A first countable linearly Lindelöf not Lindelöf topological space

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A topological space $X$ is called linearly Lindelöf if every increasing open cover of $X$ has a countable subcover. It is well known that every Lindelöf space is linearly Lindelöf. The converse implication holds only in particular cases, such as $X$ being countably paracompact or if $nw(X) < \aleph_\omega$.

Arhangel'skii and Buzyakova proved that the cardinality of a first countable linearly Lindelöf space does not exceed $2^{\aleph_0}$. Consequently, a first countable linearly Lindelöf space is Lindelöf if $\kappa_{\omega_0} > 2^{\aleph_0}$. They asked whether every linearly Lindelöf first countable space is Lindelöf in ZFC. This question is supported by the fact that all known linearly Lindelöf not Lindelöf spaces are of character at least $\kappa_{\omega_0}$. We answer this question in the negative by constructing a counterexample from $\text{MA} + \kappa_{\omega_0} < 2^{\aleph_0}$.

A modification of Alster’s Michael space that is first countable is presented.

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0. Introduction

A topological space $X$ is called linearly Lindelöf, or chain Lindelöf, if every increasing open cover of $X$ has a countable subcover. Every Lindelöf space is linearly Lindelöf, but not vice versa, as examples in [3,4,12,13,10,15] show. It is well known that $X$ is linearly Lindelöf if and only if every subset of $X$ of regular cardinality has a complete accumulation point. This criterion implies that $X$ is Lindelöf whenever the network weight of $X$ is less than $\kappa_{\omega_0}$. Hence, every linearly Lindelöf not Lindelöf space is a large one — it has network weight at least $\kappa_{\omega_0}$. On the other hand, Arhangel’skii and Buzyakova showed in [3] that the cardinality of a Tychonoff first countable linearly Lindelöf space does not exceed $2^{\aleph_0}$. Therefore, a first countable linearly Lindelöf space is Lindelöf whenever $\kappa_{\omega_0} > 2^{\aleph_0}$ (in particular, if CH holds). In fact, Arhangel’skii and Buzyakova proved a stronger result:

Theorem 1. ([3, Corollary 3.4]) Under CH, every Tychonoff $\omega_1$-Lindelöf space $X$ of countable tightness is Lindelöf.

The authors of [3] asked the following questions:

Question 1. Is it true in ZFC that every first countable linearly Lindelöf space is Lindelöf?

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Question 2. Can one drop CH in the formulation of Theorem 1?

Assuming CH, Arhangel’skii and Buzyakova were able to relax the condition of first countability:

Theorem 2. ([3, Theorem 4.2]) Assume CH, and let X be a linearly Lindelöf Tychonoff space such that \( t(X) < \aleph_\omega \). Then X is Lindelöf.

Question 3. ([3]) Does Theorem 2 remain valid if we drop CH?

A partial answer to Questions 1 and 2 was given by Arhangel’skii and Buzyakova. They constructed a linearly Lindelöf not Lindelöf space in [4, Example 15] assuming that \( 2^{\aleph_0} = 2^{\aleph_1} \). This space is of countable pseudocharacter because its topology is stronger than the usual topology of the real line. However, it is not first countable. Assuming \( MA + \aleph_1 < \mathfrak{c} \), we define a finer than usual topology on a subset of the Cantor set that makes it first countable linearly Lindelöf and not Lindelöf, thus answering Questions 1–3 in the negative.

We model our example on a linearly Lindelöf not Lindelöf space constructed by Mishchenko in [14]. That space, having non-Lindelöfness, is not first countable. Assuming GCH, Arhangel’skii and Buzyakova were able to relax the condition of first countability:

Theorem 2. ([3, Theorem 4.2]) Assume GCH, and let X be a linearly Lindelöf Tychonoff space such that \( t(X) < \aleph_\omega \). Then X is Lindelöf.

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Theorem 2. ([3, Theorem 4.2]) Assume GCH, and let X be a linearly Lindelöf Tychonoff space such that \( t(X) < \aleph_\omega \). Then X is Lindelöf.

Claim 1. \( \tilde{C}_n \) is a zero-dimensional (hence, Tychonoff) space.

Proof. Let \( U \) be an open neighborhood of \( x \) in \( \tilde{C}_n \). If \( U \) itself does not contain a clopen neighborhood of \( x \) in \( (C_n, \tau_n) \), then, by the definition of the topology \( \tau_n \), there are clopen neighborhoods \( V \) of \( x \) in \( (C_n, \tau_n) \) and a finite subset \( U' \subset \bigcup_{\alpha < \omega_1} U_{n, \alpha} \) such that \( x \in V \setminus \bigcup (U') \subset U \). Each element of \( U' \) is \( U_S \) for certain \( S \subset \omega \) that is closed in the topology of \( (C_n, \tau_n) \) and clopen in the topology of \( C_n = (C_n, \tau_n) \). Hence, \( V \setminus \bigcup (U') \) is a clopen neighborhood of \( x \) in \( \tilde{C}_n \), as required.

The following claim can be understood informally in the following way. The topology of \( \tilde{C}_n \) is stronger than that of \( C \), and a given \( x \in \tilde{C}_n \) may belong to uncountably many open sets \( U_S \) that are not open in \( C \). However, one of these \( U_S \) is the smallest one, so that \( \tilde{C}_n \) is first countable at \( x \).

Claim 2.

(a) Let \( x \in C_n \). If \( \beta \) is the smallest ordinal such that \( x \in U_{n, \beta} \) and \( x \in U_S \in U_{n, \beta} \), then every open neighborhood of \( x \) in \( \tilde{C}_n \) contains the intersection of \( U_S \) with a neighborhood of \( x \) that is open in \( C \).

(b) \( P \) is a first countable space.
Proof. (a) If $x \in U_{S_1} \subseteq U_{h \gamma}$ for some $S_1 \subseteq \omega$ and $\gamma < \omega_n$, then $\gamma \geq \beta$ due to minimality of $\beta$. Hence, $s = S_1 \setminus S$ is a finite set. Therefore, $U_s$ is an open subset of both the Cantor set and $\tilde{C}_n$, and $x \in U_s \cap U \subseteq U_{S_1}$.

Similarly, for every finite set of ordinals $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$, if $x \in \bigcap_{1 \leq i \leq k} U_{S_i}$ for some $S_i \subseteq U_{h \gamma}$, then $\gamma_i \geq \beta$ for every $1 \leq i \leq k$. Therefore, there is an open neighborhood $U$ of $x$ in $\tilde{C}_n$, which is also open in the usual topology of the Cantor set, such that $U \cap U \subseteq \bigcap_{1 \leq i \leq k} U_{S_i}$. This implies that every open neighborhood of $x$ in $\tilde{C}_n$ contains $U$ intersected with some neighborhood of $x$ that is open in the topology of the Cantor set.

(b) Since $P \subseteq \prod_{n=1}^{\infty} \tilde{C}_n$, it is enough to show that $\tilde{C}_n$ is first countable for every positive integer $n$. $\tilde{C}_n$ is first countable at every point of $\tilde{C}_n \setminus C_n$ since $\tilde{C}_n$ has the same topology at every point of $\tilde{C}_n \setminus C_n$, as the Cantor set does. First countability of $\tilde{C}_n$ at every point of $C_n$ follows directly from the fact that $C$ is first countable and item (a) of this claim. 

Claim 3. $P$ is not Lindelöf.

Proof. $P$ is not Lindelöf for the same reason as the Mishchenko’s example. The family $\{(U_k, x) \times \prod_{n \neq k} \tilde{C}_n \cap P: k \in \mathbb{N}, \alpha < \omega_n\}$ is an open cover of $P$ that does not have a subcover of cardinality less than $\kappa_n$. 

We use the following three lemmas to prove that $P$ is linearly Lindelöf. The first one probably is a fact of folklore.

Lemma 1. If $A$ and $B$ are closed subsets of a space $X$, then the set $F = A \cap B \setminus \bar{A}$ is a nowhere dense subset of $B$.

Proof. Assume the contrary, then there is an open set $U \subseteq X$ such that $B \cap U$ is a nonempty set that contains $F \cap U$ as a dense subset. Since $F$ is closed in $B$, we have that $F \cap U = B \cap U$. Then $F \subseteq A$ implies $F \cap U \subseteq A \cap U$, so $B \cap U \subseteq A \cap U$. Therefore, $(B \setminus A) \cap U = \emptyset$, which implies that $F \cap U = A \cap (B \setminus A) \cap U = \emptyset$. This contradicts the fact that $F \cap U$ is dense in $B \cap U$. 

Lemma 2 (MA). Let $(H, \tau')$ be a complete separable metrizable space, $\mu < 2^{\aleph_0}$, and suppose $\mathcal{H} = \{H_\alpha \subseteq H: \alpha < \mu\}$ is a family of closed subsets of $H$. Let $\tau'$ be a stronger topology on $H$ for which $\tau' \cup \mathcal{H}$ is a prebase. If $G$ is a closed subset of $(H, \tau')$, then $\tilde{G} \setminus G$ is a first category set in $\tilde{G}$, where both closures are taken in the topology $\tau'$ and the “first category” under $\tau'$ as well.

Proof. All the closures in this proof are taken in the coarser topology $\tau'$. Fix a countable base $\mathcal{B}$ of $(H, \tau')$. If $x \in \tilde{G} \setminus G$, then there are an open set $V \in \mathcal{B}$ and a finite subset $\mathcal{H}'$ of $\mathcal{H}$, with the intersection $\bigcap \mathcal{H}'$ denoted by $H'$, such that $x \in V \cap H'$ and $V \cap H' \cap G = \emptyset$. Because $(H, \tau')$ is a regular space, we can further assume that

$$V \cap H' \cap G = \emptyset.$$  

(1)

Apply Lemma 1 to $X = V \cap H'$ and $B = \tilde{G}$. We have that $V \cap H' \cap (\tilde{G} \setminus (V \cap H'))$ is a nowhere dense subset of $\tilde{G}$. By (1), $G \subseteq \tilde{G} \setminus (V \cap H')$, so $V \cap H' \cap G$ is a nowhere dense subset of $\tilde{G}$ as well and $x \in V \cap H' \cap G \subseteq \tilde{G} \setminus G$. Since $B$ is the countable base, and since the cardinality of the set of all finite subsets of $\mathcal{H}$ equals $\mu$, we conclude that $\tilde{G} \setminus G$ can be covered by $\mu < 2^{\aleph_0}$ nowhere dense subsets of $\tilde{G}$. Now, MA implies that $\tilde{G} \setminus G$ is a first category set in $\tilde{G}$.

Denote $P \times \prod_{n=1}^{\infty} \tilde{C}_n$, where $P$ is the set of irrational numbers with the usual topology, by $\tilde{L}$. Recall that each $\tilde{C}_n = (C_n, \tau_n)$ is a topological space whose underlying set is a Cantor set $C$. Therefore, $\tilde{L}$ is a topological space whose underlying set is $P \times C^\omega$. We denote the set $P \times C^\omega$ with the usual metrizable topology by $L$, so that $L$ is homeomorphic to $P \times C$. In the next claim and lemma, for every subset $X$ of $\tilde{L}$, $\mathcal{R}^X$ denotes the closure of $X$ if taken in the metrizable topology of $L$.

Lemma 3 (MA + $\kappa_\omega < 2^{\aleph_0}$). If $G$ is a closed subset of $\tilde{L}$, then $\tilde{G} \setminus G$ is a first category set in $\tilde{G}$. (Both closures and the “first category” are understood with respect to the coarser, metrizable topology of $L$.)

Proof. Lemma 3 follows directly from Lemma 2. 

Claim 4. $\tilde{L}$ is a Lindelöf space.

Proof. Assume that $\tilde{L}$ is not Lindelöf. Then there is $G$, a countably centered family of closed subsets of $\tilde{L}$ with an empty intersection (see [8, Theorem 3.8.3]). We can assume that the cardinality of $G$ does not exceed $w(\tilde{L}) = \kappa_\omega$. Consider $G' = \{G' \subseteq G: G' \neq \emptyset\}$. $G'$ has a nonempty intersection, $B$, since $G'$ is countably centered and $L$ is Lindelöf. Because $L$ is second-countable, there is a countable subset $G''$ of $G'$ such that $\bigcap G'' = B$. Since $G$ is countably centered, we can assume that $\tilde{C} \subseteq B$ for some $G \in G$. Lemma 3 implies that $\tilde{C} \setminus G$ is a first category set in $B$. We can assume without loss of generality that $T \subseteq G$ for every $T \subset B$. Then $U \setminus T = B \setminus T$ is a first category in $B$ for every $T \subseteq G$, $|G| < \kappa_\omega < 2^{\aleph_0}$, so $B \setminus (\bigcap G) = \bigcup_{T \subseteq G} (B \setminus T)$ is a first category set in $B$ as well by Martin’s axiom. $B$ is not a first category set in itself since it is a complete space, so $\bigcap G$ is a nonempty set, a contradiction. The claim is proved.
Claim 5. \(P\) is linearly Lindelöf.

Proof. Since \(w(C_n) = w(\tilde{C}_n) = \aleph_n\), we have that \(w(P) = \aleph_\omega\). Therefore, it is enough to show that every subset of \(P\) of cardinality \(\aleph_k\) has a complete accumulation point in \(P\). Let \(A\) be a subset of \(P\) of cardinality \(\aleph_k\) for some \(k \in \mathbb{N}\). We use the fact that \(P\) is a subspace of \(\prod_{\Omega=1}^{\aleph_n} \tilde{C}_n\).

Since \(P\) is a union of countably many sets \(\prod_{\Omega=1}^{\aleph_n} \tilde{C}_n \times \prod_{\Omega=1}^{\aleph_{k+1}} C_n\), there is \(k' > k\) such that \(|A \cap (\prod_{\Omega=1}^{\aleph_n} \tilde{C}_n \times \prod_{\Omega=1}^{\aleph_{k+1}} C_n)| = \aleph_k\). We can assume without loss of generality that \(A \subset \prod_{\Omega=1}^{\aleph_n} \tilde{C}_n \times \prod_{\Omega=1}^{\aleph_{k+1}} C_n\) for this \(k'\). For every \(n \in \mathbb{N}\), \(C_n\) is a union of the increasing family \(\{U_{n,\alpha} : \alpha < \aleph_n\}\) of cofinality \(\aleph_n\). Therefore, for every \(n \geq k'\) there is an ordinal \(\alpha_n < \omega_n\) such that the \(n\)th projection of \(A\) is a subset of \(U_{n,\alpha_n}\). This means that \(A \subset K\), where \(K\) stands for \(\prod_{\Omega=1}^{\aleph_n} \tilde{C}_n \times \prod_{\Omega=1}^{\aleph_{k+1}} U_{n,\alpha_n}\).

Recall that each \(U_{n,\alpha_n}\) is a union of a countable family \(U_{n,\alpha_n}\) of clopen subsets of \(\tilde{C}_n\). Thus, \(K\) is an \(\mathcal{F}_{\sigma\delta}\) subset of \(\prod_{\Omega=1}^{\aleph_n} \tilde{C}_n\). It is well known that an \(\mathcal{F}_{\sigma\delta}\) subset of an arbitrary space \(X\) is a continuous image of a closed subset of \(P \times X\). Claim 4 states that \(L = P \times \prod_{\Omega=1}^{\aleph_n} \tilde{C}_n\) is Lindelöf, so any closed subset of \(L\) is Lindelöf too, hence \(K\) is Lindelöf as a continuous image of a Lindelöf space. This implies that \(A\) has a complete accumulation point in \(K\). Since \(K \subset P\), we conclude that \(A\) has a complete accumulation point in \(P\), so \(P\) is a linearly Lindelöf space. \(\square\)

Claims 1–3 and 5 imply our result:

**Theorem 3 (MA + \(\aleph_\omega < 2^{\aleph_0}\)). There is a finer than usual topology on the Cantor set (and on the real line) that is Tychonoff, first countable, linearly Lindelöf, and not Lindelöf.**

2. A first countable Michael space from MA

Alster constructed in [1] a Michael space (that is, a Lindelöf space whose product with the set of the irrational numbers \(P\) is not Lindelöf) by a suitable refinement of the topology of the Cantor set. The referee noted that a refinement similar to the one used in our paper provides a Michael space that has the advantage of being first countable.

Alster’s construction goes as follows. Consider \(P\) as a subset of a metrizable compactification \(Y\). (Without loss of generality, we can assume that \(Y = \text{the countable power of } \omega + 1\).) For every \(x \in P\) and \(i \in \omega\), denote the \(i\)th coordinate of \(x\) by \(x_i\). Assume Martin’s Axiom, then there is a scale \(\{x_\alpha \in P : \alpha < \epsilon\}\) in \(P\) of cardinality \(\epsilon\). For every \(x \in P\), denote the set \(\{y \in P : y_i < x_i \text{ for every } i < \omega\}\) by \(V_\epsilon\). For every \(\alpha < \epsilon\), let \(V_\alpha\) be the set of all \(x \in P\) such that \(x \in P\) and \(x\) coincides with \(x_\alpha\) on a countable set, and let \(V_\epsilon = \bigcup V_\alpha\). Alster refines the metrizable topology of \(Y\) by declaring all \(V_\alpha\) to be clopen and proves that the resulting space is a Michael space. It is easy to see that this space is not first countable. Let \(\tau\) be a usual metrizable topology on \(Y\). Denote the topology on \(Y\) for which \(\tau \cup \{\bigcup \{V_\alpha : \alpha < \epsilon\}\}\) is a prebase by \(\tau'\). Then \(\tau'\) is a first countable topology that is finer then Alster’s topology, and \((Y, \tau')\) is a Michael space.

**Theorem 4 (MA). \((Y, \tau')\) is a Michael space.**

Proof. Observe that the set \(Y'' = Y \setminus P\) has the same topology as a subset of \((Y, \tau')\) and of the Cantor set \(Y\). Hence, if \(U\) is an open cover of \((Y, \tau')\), then there is a set \(K \subset P\) that is compact in the metrizable topology and that is covered by a countable subset of \(U\). Because \(\{x_\alpha : \alpha < \epsilon\}\) is a scale, there is a \(\beta < \epsilon\) and \(x' \in P\) that coincides with \(x_\beta\) on a countable set such that \(K \subset V_{x'}\). The set \(V_{x'}\) is Lindelöf as a subspace of \((Y, \tau')\) by Lemma 2, therefore \(K\) is covered by a countable subset of \(U\). This proves that \(U\) contains a countable subcover, so \((Y, \tau')\) is Lindelöf. \(\square\)

\((Y, \tau') \times P\) is not Lindelöf by [1, Lemma 4] since \((Y, \tau')\) has a finer topology than Alster’s example.

3. Closing remarks and questions

We carried out our construction in a model of \(MA + \aleph_\omega < 2^{\aleph_0}\). It is well known that if \(\tau < 2^{\aleph_0}\), then \(2^\tau = 2^{\aleph_0}\) in every model of MA. Therefore, \(2^{\aleph_0} = 2^{\aleph_0}\) in our model. MA was used twice: first, to ensure that \(\omega\) contains some uncountable systems of subsets ordered by inclusion modulo finite sets, and then to make sure that a complete separable metric space cannot be represented as a union of less than \(2^{\aleph_0}\) nowhere dense sets. Both times, a weaker form of MA would suffice (see [9]). It would be of interest to see whether an assumption of any form of Martin’s Axiom could be dropped altogether, so that the example can be constructed from an assumption on cardinal arithmetic only.

**Question 4. Does the existence of a first countable linearly Lindelöf not Lindelöf space follow from \(\aleph_\omega < 2^{\aleph_0}\)? From \(2^{\aleph_0} = 2^{\aleph_0}\)?**

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3 The author is grateful to M.V. Matveev for a prompt that streamlined this proof.

4 Note that a similar reasoning proves the following statement: for every set \(S \subset \mathbb{N}\) and \(\alpha_n < \aleph_n\), the space \(\prod_{\Omega \in S} U_{\Omega,\alpha_n} \times \prod_{\Omega \in \mathbb{N} \setminus S} \tilde{C}_n\) is Lindelöf. An analogous statement related to Mishchenko’s example is: for every set \(S \subset \mathbb{N}\) and \(\alpha_n < \aleph_n\), the space \(\prod_{\Omega \in S} (\alpha_n + 1) \times \prod_{\Omega \in \mathbb{N} \setminus S} (\alpha_n + 1)\) is compact.
A linearly Lindelöf not Lindelöf space of countable pseudocharacter was constructed in [4] from $2^{\aleph_\omega} = 2^{\aleph_0}$.

**Question 5.** Does the existence a linearly Lindelöf not Lindelöf space of countable pseudocharacter follow from $\aleph_\omega < 2^{\aleph_0}$?

The linearly Lindelöf not Lindelöf spaces described in [14] and [3] are pseudocompact. Our example is not pseudocompact since a one-to-one continuous map from a pseudocompact space onto a first countable space is necessarily a homeomorphism (see [6]).

**Question 6.** Is there a pseudocompact first countable linearly Lindelöf not Lindelöf space?

Since every Lindelöf pseudocompact space is compact, the last question is equivalent to the following one: is there a pseudocompact first countable linearly Lindelöf not compact space? Note that Question 6 has a negative answer under CH even if we weaken linearly Lindelöfness to $\omega_1$-Lindelöfness (see [3, Corollary 3.4 and Theorem 4.2] and [7]). On the other hand, Piotr Koszmider constructed in [11] a first countable, initially $\omega_1$-compact not compact (hence, pseudocompact $\omega_1$-Lindelöf not linearly Lindelöf), normal space.

Our example is Hewitt-complete (as a one-to-one continuous preimage of a metrizable space), and so are the examples in [4] and [10].

**Question 7.** Is there a ZFC linearly Lindelöf not Lindelöf Hewitt-complete space?

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**References**