JOURNAL OF ALGEBRA 35, 123–138 (1975)

A Grothendieck Ring of Higher Degree Forms

D. K. HARRISON*

Department of Mathematics, University of Oregon, Eugene, Oregon 97403

Communicated by N. Jacobson

Received October 10, 1973

The purpose of this paper is to lay the ground work for a treatment of higher degree forms which is analogous to the treatment initiated by Witt of quadratic forms (see [3]).

In our first section we set up notation and state a cancellation theorem which generalizes Witt's cancellation theorem to arbitrary degree greater than one for fields whose characteristic is zero or larger than the degree. In the second section we prove this together with the uniqueness of the decomposition of a form into a direct sum of indecomposable forms. In the third section we put the forms together to get a sort of Grothendieck ring of forms. In the fourth section we prove that over an algebraically closed field, the tensor product of two indecomposable forms is indecomposable. This means that the Grothendieck ring is just a semigroup ring—the semigroup being the set of additively indecomposable forms with tensor product as operation. In the case of a field which is not algebraically closed, we define the center of a form, show that a nonsingular form is a sort of generalized separable algebra, and show how to reduce considerations, at least for nonsingular forms, to forms which stay indecomposable over any field extension.

We are mainly interested in forms over fields which have characteristic zero or sufficiently large. For these there is no loss of generality in assuming no variable can be removed (i.e., have its coefficients made zero) by a linear change of variables; such forms are called nondegenerate. Sometimes it seems appropriate to consider forms over Noetherian rings, in which case we restrict attention to those forms which are nondegenerate when considered modulo any maximal ideal. We call these nondegenerate, but use an equivalent and easier to handle definition for them.

* Written with the support of NSF Grant No. GP-32842.

1. Notation and Terminology

If A is a set, A^n demotes $A \times A \times \cdots \times A$ (n copies). Let F be a field. For n and r nonnegative integers, we wish to consider forms of degree r in n indeterminants with coefficients in F; in other words, elements of the polynomial ring $F[X_1, X_2, ..., X_n]$ which are sums of monomials of degree r. Two forms f and g are called *equivalent* and we write $f \cong g$ if f can be gotten from g by a reversible linear change of variables; i.e., if f and g have the same number of indeterminants n and there exists an invertable n by n matrix $[\alpha_{i,j}]$ with entries in F with

$$f(\Sigma\alpha_{1,j}X_j, \Sigma\alpha_{2,j}X_j, ..., \Sigma\alpha_{n,j}X_j) = g(X_1, X_2, ..., X_n).$$

We shall only consider forms of degree r where the characteristic of F is zero or greater than r. In other words, we assume $r! \neq 0$ in F where $r!$ denotes $1 + 1 + \cdots + 1$ ($1 \cdot 2 \cdot 3 \cdots r$ copies). With this assumption, if f and g are forms of degree r in n indeterminants, $f(a_1, ..., a_n) = g(a_1, ..., a_n)$ for all $a_1, ..., a_n \in F$ if and only if $f = g$. Also, for f a form of degree r, there exists a unique symmetric multilinear map θ_f from r copies of F^n to F (i.e., from $(F^n)^r$ to F) with

$$\theta_f((a_1, ..., a_n), (a_1, ..., a_n), ..., (a_1, ..., a_n)) = f(a_1, ..., a_n)$$

for all $a_1, ..., a_n \in F$ (that this exists is a tedious induction). Here F^n is made into a vector space in the usual way. Any vector space of dimension n over F is isomorphic to F^n, so F^n may be replaced simply by a vector space of dimension n. By a *symmetric space of degree* r over F we mean a pair (V, θ), where V is a finite dimensional vector space over F and θ is a symmetric multilinear map from V^r to F. Two symmetric spaces (V, θ) and (V', θ') are called *isomorphic* if there is a bijective linear map t from V to V' with $\theta'(t(v_1), t(v_2), ..., t(v_r)) = \theta(v_1, v_2, ..., v_r)$ for all $v_1, v_2, ..., v_r \in V$. Such a t is called an *isomorphism* from (V, θ) to (V', θ'). Thus since $r! \neq 0$ in F, the equivalence classes of forms of degree r, and the isomorphism classes of symmetric spaces of degree r are in bijective correspondence. A symmetric space (V, θ) corresponds to the set of all forms equivalent to f where f is defined by choosing a basis $v_1, v_2, ..., v_n$ of V and letting f be that form with

$$\theta(x_1v_1 + \cdots + x_nv_n, x_1v_1 + \cdots + x_nv_n, ..., x_1v_1 + \cdots + x_nv_n) = f(x_1, x_2, ..., x_n)$$

for all $x_1, x_2, ..., x_n \in F$. We write $f \leftrightarrow (V, \theta)$ by $v_1, v_2, ..., v_n$ if the above displayed formula holds for the particular basis $v_1, v_2, ..., v_n$ of V.

Let (V, θ) be a symmetric space of degree r. Let $r \geqslant 1$. Let $v \in V$. We define a symmetric multilinear map $\theta^{(v)}$ from $V^{(r-1)}$ to V by

$$\theta^{(v)}(v_1, v_2, ..., v_{r-1}) = \theta(v, v_1, v_2, ..., v_{r-1})$$

for all $v_1, v_2, \ldots, v_{r-1} \in V$. We call $(V, \theta^{(v)})$ the *derivative of* (V, θ) *in the direction of* v, since it can be checked that if $f \leftrightarrow (V, \theta)$ by $v_1, v_2, \ldots, v_n$, and $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ with $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, then $\Sigma(1/r)\, \alpha_i\, \partial f/\partial X_i \leftrightarrow (V, \theta^{(v)})$ by $v_1, v_2, \ldots, v_n$.

Let $f \leftrightarrow (V, \theta)$ by $v_1, v_2, \ldots, v_n$. Let K be an extension field of F. It is usual to think of $F[X_1, \ldots, X_n]$ as a subring of $K[X_1, \ldots, X_n]$ and so f can be thought of as an element in $K[X_1, \ldots, X_n]$. One checks there exists a unique symmetric K-multilinear map θ_K from $(K \otimes_F V)^r$ to K with

$$\theta_K(1 \otimes u_1, 1 \otimes u_2, \ldots, 1 \otimes u_r) = \theta(u_1, u_2, \ldots, u_r) \quad \text{for all } u_1, u_2, \ldots, u_r \in V.$$

One checks $f \leftrightarrow (K \otimes_F V, \theta_K)$ by $1 \otimes v_1, 1 \otimes v_2, \ldots, 1 \otimes v_n$. Let Ω be the algebraic closure of F. A *zero* of f is a nonzero element $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Omega^n$ with $f(\lambda_1, \lambda_2, \ldots, \lambda_n) = 0$; in other words, a nonzero element $v = \lambda_1 \otimes v_1 + \lambda_2 \otimes v_2 + \cdots + \lambda_n \otimes v_n$ of $\Omega \otimes_F V$ with $\theta_\Omega(v, v, \ldots, v) = 0$. This zero is singular or simple according as whether $\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n$ when evaluated at $x_1 = \lambda_1, \ldots, x_n = \lambda_n$, are all zero or not. This is the same as asking whether $\theta_\Omega^{(1\otimes v_1)}, \theta_\Omega^{(1\otimes v_2)}, \ldots, \theta_\Omega^{(1\otimes v_n)}$ when evaluated at $(v, v, \ldots, v)$ are all zero or not. Since $1 \otimes v_1, \ldots, 1 \otimes v_n$ are a basis of $\Omega \otimes_F V$, this is the same as asking whether $\theta_\Omega(u, v, v, \ldots, v) = 0$ for all $u \in \Omega \otimes_F V$. Thus the symmetric space (v, θ) is *nonsingular* (meaning it corresponds to an equivalence class of nonsingular forms) if and only if the only $v \in \Omega \otimes_F V$ with $\theta_\Omega(u, v, \ldots, v) = 0$ for all $u \in \Omega \otimes_F V$ is $v = 0$. We call (v, θ) *nondegenerate* if the only $v \in V$ with $\theta(v, w, w, \ldots, w) = 0$ for all $w \in V$ is $v = 0$ (or equivalently $\theta(v, w_1, w_2, \ldots, w_{r-1}) = 0$ for all $w_1, w_2, \ldots, w_{r-1} \in V$ implies $v = 0$). For quadratic forms (i.e., $r = 2$), one can check that nonsingular and nondegenerate are equivalent concepts. If $f \leftrightarrow (V, \theta)$ by $v_1, v_2, \ldots, v_n$, one can check that f is degenerate (meaning (V, θ) is degenerate) if and only if there is a form g equivalent to f with $g(X_1, X_2, \ldots, X_n) = g(X_1, X_2, \ldots, X_{n-1}, 0)$ (i.e., the indeterminant X_n can be "removed"). If $h(X_1, X_2, \ldots, X_n)$ and $k(X_1, X_2, \ldots, X_m)$ are forms of degree r, we write $h \oplus k$ for $h(X_1, X_2, \ldots, X_n) + k(X_{n+1}, X_{n+2}, \ldots, X_{n+m})$. This operation preserves equivalence of forms. One can check that if $r \neq 0$ and if f is any form of degree r then there exists a nondegenerate form h and a zero form k (meaning $k(X_1, X_2, \ldots, X_m) = 0$) with $f = h \oplus k$, and both h and k are unique up to equivalence with this property. The forms of degree 0 and 1 are trivial, se we may ignore them. Hence there is no loss of generality in restricting attention to nondegenerate forms.

Our aim now is to prove a generalization of Witt's theorem for quadratic forms; namely, if f, g, and h are forms of degree r with $r > 1$ and if $f \oplus g \simeq f \oplus h$, then $g \simeq h$ (if $r! \neq 0$). Using the decomposition of the last paragraph, there is no loss of generality in assuming f, g, and h are all nondegenerate. The case $r = 2$ is exactly Witt's theorem ([3]), so we may assume $r \geqslant 3$. First we give the general result:

LEMMA 1.1. *Let (V, θ) be a symmetric space of degree r over a field F. Assume $r > 1$. Then (V, θ) is nondegenerate if and only if there is a positive m and elements $a_i(j) \in V$, $i = 1,..., m$, $j = 1,..., r$ with*

$$\Sigma\theta(a_i(1), a_i(2),..., a_i(r-1), v)\, a_i(r) = v$$

for all $v \in V$.

Proof. If this formula holds and $\theta(w_1, w_2,..., w_{r-1}, v) = 0$ for all $w_1,..., w_{r-1} \in V$, then $v = 0$. Conversely, suppose (V, θ) is nondegenerate. Let V^* be $\mathrm{Hom}_F(V, F)$. For S a subspace of V^*, let $S^\perp = \{a \in V \mid f(a) = 0\ \forall f \in S\}$. One checks $S^\perp = \{0\}$ if and only if $S = V^*$. Choose a dual basis $f_1,..., f_n \in V^*$, $v_1,..., v_n \in V$ of V (so $\sum f_i(v)\, v_i = v$ for all $v \in V$). Define a map Γ from V^{r-1} to V^* by $\Gamma(b_1, b_2,..., b_{r-1})(c) = \theta(b_1, b_2,..., b_{r-1}, c)$ for all $b_1,..., b_{r-1} \in V$ and $c \in V$. Then if S is the image of Γ, $S^\perp = \{0\}$ since (V, θ) is nondegenerate. But $f_1,..., f_n \in S$ so for $j = 1, 2,..., n$ there exist $b_{i,j,k}$ in V, $k = 1,..., r-1$, $i = 1,..., s_j$ with

$$\Sigma\theta(b_{i,j,1}, b_{i,j,2},..., b_{i,j,r-1}, v) = f_j(v)$$

for all $v \in V$ and all j. We can replace each s_j by a larger number by simply letting $b_{i,j,k} = 0$ if $i > s_j$.

Thus

$$\Sigma\Sigma\theta(b_{i,j,1}, b_{i,j,2},..., b_{i,j,r-1}, v)\, v_{i,j} = v$$

for all $v \in V$, where $v_{i,j}$ is v_j for all i. Reindexing the appropriate elements proves the lemma.

2. Cancellation and Decomposition Results

In this section we introduce quickly more concepts and prove the cancellation theorem just stated. To avoid being mired in details we leave most of the computations to the reader. Since any attempt to get information about equivalence classes of forms over a field must involve the structure of the field itself, and thus of its subrings and their factor rings, we work in a more general but less motivated situation than a field in which $r! \neq 0$.

Let R be a Noetherian ring. Let r be an integer with $r > 1$. By a *symmetric space of degree r over R* we mean a pair (E, θ) where E is a finitely generated R-module and θ is a symmetric multilinear map from E^r to R. Let (E, θ)

and (U, ψ) be symmetric spaces of degree r over R. Let h be a ring homomorphism from R to a Noetherian ring S. Then S is an R-module by defining $\alpha \cdot s$ to be $h(\alpha) \cdot s$ for $\alpha \in R$, $s \in S$. We define a map $\theta \oplus \psi$ from $(E \oplus U)^r$ to R by

$$\theta \oplus \psi(e_1 + u_1, e_2 + u_2, \dots, e_r + u_r) = \theta(e_1, e_2, \dots, e_r) + \theta(u_1, u_2, \dots, u_r)$$

for all $e_1, \dots, e_r \in E$, $u_1, \dots, u_r \in U$. One checks there is a unique multilinear map $\theta \otimes \psi$ from $(E \otimes_R U)^r$ to R with

$$\theta \otimes \psi(e_1 \otimes u_1, e_2 \otimes u_2, \dots, e_r \otimes u_r) = \theta(e_1, e_2, \dots, e_r) \cdot \psi(u_1, u_2, \dots, u_r)$$

for all $e_1, \dots, e_r \in E$, $u_1, \dots, u_r \in U$. One checks there is a unique S-multilinear map θ_S from $(S \otimes_R E)^r$ to S with

$$\theta_S(s_1 \otimes e_1, s_2 \otimes e_2, \dots, s_r \otimes e_r) = s_1 \cdot s_2 \cdots s_r \cdot \theta(e_1, e_2, \dots, e_r)$$

for all $s_1, \dots, s_r \in S$, $e_1, \dots, e_r \in E$. One checks that $(E \oplus U, \theta \oplus \psi)$, $(E \otimes_R U, \theta \otimes \psi)$, and $(S \otimes_R E, \theta_S)$ are symmetric spaces of degree r which we denote by $(E, \theta) \oplus (U, \psi)$, $(E, \theta) \otimes_R (U, \psi)$, and $S \otimes_R (E, \theta)$, respectively. We let 0 denote $(0, \Gamma)$, where 0 is the zero R-module and Γ is the map from 0^r to R with $\Gamma(0, 0, \dots, 0) = 0$. We let 1 denote (R, Δ), where R is the free R-module of rank 1 and $\Delta : R^r \to R$ is defined by $\Delta(\alpha_1, \alpha_2, \dots, \alpha_r) = \alpha_1 \cdot \alpha_2 \cdots \alpha_r$ for all $\alpha_1, \dots, \alpha_r \in R$. An *isomorphism* from (E, θ) to (U, ψ) is defined to be a bijective R-linear map t from E to U with $\psi(t(e_1), t(e_2), \dots, t(e_r)) = \theta(e_1, e_2, \dots, e_r)$ for all $e_1, e_2, \dots, e_r \in E$. If there exists an isomorphism from (E, θ) to (U, ψ) we write $(E, \theta) \simeq (U, \psi)$. Let (L, Λ) be a symmetric space of degree r over R. Let k be a ring homomorphism from S to a Noetherian ring T.

PROPOSITION 2.1.

$$((E, \theta) \oplus (U, \psi)) \oplus (L, \Lambda) \simeq (E, \theta) \oplus ((U, \psi) \oplus (L, \Lambda)),$$
$$(E, \theta) \oplus (U, \psi) \simeq (U, \psi) \oplus (E, \theta), \qquad 0 \oplus (U, \psi) \simeq (U, \psi),$$
$$((E, \theta) \otimes_R (U, \psi)) \otimes_R (L, \Lambda) \simeq (E, \theta) \otimes_R ((U, \psi) \otimes_R (L, \Lambda)),$$
$$(E, \theta) \otimes_R (U, \psi) \simeq (U, \psi) \otimes_R (E, \theta), \qquad 1 \otimes_R (U, \psi) \simeq (U, \psi),$$
$$(E, \theta) \otimes_R ((U, \psi) \oplus (L, \Lambda)) \simeq ((E, \theta) \otimes_R (U, \psi) \oplus ((E, \theta) \otimes_R (L, \psi)),$$
$$S \otimes_R ((U, \psi) \oplus (L, \Lambda)) \simeq (S \otimes_R (U, \psi) \oplus (S \otimes_R (L, \Lambda)),$$
$$S \otimes_R ((U, \psi) \otimes_R (L, \Lambda)) \simeq (S \otimes_R (U, \psi)) \otimes_S (S \otimes_R (L, \Lambda)),$$
$$S \otimes_R 1 \simeq 1, \quad T \otimes_S (S \otimes_R (E, \theta)) \simeq T \otimes_R (E, \theta), \quad R \otimes_R (E, \theta) \simeq (E, \theta).$$

The proof is a very tedious checking that the natural isomorphisms between the modules involved are isomorphisms in our sense; we leave it to the reader.

We call a symmetric space (E, θ) of degree r over R *nondegenerate* if there exists elements $a_i(j) \in E, j = 1,..., r, i = 1,..., m$ (for some m) with

$$\Sigma\theta(a_i(1), a_i(2),..., a_i(r-1), e)\, a_i(r) = e$$

for all $e \in E$. If R is a field with $r! \neq 0$, by Lemma 1.1 this concept is equivalent to the one defined earlier, while if $r = 2$, one checks it is equivalent to the usual one for quadratic forms. If $b_k(s) \in U$, $k = 1,..., r$, $s = 1,..., q$ (for some q) with

$$\Sigma\theta(b_k(1), b_k(2),..., b_k(r-1), u)\, b_k(r) = u$$

for all $u \in U$, one checks that the $a_i(j) \otimes b_k(s)$, the $1 \otimes a_i(j)$, and the $a_i(j) + 0$ with the $0 + b_k(s)$ gives a similar set for $(E, \theta) \otimes_R (U, \psi)$, $S \otimes_R (E, \theta)$, and $(E, \theta) \oplus (U, \psi)$, respectively (using $r > 1$). Hence being nondegenerate is preserved by $\otimes$, $S\otimes$, and $\oplus$. We call (E, θ) *decomposable* if there exist nonzero symmetric spaces (M, Φ), (N, Δ) with $(E, \theta) \cong (M, \Phi) \oplus (N, \Delta)$. One checks that if this is so and (E, θ) is nondegenerate, then both (M, Φ) and (N, Δ) are nondegenerate. We call (E, θ) *indecomposable* if it is nonzero and not decomposable. If A is a submodule of E, we let $A^\perp$ denote $\{e \in E \mid \theta(a, e, e_1,..., e_{r-2}) = 0$ for all $a \in A$ and all $e_1,..., e_{r-2} \in E\}$. A with the restriction of θ to A^r is a symmetric space, which we denote by (A, θ_A). If $(E, \theta) \cong (M, \Phi) \oplus (N, \Delta)$, then there are submodules A and B of E with $A + B = E$, $A \cap B = 0$, $A \subseteq B^\perp$, $(A, \theta_A) \cong (M, \Phi)$, and $(B, \theta_B) \cong (N, \Delta)$. Conversely, if A and B are submodules of E with $A + B = E$, $A \cap B = 0$, $A \subseteq B^\perp$, then $(E, \theta) \cong (A, \theta_A) \oplus (B, \theta_B)$. We say E is the *orthogonal* sum of submodules $A_1, A_2,..., A_s$ if E is a direct sum of these submodules and $A_i \subseteq A_j^\perp$ for $i \neq j$, $i, j = 1,..., s$.

LEMMA 2.2. *Suppose (E, θ) is non-degenerate and $r \geqslant 3$. Suppose A, B are submodules of E with $A + B = E$, $A \cap B = 0$, and $A \subseteq B^\perp$. Then $A = B^\perp$ and $B = A^\perp$. If C, D are submodules of E with $C + D = E$, $C \cap D = 0$, and $C \subseteq D^\perp$, then $C = (C \cap A) + (C \cap B)$, $(C \cap A) \cap (C \cap B) = 0$, and $(C \cap A) \subseteq (C \cap B)^\perp$.*

Proof. Since E is nondegenerate one checks $E^\perp = 0$. Let $x \in B^\perp$. We can write $x = a_0 + b_0$ with $a_0 \in A$, $b_0 \in B$. Thus

$$\begin{aligned} 0 &= \theta(x, b, e_1,..., e_{r-2}) = \theta(a_0 + b_0, b, e_1,..., e_{r-2}) \\ &= \theta(b_0, b, e_1,..., e_{r-2}) = \theta(b_0, a + b, e_1,..., e_{r-2}) \end{aligned}$$

for all $a \in A$, $b \in B$, $e_1,..., e_{r-2} \in E$. Thus $b_0 \in E^\perp$ so $b_0 = 0$ and $x \in A$.

Thus $A = B^{\perp}$. $A \subseteq B^{\perp}$ implies $B \subseteq A^{\perp}$ so similarly we get $B = A^{\perp}$. Also $C = D^{\perp}$. Let $c \in C$. We write $c = a_1 + b_1$ with $a_1 \in A$, $b_1 \in B$. We will show $a_1 \in D^{\perp} = C$. For $d \in D$, $e_3,\ldots, e_r \in E$, we write $d = a_2 + b_2$, $e_3 = a_3 + b_3,\ldots,e_r = a_r + b_r$ with $a_2 ,\ldots, a_r \in A$, $b_2 ,\ldots, b_r \in B$. Then

$$\begin{aligned}\theta(a_1 , d, e_3 ,\ldots, e_r) &= \theta(a_1 , a_2 + b_2 ,\ldots, a_r + b_r) = \theta(a_1 , a_2 ,\ldots, a_r) + 0\\ &= \theta(a_1 , a_2 ,\ldots, a_r) + \theta(b_1 , b_2 , 0,\ldots, 0)\\ &= \theta(a_1 + b_1 , a_2 + b_2 , a_3 ,\ldots, a_r) = \theta(c, d, a_3 ,\ldots, a_r) = 0.\end{aligned}$$

A similar argument gives $b_1 \in D^{\perp} = C$. The rest can be checked quickly.

PROPOSITION 2.3. *Suppose* (E, θ) *is a nonzero nondegenerate symmetric space of degree* $r \geqslant 3$ *over a Noetherian ring* R. *Then there exist finitely many nondegenerate indecomposable symmetric spaces* $(U_1 , \psi_1),\ldots, (U_s , \psi_s)$, *which are unique up to isomorphism and order, with* $(E, \theta) \cong (U_1 , \psi_1) \oplus \cdots \oplus (U_s , \psi_s)$.

Proof. We call a subspace A of E a *summand* if $A + A^{\perp} = E$ and $A \cap A^{\perp} = 0$. If A and C are summands, then $A \cap C$ is a summand (with $(A \cap C)^{\perp} = C \cap A^{\perp} + C^{\perp} \cap A + C^{\perp} \cap A^{\perp}$) by Lemma 2.2. Also $A \subseteq C$ if and only if $A^{\perp} \supseteq C^{\perp}$. Since R is Noetherian and E is finitely generated, E is Noetherian. Hence by an easy induction we can write E as an orthogonal sum of finitely many nonzero indecomposable summands $A_1 , A_2 ,\ldots, A_s$. If C is any nonzero indecomposable summand, then by a repeated application of Lemma 2.2, C is an orthogonal sum of $C \cap A_1$, $C \cap A_2 ,\ldots, C \cap A_s$. Hence there is an i with $C = C \cap A_i = A_i$. Thus E is the orthogonal sum of its nonzero indecomposable summands. This proves the uniqueness as well as the existence.

PROPOSITION 2.4. *Suppose* $(E, \theta), (U, \psi)$, *and* (L, Λ) *are nondegenerate symmetric spaces of degree* $r \geqslant 3$ *over a Noetherian ring* R. *Suppose*

$$(E, \theta) \oplus (U, \psi) \cong (E, \theta) \oplus (L, \Lambda).$$

Then $(U, \psi) \cong (L, \Lambda)$.

Proof. Write $(E, \theta), (U, \psi)$, and (L, Λ) as a sum of nonzero indecomposable spaces, Then by the uniqueness part of the last proposition, the result follows.

Note 2.5. If we call a symmetric space (E, θ) *regular* if $v \in V$, $\theta(v, v_2 ,\ldots, v_r) = 0$ for all $v_2 ,\ldots, v_r \in V$, imply $v = 0$, then Propositions 2.3 and 2.4 hold (with the same proofs) with "regular" substituted for each appearance of "nondegenerate." If R is a field (or even a Dedekind domain), every symmetric space is uniquely (up to isomorphism) a direct sum of a zero symmetric space and a regular one. Hence, for R a Dedekind domain (e.g., a field), the conclusions of both Propositions 2.3 and 2.4 hold without any restrictions placed on the symmetric spaces involved (other than $r \geqslant 3$), since cancellation holds for modules (finitely generated) over a Dedekind domain.

We give an example of a class of symmetric spaces, omitting details. A commutative algebra A over R is strongly separable if A is separable and is finitely and projective as an R-module. This is equivalent to existence of $f \in \mathrm{Hom}_R(A, R)$ and $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in A$ with $\Sigma a_i \cdot b_i = 1$ and $\Sigma f(c \cdot a_i)\, b_i = c$ for all $c \in A$. If such exists, then the f is unique and is the trace. We define $\theta_{(A)}$ from A^r to R by $\theta_{(A)}(c_1, c_2, \ldots, c_r) = f(c_1 \cdot c_2 \cdots c_r)$ for all $c_1, c_2, \ldots, c_r \in A$. Then one checks $(A, \theta_{(A)})$ is a nondegenerate symmetric space. If B is another commutative strongly separable algebra over R, and if $r \geqslant 3$ one can show $(A, \theta_{(A)}) \cong (B, \theta_{(B)})$ if and only if $A \cong B$. The commutative strongly separable algebras are closed under direct sum, tensor product, and scalar extension and the above map to symmetric spaces preserves these operations. Hence we can think of each commutative strongly separable algebra as a special type of nondegenerate symmetric space of degree r (for each $r \geqslant 3$). Let A be such an algebra, with trace t. For (V, θ) a symmetric space over A, we can generalize W. Scharlau for $r = 2$ and note that $(V, t \circ \theta)$ is a symmetric space over R. With the two lemmas that follow one can check that if (V, θ) is nondegenerate over A, then $(V, t \circ \theta)$ is nondegenerate over R.

LEMMA 2.6. *Let (E, θ) be a symmetric space over a Noetherian ring R. Then (E, θ) is nondegenerate if and only if $((R/M) \otimes_R E, \theta_{(R/M)})$ is nondegenerate for each maximal ideal M of R, and E is finitely generated projective as an R-module.*

Proof. Define a map Δ from $E \otimes_R \cdots \otimes_R E$ (r-copies) to $\mathrm{Hom}_R(E, E)$ by $\Delta(e_1 \otimes e_2 \otimes \cdots \otimes e_r)(e) = \theta(e_1, e_2, \ldots, e_{r-1}, e)\, e_r$ for all $e_1, \ldots, e_r \in E$ and linearity. One checks Δ is surjective if and only if E is nondegenerate. The result follows from Proposition 11, p. 113 of [1].

LEMMA 2.7. *Let K be a finite field extension of a field F. Let $f \in \mathrm{Hom}_F(K, F)$ be nonzero. Let (E, θ) be a nondegenerate symmetric space over K of degree $r > 1$. Then $(E, f \circ \theta)$ is a nondegenerate symmetric space over F.*

Proof. Suppose $e \in E$ is nonzero. Then there are $e_2, \ldots, e_r \in E$ with $\theta(e, e_2, \ldots, e_r) \neq 0$. $f \neq 0$ so there is a $b \in K$ with $f(b) \neq 0$. K is a field so there is an $a \in K$ with $a \cdot \theta(e, e_2, \ldots, e_r) = b$. Thus

$$(f \circ \theta)(e, a \cdot e_2, e_3, \ldots, e_r) \neq 0$$

Hence $(E, f \circ \theta)$ is nondegenerate.

3. The Ring of Higher Degree Forms

In this section we develop a partially ordered ring whose positive elements are the equivalence classes of forms of degree r. We work over a Noetherian ring R.

Forms and equivalence of forms over R are defined as in Section 1 with F replaced by R. We call a form of degree r

$$f = \Sigma\alpha_{(i_2,i_2,\ldots,i_n)}X_1^{i_1}X_2^{i_2}\cdots X_n^{i_n}$$

(where the sum is over all nonnegative integers $i_1,\ldots, i_n$ which add to r) with coefficients in R *balanced* if for each $i_1, i_2,\ldots, i_n$ in the above sum, there exists a $\beta_{(i_1,i_2,\ldots,i_n)} \in R$ with

$$(r!/i_1!\cdot i_2!\cdots i_n!)\,\beta_{(i_1,i_2,\ldots,i_n)} = \alpha_{(i_1,i_2,\ldots,i_n)}\,.$$

We call the above form f *nondegenerate*, if for each $i = 1, 2,\ldots, n$, we can write $r!X_i$ as a linear combination (with coefficients in R) of the

$$\partial^{r-1}f/\partial X_{i_1}\,\partial X_{i_2}\cdots\partial X_{i_{r-1}}, i_1, i_2,\ldots, i_{r-1} = 1, 2,\ldots, n.$$

If $r!$ is not a zero-divisor in R, then it is easy to check that as in the field case, the isomorphism classes of free symmetric spaces over R of degree r correspond bijectively with the equivalence classes of balanced forms of degree r with coefficients in R. Moreover, a form is nondegenerate if and only if the corresponding symmetric space is nondegenerate. This motivates the following notation. We let $P_r(R)$ denote the class of all isomorphism classes (or isomorphism types if one prefers) of free symmetric spaces of degree r over R. For (E, θ) a symmetric space which is free (meaning E is free), we let $[(E, \theta)]$ be the isomorphism class of (E, θ). Then $[(E, \theta)] = [(U, \psi)]$ if and only if $(E, \theta) \cong (U, \psi)$. One can check that $P_r(R)$ is a set, and $\otimes$ and $\oplus$ induce operations on $P_r(R)$ which we denote by $\cdot$ and $+$, respectively. If $r \geqslant 3$, $+$ satisfies the cancellation law by Proposition 2.4, so $P_r(R)$ with $+$ can be embedded in a unique way in an abelian group $L_r(R)$ so that every element in $L_r(R)$ is a difference of elements in $P_r(R)$. The operation $\cdot$ on $P_r(R)$ can be extended in a unique way to $L_r(R)$ so that $L_r(R)$ with the resulting operation is a commutative ring with 1. For $x, y \in L_r(R)$ we write $x \geqslant y$ if $x - y \in P_r(R)$. Then $L_r(R)$ is a partially ordered ring, and every element in $L_r(R)$ can be written as the difference of two positive elements. $P_r(R)$ can be recovered from $L_r(R)$ as the set of $x \geqslant 0$. Determining the structure of $L_r(R)$ is the same as finding all isomorphism classes of nondegenerate symmetric spaces of degree r over R. If h is a ring homomorphism from R to a Noetherian ring S, with Proposition 2.1 one checks that $S \otimes_R (\ )$

induces an order preserving ring homomorphism from $L_r(R)$ to $L_r(S)$. One checks that $L_r(\ )$ is a functor from the category of Noetherian rings to the category of partially ordered commutative rings, if $r \geqslant 3$. If $r = 2$, cancellation of $\oplus$ still often holds for R (see [2]) and we proceed as above, but if it does not hold we replace isomorphism by stable isomorphism (see [2]) and get a functor $L_2(\ )$ which is the same as that considered in [2]. By this same procedure, we can extend the functor $L_r(\ )(r \geqslant 2)$ to not-necessarily-Noetherian commutative rings.

PROPOSITION 3.1. *Let $h: R \to S$ be a surjective ring homomorphism between Noetherian rings R and S. Let $r \geqslant 3$. Then $L_r(h): L_r(R) \to L_r(S)$ is surjective.*

Proof. Let h_* be $L_r(h)$. We show that if $a \in P_r(S)$ is of rank n, then $a + n1 = h_*(b)$ for some $b \in P_r(R)$. Then if $x = a_1 - a_2$ is any element in $L_r(h)$ and $a_1 + n_1 = h_*(b_1)$, $a_2 + n_2 1 = h_*(b_2)$, we have $x = h_*(b_1 - b_2 + (n_2 - n_1)1)$. Thus the result follows from the next two lemmas.

LEMMA 3.2. *Let (U, ψ) be a symmetric space of degree r over a ring S. For $u_1, u_2, \ldots, u_{r-1} \in U$ let $h = h_{(u_1,\ldots,u_{r-1})}$ be the element in $\mathrm{Hom}_S(U, S)$ where $h(u) = \psi(u_1, \ldots, u_{r-1}, u)$ for all $u \in U$. Let Δ be the map from $U \otimes \cdots \otimes U$ ($r - 1$ copies) to $\mathrm{Hom}_S(U, S)$, where $\Delta(u_1 \otimes u_2 \otimes \cdots \otimes u_{r-1}) = h_{(u_1,\ldots,u_{r-1})}$ for all $u_1, \ldots, u_{r-1} \in U$. Then (U, ψ) is nondegenerate if and only if U is projective and Δ is surjective.*

Proof. This is easily checked using the dual basis lemma.

LEMMA 3.3. *Let the notation be as in Proposition 3.1. Let (U, ψ) be a non-degenerate free symmetric space of degree r over S. Let n be the rank of U. Let $(S, 1)$ be the symmetric space where $1(s_1, \ldots, s_r) = s_1 \cdot s_2 \cdots s_r$ for all $s_1, \ldots, s_r \in S$, and let $(S^n, 1^n)$ be $(S, 1) \oplus \cdots \oplus (S, 1)$ (n-copies). Assume $r \geqslant 3$. Then there is a nondegenerate symmetric space (V, θ) over R with $S \otimes_R (V, \theta) \cong (U, \psi) \oplus (S^n, 1^n)$.*

Proof. Let $u_1, \ldots, u_n$ be a basis of U over S. Let $A = \{u_1, \ldots, u_n\}$ be this basis. For each $c \in A$ let $f_c \in \mathrm{Hom}_S(U, S)$ be defined by linearity and $f_c(a) = \delta_{c,a}$ (the Kronecker delta) for all $a \in A$. By Lemma 3.2 there exist $\lambda_{a_1,\ldots,a_{r-1}}(c) \in S$ for all $a_1, \ldots, a_{r-1} \in A$ with

$$\Sigma_{a_1}\Sigma_{a_2} \cdots \Sigma_{a_{r-1}}\lambda_{a_1,\ldots,a_{r-1}}(c)\, \Delta(a_1 \otimes \cdots \otimes a_{r-1}) = f_c .$$

Thus for each $e \in A$

$$\Sigma_{a_1}\Sigma_{a_2} \cdots \Sigma_{a_{r-1}}\lambda_{a_1,\ldots,a_{r-1}}(c)\, \psi(a_1, \ldots, a_{r-1}, e) = \delta_{c,e} .$$

Since h is surjective there is a map $t: S \to R$ with $t(1) = 1$, $t(0) = 0$ and $h(t(s)) = s$ for all $s \in S$. Writing $\beta_{a_1,\ldots,a_{r-1}}(c)$ for $t(\lambda_{a_1,\ldots,a_{r-1}}(c))$ and $\theta(a_1, \ldots, a_{r-1}, e)$ for $t(\psi(a_1, \ldots, a_{r-1}, e))$ we have

$$\Sigma_{a_1}\Sigma_{a_2} \cdots \Sigma_{a_{r-1}}\beta_{a_1,\ldots,a_{r-1}}(c)\, \theta(a_1, \ldots, a_{r-1}, e) = \delta_{c,e} - \alpha_e(c)$$

for all $c, e \in A$, where $\alpha_e(c)$ is an element in the kernel of h. Let A^p be a set whose elements are in bijective correspondence with A by $a \mapsto a^p$, and which is disjoint from A. Let V be a free module over R with the elements of $A \cup A^p$ as a basis. Define a map θ: $V \times \cdots \times V \to R$ by linearity, symmetry, and

$$\theta(a_1^p, a_2^p, \ldots, a_r^p) = \delta_{a_1,a_2,\ldots,a_r} \qquad \text{(Kronecker delta)},$$

$$\theta(a_1^p, a_2^p, x_3, \ldots, x_{r-1}, b) = 0,$$

$$\theta(a_1, a_2, \ldots, a_{r-1}, b^p) = \alpha_{a_1}(b)\, \delta_{a_1,\ldots,a_{r-1}},$$

$$\theta(a_1, a_2, \ldots, a_{r-1}, a_r) = t(\psi(a_1, a_2, \ldots, a_{r-1}, a_r)),$$

for all $a_1, \ldots, a_r$, $b \in A$ and all $x_3, \ldots, x_{r-1} \in A \cup A^p$. One checks (V, θ) is a symmetric space over R with $S \otimes_R (V, \theta) \cong (U, \psi) \oplus (S^n, 1^n)$. To show that (V, θ) is nondegenerate, we use Lemma 3.2. For $x \in A \cup A^p$, define $f_x \in \mathrm{Hom}_r(V, R)$ by linearity and $f_x(y) = \delta_{x,y}$ (Kronecker delta) for all $y \in A \cup A^p$. These f_x form a basis of $\mathrm{Hom}_R(V, R)$ so it is enough to show each f_x is in the image of the Δ of Lemma 3.2. If $x \in A^p$, $\theta(x, x, \ldots, x, y) = \delta_{x,y} = f_x(y)$ for all $y \in A \cup A^p$. Hence let $x = c \in A$. Then

$$\Sigma_{a_1} \cdots \Sigma_{a_{r-1}}\beta_{a_1,\ldots,a_{r-1}}(c)\, \theta(a_1, \ldots, a_{r-1}, y) + \Sigma_a \theta(a, \ldots, a, c^p, y)$$
$$+ \Sigma_a(\Sigma_d - \beta_{d,d,\ldots,d}(c)\, \alpha_d(a))\, \theta(a^p, \ldots, a^p, y)$$

can be calculated to be $f_c(y)$ for all $y \in A \cup A^p$. By Lemma 3.2 (V, θ) is nondegenerate. This Lemma 3.3 and Proposition 3.1 follow.

4. Absolutely Indecomposable Forms, Their Tensor Products, and Nonsingular Forms

Let (E, θ) be a symmetric space of degree r over a Noetherian ring R. Assume $r \geqslant 3$. We consider the set of all $f \in \mathrm{Hom}_R(E, E)$ such that

$$\theta(f(e_1), e_2, \ldots, e_r) = \theta(e_1, f(e_2), e_3, \ldots, e_r)$$

for all $e_1, e_2, \ldots, e_r \in E$. We call this the *center* of (E, θ) and denote it by $Z(E, \theta)$, or simply $Z(E)$.

PROPOSITION 4.1. *Suppose (E, θ) is nondegenerate (actually regular is enough). Then $Z(E)$, with composition, is a commutative algebra. This algebra has no idempotents but 0 and 1 if and only if (E, θ) is indecomposable.*

Proof. Let $f, g \in Z(E)$. One checks

$$\theta(f(v_1), v_2, \ldots, v_r) = \theta(v_1, v_2, \ldots, f(v_i), \ldots, v_r)$$

for all $v_1, \ldots, v_r \in E$ and all $i = 1, 2, \ldots, r$. Hence $\theta(g(f(v_1)), v_2, \ldots, v_r) = \theta(f(v_1), g(v_2), v_3, \ldots, v_r) = \theta(v_1, g(v_2), f(v_3), \ldots, v_r) = \theta(g(v_1), v_2, f(v_3), \ldots, v_r) = \theta(f(g(v_1)), v_2, v_3, \ldots, v_r)$, so $\theta(g(f(v_1)) - f(g(v_1)), v_2, \ldots, v_r) = 0$ for all $v_2, \ldots, v_r \in E$. Hence $g(f(v_1)) = f(g(v_1))$ for all $v_1 \in E$. Now

$$\begin{aligned}\theta(f(g(v_1)), v_2, \ldots, v_r) &= \theta(g(v_1), f(v_2), \ldots, v_r)\\ &= \theta(v_1, g(f(v_2), \ldots, v_r) = \theta(v_1, f(g(v_2)), \ldots, v_r)\end{aligned}$$

so $f \circ g \in Z(E)$. That $Z(E)$ is an algebra is now easily checked. If f is an idempotent in this algebra, and I is the image of f, and K is the kernel of f, then one checks $(E, \theta) \cong (I, \theta|_I) \oplus (K, \theta|_K)$ where $\theta|_I$, $\theta|_K$ denote the restriction of θ to I^r and K^r, respectively. Conversely, if (E, θ) is decomposable, the E is internally a direct sum of summands which are perpendicular to each other in the sense of Section 2. If we let h be the projection onto the first summand, we can check that h is an idempotent in $Z(E)$.

PROPOSITION 4.2. *Suppose R is a field (actually a principal ideal domain is enough). Suppose (V, θ) and (U, ψ) are nondegenerate symmetric spaces over R of degree r with $r \geq 3$. Then $Z(V \otimes_R U) \cong Z(V) \otimes_R Z(U)$ by a natural algebra isomorphism.*

Proof. For generality, assume R is a principal ideal domain. By Lemma 3.2, both V and U are projective, and thus free. Also they are both regular. One checks that $Z(V)$ is a pure submodule of $\mathrm{Hom}_R(V, V)$. Hence any basis of $Z(V)$ may be extended to a basis of $\mathrm{Hom}_R(V, V)$. Let $f_1, \ldots, f_s$ be a basis of $Z(V)$, and extend this to a basis $f_1, \ldots, f_s, f_{s+1}, \ldots, f_n$ of $\mathrm{Hom}_R(V, V)$. Let $g_1, \ldots, g_t$ be a basis of $Z(U)$, and extend this to a basis $g_1, \ldots, g_t, g_{t+1}, \ldots, g_m$ of $\mathrm{Hom}_R(U, U)$. Then $\{f_i \otimes g_j \mid i = 1, \ldots, n, j = 1, \ldots, m\}$ is a basis of $\mathrm{Hom}_R(V \otimes_R U, V \otimes_R U)$ (which is isomorphic to $\mathrm{Hom}_R(V, V) \otimes \mathrm{Hom}_R(U, U)$). One checks that $\{f_i \otimes g_j \mid i = 1, \ldots, s, j = 1, \ldots, t\}$ are all in $Z(V \otimes U)$. Let

$$k = \Sigma_i \Sigma_j \alpha_{j,i} f_i \otimes g_j$$

be any element in $Z(V \otimes_R U)$, where the $\alpha_{i,j}$ are in R and $i = 1, \ldots, n, j = 1, \ldots, m$. For all $v_1, \ldots, v_r \in V$, $u_1, \ldots, u_r \in U$,

$$\begin{aligned}&\theta \otimes \psi(k(v_1 \otimes u_1), v_2 \otimes u_2, \ldots, v_r \otimes u_r)\\ &\quad = \theta \otimes \psi(v_1 \otimes u_1, k(v_2 \otimes u_2), \ldots, v_r \otimes u_r),\end{aligned}$$

so

$$\Sigma_i\Sigma_j\alpha_{i,j}\theta(f_i(v_1), v_2, \ldots, v_r) \cdot \psi(g_i(v_1), v_2, \ldots, v_r)$$
$$= \Sigma_i\Sigma_j\alpha_{i,j}\theta(v_1, f(v_2), \ldots, v_r) \cdot \psi(v_1, g(v_2), \ldots, v_r)$$

so

$$\psi(\Sigma_i\Sigma_j\alpha_{i,j}\theta(f_i(v_1), \ldots, v_r)\, g_j(u_1), u_2, \ldots, u_r)$$
$$= \psi(u_1, \Sigma_i\Sigma_j\alpha_{i,j}\theta(v_1, f_i(v_2), \ldots, v_r)\, g_j(u_2), \ldots, u_r).$$

Letting $h_1 = \Sigma_i\Sigma_j\alpha_{i,j}\theta(f_i(v_1), \ldots, v_r)g_j$, $h_2 = \Sigma_i\Sigma_j\alpha_{i,j}\theta(v_1, f_i(v_2), \ldots, v_r)g_j$, we have

$$\psi(h_1(u_1), u_2, \ldots, u_r) = \psi(u_1, h_2(u_2), \ldots, u_r).$$

For all $u_1, \ldots, u_r \in U$. Thus $\psi(h_2(u_1), u_2, u_3, \ldots, u_r) = \psi(u_3, h_2(u_1), u_2, \ldots, u_r) = \psi(h_1(u_3), u_1, u_2, \ldots, u_r) = \psi(h_1(u_3), u_2, u_1, \ldots, u_r) = \psi(u_3, h_2(u_2), u_1, \ldots, u_r) = \psi(u_1, h_2(u_2), u_3, \ldots, u_r)$ so $h_2 \in Z(U)$. Hence the equation above gives $\psi(h_1(u_1)\, u_2, \ldots, u_r) = \psi(h_2(u_1), u_2, \ldots, u_r)$, and since ψ is regular we get $h_1 = h_2$. This means that $h_1 \in Z(U)$, so $\Sigma_i\alpha_{i,j}\theta(f_i(v_1), \ldots, v_r) = 0$ for $j > t$ (this is the coefficient of g_j in h_1). Thus for $j > t$, $\theta(\Sigma_i\alpha_{i,j}f_i(v_1), v_2, \ldots, v_r) = 0$ for all $v_2, \ldots, v_r$. Since (V, θ) is regular, $\Sigma_i\alpha_{i,j}f_i = 0$ so $\alpha_{i,j} = 0$ for all i and $j > t$. Similarly, $\alpha_{i,j} = 0$ for all j and $i > s$. Hence $k \in Z(V) \otimes_R Z(U)$.

Proposition 4.3. *Suppose R is a field (actually a principal ideal domain is enough). Suppose (V, θ) is a non-degenerate symmetric space over R of degree r with $r \geqslant 3$. Suppose S is a flat R-algebra. Then $Z(S \otimes_R V) \cong S \otimes_R Z(V)$ by a natural algebra isomorphism.*

Proof. Extend a basis $f_1, \ldots, f_t$ of $Z(V)$ to a basis $f_1, \ldots, f_t, \ldots, f_m$ of $\mathrm{Hom}_R(V, V)$. Since V is free as an R-module (or since S is flat),

$$S \otimes_R \mathrm{Hom}_R(V, V) = \mathrm{Hom}_S(S \otimes_R V, S \otimes_R V).$$

Hence $1 \otimes f_1, \ldots, 1 \otimes f_m$ is a basis of $\mathrm{Hom}(S \otimes_R V, S \otimes_R V)$. One checks that $1 \otimes f_1, \ldots, 1 \otimes f_t$ are each in $Z(S \otimes_R V)$. Let $k = \Sigma_j s_j \otimes f_j$, where $j = 1, \ldots, m$, be any element in $Z(S \otimes_R V)$. $s_1, \ldots, s_m$ generate an R-module which is free; let $a_1, \ldots, a_n$ be a basis of this submodule of S. Then $k = \Sigma_i\Sigma_j\alpha_{i,j}a_i \otimes f_j$, where the $\alpha_{i,j}$ are in R. For all $v_1, v_2, \ldots, v_r \in V$,

$$\theta_S(k(v_1), v_2, \ldots, v_r) = \theta_S(v_1, k(v_2), \ldots, v_r)$$

so

$$\Sigma_i\Sigma_j\alpha_{i,j}a_i\theta(f_j(v_1), v_2, \ldots, v_r) = \Sigma_i\Sigma_j\alpha_{i,j}a_i\theta(v_1, f_j(v_2), \ldots, v_r).$$

Equating coefficients of a_i, we get for each i that

$$\theta(\Sigma_j \alpha_{i,j} f_j(v_1), v_2, \ldots, v_r) = \theta(v_1, \Sigma_j \alpha_{i,j} f_j(v_2), \ldots, v_r),$$

so $\Sigma_j \alpha_{i,j} f_j \in Z(V)$. Hence $k = \Sigma_i a_i \otimes (\Sigma_j \alpha_{i,j} f_j) \in S \otimes_R Z(V)$. Thus $S \otimes_R Z(V) = Z(S \otimes_R V)$ and the proposition is proved.

PROPOSITION 4.4. *Let F be a field. Let $r \geqslant 3$. Then the tensor product of nondegenerate indecomposable symmetric spaces of degree r over F is always indecomposable if and only if F is separably closed.*

Proof. Suppose F is separably closed. Let (V, θ), (U, ψ) be nondegenerate indecomposable symmetric spaces of degree r over F. By Proposition 4.1, both $Z(V)$ and $Z(U)$ have no idempotents but 0 and 1. Since idempotents can be lifted, we let rad $Z(V)$, rad $Z(U)$ be the corresponding radicals and have that both $Z(V)/\text{rad } Z(V)$ and $Z(U)/\text{rad } Z(U)$ have no idempotents but 0 and 1, and thus are fields. They must be purely inseparable field extensions of F, so their tensor product is a local algebra and thus has no idempotents but 0 and 1. $Z(V) \otimes \text{rad } Z(U) + \text{rad } Z(V) \otimes Z(U)$ is a nilpotent ideal in $Z(V) \otimes Z(U)$ with factor ring isomorphic to $(Z(V)/\text{rad } Z(V)) \otimes (Z(U)/\text{rad } Z(U))$, so since idempotents can be lifted, $Z(V) \otimes Z(U)$ has no idempotents but 0 and 1. Now by Propositions 4.2 and 4.1 we get $(V \otimes_R U, \theta \otimes \psi)$ is indecomposable.

Conversely, suppose F is not separably closed. Then F has a proper normal separable field extension K. As in the end of Section 2, K may be identified with a nondegenerate symmetric space of degree r over F. One checks $Z(K) = K$, so K is indecomposable. But $K \otimes_F K \simeq K \oplus K \oplus \cdots \oplus K$ ($[K : F]$-copies) which is certainly not indecomposable. The proposition is proved.

Classically the most important field is C, the complex numbers. The above proposition says that $L_r(C)$ is isomorphic as a partially ordered ring to the semigroup ring $Z(I_r(C))$, where $I_r(C)$ is the semigroup of isomorphism classes of indecomposable equivalence classes of forms of degree r over C with tensor product. We know nothing about the structure of $I_r(C)$; for example, we ask whether this semigroup is free. In order to make this construction functorial, we call an indecomposable symmetric space over an arbitrary field F *absolutely indecomposable* if it remains indecomposable under every field extension of F. By Proposition 4.3, a nondegenerate symmetric space (V, θ) over F is absolutely indecomposable if and only if $Z(V)/\text{rad } Z(V)$ is a purely inseparable field extension of F. The tensor product of two absolutely indecomposable nondegenerate symmetric spaces is absolutely indecomposable, so the isomorphism classes of these spaces, with tensor product, forms a monoid which we denote be $I_r(F)$. Using Proposition 4.3, one checks that $I_r(\ )$ is a functor from fields to commutative monoids.

In Section 1 we saw that a symmetric space (V, θ) of degree r over a field F in which $r! \neq 0$, is nonsingular if and only if $v \in \bar{F} \otimes_F V$, $\theta_{\bar{F}}(v, v, \ldots, v, u) = 0$ for all $u \in \bar{F} \otimes_F V$, imply $v = 0$, where $\bar{F}$ is the algebraic closure of F. We use this terminology even if $r! = 0$. We note that if (V, θ) is nonsingular, then it is nondegenerate and $Z(V)$ is a separable algebra (for if $0 \neq f \in Z(\bar{F} \otimes_F V)$ with $f^2 = 0$, then there is a $v \in \bar{F} \otimes_F V$ with $f(v) \neq 0$, but for all $u \in \bar{F} \otimes_F V$ $\theta_{\bar{F}}(f(v), f(v), \ldots, f(v), u) = \theta_{\bar{F}}(v, f^2(v), \ldots, f(v), u) = 0$). If in addition (V, θ) is indecomposable (note a symmetric space is nonsingular if and only if each indecomposable part is nonsingular), then $Z(V)$ is a finite field extension of F. Suppose this is the case, and write K for $Z(V)$. Let $t = t_{K/F}$ be the trace of K over F. In the end of Section 2 we saw K with t gives a nondegenerate symmetric space of each degree greater than one. Taking degree two, one checks that this means $K \simeq \mathrm{Hom}_F(K, F)$ by $a \mapsto f_a$, where $f_a(b) = t(a \cdot b)$ for all $b \in K$. For $v_1, v_2, \ldots, v_r \in V$, $b \mapsto t(\theta(bv_1, v_2, \ldots, v_r))$ is a homomorphism from K to F, so there exists a unique element $\Gamma(v_1, v_2, \ldots, v_r) \in K$ with

$$t(b \cdot \Gamma(v_1, v_2, \ldots, v_r)) = t(\theta(bv_1, v_2, \ldots, v_r))$$

for all $b \in K$. One checks that (V, Γ) is a symmetric space of degree r over K, and $(V, t \circ \Gamma)$ (as in Section 2) is (V, θ). One can also checks that $Z(V, \Gamma) = K$; we call such a symmetric space *central* (since its center is as small as possible; note a nonsingular form is central if and only if it is absolutely indecomposable). By first principals, (V, Γ) is unique up to semilinear isomorphism (meaning up to isomorphism and the effect of an automorphism of K over F).

PROPOSITION 4.5. *Let F be a field and r be an integer with $r \geqslant 3$. Let K be a finite separable field extension of F. Let $t_{K/F}$ be the trace map from K to F. Let (U, ψ) be a nonsingular central symmetric space of degree r over K. Then $(U, t_{K/F} \circ \psi)$ is a nonsingular indecomposable symmetric space of degree r over F. Conversely, every nonsingular indecomposable symmetric space of degree r over F can be realized in this fashion with a unique (up to isomorphism) K and a unique (up to semilinear isomorphism over F) (U, ψ).*

Proof. Some details remain. If (U, ψ) is central over K, we must check that $(U, t_{K/F} \circ \psi)$ has center exactly K. If f is in this center, then for all $v_1, \ldots, v_r \in U$ and $b, c \in K$,

$$\begin{aligned} t_{K/F}(c \cdot \psi(f(bv_1), v_2, \ldots, v_r)) &= t_{K/F}(\psi(f(bv_1), v_2, \ldots, cv_r)) \\ &= t_{K/F}(\psi(bv_1, f(v_2), \ldots, cv_r)) = t_{K/F}(\psi(v_1, f(v_2), \ldots, bcv_r)) \\ &= t_{K/F}(\psi(f(v_1), v_2, \ldots, bcv_r) = t_{K/F}(c \cdot \psi(bf(v_1), v_2, \ldots, v_r)) \end{aligned}$$

so $\psi(f(bv_1), v_2, ..., v_r) = \psi(bf(v_1), v_2, ..., v_r)$ so f is K-linear. Similarly we check f is in the center of (U, ψ) and thus is multiplication by an element in K. The Proposition now follows from the next two lemmas.

LEMMA 4.6. *Let K be a finite separable field extension of a field F. Let $\bar{F}$ be the algebraic closure of K, and let $\sigma_1, ..., \sigma_n$ be the distinct F-algebra isomorphisms from K into $\bar{F}$. For $i = 1, ..., n$, let $\bar{F}_i$ be the field extension σ_i (which is an algebraic closure of K). Let (U, ψ) be a symmetric space of degree r over K. Let t be the trace map from K to F. Then*

$$\bar{F} \otimes_F (U, t \circ \psi) \cong \bar{F}_1 \otimes_K (U, \psi) \oplus \cdots \oplus \bar{F}_n \otimes_K (U, \psi).$$

Proof. By Galois theory we know $\bar{F} \otimes_F K \cong \Sigma_i \bar{F}_i$ by $x \otimes a \to \Sigma x_i \sigma(a)$ for $x \in \bar{F}$, $a \in K$, and this is a K-algebra isomorphism. Applying ( ) $\otimes_K U$ to both sides we get

$$\bar{F} \otimes_F U \cong (\bar{F} \otimes_K K) \otimes_K U \cong \Sigma_i \bar{F}_i \otimes_K U$$

by $x \otimes u \mapsto (x \otimes 1) \otimes u \to \Sigma x \otimes u$ for $x \in \bar{F}$, $u \in U$. For $x_1, ..., x_r \in \bar{F}$, $u_1, ..., u_r \in U$, $(t \circ \psi)_{\bar{F}}(x_1 \otimes u_1, ..., x_r \otimes u_r) = x_1 \cdots x_r t(\psi(u_1, ..., u_r))$ and

$$\begin{aligned}(\Sigma\psi_i)(\Sigma x_1 \otimes u_1, \Sigma x_2 \otimes u_2, ..., \Sigma x_r \otimes u_r) &= \Sigma x_1 x_2 \cdots x_r \sigma_i(\psi(u_1, ..., u_r)) \\ &= x_1 x_2 \cdots x_r t(\psi(u_1, ..., u_r)).\end{aligned}$$

LEMMA 4.7. *Let the notation be as in Lemma 4.6. Then (U, ψ) is nonsingular if and only if $(U, t \circ \psi)$ is nonsingular.*

Proof. By Lemma 4.6, $\bar{F} \otimes_F (U, t \circ \psi)$ is nonsingular if and only if each $\bar{F}_i \otimes_K (U, \psi)$ is nonsingular. Each of these is $\bar{K} \otimes_K (U, \psi)$, which is nonsingular if and only if (U, ψ) is nonsingular.

ACKNOWLEDGMENTS

We give thanks to Michael Gilpin, Gary Fowler, Joel Cunningham, and Donald Coleman for numerous helpful conversations.

REFERENCES

1. N. BOURBAKI, "Elements de Mathematique 27," Algèbre Commutative, Chapt. 1 and 2, Hermann, Paris, 1961.
2. M. KNEBUSCH, Grothendieck und Wittringe von nichtausgearteten symmetrischem Bilinearformen, Sitzungsber. Heidelb. Akad. Wiss. Math.-naturw. Kl. 1969/70, 3. Abh.
3. E. WITT, Theorie der quadratischen Formen in bebliebigen Korpern, *J. Reine Angew. Math.* **176** (1937), 31–44.
4. D. K. HARRISON, Commutative Non-associative Algebras and Cubic Forms, *J. Algebra* **32** (1974), 518–528.