeo equality (see, for example, [Petrov, 1975, chapter III, Section 5])

$$\boldsymbol{E}_{\pi,\xi_n}\left|\left(\frac{\alpha^{(1)}_{n+1}}{\exp X_{n+1}} - 1\right) + \cdots + \left(\frac{\alpha^{(\xi_n)}_{n+1}}{\exp X_{n+1}} - 1\right)\right|^p \leqslant c(p)\xi_n^{p/2}\boldsymbol{E}_\pi\left(\frac{\alpha_{n+1}}{\exp X_{n+1}}\right)^p, \tag{23}$$

where $c(p)>0$ depends on p only. Thus,

$$\boldsymbol{E}_\pi\left|\frac{\xi_{n+1}}{\exp S_{n+1}}-\frac{\xi_n}{\exp S_n}\right|^p \leqslant c(p)\boldsymbol{E}_\pi\left(\frac{\alpha_{n+1}}{\exp X_{n+1}}\right)^p \mathrm{e}^{-pS_n}\boldsymbol{E}_\pi\xi_n^{p/2}$$
$$\leqslant c(p)\boldsymbol{E}_\pi\left(\frac{\alpha_{n+1}}{\exp X_{n+1}}\right)^p \mathrm{e}^{-(p-[æ])S_n}\boldsymbol{E}_\pi\left(\frac{\xi_n}{\exp S_n}\right)^{[æ]}, \quad (24)$$

where the last inequality follows from (20). The random variables

$$\boldsymbol{E}_\pi\left(\frac{\alpha_{n+1}}{\exp X_{n+1}}\right)^p \quad \text{and} \quad \mathrm{e}^{-(p-[æ])S_n}\boldsymbol{E}_\pi\left(\frac{\xi_n}{\exp S_n}\right)^{[æ]}$$

are independent. Using this fact and taking into account (19) and Lemma 4 we deduce from (24) that

$$\boldsymbol{E}\left|\frac{\xi_{n+1}}{\exp S_{n+1}}-\frac{\xi_n}{\exp S_n}\right|^p \leqslant c(p)L_1\frac{([æ]!)^2}{2^{[æ]}}\boldsymbol{E}(\mathrm{e}^{-(p-[æ])S_n}\sigma_{[æ]}(n)).$$

Hence by Lemma 5 we conclude that

$$\boldsymbol{E}\left|\frac{\xi_{n+1}}{\exp S_{n+1}}-\frac{\xi_n}{\exp S_n}\right|^p \leqslant c(p)\frac{([æ]!)^2}{2^{[æ]}}L_1^{[æ]}n^{[æ]-1}q^{n-[æ]+1}.$$

Letting $C_p=c(p)([æ]!)^2 2^{-[æ]}L_1^{[æ]}q^{-[æ]+1}$ we obtain the statement of Lemma 6 for $æ\geqslant 2$.

Using the Bahr–Esseen inequality (see, for example, [Petrov, 1975, chapter III, Section 6]) for the case $æ\in[1,2)$, we obtain instead of (23) the inequality

$$\boldsymbol{E}_{\pi,\xi_n}\left|\left(\frac{\alpha_{n+1}^{(1)}}{\exp X_{n+1}}-1\right)+\cdots+\left(\frac{\alpha_{n+1}^{(\xi_n)}}{\exp X_{n+1}}-1\right)\right|^p \leqslant 4\xi_n\boldsymbol{E}_\pi\left(\frac{\alpha_{n+1}}{\exp X_{n+1}}\right)^p.$$

This inequality and relation (22) imply

$$\boldsymbol{E}_\pi\left|\frac{\xi_{n+1}}{\exp S_{n+1}}-\frac{\xi_n}{\exp S_n}\right|^p \leqslant 4\boldsymbol{E}_\pi\left(\xi_n\mathrm{e}^{-pS_n}\right)\boldsymbol{E}_\pi\left(\frac{\alpha_{n+1}}{\exp X_{n+1}}\right)^p$$
$$=4\mathrm{e}^{-(p-1)S_n}\boldsymbol{E}_\pi\left(\frac{\alpha_{n+1}}{\exp X_{n+1}}\right)^p.$$

Hence, taking into account independence of the random variables

$$\mathrm{e}^{-(p-1)S_n} \quad \text{and} \quad \boldsymbol{E}_\pi\left(\frac{\alpha_{n+1}}{\exp X_{n+1}}\right)^p,$$

condition (19) and relation (21), we have

$$\boldsymbol{E}\left|\frac{\xi_{n+1}}{\exp S_{n+1}}-\frac{\xi_n}{\exp S_n}\right|^p \leqslant 2L_1q^n.$$

Lemma 6 is proved. □

We assume that, for $T_x=+\infty$

$$\frac{\xi_{T_x}}{\exp S_{T_x}}=\lim_{k\to\infty}\frac{\xi_k}{\exp S_k} \quad (25)$$

(this limit exists a.s. for any branching process in random environment).

Lemma 7. *The sets*

$$A=\left\{\left(\frac{\xi_k}{\exp S_k}\right)^{æ},\ k\in\mathbf{N}_0\right\}\quad and\quad B=\left\{\left(\frac{\xi_{T_x}}{\exp S_{T_x}}\right)^{æ},\ x\in[0,+\infty)\right\}$$

of random variables are uniformly integrable for $æ\in(0,1)$. *This is true for* $æ\geqslant 1$ *as well, if condition* (19) *is satisfied.*

Proof. First we consider the case $æ\in(0,1)$. Set A is uniformly integrable since, for all $k\in\mathbf{N}_0$ $\boldsymbol{E}(\xi_k\exp(-S_k))=1$. Fix a positive number ε such that $æ+\varepsilon<1$. For any fixed environment $\{\pi_n\}$ the sequence $\{(\xi_k/\exp S_k)^{æ+\varepsilon},\ k\in\mathbf{N}_0\}$ is a uniformly integrable supermartingale in view of $\boldsymbol{E}_\pi(\xi_k\exp(-S_k))=1$ for all $k\in\mathbf{N}_0$. Since T_x is a stopping time for $x\in[0,+\infty)$, we conclude that for fixed environment $\{\pi_n\}$ the process $\{(\xi_{T_x}/\exp S_{T_x})^{æ+\varepsilon},\ x\in[0,+\infty)\}$ is a supermartingale (see, for example, [Elliott, 1982, chapter 3]). Therefore

$$\boldsymbol{E}_\pi\left(\frac{\xi_{T_x}}{\exp S_{T_x}}\right)^{æ+\varepsilon}\leqslant\boldsymbol{E}_\pi\left(\frac{\xi_{T_0}}{\exp S_{T_0}}\right)^{æ+\varepsilon}=1.$$

It is not difficult to show that

$$\boldsymbol{E}\left(\frac{\xi_{T_x}}{\exp S_{T_x}}\right)^{æ+\varepsilon}=\boldsymbol{E}\boldsymbol{E}_\pi\left(\frac{\xi_{T_x}}{\exp S_{T_x}}\right)^{æ+\varepsilon},$$

where for $T_x=+\infty$ the ratio on the left-hand side is defined by (25) and the same value on the right-hand side is defined as the limit of $\xi_k/\exp S_k$ as $k\to\infty$ provided the environment is fixed. Hence, for all $æ\in(0,1)$,

$$\boldsymbol{E}\left(\frac{\xi_{T_x}}{\exp S_{T_x}}\right)^{æ+\varepsilon}\leqslant 1$$

implying the uniform integrability of B.

Let now $æ\geqslant 1$ and condition (19) be satisfied. Without loss of generality we suppose that condition (20) is satisfied too. By Lemma 6, for $k\in\mathbf{N}_0$,

$$\boldsymbol{E}\left(\frac{\xi_k}{\exp S_k}\right)^p\leqslant L_2,\tag{26}$$

where L_2 is a positive constant independent of k. This provides the uniformly integrability of A. Finally, for a fixed environment, $\{(\xi_k/\exp S_k),\ k\in\mathbf{N}_0\}$ is a non-negative martingale and the Doob inequality gives, for all $k\in\mathbf{N}_0$,

$$\boldsymbol{E}_\pi\left(\sup_{n\leqslant k}\frac{\xi_n}{\exp S_n}\right)^p\leqslant\left(\frac{p}{p-1}\right)^p\boldsymbol{E}_\pi\left(\frac{\xi_k}{\exp S_k}\right)^p.$$

Therefore, for $k\in\mathbf{N}_0$

$$\boldsymbol{E}\left(\sup_{n\leqslant k}\frac{\xi_n}{\exp S_n}\right)^p\leqslant\left(\frac{p}{p-1}\right)^p\boldsymbol{E}\left(\frac{\xi_k}{\exp S_k}\right)^p.$$

Whence, passing to the limit as $k\to\infty$ and taking into account relation (26), we obtain

$$\boldsymbol{E}\left(\sup_{n\in\mathbf{N}_0}\frac{\xi_n}{\exp S_n}\right)^p\leqslant\left(\frac{p}{p-1}\right)^p L_2.\tag{27}$$

Observe that for all $x \in [0, +\infty)$

$$\frac{\xi_{T_x}}{\exp S_{T_x}} \leqslant \sup_{n \in \mathbf{N}_0} \frac{\xi_n}{\exp S_n}. \tag{28}$$

Relations (27) and (28) imply that, for all $x \in [0, +\infty)$,

$$\boldsymbol{E}\left(\frac{\xi_{T_x}}{\exp S_{T_x}}\right)^p \leqslant \left(\frac{p}{p-1}\right)^p L_2.$$

This gives the desired uniform integrability of B. Lemma 7 is proved. □

4. Representation of the event $\{T_x < +\infty\}$

Since $\{\xi_n\}$ is a supercritical branching process in random environment

$$\lim_{n \to \infty} \frac{\xi_n}{\exp S_n} = W < +\infty \tag{29}$$

exists a.s. Taking into account the assumption of Theorem 1 and Lemma 3 we have

$$\boldsymbol{E}(\xi_1 \ln^+ \xi_1 \exp(-X_1)) < +\infty. \tag{30}$$

It means (see [Tanny, 1988, Theorem 2]) that

$$\boldsymbol{P}\left(\lim_{n \to \infty} \frac{\xi_n}{\exp S_n} = 0\right) = \boldsymbol{P}(\exists n \in \mathbf{N} : \xi_n = 0) < 1. \tag{31}$$

Relations (29) and (31) yield

$$\begin{aligned} \boldsymbol{P}\left(+\infty > \lim_{n \to \infty} \frac{\xi_n}{\exp S_n} = W > 0\right) &= \boldsymbol{P}\left(\lim_{n \to \infty} \xi_n = +\infty\right) \\ &= \lim_{k \to \infty} \boldsymbol{P}(\xi_k > 0) > 0. \end{aligned} \tag{32}$$

Consider the random event

$$D = \left\{\lim_{n \to \infty} \xi_n = +\infty\right\}.$$

Clearly, the inequality $\lim_{n \to \infty}(\xi_n \exp(-S_n)) > 0$ implies $\lim_{n \to \infty} \xi_n = +\infty$ a.s. Hence, by relation (32)

$$\boldsymbol{P}\left(+\infty > \lim_{n \to \infty} \frac{\xi_n}{\exp S_n} = W > 0,\ D\right) = \boldsymbol{P}(D) > 0. \tag{33}$$

Fix an arbitrary $\varepsilon \in (0, 1)$ and, for $k \in \mathbf{N}$, consider the event

$$D(k) = \left\{\lim_{n \to \infty} \xi_n = +\infty,\ \sup_{n:\ n \geqslant k} \left|\frac{\exp S_n}{\xi_n} - \frac{\exp S_k}{\xi_k}\right| \leqslant \varepsilon \frac{\exp S_k}{\xi_k}\right\}.$$

Lemma 8. *Let condition* (30) *hold. Then*

$$\lim_{k \to \infty} \boldsymbol{P}(D(k)) = \boldsymbol{P}(D).$$

Proof. Let δ be positive and less than $\boldsymbol{P}(D)$. By (33) there exists $c > 0$ such that

$$\boldsymbol{P}(1/W \geqslant c,\ D) \geqslant \boldsymbol{P}(D) - \delta. \tag{34}$$

Applying (33) once again we obtain

$$\lim_{k\to\infty} \boldsymbol{P}\left(\sup_{n:\ n\geqslant k}\left|\frac{\exp S_n}{\xi_n}-\frac{\exp S_k}{\xi_k}\right|\leqslant\frac{\varepsilon c}{2},\ D\right)=\boldsymbol{P}(D). \tag{35}$$

By (34) and (35) we conclude that

$$\underline{\lim}_{k\to\infty} \boldsymbol{P}\left(\sup_{n:\ n\geqslant k}\left|\frac{\exp S_n}{\xi_n}-\frac{\exp S_k}{\xi_k}\right|\leqslant\frac{\varepsilon}{2W},\ W>0,\ D\right)\geqslant\boldsymbol{P}(D)-\delta. \tag{36}$$

But if

$$\sup_{n:\ n\geqslant k}\left|\frac{\exp S_n}{\xi_n}-\frac{\exp S_k}{\xi_k}\right|\leqslant\frac{\varepsilon}{2W}$$

then

$$\left|\frac{1}{W}-\frac{\exp S_k}{\xi_k}\right|\leqslant\frac{\varepsilon}{2W}\ \text{a.s.}\ \Rightarrow\ \frac{\exp S_k}{\xi_k}\geqslant\frac{1}{W}\left(1-\frac{\varepsilon}{2}\right)\ \text{a.s.} \tag{37}$$

Relations (36) and (37) imply

$$\underline{\lim}_{k\to\infty} \boldsymbol{P}\left(\sup_{n:\ n\geqslant k}\left|\frac{\exp S_n}{\xi_n}-\frac{\exp S_k}{\xi_k}\right|\leqslant\frac{\varepsilon}{2(1-\varepsilon/2)}\frac{\exp S_k}{\xi_k},\ D\right)\geqslant\boldsymbol{P}(D)-\delta.$$

Since $\varepsilon\in(0,1)$, we obtain $\underline{\lim}_{k\to\infty}\boldsymbol{P}(D(k))\geqslant\boldsymbol{P}(D)-\delta$. On the other hand, $\overline{\lim}_{k\to\infty}\boldsymbol{P}(D(k))\leqslant\boldsymbol{P}(D)$. Since δ is arbitrary, Lemma 8 follows. □

For $x\in(0,+\infty)$, $k\in\mathbf{N}_0$ and $\varepsilon\in(0,1)$ consider the event

$$D_{x,k}=\left\{k<T_x<+\infty,\ \lim_{n\to\infty}\xi_n=+\infty,\ \sup_{n:\ n\geqslant k}\left|\frac{\exp S_n}{\xi_n}-\frac{\exp S_k}{\xi_k}\right|\leqslant\varepsilon\frac{\exp S_k}{\xi_k}\right\}.$$

Lemma 9. *If condition* (30) *holds, then*

$$\overline{\lim}_{k\to\infty}\ \overline{\lim}_{x\to+\infty}\ |\boldsymbol{P}(\xi_k>0)-\boldsymbol{P}(D_{x,k})|=0.$$

Proof. It is obviously that for all fixed $k\in\mathbf{N}_0$

$$\lim_{x\to+\infty}\boldsymbol{P}(T_x\leqslant k)=\lim_{x\to+\infty}\boldsymbol{P}\left(\max_{n\leqslant k}\xi_n>x\right)=0. \tag{38}$$

Therefore

$$\lim_{x\to+\infty}\boldsymbol{P}(k<T_x<+\infty,\ D)=\boldsymbol{P}(D) \tag{39}$$

and in view of Lemma 8

$$\overline{\lim}_{k\to\infty}\ \overline{\lim}_{x\to+\infty}\ |\boldsymbol{P}(k<T_x<+\infty,\ D)-\boldsymbol{P}(D_{x,k})|=0. \tag{40}$$

Since

$$\begin{aligned}|\boldsymbol{P}(\xi_k>0)-\boldsymbol{P}(D_{x,k})|&\leqslant|\boldsymbol{P}(\xi_k>0)-\boldsymbol{P}(D)|\\&\quad+|\boldsymbol{P}(D)-\boldsymbol{P}(k<T_x<+\infty,\ D)|+|\boldsymbol{P}(k<T_x<+\infty,\ D)-\boldsymbol{P}(D_{x,k})|,\end{aligned}$$

relations (32), (39) and (40) imply the statement of Lemma 9. □

For $x\in(0,+\infty)$ consider the event

$$D_x=\{T_x<+\infty\}.$$

Lemma 10. *If condition* (30) *is valid*, *then*

$$\lim_{x\to+\infty} \boldsymbol{P}(D_x) = \boldsymbol{P}(D).$$

Proof. It is clear that, for all $k \in \mathbf{N}$,

$$\boldsymbol{P}(D_x) = \boldsymbol{P}(T_x < +\infty,\ \xi_k > 0) + \boldsymbol{P}(T_x < +\infty,\ \xi_k = 0),$$
$$\boldsymbol{P}(D) = \boldsymbol{P}\left(T_x < +\infty,\ \lim_{n\to\infty} \xi_n = +\infty\right).$$

Therefore,

$$0 \leqslant \boldsymbol{P}(D_x) - \boldsymbol{P}(D) \leqslant \boldsymbol{P}(T_x < +\infty,\ \xi_k = 0) + \boldsymbol{P}(T_x < +\infty,\ \xi_k > 0) - \boldsymbol{P}\left(T_x < +\infty,\ \lim_{n\to\infty} \xi_n = +\infty\right). \tag{41}$$

Since $\boldsymbol{P}(T_x < +\infty,\ \xi_k = 0) \leqslant \boldsymbol{P}(\max_{n\leqslant k} \xi_n > x)$, it follows from relation (38) that

$$\overline{\lim_{x\to+\infty}}\ \boldsymbol{P}(T_x < +\infty,\ \xi_k = 0) = 0. \tag{42}$$

Further we have

$$0 \leqslant \boldsymbol{P}(T_x < +\infty,\ \xi_k > 0) - \boldsymbol{P}\left(T_x < +\infty,\ \lim_{n\to\infty} \xi_n = +\infty\right) \leqslant \boldsymbol{P}(\xi_k > 0) - \boldsymbol{P}\left(\lim_{n\to\infty} \xi_n = +\infty\right).$$

Hence, using relation (32), we obtain

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{x\to+\infty}} \left|\boldsymbol{P}(T_x < +\infty,\ \xi_k > 0) - \boldsymbol{P}\left(T_x < +\infty,\ \lim_{n\to\infty} \xi_n = +\infty\right)\right| = 0. \tag{43}$$

Relations (41)–(43) imply Lemma 10. □

Lemma 11. *If condition* (30) *holds*, *then*

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{x\to+\infty}} |\boldsymbol{P}(D_x) - \boldsymbol{P}(D_{x,k})| = 0.$$

Proof. It is clear that

$$|\boldsymbol{P}(D_x) - \boldsymbol{P}(D_{x,k})| \leqslant |\boldsymbol{P}(D_x) - \boldsymbol{P}(D)| + |\boldsymbol{P}(D) - \boldsymbol{P}(\xi_k > 0)| + |\boldsymbol{P}(\xi_k > 0) - \boldsymbol{P}(D_{x,k})|.$$

Combining Lemmas 9, 10 and relation (32) gives Lemma 11. □

5. Accomplishment of the proof of Theorem 1

First we consider

$$M_2(x,k) = \boldsymbol{E}\left(\left(\left(\frac{\xi_{T_x}}{\exp S_{T_x}}\right)^{æ} - \left(\frac{\xi_k}{\exp S_k}\right)^{æ}\right)\left(\frac{x}{\xi_{T_x}}\right)^{æ};\ D_x\right)$$

(recall (15)) and show that

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{x\to+\infty}} |M_2(x,k)| = 0. \tag{44}$$

By the definition of T_x

$$\xi_{T_x} > x \tag{45}$$

if $T_x < +\infty$. On the other hand, if $\lim_{n\to\infty} \xi_n = +\infty$, then $T_x < +\infty$ and $\lim_{x\to+\infty} T_x = +\infty$. Hence in virtue of (29)

$$\lim_{x\to+\infty} \frac{\xi_{T_x}}{\exp S_{T_x}} = W \quad \text{a.s.} \tag{46}$$

This relation remains true if $\xi_n = 0$ for some $n \in \mathbf{N}$, since in this case $T_x = +\infty$ for large x and by definition $\xi_{T_x}/\exp S_{T_x} = \lim_{k\to\infty} \xi_k/\exp S_k = 0$ given $T_x = +\infty$.

Observe, that by Lemma 7 and relation (29)

$$\boldsymbol{E} W^{æ} < +\infty. \tag{47}$$

It follows from (45) that:

$$|M_2(x,k)| \leqslant \boldsymbol{E} \left| \left(\frac{\xi_{T_x}}{\exp S_{T_x}} \right)^{æ} - \left(\frac{\xi_k}{\exp S_k} \right)^{æ} \right|.$$

Passing to limit first as $x \to +\infty$ and then as $k \to \infty$ in the right-hand side, and taking into account Lemma 7 and relations (29), (46) and (47), we obtain 0, that proves (44).

Now we consider the term

$$M_1(x,k) = \boldsymbol{E} \left(\left(\frac{\xi_k}{\exp S_k} \right)^{æ} \left(\frac{x}{\xi_{T_x}} \right)^{æ} ;\ D_x \right).$$

Put

$$M(x,k) = \boldsymbol{E} \left(\left(\frac{\xi_k}{\exp S_k} \right)^{æ} \left(\frac{x}{\xi_{T_x}} \right)^{æ} ;\ D_{x,k} \right).$$

Obviously,

$$0 \leqslant M_1(x,k) - M(x,k) \leqslant \boldsymbol{E} \left(\left(\frac{\xi_k}{\exp S_k} \right)^{æ} ;\ D_x \setminus D_{x,k} \right).$$

Whence, taking into account Lemmas 7 and 12, we see that

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{x\to+\infty}}\ |M_1(x,k) - M(x,k)| = 0. \tag{48}$$

Denote by $\boldsymbol{E}_k$ and $\boldsymbol{P}_k$ the conditional expectation and conditional probability given $\pi_0, \pi_1, \ldots, \pi_k, \xi_0,\ \xi_1, \ldots, \xi_k$. Then

$$M(x,k) = \boldsymbol{E} \left(\left(\frac{\xi_k}{\exp S_k} \right)^{æ} \boldsymbol{E}_k \left(\left(\frac{x}{\xi_{T_x}} \right)^{æ} ;\ D_{x,k} \right) \right). \tag{49}$$

It is clear that

$$\begin{aligned} \boldsymbol{E}_k \left(\left(\frac{x}{\xi_{T_x}} \right)^{æ} ;\ D_{x,k} \right) &= \int_0^1 \left(\boldsymbol{P}_k(D_{x,k}) - \boldsymbol{P}_k \left(\left(\frac{x}{\xi_{T_x}} \right)^{æ} \leqslant z;\ D_{x,k} \right) \right) \mathrm{d}z \\ &= \int_0^1 (\boldsymbol{P}_k(D_{x,k}) - \boldsymbol{P}_k(\xi_{T_x} > x z^{-1/æ};\ D_{x,k}))\, \mathrm{d}z \\ &= \int_1^{+\infty} (\boldsymbol{P}_k(D_{x,k}) - \boldsymbol{P}_k(\xi_{T_x} > xy;\ D_{x,k})) \frac{æ\, \mathrm{d}y}{y^{æ+1}}. \end{aligned} \tag{50}$$

If $D_{x,k}$ occurs, then $k < T_x < +\infty$ and, therefore,

$$\boldsymbol{P}_k(\xi_{T_x} > xy;\ D_{x,k}) = \sum_{l=k+1}^{\infty} \boldsymbol{P}_k(\xi_l > xy,\ T_x = l;\ D_{x,k}). \tag{51}$$

Observe now that by the definition of $D_{x,k}$, for $y \geqslant 1,\ l = k+2, k+3, \ldots,$

$$\begin{aligned}
&\boldsymbol{P}_k(\xi_l > xy,\ T_x = l;\ D_{x,k}) \\
&\quad = \boldsymbol{P}_k(\xi_{k+1} \leqslant x, \ldots, \xi_{l-1} \leqslant x, \xi_l > xy;\ D_{x,k}) \\
&\quad = \boldsymbol{P}_k\left(\mathrm{e}^{S_{k+1}} \leqslant x\frac{\mathrm{e}^{S_{k+1}}}{\xi_{k+1}}, \ldots, \mathrm{e}^{S_{l-1}} \leqslant x\frac{\mathrm{e}^{S_{l-1}}}{\xi_{l-1}},\ \mathrm{e}^{S_l} > xy\frac{\mathrm{e}^{S_l}}{\xi_l};\ D_{x,k}\right) \\
&\quad \leqslant \boldsymbol{P}_k\left(\mathrm{e}^{S_{k+1}} \leqslant (1+\varepsilon)x\frac{\mathrm{e}^{S_k}}{\xi_k}, \ldots, \mathrm{e}^{S_{l-1}} \leqslant (1+\varepsilon)x\frac{\mathrm{e}^{S_k}}{\xi_k},\ \mathrm{e}^{S_l} > (1-\varepsilon)xy\frac{\mathrm{e}^{S_k}}{\xi_k};\ D_{x,k}\right).
\end{aligned}$$

Put $S_0'=0,\ S_1'=S_{k+1}-S_k,\ S_2'=S_{k+2}-S_k, \ldots$ Clearly, $\{S_n'\}$ generates a random walk which has the same distribution as $\{S_n\}$ but is independent of $\pi_0, \pi_1, \ldots, \pi_k,\ \xi_0, \xi_1, \ldots, \xi_k$. Thus, we have established that, for $y \geqslant 1\ l = k+2, k+3, \ldots,$

$$\begin{aligned}
&\boldsymbol{P}_k(\xi_l > xy,\ T_x = l;\ D_{x,k}) \\
&\quad \leqslant \boldsymbol{P}_k\left(S_1' \leqslant \ln\frac{(1+\varepsilon)x}{\xi_k}, \ldots, S_{l-1-k}' \leqslant \ln\frac{(1+\varepsilon)x}{\xi_k},\ S_{l-k}' > \ln\frac{(1-\varepsilon)xy}{\xi_k};\ D_{x,k}\right)
\end{aligned} \tag{52}$$

(for $l = k+1$ we have to replace the right-hand side of relation (52) with $\boldsymbol{P}_k(S_1' > \ln((1-\varepsilon)xy/\xi_k);\ D_{x,k})$). It follows from (51) and (52) that, for $y \geqslant 1$,

$$\begin{aligned}
&\boldsymbol{P}_k(\xi_{T_x} > xy;\ D_{x,k}) \\
&\quad \leqslant \boldsymbol{P}_k\left(\exists\ l \in \mathbf{N} : S_1' \leqslant \ln\frac{(1+\varepsilon)x}{\xi_k}, \ldots, S_{l-1}' \right. \\
&\qquad\qquad \left. \leqslant \ln\frac{(1+\varepsilon)x}{\xi_k},\ S_l' > \ln\frac{(1-\varepsilon)xy}{\xi_k};\ D_{x,k}\right).
\end{aligned} \tag{53}$$

If $y > (1+\varepsilon)/(1-\varepsilon)$, then the event

$$\left\{\exists\ l \in \mathbf{N} : S_1' \leqslant \ln\frac{(1+\varepsilon)x}{\xi_k}, \ldots, S_{l-1}' \leqslant \ln\frac{(1+\varepsilon)x}{\xi_k},\ S_l' > \ln\frac{(1-\varepsilon)xy}{\xi_k}\right\}$$

means that the first overshoot of the random walk $\{S_n'\}$ over the level $\ln((1+\varepsilon)x/\xi_k)$ is greater than

$$\ln\frac{(1-\varepsilon)xy}{\xi_k} - \ln\frac{(1+\varepsilon)x}{\xi_k} = \ln\frac{(1-\varepsilon)y}{1+\varepsilon}.$$

Let $\chi(t)$ be the first overshoot of $\{S_n'\}$ over the level t. It follows from (53) that, for $y > (1+\varepsilon)/(1-\varepsilon)$,

$$\boldsymbol{P}_k(\xi_{T_x} > xy;\ D_{x,k}) \leqslant \boldsymbol{P}_k\left(\chi\left(\ln\frac{(1+\varepsilon)x}{\xi_k}\right) > \ln\frac{(1-\varepsilon)y}{1+\varepsilon};\ D_{x,k}\right). \tag{54}$$

It is known (see, for example, [Feller, 1971, chapter XI]), that if the distribution of S_1' is non-lattice (it is provided by the assumption of Theorem 1) and $E|S_1'| < +\infty$ (it holds in view of condition (3) and Lemma 1), then for any $u \in (0, +\infty)$ there exists

$$\lim_{t \to +\infty} \boldsymbol{P}(\chi(t) > u) = G(u), \tag{55}$$

where $1-G(u)$ is the distribution function of an absolutely continuous probability measure. Since $\{S_n'\}$ is independent of $\pi_0,\pi_1,\ldots,\pi_n,\ \xi_0,\xi_1,\ldots,\xi_n$, relation (55) implies, for $y>(1+\varepsilon)/(1-\varepsilon),\ \xi_k>0$,

$$\lim_{x\to+\infty}\boldsymbol{P}_k\left(\chi\left(\ln\frac{(1+\varepsilon)x}{\xi_k}\right)>\ln\frac{(1-\varepsilon)y}{1+\varepsilon}\right)=G\left(\ln\frac{(1-\varepsilon)y}{1+\varepsilon}\right). \tag{56}$$

It follows from (50) and (54) that:

$$\begin{aligned}
M(x,k)\geqslant{}&\boldsymbol{E}\left(\left(\frac{\xi_k}{\exp S_k}\right)^{æ}\int_{(1+\varepsilon)/(1-\varepsilon)}^{+\infty}\left(\boldsymbol{P}_k(D_{x,k})\right.\right.\\
&\left.\left.-\boldsymbol{P}_k\left(\chi\left(\ln\frac{(1+\varepsilon)x}{\xi_k}\right)>\ln\frac{(1-\varepsilon)y}{1+\varepsilon};\ D_{x,k}\right)\right)\frac{æ\,\mathrm{d}y}{y^{æ+1}}\right)\\
\geqslant{}&\boldsymbol{E}\left(\left(\frac{\xi_k}{\exp S_k}\right)^{æ}\int_{\frac{(1+\varepsilon)}{(1-\varepsilon)}}^{+\infty}\left(1-\boldsymbol{P}_k\left(\chi\left(\ln\frac{(1+\varepsilon)x}{\xi_k}\right)\right.\right.\right.\\
&\left.\left.\left.>\ln\frac{(1-\varepsilon)y}{1+\varepsilon}\right)\right)\frac{æ\,\mathrm{d}y}{y^{æ+1}}\right)\\
&-\boldsymbol{E}\left(\left(\frac{\xi_k}{\exp S_k}\right)^{æ}\int_{(1+\varepsilon)/(1-\varepsilon)}^{+\infty}\boldsymbol{P}_k(\bar{D}_{x,k})\frac{æ\,\mathrm{d}y}{y^{æ+1}}\right),
\end{aligned} \tag{57}$$

where $\bar{D}_{x,k}$ is the complementary event to $D_{x,k}$. In virtue of Lemmas 7 and 10

$$\begin{aligned}
&\overline{\lim_{k\to\infty}}\ \overline{\lim_{x\to+\infty}}\ \boldsymbol{E}\left(\left(\frac{\xi_k}{\exp S_k}\right)^{æ}\int_{(1+\varepsilon)/(1-\varepsilon)}^{+\infty}\boldsymbol{P}_k(\bar{D}_{x,k})\frac{æ\,\mathrm{d}y}{y^{æ+1}}\right)\\
&\leqslant\overline{\lim_{k\to\infty}}\ \overline{\lim_{x\to+\infty}}\ \boldsymbol{E}\left(\left(\frac{\xi_k}{\exp S_k}\right)^{æ};\ \bar{D}_{x,k}\right)\\
&=\overline{\lim_{k\to\infty}}\ \overline{\lim_{x\to+\infty}}\ \boldsymbol{E}\left(\left(\frac{\xi_k}{\exp S_k}\right)^{æ};\ \{\xi_k>0\}\cap\bar{D}_{x,k}\right)=0.
\end{aligned} \tag{58}$$

Applying Lemma 7 and recalling relations (29) and (56) we obtain

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{x\to+\infty}\boldsymbol{E}\left(\left(\frac{\xi_k}{\exp S_k}\right)^{æ}\int_{(1+\varepsilon)/(1-\varepsilon)}^{+\infty}\left(1-\boldsymbol{P}_k\left(\chi\left(\ln\frac{(1+\varepsilon)x}{\xi_k}\right)\right.\right.\right.\\
&\left.\left.\left.>\ln\frac{(1-\varepsilon)y}{1+\varepsilon}\right)\right)\frac{æ\,\mathrm{d}y}{y^{æ+1}}\right)\\
&=\boldsymbol{E}W^{æ}\int_{(1+\varepsilon)/(1-\varepsilon)}^{+\infty}\left(1-G\left(\ln\frac{(1-\varepsilon)y}{1+\varepsilon}\right)\right)\frac{æ\,\mathrm{d}y}{y^{æ+1}}.
\end{aligned} \tag{59}$$

It follows from (57)–(59) that

$$\underline{\lim_{k\to\infty}}\ \underline{\lim_{x\to+\infty}}\ M(x,k)\geqslant\boldsymbol{E}W^{æ}\int_{(1+\varepsilon)/(1-\varepsilon)}^{+\infty}\left(1-G\left(\ln\frac{(1-\varepsilon)y}{1+\varepsilon}\right)\right)\frac{æ\,\mathrm{d}y}{y^{æ+1}}. \tag{60}$$

Now we establish an upper bound for $M(x,k)$. Similarly to (52) we find that, for $y\geqslant 1,\ l=k+2,k+3,\dots,$

$$\boldsymbol{P}_k(\xi_l > xy,\ T_x = l;\ D_{x,k}) \geqslant \boldsymbol{P}_k\left(S_1'\leqslant \ln\frac{(1-\varepsilon)x}{\xi_k},\dots,S_{l-1-k}'\leqslant \ln\frac{(1-\varepsilon)x}{\xi_k},\ S_{l-k}' > \ln\frac{(1+\varepsilon)xy}{\xi_k};\ D_{x,k}\right) \tag{61}$$

(for $l=k+1$ we have to replace on right-hand side of relation (61) with $\boldsymbol{P}_k(S_1' > \ln((1+\varepsilon)xy/\xi_k;\ D_{x,k}))$. It is clear from (51) and (61) that, for $y\geqslant 1$,

$$\boldsymbol{P}_k(\xi_{T_x} > xy;\ D_{x,k})\geqslant \boldsymbol{P}_k\left(\exists\ l\in\mathbf{N}: S_1'\leqslant \ln\frac{(1-\varepsilon)x}{\xi_k},\dots,S_{l-1}'\leqslant \ln\frac{(1-\varepsilon)x}{\xi_k},\right.$$
$$\left. S_l' > \ln\frac{(1+\varepsilon)xy}{\xi_k};\ D_{x,k}\right) = \boldsymbol{P}_k\left(\chi\left(\ln\frac{(1-\varepsilon)x}{\xi_k}\right) > \ln\frac{(1+\varepsilon)y}{1-\varepsilon};\ D_{x,k}\right). \tag{62}$$

Relations (49), (50) and (62) imply

$$M(x,k)\leqslant \boldsymbol{E}\left(\left(\frac{\xi_k}{\exp S_k}\right)^{æ}\int_1^{+\infty}\left(\boldsymbol{P}_k(D_{x,k})\right.\right.$$
$$\left.\left. -\boldsymbol{P}_k\left(\chi\left(\ln\frac{(1-\varepsilon)x}{\xi_k}\right) > \ln\frac{(1+\varepsilon)y}{1-\varepsilon};\ D_{x,k}\right)\right)\frac{æ\,\mathrm{d}y}{y^{æ+1}}\right)$$
$$\leqslant \boldsymbol{E}\left(\left(\frac{\xi_k}{\exp S_k}\right)^{æ}\int_1^{+\infty}\left(1-\boldsymbol{P}_k\left(\chi\left(\ln\frac{(1-\varepsilon)x}{\xi_k}\right)\right.\right.\right.$$
$$\left.\left.\left. > \ln\frac{(1+\varepsilon)y}{1-\varepsilon}\right)\right)\frac{æ\,\mathrm{d}y}{y^{æ+1}}\right). \tag{63}$$

Using Lemma 7 and relations (29) and (55) we obtain

$$\lim_{k\to\infty}\lim_{x\to+\infty}\boldsymbol{E}\left(\left(\frac{\xi_k}{\exp S_k}\right)^{æ}\int_1^{+\infty}\left(1-\boldsymbol{P}_k\left(\chi\left(\ln\frac{(1-\varepsilon)x}{\xi_k}\right)\right.\right.\right.$$
$$\left.\left.\left. > \ln\frac{(1+\varepsilon)y}{1-\varepsilon}\right)\right)\frac{æ\,\mathrm{d}y}{y^{æ+1}}\right) = \boldsymbol{E}W^{æ}\int_1^{+\infty}\left(1-G\left(\ln\frac{(1+\varepsilon)y}{1-\varepsilon}\right)\right)\frac{æ\,\mathrm{d}y}{y^{æ+1}}. \tag{64}$$

It follows from (63) and (64) that:

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{x\to+\infty}}\ M(x,k)\leqslant \boldsymbol{E}W^{æ}\int_1^{+\infty}\left(1-G\left(\ln\frac{(1+\varepsilon)y}{1-\varepsilon}\right)\right)\frac{æ\,\mathrm{d}y}{y^{æ+1}}. \tag{65}$$

Passing to the limit as $\varepsilon\downarrow 0$ in (60) and (65) we have

$$\underline{\lim_{k\to\infty}}\ \underline{\lim_{x\to+\infty}}\ M(x,k) = \overline{\lim_{k\to\infty}}\ \overline{\lim_{x\to+\infty}}\ M(x,k) = \boldsymbol{E}W^{æ}\int_1^{+\infty}(1-G(\ln y))\frac{æ\,\mathrm{d}y}{y^{æ+1}}. \tag{66}$$

Relations (15), (44), (48) and (66) imply

$$\boldsymbol{E}W^{æ}\int_1^{+\infty}(1-G(\ln y))\frac{æ\,\mathrm{d}y}{y^{æ+1}}\leqslant \underline{\lim_{x\to+\infty}}\ \boldsymbol{E}\left(\left(\frac{\xi_{T_x}}{\exp S_{T_x}}\right)^{æ}\left(\frac{x}{\xi_{T_x}}\right)^{æ};\ D_x\right)$$
$$\leqslant \overline{\lim_{x\to+\infty}}\ \boldsymbol{E}\left(\left(\frac{\xi_{T_x}}{\exp S_{T_x}}\right)^{æ}\left(\frac{x}{\xi_{t_x}}\right)^{æ};\ D_x\right)\leqslant \boldsymbol{E}W^{æ}\int_1^{+\infty}(1-G(\ln y))\frac{æ\,\mathrm{d}y}{y^{æ+1}}.$$

Thus, we have

$$\lim_{x\to+\infty} \boldsymbol{E}\left(\left(\frac{\xi_{T_x}}{\exp S_{T_x}}\right)^{æ}\left(\frac{x}{\xi_{T_x}}\right)^{æ};\ T_x<+\infty\right) = \boldsymbol{E}W^{æ}\int_1^{+\infty}(1-G(\ln y))\frac{æ\,\mathrm{d}y}{y^{æ+1}}. \tag{67}$$

Recall that $\boldsymbol{E}W^{æ}$ is finite (see (47)). We consider the second factor on the right-hand side of (67). Let χ be a random variable to which $\chi(t)$ converges in distribution as $t\to+\infty$ (see (55)). Then $\boldsymbol{P}(\chi>u)=G(u)$ and

$$\begin{aligned}\int_1^{+\infty}(1-G(\ln y))\frac{æ\,\mathrm{d}y}{y^{æ+1}} &= \int_1^{+\infty}\boldsymbol{P}(\chi\leqslant\ln y)\frac{æ\,\mathrm{d}y}{y^{æ+1}}\\ &= \int_0^{+\infty}\boldsymbol{P}(\chi\leqslant u)æ\,\mathrm{e}^{-æu}\,\mathrm{d}u\\ &= \int_0^{+\infty}\mathrm{e}^{-æu}\,\mathrm{d}\boldsymbol{P}(\chi\leqslant u) = \boldsymbol{E}\mathrm{e}^{-æ\chi}. \end{aligned} \tag{68}$$

Combining (14), (67), and (68) we see that, as $x\to+\infty$,

$$\boldsymbol{E}(\exp(-æS_{T_x});\ T_x<+\infty) \sim \boldsymbol{E}W^{æ}\boldsymbol{E}\mathrm{e}^{-æ\chi}x^{-æ}.$$

Thus, Theorem 1 is proved and the constant in (5) has the form

$$K = \boldsymbol{E}W^{æ}\boldsymbol{E}\mathrm{e}^{-æ\chi},$$

where W and χ are the random variables defined by the branching process in the conjugate random environment.

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