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On the irregularity of bipartite graphs☆

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Abstract

The imbalance of an edge uv in a graph G is defined as $|d(u) - d(v)|$, where $d(u)$ denotes the degree of u. The irregularity of G, denoted $\mathrm{irr}(G)$, is the sum of the edge imbalances taken over all edges in G. We determine the structure of bipartite graphs having maximum possible irregularity with given cardinalities of the partite sets and given number of edges. We then derive a corresponding result for bipartite graphs with given cardinalities of the partite sets and determine an upper bound on the irregularity of these graphs. In particular, we show that if G is a bipartite graph of order n with partite sets of equal cardinalities, then $\mathrm{irr}(G) \leqslant n^3/27$, while if G is a bipartite graph with partite sets of cardinalities n_1 and n_2, where $n_1 \geqslant 2n_2$, then $\mathrm{irr}(G) \leqslant \mathrm{irr}(K_{n_1,n_2})$.

MSC: 05C35

Keywords: Bipartite; Edge imbalance; Graph irregularity

1. Introduction

We consider finite, simple and undirected graphs $G = (V, E)$ with vertex set V, edge set E, order $|V|$ and size $|E|$. The neighbourhood and the degree of a vertex $u \in V$ in the graph G are denoted by $N_G(u)$ and $d_G(u)$, respectively. If G is a bipartite graph and a vertex u in one partite set of G is adjacent to all vertices of the other partite set of G, then u is called *universal*.

In [2] Albertson defines the *imbalance* of an edge $e = uv \in E$ as $|d_G(u) - d_G(v)|$ and the *irregularity* of the graph G as

$$\mathrm{irr}(G) = \sum_{uv \in E} |d_G(u) - d_G(v)|.$$

He gives upper bounds on the irregularity of bipartite, triangle-free and general graphs. In [2] he claims that a bipartite graph of given order and size has maximum possible irregularity if it is as close to a complete bipartite graph as possible and that the irregularity of a bipartite graph of given order is maximum if the graph is complete bipartite. Albertson

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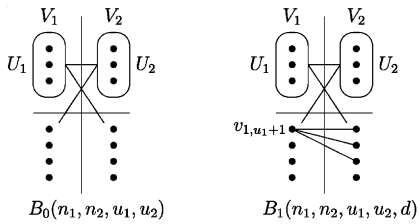

Fig. 1.

gives no formal proof of his claims but motivates them with the following argument:

> "If G is bipartite, then G must be a subgraph of $K_{t,n-t}$ for some t. We assume that $1 \leqslant t \leqslant n/2$. In such a graph the maximum degree is $\leqslant n-t$. The irregularity will be maximized by having as many edges as possible have the maximum possible imbalance."

We think that this argument does not imply his claims. The problem is that the maximum possible imbalance in a subgraph of $K_{t,n-t}$ for $1 \leqslant t \leqslant n/2$ would be $n-t-1$ and maximizing the number of edges having this imbalance would lead to the union of a star $K_{1,n-t}$ and $t-1$ isolated vertices. Obviously, these graphs have irregularity $(n-t-1)(n-t)$ which is not maximum (for $n \geqslant t+4$ consider for example the union of $K_{2,n-t}$ and $t-2$ isolated vertices which has larger irregularity).

The purpose of the present paper is to provide formal proofs for some of the claims in [2]. In Section 2 we determine the structure of the graphs having maximum possible irregularity among all bipartite graphs with given cardinalities of the partite sets and given number of edges. In Section 3 we derive a corresponding result for bipartite graphs with given cardinalities of the partite sets. In Section 4 we prove an upper bound on the irregularity of these graphs. For more references about similar and alternative measures of the irregularity of a graph we refer the reader to [1,3–6].

Before we proceed to the results we define two special classes of bipartite graphs.

Definition 1.1. Let n_1, n_2, u_1 and u_2 be non-negative integers such that $\max\{u_1, u_2\} \leqslant \min\{n_1, n_2\}$. The bipartite graph $B_0(n_1, n_2, u_1, u_2)$ has partite sets V_1 of cardinality n_1 and V_2 of cardinality n_2. Let $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\}$ and $U_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,u_i}\} \subseteq V_i$ for $i \in \{1, 2\}$. For $1 \leqslant i \leqslant n_1$ and $1 \leqslant j \leqslant n_2$ the edge set of $B_0(n_1, n_2, u_1, u_2)$ contains the edge $v_{1,i}v_{2,j}$ if and only if either $v_{1,i} \in U_1$ or $v_{2,j} \in U_2$. The class of all such graphs $B_0(n_1, n_2, u_1, u_2)$ is denoted by $\mathscr{B}_0$. A graph in the family $\mathscr{B}_0$ is illustrated in Fig. 1.

Note that the set $U_1 \cup U_2$ in Definition 1.1 contains only universal vertices of $B_0(n_1, n_2, u_1, u_2)$ but not necessarily all. This can be seen for example by considering the complete bipartite graph K_{n_1,n_2} which belongs to $\mathscr{B}_0$. For $n_1 > n_2$ we have $K_{n_1,n_2} = B_0(n_1, n_2, u_1, n_2)$ for all choices of u_1 such that $u_1 \leqslant n_2$ and for $n_1 = n_2$ we have $K_{n_1,n_2} = B_0(n_1, n_2, u_1, u_2)$ for all choices of u_1 and u_2 such that $\max\{u_1, u_2\} = \min\{n_1, n_2\} = n_1 = n_2$. In fact, $U_1 \cup U_2$ does not contain all universal vertices of $B_0(n_1, n_2, u_1, u_2)$ if and only if either $u_1 = n_1$ and $u_2 < n_2$ or $u_1 < n_1$ and $u_2 = n_2$.

Definition 1.2. Let n_1, n_2, u_1, u_2 and d be non-negative integers such that $u_1 \leqslant n_1 - 1$, $\max\{u_1+1, u_2+1\} \leqslant d \leqslant n_2 - 1$ and if $u_2 \neq 0$, then $d \leqslant n_1 - 1$. The bipartite graph $B_1(n_1, n_2, u_1, u_2, d)$ has partite sets V_1 of cardinality n_1 and V_2 of cardinality n_2. Let $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\}$ and $U_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,u_i}\} \subseteq V_i$ for $i \in \{1, 2\}$. For $1 \leqslant i \leqslant n_1$ and $1 \leqslant j \leqslant n_2$ the edge set of $B_1(n_1, n_2, u_1, u_2, d)$ contains the edge $v_{1,i}v_{2,j}$ if and only if either $v_{1,i} \in U_1$ or $v_{2,j} \in U_2$ or $i = u_1 + 1$ and $1 \leqslant j \leqslant d$. The class of all such graphs $B_1(n_1, n_2, u_1, u_2, d)$ is denoted by $\mathscr{B}_1$. A graph in the family $\mathscr{B}_1$ is illustrated in Fig. 1.

We observe that in Definition 1.1 the vertices in $U_1 \cup U_2$ have degree at least as large as the vertices in $(V_1 \cup V_2) \setminus (U_1 \cup U_2)$, while in Definition 1.2 the vertices in $U_1 \cup U_2$ have degree strictly larger than the vertices in $(V_1 \cup V_2) \setminus (U_1 \cup U_2)$.

2. The structure of extremal graphs

Let n_1, n_2 and $m \leqslant n_1 n_2$ be non-negative integers. Throughout this section let $G = (V, E)$ be a bipartite graph with partite sets V_1 of cardinality n_1 and V_2 of cardinality n_2 and $m = |E|$ edges. We assume that G has maximum possible irregularity given these conditions, i.e., $\mathrm{irr}(G) \geqslant \mathrm{irr}(H)$ for all bipartite graphs H with partite sets of cardinalities n_1 and n_2 and m edges.

Lemma 2.1. *Let G be as specified at the beginning of this section. Let $\tilde{U}_1 \subseteq V_1$ and $\tilde{U}_2 \subseteq V_2$ be sets of universal vertices such that $d_G(u) \geqslant d_G(v)$ for all $u \in \tilde{U}_1 \cup \tilde{U}_2$ and $v \in V \backslash (\tilde{U}_1 \cup \tilde{U}_2)$. (Note that $\tilde{U}_1 \cup \tilde{U}_2$ is not assumed to contain all universal vertices of G.) Let $x \in V_1 \backslash \tilde{U}_1$ be such that $d_G(x) \geqslant d_G(v)$ for all $v \in V \backslash (\tilde{U}_1 \cup \tilde{U}_2)$. Then the following holds.*

(i) *There are no two vertices $y \in V_2 \backslash \tilde{U}_2$ and $z \in V_1 \backslash (\{x\} \cup \tilde{U}_1)$ such that $xy \notin E$ and $yz \in E$.*
(ii) *There are no three vertices $w, y \in V_2 \backslash \tilde{U}_2$ and $z \in V_1 \backslash (\{x\} \cup \tilde{U}_1)$ such that $d_G(w) = |\tilde{U}_1|$ and $yz \in E$.*

Proof. Let $\tilde{u}_1 = |\tilde{U}_1|$ and $\tilde{u}_2 = |\tilde{U}_2|$. To prove (i) we assume for contradiction that such vertices y, z exist. If $G' = (V, \{xy\} \cup (E \backslash \{yz\}))$, then going from G to G' the total imbalance on the edges between x and $\tilde{U}_2$ drops by at most $\tilde{u}_2$, the total imbalance on the edges between x and $V_2 \backslash (\tilde{U}_2 \cup \{y\})$ grows by $d_G(x) - \tilde{u}_2$, the edge xy has imbalance $d_G(x) + 1 - d_G(y)$, the total imbalance on the edges between z and $\tilde{U}_2$ grows by $\tilde{u}_2$ and the total imbalance on the edges between z and $V_2 \backslash (\tilde{U}_2 \cup \{y\})$ drops by at most $d_G(z) - \tilde{u}_2 - 1$. This implies that

$$\begin{aligned}
\mathrm{irr}(G') - \mathrm{irr}(G) &\geqslant -\tilde{u}_2 + (d_G(x) - \tilde{u}_2) + (d_G(x) + 1 - d_G(y)) \\
&\quad + \tilde{u}_2 - (d_G(z) - \tilde{u}_2 - 1) - |d_G(y) - d_G(z)| \\
&= 2d_G(x) + 2 - d_G(y) - d_G(z) - |d_G(y) - d_G(z)| \\
&\geqslant 2,
\end{aligned}$$

which is a contradiction. Hence no such vertices exist and (i) is proved.

To prove (ii) we assume for contradiction that such vertices w, y, z exist. By (i), $xy \in E$. If $G' = (V, \{xw\} \cup (E \backslash \{yz\}))$, then going from G to G' the edge xw has imbalance $d_G(x) + 1 - \tilde{u}_1 - 1$, the imbalance on the edge xy increases by 2, the total imbalance on the edges between x and $\tilde{U}_2$ drops by at most $\tilde{u}_2$, the total imbalance on the edges between x and $V_2 \backslash (\tilde{U}_2 \cup \{w, y\})$ grows by $d_G(x) - \tilde{u}_2 - 1$, the total imbalance on the edges between y and $\tilde{U}_1$ grows by $\tilde{u}_1$, the total imbalance on the edges between y and $V_1 \backslash (\tilde{U}_1 \cup \{x, z\})$ drops by at most $d_G(y) - \tilde{u}_1 - 2$, the total imbalance on the edges between z and $\tilde{U}_2$ grows by $\tilde{u}_2$, the total imbalance on the edges between z and $V_2 \backslash (\tilde{U}_2 \cup \{y\})$ drops by at most $d_G(z) - \tilde{u}_2 - 1$, and the total imbalance on the edges between w and $\tilde{U}_1$ drops by at most $\tilde{u}_1$. This implies that

$$\begin{aligned}
\mathrm{irr}(G') - \mathrm{irr}(G) &\geqslant (d_G(x) + 1 - \tilde{u}_1 - 1) + 2 - \tilde{u}_2 + (d_G(x) - \tilde{u}_2 - 1) + \tilde{u}_1 - (d_G(y) - \tilde{u}_1 - 2) \\
&\quad + \tilde{u}_2 - (d_G(z) - \tilde{u}_2 - 1) - \tilde{u}_1 - |d_G(y) - d_G(z)| \\
&= 2d_G(x) + 4 - d_G(y) - d_G(z) - |d_G(y) - d_G(z)| \\
&\geqslant 4,
\end{aligned}$$

which is a contradiction. Hence no such vertices exist and also (ii) is proved. □

The next result characterizes the structure of the extremal graphs.

Theorem 2.2. *Let G be as specified at the beginning of this section. Then, $G \in \mathscr{B}_0 \cup \mathscr{B}_1$.*

Proof. Let U_1 and U_2 be the sets of all (!) universal vertices in V_1 and V_2, respectively. Let $u_1 = |U_1|$ and $u_2 = |U_2|$. We consider two cases.

Case 1: All edges of G are incident with a vertex in $U_1 \cup U_2$.

If $d_G(u) \geqslant d_G(v)$ for all $u \in U_1 \cup U_2$ and $v \in V \backslash (U_1 \cup U_2)$, then $G = B_0(n_1, n_2, u_1, u_2) \in \mathscr{B}_0$. Hence we may assume without loss of generality that there is a vertex $x \in V_1 \backslash U_1$ such that $d_G(x) > d_G(u) = n_1$ for all $u \in U_2 \neq \emptyset$ and $d_G(x) \geqslant d_G(v)$ for all $v \in V \backslash (U_1 \cup U_2)$.

If $n_1 = u_1 + 1$, then $G = B_1(n_1, n_2, u_1, 0, d) \in \mathscr{B}_1$ where $u_1 + 2 \leqslant d = d_G(x) \leqslant n_2 - 1$.

If $n_1 \geqslant u_1 + 2$, then let $z \in V_1 \backslash (U_1 \cup \{x\})$ and $y \in U_2$. Since x is not universal, there is a vertex $w \in V_2 \backslash U_2$. Clearly, $d_G(w) = u_1$. Now Lemma 2.1 (ii) (for $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = \emptyset$) implies a contradiction to the choice of G. This completes the proof for Case 1.

Case 2: There is an edge $x_1x_2 \in E$ such that $x_1 \in V_1 \backslash U_1$ and $x_2 \in V_2 \backslash U_2$.

We may assume without loss of generality that $d_G(x_1) \geqslant d_G(v)$ for all $v \in V \backslash (U_1 \cup U_2)$. Since $x_1 \notin U_1$, $d_G(x_1) \leqslant n_2 - 1$.

First, we assume that there is an edge $yz \in E$ with $y \in V_2 \backslash U_2$ and $z \in V_1 \backslash (U_1 \cup \{x_1\})$. Lemma 2.1 (i) (if $d_G(x_1) < n_1$ for $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = U_2$ and if $d_G(x_1) \geqslant n_1$ for $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = \emptyset$) implies that $x_1y \in E$. Since x_1 is not universal, there is a vertex $w \in V_2 \backslash (U_2 \cup \{y\})$ with $x_1w \notin E$. By Lemma 2.1 (i) (if $d_G(x_1) < n_1$ for $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = U_2$ and if $d_G(x_1) \geqslant n_1$ for $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = \emptyset$), $d_G(w) = u_1$. Now Lemma 2.1 (ii) (if $d_G(x_1) < n_1$ for $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = U_2$ and if $d_G(x_1) \geqslant n_1$ for $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = \emptyset$) implies a contradiction to the choice of G. Hence all edges of G are incident with at least one vertex in $\{x_1\} \cup U_1 \cup U_2$.

By the choice of x_1, we have $u_1 + 1 = d_G(x_2) \leqslant d_G(x_1)$ and $u_2 + 1 \leqslant d_G(x_1)$.

If $u_2 = 0$ or $d_G(x_1) \leqslant n_1 - 1$, then $G = B_1(n_1, n_2, u_1, u_2, d_G(x_1)) \in \mathscr{B}_1$. Hence we assume that $u_2 \neq 0$ and $d_G(x_1) \geqslant n_1$. Let $y \in U_2$. Since x_1 is not universal, there is a vertex $w \in V_2 \backslash (U_2 \cup \{y\})$ with $x_1w \notin E$. By Lemma 2.1 (i) (for $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = \emptyset$), $d_G(w) = u_1$. Since x_2 is not universal, there is a vertex $z \in V_1 \backslash (U_1 \cup \{x_1\})$ with $x_2z \notin E$. Now Lemma 2.1 (ii) (for $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = \emptyset$) implies a contradiction to the choice of G. This completes the proof for Case 2. $\square$

Even though Theorem 2.2 considerably restricts the structure of the extremal graphs it is not easy to derive upper bounds on the irregularity from it. To do so one has to maximize a complicated function depending on several integer variables. We will do this for some special cases in the following sections.

3. Extremal graphs with arbitrary size

Let n_1 and n_2 be non-negative integers. Throughout this section let $G = (V, E)$ be a bipartite graph with partite sets V_1 of cardinality n_1 and V_2 of cardinality n_2. We assume that G has maximum possible irregularity given these conditions, i.e., $\mathrm{irr}(G) \geqslant \mathrm{irr}(H)$ for all bipartite graphs H with partite sets of cardinalities n_1 and n_2. *Note that we do not specify the number of edges of G.*

Theorem 3.1. *Let G be as specified at the beginning of this section. Then* $G \in \mathscr{B}_0$.

Proof. By Theorem 2.2, we have $G \in \mathscr{B}_0 \cup \mathscr{B}_1$. We assume $G = B_1(n_1, n_2, u_1, u_2, d) \in \mathscr{B}_1 \backslash \mathscr{B}_0$ and consider different cases.

Case 1: $u_1 = u_2 = 0$.

Clearly, $\mathrm{irr}(G) < \mathrm{irr}(B_0(n_1, n_2, 1, 0))$ which is a contradiction to the choice of G.

Case 2: $u_1 \neq 0$ and $u_2 = 0$.

We have $u_1 + 1 \leqslant d \leqslant n_2 - 1$ and

$$\begin{aligned}
\mathrm{irr}(G) &= \mathrm{irr}(B_0(n_1, n_2, u_1, 0)) - du_1 + d(d - u_1 - 1) \\
&= \mathrm{irr}(B_0(n_1, n_2, u_1, 0)) + (d - (u_1 + \tfrac{1}{2}))^2 - (u_1 + \tfrac{1}{2})^2 \\
&< \mathrm{irr}(B_0(n_1, n_2, u_1, 0)) + (n_2 - (u_1 + \tfrac{1}{2}))^2 - (u_1 + \tfrac{1}{2})^2 \\
&= \mathrm{irr}(B_0(n_1, n_2, u_1 + 1, 0))
\end{aligned}$$

which is a contradiction to the choice of G.

Case 3: $u_1 = 0$ and $u_2 \neq 0$.

We have $u_2 + 1 \leqslant d \leqslant \min\{n_1 - 1, n_2 - 1\} = \min\{n_1, n_2\} - 1$. If H arises from G by joining x to a non-neighbour of x in V_2, then

$$\begin{aligned}\mathrm{irr}(G) &= \mathrm{irr}(B_0(n_1, n_2, 0, u_2)) - (d - u_2)u_2 + (d - 1)(d - u_2)\\ &= \mathrm{irr}(B_0(n_1, n_2, 0, u_2)) + (d - (u_2 + \tfrac{1}{2}))^2 - (u_2 + \tfrac{1}{2})^2 + u_2^2 + u_2\\ &< \mathrm{irr}(B_0(n_1, n_2, 0, u_2)) + ((d + 1) - (u_2 + \tfrac{1}{2}))^2 - (u_2 + \tfrac{1}{2})^2 + u_2^2 + u_2\\ &= \mathrm{irr}(H)\end{aligned}$$

which is a contradiction to the choice of G.

Case 4: $u_1, u_2 \neq 0$.

We have $\max\{u_1 + 1, u_2 + 1\} \leqslant d \leqslant \min\{n_1 - 1, n_2 - 1\} = \min\{n_1, n_2\} - 1$ and

$$\begin{aligned}\mathrm{irr}(G) &= \mathrm{irr}(B_0(n_1, n_2, u_1, u_2)) - (d - u_2)u_1 - (d - u_2)u_2 + (d - u_2)(d - u_1 - 1)\\ &= \mathrm{irr}(B_0(n_1, n_2, u_1, u_2)) + (d - (u_1 + u_2 + \tfrac{1}{2}))^2 - (u_1 + u_2 + \tfrac{1}{2})^2 + 2u_1u_2 + u_2^2 + u_2.\end{aligned}$$

Let H arise from G by joining x to a non-neighbour of x in V_2. Since

$$\begin{aligned}\mathrm{irr}(B_0(n_1, n_2, u_1, u_2)) = \mathrm{irr}(B_0(n_1, n_2, u_1, u_2)) &+ (u_2 - (u_1 + u_2 + \tfrac{1}{2}))^2\\ &- (u_1 + u_2 + \tfrac{1}{2})^2 + 2u_1u_2 + u_2^2 + u_2\end{aligned}$$

and

$$\mathrm{irr}(H) = \mathrm{irr}(B_0(n_1, n_2, u_1, u_2)) + ((d + 1) - (u_1 + u_2 + \tfrac{1}{2}))^2 - (u_1 + u_2 + \tfrac{1}{2})^2 + 2u_1u_2 + u_2^2 + u_2$$

we obtain that either $\mathrm{irr}(G) < \mathrm{irr}(B_0(n_1, n_2, u_1, u_2))$ or $\mathrm{irr}(G) < \mathrm{irr}(H)$ which is a contradiction to the choice of G. This completes the proof. □

4. Maximum values of the irregularity

Theorem 4.1. *Let the bipartite graph $G = (V, E)$ have partite sets of cardinalities $n_1 \geqslant n_2$, respectively. Then*

$$\mathrm{irr}(G) \leqslant u_1u_2(n_1 - n_2) + u_1(n_2 - u_2)(n_2 - u_1) + u_2(n_1 - u_1)(n_1 - u_2), \tag{1}$$

where u_1 and u_2 are given by

$$u_1 = \tfrac{1}{2}(n_2 - u_2) \quad \textit{and} \tag{2}$$

$$u_2 = \begin{cases} n_2 - \frac{4}{3}n_1 + \sqrt{\frac{28}{9}n_1^2 - \frac{8}{3}n_1n_2} & \textit{if } n_1 < 2n_2,\\ n_2 & \textit{if } n_1 \geqslant 2n_2.\end{cases} \tag{3}$$

Proof. We assume that G has maximum possible irregularity given the cardinalities of the partite sets. By Theorem 3.1, $G = B_0(n_1, n_2, u_1', u_2') \in \mathscr{B}_0$ for some non-negative integers u_1' and u_2' with $\max\{u_1', u_2'\} \leqslant \min\{n_1, n_2\}$. This implies that (1) holds with equality for $(u_1, u_2) = (u_1', u_2')$.

We will now prove that the right-hand side of (1) is maximized for non-negative real numbers u_1 and u_2 with $\max\{u_1, u_2\} \leqslant \min\{n_1, n_2\}$, if u_1 and u_2 are as specified in (2) and (3). Let

$$\begin{aligned}f(u_1, u_2) &= u_1u_2(n_1 - n_2) + u_1(n_2 - u_2)(n_2 - u_1) + u_2(n_1 - u_1)(n_1 - u_2)\\ &= -(n_2 - u_2)u_1^2 + (n_2 - u_2)^2u_1 + u_2n_1^2 - u_2^2n_1.\end{aligned}$$

We have $(\partial/\partial u_1) f(u_1, u_2) = (n_2 - u_2)(n_2 - u_2 - 2u_1)$. If $n_2 - u_2 \neq 0$, then $\partial f/\partial u_1 = 0$ implies (2). If $n_2 - u_2 = 0$, then f is independent of u_1 and we may assume that (2) holds. Note that $\frac{1}{2}(n_2 - u_2) \leqslant \min\{n_1, n_2\}$.

Substituting in f the value for u_1 given by (2) we obtain

$$f\left(\tfrac{1}{2}(n_2 - u_2), u_2\right) = \tfrac{1}{4}(n_2 - u_2)^3 + u_2 n_1^2 - u_2^2 n_1$$

and thus

$$\frac{\partial}{\partial u_2} f\left(\frac{1}{2}(n_2 - u_2), u_2\right) = -\frac{3}{4}(n_2 - u_2)^2 + n_1^2 - 2u_2 n_1.$$

$(\partial/\partial u_2) f(\frac{1}{2}(n_2 - u_2), u_2) = 0$ implies that $u_2 = n_2 - \frac{4}{3}n_1 \pm \sqrt{\frac{28}{9}n_1^2 - \frac{8}{3}n_1 n_2}$. Since $n_2 - \frac{4}{3}n_1 - \sqrt{\frac{28}{9}n_1^2 - \frac{8}{3}n_1 n_2} < 0$ and the coefficient of u_2^3 in $f(\frac{1}{2}(n_2 - u_2), u_2)$ is negative, we obtain that $u_2 = \min\left\{n_2, n_2 - \frac{4}{3}n_1 + \sqrt{\frac{28}{9}n_1^2 - \frac{8}{3}n_1 n_2}\right\}$ which easily implies (3). $\square$

We illustrate Theorem 4.1 by two immediate corollaries.

Corollary 4.2. *Let the bipartite graph G have partite sets of cardinalities n_1 and n_2, where $n_1 = n_2$. Then,* $\mathrm{irr}(G) \leqslant 8n_1^3/27$.

If $n_1 \equiv 0 (\mathrm{mod}\, 3)$, then $\mathrm{irr}(B_0(n_1, n_1, n_1/3, n_1/3)) = 8n_1^3/27$. Hence Corollary 4.2 is essentially best-possible.

Corollary 4.3. *Let the bipartite graph G have partite sets of cardinalities n_1 and n_2, where $n_1 \geqslant 2n_2$. Then,* $\mathrm{irr}(G) \leqslant \mathrm{irr}(K_{n_1,n_2}) = n_1 n_2 (n_1 - n_2)$.

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