Young–Fibonacci insertion, tableauhedron and Kostka numbers

Janvier Nzeutchap

LITIS EA 4108 (Laboratoire d’Informatique, de Traitement de l’Information et des Systèmes), Avenue de l’Université, 76800 Saint Etienne du Rouvray, France

A R T I C L E   I N F O

Article history:
Received 16 April 2007
Available online 6 August 2008

Keywords:
Schensted–Fomin
Young–Fibonacci
Kostka number
Permutohedron
Tableauhedron
Okada’s algebra

A B S T R A C T

This work is first concerned with some properties of the Young–Fibonacci insertion algorithm and its relation with Fomin’s growth diagrams. It also investigates a relation between the combinatorics of Young–Fibonacci tableaux and the study of Okada’s algebras associated to the Young–Fibonacci lattice. The original algorithm was introduced by Roby and we redefine it in such a way that both the insertion and recording tableaux of any permutation are conveniently interpreted as saturated chains in the Young–Fibonacci lattice. Using our conventions, we give a simpler proof of a property of Killpatrick’s evacuation algorithm for Fibonacci tableaux. It also appears that this evacuation is no longer needed in making Roby’s and Fomin’s constructions coincide. We provide the set of Young–Fibonacci tableaux of size \( n \) with a structure of graded poset called tableauhedron, induced by the weak order of the symmetric group, and realized by transitive closure of elementary transformations on tableaux. We show that this poset gives a combinatorial interpretation of the coefficients of the transition matrix from the analogue of complete symmetric functions to analogue of the Schur functions in Okada’s algebra associated to the Young–Fibonacci lattice. We prove a similar result relating usual Kostka numbers with four partial orders on Young tableaux, studied by Melnikov and Taskin.

© 2008 Elsevier Inc. All rights reserved.

Contents

1. Introduction ......................................................... 144

E-mail address: janvier.nzeutchap@univ-mlv.fr.
1. Introduction

The Young's lattice \((\mathcal{Y}L)\) is defined on the set of partitions of positive integers, with cover relations given by the natural inclusion order. The differential poset nature of this graph was generalized by Fomin who introduced graph duality \([11]\). With this extension he introduced \([13]\) a generalization of the classical Robinson–Schensted \([2]\) algorithm, giving a general scheme for establishing bijective correspondences between couples of saturated chains in dual graded graphs, and permutations of the symmetric group. This approach naturally leads to the Robinson–Schensted insertion algorithm.

Roby \([15]\) gave an insertion algorithm analogous to the Schensted correspondence, which maps a permutation \(\sigma\) onto a couple made of a standard Young–Fibonacci tableau \(P(\sigma)\) and a standard path tableau \(Q(\sigma)\). Roby's path tableau \(Q(\sigma)\) is canonically interpreted as a saturated chain in the Fibonacci lattice \(Z(1)\) introduced by Stanley \([10]\) and also by Fomin \([12]\). Roby also showed that Fomin’s approach is partially equivalent to his construction. Indeed, only the saturated chain \(\hat{Q}\) obtained from Fomin’s growth diagram has an interpretation as the path tableau \(Q(\sigma)\), while there seems to be no way to translate Roby's insertion tableau \(P(\sigma)\) into its equivalent chain \(\hat{P}\). Contrarily to the approach of Killpatrick \([8]\) who has used an evacuation to relate the two constructions of Roby and Fomin, we show that with a suitable mechanism for converting a saturated chain of the Young–Fibonacci lattice into a Young–Fibonacci tableau, Roby's construction naturally coincides with Fomin's description using growth diagrams.

There are many similarities between Young's lattice \((\mathcal{Y}L)\) and Young–Fibonacci lattice \((\mathcal{Y}FL)\), and this motivates the investigation of combinatorial as well as algebraic interpretations of \(\mathcal{Y}FL\). Some of these interpretations were found and stated by Okada \([14]\) as follows:

1. There is a family of algebras \((\mathcal{S}_n)_{n \in \mathbb{N}}\) whose Bratteli diagram is \(\mathcal{Y}FL\), the same way as \(\mathcal{Y}L\) is the Bratteli diagram of \((\mathcal{K}[\mathcal{S}_n])_{n \in \mathbb{N}}\);
2. There is a graded algebra \(\mathcal{Y}F\mathcal{S}\mathcal{Y}m\) whose basis elements are indexed by \(\mathcal{Y}FL\), and such that multiplication by the distinguished element \(s_1\) is described by the cover relations in \(\mathcal{Y}FL\). This is an analogue of the Pieri rule for the multiplication of Schur functions.

So Okada's algebra is a clone of the algebra of symmetric functions. It is known that Young tableaux play an important role in the representation theory of the symmetric group, as well as in the study of the algebra of symmetric functions. We are interested in the relations between the combinatorics of Young–Fibonacci tableaux and Okada’s algebras associated to the Young–Fibonacci lattice.

The paper is organized as follows. In Section 2.1 we recall the definition of the Young–Fibonacci lattice, then in Section 2.2 we define a mechanism for converting a saturated chain in this lattice into...
a standard Young–Fibonacci tableau. In the same section, we also redefine Roby’s algorithm, in such a way that both the insertion and recording tableaux of any permutation will have an interpretation in terms of saturated chain in the Young–Fibonacci lattice. In Section 3.1 we relate Roby’s algorithm with Fomin’s construction using growth diagrams and we also comment Killpatrick’s evacuation for Fibonacci tableaux.

In Section 4, we define Young–Fibonacci numbers who are some sort of analogues of Kostka numbers, and we point out one of their relation with usual Fibonacci numbers.

In Section 5 we define and we study some properties of a poset on Young–Fibonacci tableaux. This poset turns out to be a model for the interpretation as well as the computation of another analogue of Kostka numbers, introduced by Okada [14] in the course of his study of $\text{YF} \text{Sym}$. We prove this result is Section 6, and in the last section of the paper we prove a similar result relating usual Kostka numbers with four posets on Young tableaux studied by Melnikov [1] and Taskin [9].

2. The Young–Fibonacci insertion algorithm

2.1. The Young–Fibonacci lattice

A Fibonacci word of size $n$, also called Fibonacci diagram or snakeshape, is a column by column graphical representation of a composition of the integer $n$, with each part equal to 1 or 2. The number of such compositions is the $n$th Fibonacci number. For example,

$w = 22121 = \begin{array}{|c|c|c|c|} \hline & & & \\
\hline & & & \\
\hline & & & \\
\hline & & & \\
\hline \end{array}$ is a snakeshape of size $n = 8$.

A partial order is defined on the set of all snakeshapes, in such a way to obtain an analogue of Young’s lattice. This lattice is called the Young–Fibonacci lattice ($\text{YFL}$), it was introduced by Stanley [10] and also by Fomin [12]. As we will see in the sequel, there are quite many similarities between the two lattices, as well as between the two underlying combinatorics of tableaux. The cover relations in $\text{YFL}$ are given below, for any snakeshape $u$:

1. $u$ is covered by the snakeshape obtained by attaching a single box just in front;
2. $u$ is covered by the snakeshape obtained by adding a single box on top of its first single-boxed column, reading $u$ from left to right;
3. if $u$ starts with a series of two-boxed columns, then it is covered by all snakeshapes obtained by inserting a single-boxed column just after any of those columns.

The rank $|u|$ of a snakeshape $u$ is the sum of digits of the corresponding Fibonacci word. Its length will be denoted $\ell(u)$. Let $u$ and $v$ be two snakeshapes such that $v$ covers $u$ in $\text{YFL}$, the cell added to $u$ to obtain $v$ is an inner corner of $v$, it is also called an outer corner of $u$.

Remark 2.1. Young–Fibonacci tableaux ($\text{YFT}$) will naturally appear as numberings of snakeshapes, satisfying certain conditions described in the sequel, the same way as Young tableaux are numberings of partitions of integers with prescribed numbering conditions. The numbering conditions of Young–Fibonacci tableaux are deduced from the description of the Young–Fibonacci insertion algorithm.

Fig. 1 is a pictorial representation of a finite realization of $\text{YFL}$, from rank 0 up to rank $n = 4$.

Now let us look at the problem of converting a saturated chain of $\text{YFL}$ into a standard Young–Fibonacci tableau.

2.2. Young–Fibonacci tableaux

In the Young’s lattice ($\text{YL}$), any saturated chain starting at the empty partition can be canonically converted into a standard Young tableau, and this representation is convenient in many ways. It consists in labeling the boxes as their occur in the chain. As already observed by Roby [15], one question which presents itself is to do the same in the Young–Fibonacci lattice ($\text{YFL}$) for any saturated chain
starting at the empty snakeshape $\emptyset$. The need of such a conversion mechanism will appear in Section 3.1 in the interpretation of two saturated chains obtained from a growth diagram.

One may also use the canonical labeling to convert a saturated chain of $\mathcal{YFL}$ into a tableau, but Roby had already pointed out that one major problem with this canonical labeling is that except for the trivial rule that each element in the top row must be greater than the one below it, no other obvious rules govern what numberings are allowed for a given shape. We suggest that one first defines simple rules governing what numberings are allowed for a given shape, so that it be easy to decide if a numbering of a snakeshape is a legitimate Young–Fibonacci tableau or not. The convention we use is described in the next section.

2.3. Converting a chain of $\mathcal{YFL}$ into a standard Young–Fibonacci tableau

Since we do not use the same conventions as Roby [15] and Fomin [11], let us give the following definition of Young–Fibonacci tableaux.

**Definition 2.2.** A numbering of a snakeshape with distinct nonnegative integers is a standard Young–Fibonacci tableau ($\mathcal{SYFT}$) under the following conditions:

1. entries are strictly increasing in columns;
2. the maximal entry of each column has no entry greater than itself on its right.

To convert a chain of $\mathcal{YFL}$ into a standard $\mathcal{YFT}$, one will follow the canonical approach as far as the new box added to the chain lies in the first column. Example with the saturated chain $\mathcal{Q} = (\emptyset, 1, 2, 12, 22, 221, 2211, 21211)$; the sub-chain $(\emptyset, 1, 2, 12, 22)$ is converted as follows:

$$
\emptyset \rightarrow \begin{array}{c}
1 \\
\end{array} \rightarrow \begin{array}{c}
2 \\
1 \\
\end{array} \rightarrow \begin{array}{c}
2 \\
3 \\
1 \\
\end{array} \rightarrow \begin{array}{c}
4 \\
2 \\
3 \\
1 \\
\end{array}
$$

Moving from the shape 22 to the shape 221 in $\mathcal{YFL}$, one has inserted a box just after a two-boxed column of the previous shape. In such a situation, one will move the entry on top in that column into the newly created box, and then shift the other entries of the top row to the right. Finally, if $n$ is the largest entry in the partial tableau obtained, then label the box on top in the first column with $(n + 1)$. So the conversion started above keeps on as follows.
It easily follows from the description above that this mechanism produces only legitimate Young–Fibonacci tableaux (Definition 2.2), and the conversion is reversible, that is given a Young–Fibonacci tableau \( t \), one can recover the corresponding chain in the lattice.

Now another question which presents itself is how to count standard Young–Fibonacci tableaux of a given shape \( u \neq \emptyset \), where \( n \) 

/!

\( \text{Ext}(P) = n! \prod_{i=1}^{n} d_i \) (2.1)

where \( v_i \) is the vertex labeled \( i \) and \( d_i \) is the number of vertices \( v \) such that \( v \leq_P v_i \). This formula is due to Knuth [3], and since any snakeshape \( u \) can be canonically assimilated to a poset \( P_u \), then we have the following.

**Proposition 2.3.** Standard Young–Fibonacci tableaux of a given shape are counted by the hook-length formula for binary trees.

To apply the formula to a snakeshape \( u \), count it cells from right to left and from bottom to top, labeling the first box and each box appearing in the bottom row of any two-boxed column. The number of standard \( YFT \) of the given shape is the product of all the labels obtained.

**Example 2.4.** Let us consider \( u = 2212 \).

\[
\begin{array}{c}
22 \rightarrow 221 : \\
\begin{array}{c}
221 \rightarrow 2211 : \\
2211 \rightarrow 21211 :
\end{array}
\end{array}
\]

2.4. Redefining the Young–Fibonacci insertion algorithm

The Young–Fibonacci insertion algorithm is a Schensted-like correspondence mapping each permutation \( \sigma \) onto a couple \( (P(\sigma), Q(\sigma)) \) of standard Young–Fibonacci tableaux of the same shape. We refer the reader to [8,15] for a description of the original algorithm; below is a geometric description of the one we consider.

**Definition 2.5.** The insertion tableau \( P(\sigma) \) is geometrically built by reading \( \sigma \) from right to left, matching any of the numbers encountered and not yet matched with the largest number not yet matched on its left, if any, as long as the latter is greater than the former. Then place the numbers in
any matched pair in a column of height 2 with the larger number on top and any unmatched number in a column of height 1. The recording tableau $Q(\sigma)$ records the positions in $\sigma$ of the entries of $P(\sigma)$, but in reverse order.

**Example 2.6.** Insertion and recording tableaux of the permutation $\sigma = 2715643$.

![Example Tableau](image)

\[
P(\sigma) = \begin{array}{cccccc}
7 & 6 & 2 & \\
3 & 4 & 5 & 1 \\
2 & 5 & 4 & 1
\end{array}
\quad \text{and} \quad
Q(\sigma) = \begin{array}{cccc}
7 & 6 & 3 & \\
2 & 5 & 4 & 1
\end{array}
\]

**Remark 2.7.** That both $P(\sigma)$ and $Q(\sigma)$ are standard Young–Fibonacci tableaux (Definition 2.2) follows from the geometric description of the algorithm, which is not the case for the original algorithm. Indeed, using Roby’s description of the insertion algorithm, the insertion tableau $P_{\text{roby}}(\sigma)$ is obtained from our own $P(\sigma)$ by inverting the order of entries in each column. The recording tableau $Q_{\text{roby}}(\sigma)$ which follows does not satisfy Definition 2.2.

\[
Q_{\text{roby}}(\sigma) = \begin{array}{ccc}
3 & 7 & 4 \\
2 & 6 & 5 \\
1
\end{array}
\]

The definition of $Q(\sigma)$ we adopt is inspired from the hypoplactic [4] and sylvester [6] insertion algorithms, where $Q(\sigma)$ also records the positions in $\sigma$ of the labels of $P(\sigma)$. With this definition, some essential properties of the Young–Fibonacci correspondence have a much easier combinatorial proof, which is not always the case in [15]. For example, let us recall the Schützenberger property.

**Theorem 2.8.** (See [15].) For any permutation $\sigma$, $P(\sigma^{-1}) = Q(\sigma)$.

**Proof.** $P(\sigma)$ and $Q(\sigma)$ are also obtained by drawing the permutation matrix of $\sigma$. Then from right to left draw a set of broken lines to join the rightmost and uppermost symbols 1 of the matrix. Then remove these symbols from the matrix. $P(\sigma)$ corresponds to reading vertical coordinates of the rightmost and uppermost symbol 1 in that order, for all the broken lines. As for $Q(\sigma)$ it corresponds to reading horizontal coordinates of the uppermost and rightmost symbol 1 in the same order. It is not difficult to see that the construction for $\sigma^{-1}$ is obtained by transposing the one for $\sigma$, hence the result. The example below corresponds to $\sigma = 4173265$. \qed

![Diagram](image)

Another fundamental property of the algorithm which is easily proved using Definition 2.5 follows.

**Theorem 2.9.** (See [15].) Let $\sigma$ be an involution of the symmetric group, then the cycle decomposition of $\sigma$ is the column reading of its insertion tableau $P(\sigma)$. 

Let $\mathcal{YFC}(t)$ denote the equivalence class made of permutations having $t$ as insertion tableau, then $\mathcal{YFC}(t)$ has at least three canonical elements. The first canonical element is its canonical involution, that is the only involution the cycles of which coincide with the columns of $t$, as stated in Theorem 2.9. The two other canonical elements are the maximal (resp. minimal) element for the lexicographical order. We will make use of these elements in Section 5.

**Lemma 2.10.** Let $t$ be a Young–Fibonacci tableau, $w_1$ the left-to-right reading word of its top row and $w_2$ the right-to-left reading word of its bottom row, then $w_1, w_2$ (where denotes the usual concatenation of words) is the maximal element (for the lexicographical order) of $\mathcal{YFC}(t)$, it is denoted $w^t_{\max}$.

**Lemma 2.11.** The word consisting of the right-to-left and up-down column reading of $t$ is the minimal element (for the lexicographical order) of $\mathcal{YFC}(t)$, it is denoted $w^t_{\min}$.

**Proof.** Easy from the description of the Young–Fibonacci insertion algorithm. □

We will see (Theorem 5.11) that $\mathcal{YFC}(t)$ is the set of linear extensions of a given poset, and additionally, $\mathcal{YFC}(t)$ is an interval of the weak order on the symmetric group (Theorem 5.14).

An example is given with the tableau $t$ below.

$$
\begin{array}{cccc}
8 & 6 & 3 \\
4 & 2 & 5 & 1
\end{array}
$$

Its canonical involution is $(13)(26)(48)(5)(7) = 36185274$, the maximal canonical element is $86315274$, and the minimal one is $31562784$.

### 3. Young–Fibonacci insertion and growth in differential posets

In this section we show that with the modification we have introduced in Roby’s original insertion algorithm, together with the conversion mechanism discussed in Section 2.3, the Young–Fibonacci insertion algorithm naturally coincides with Fomin’s approach using growth diagrams. So we claim that Killpatrick’s evacuation for Fibonacci tableaux [8] is no longer needed in making the two constructions coincide. Nevertheless we give a simpler proof of Killpatrick’s theorem relating Roby’s original algorithm to Fomin’s one. We will later need this evacuation in the proof of Theorem 6.2 which gives a combinatorial interpretation of Okada’s analogue of Kostka numbers.

Let us recall that Fomin’s construction with growth diagrams consists in using some local rules to fill a diagram, giving rise to a pair of saturated chains in $\mathcal{YFL}$. For any permutation $\sigma$, the growth diagram $d(\sigma)$ is build the following way. First draw the permutation matrix of $\sigma$; fill the left and lower border of $d(\sigma)$ with the empty snakeshape $\emptyset$. The rest of the construction is iterative; $d(\sigma)$ is filled from the lower left corner to the upper right corner, following the diagonal. At each step and for any configuration as pictured in Fig. 2, $z$ is obtained by applying the local rules to the vertices $t, x, y$ and the permutation matrix element $\alpha$. We refer the reader to [13] for more details on this construction.
Algorithm 1: local rules for the natural $r$-correspondence in $YFL$

1: if $x \neq y$ and $y \neq t$ then
2: $z$ if obtained from $t$ by appending a two-boxed column just in front
3: else
4: if $x = y = t$ and $\alpha = 1$ then
5: $z$ is obtained from $t$ by appending a single-boxed column just in front
6: else
7: $z$ is defined in such a way that the edge $b_i$ is degenerated whenever $a_i$ is degenerated, and vice-versa
8: end if
9: end if

3.1. Equivalence between Roby’s and Fomin’s constructions

Let us build Fomin’s growth diagram for the permutation $\sigma = 2715643$ (see Fig. 3).

We get the paths $\hat{Q} = (\emptyset, 1, 11, 21, 22, 2112, 2112, 2212)$ and $\hat{P} = (\emptyset, 1, 2, 12, 22, 212, 222, 2212)$ on the upper and right border respectively. Now let us convert them into Young–Fibonacci tableaux, using the mechanism discussed in Section 2.3.

$$\emptyset \rightarrow \begin{array}{c} 1 \\ \end{array} \rightarrow \begin{array}{c} 2 \\ 1 \\ \end{array} \rightarrow \begin{array}{c} 3 \\ 2 \\ 1 \\ \end{array} \rightarrow \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \\ \end{array} \rightarrow \begin{array}{c} 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ \end{array} \rightarrow \begin{array}{c} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ \end{array} \rightarrow \begin{array}{c} 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ \end{array} = \hat{Q}(\sigma) = Q(\sigma)$$

$$\emptyset \rightarrow \begin{array}{c} 1 \\ \end{array} \rightarrow \begin{array}{c} 2 \\ \end{array} \rightarrow \begin{array}{c} 3 \\ \end{array} \rightarrow \begin{array}{c} 4 \\ \end{array} \rightarrow \begin{array}{c} 5 \\ \end{array} \rightarrow \begin{array}{c} 6 \\ \end{array} \rightarrow \begin{array}{c} 7 \\ \end{array} = \hat{P}(\sigma) = P(\sigma)$$

So as we can see on this example, the two constructions naturally coincide.

Remark 3.1. Let us mention that because Roby used the canonical labeling to convert a chain into a tableau, there seemed to be no way to convert the chain $\hat{P}$ into its equivalent tableau $P(\sigma)$. Killpatrick’s algorithm was then an approach to relate $\hat{P}$ with $P(\sigma)$. Our own approach has then consisted in the introduction of a modification of the original algorithm, and a new labeling process.
Theorem 3.2. Let \((\hat{P}(\sigma), \hat{Q}(\sigma))\) be the pair of Young–Fibonacci tableaux obtained from \(\sigma\) using Fomin’s growth diagram and let \((P(\sigma), Q(\sigma))\) be the Young–Fibonacci insertion and recording tableaux using Roby’s insertion modified (Definition 2.5), then \(\hat{P}(\sigma) = P(\sigma)\) and \(\hat{Q}(\sigma) = Q(\sigma)\).

Proof. The equality \(\hat{P}(\sigma) = P(\sigma)\) follows from that any snakeshape \(\hat{P}_k\) appearing in \(\hat{P}\) is the shape of the tableau \(P(\sigma/_{[1..k]}\) where \(\sigma/_{[1..k]}\) is the restriction of \(\sigma\) to the interval \([1..k]\). Indeed, the path \(\hat{P}\) is obtained by applying to \(P(\sigma)\) the reverse process of the one described in Section 2.3. In so doing, the cell added to \(\hat{P}_k\) to get \(\hat{P}_{k+1}\) lies in the first column when either \(\sigma/_{[1..k+1]}\) ends with the letter \(k+1\) or \(\sigma/_{[1..k]}\) does not end with the letter \(k+1\) but \(\sigma/_{[1..k]}\) ends with the letter \(k\). A quite similar reasoning is used to prove the equality \(\hat{Q}(\sigma) = Q(\sigma)\).

3.2. Another viewpoint of Killpatrick’s evacuation for Young–Fibonacci tableaux

For a tableau \(t\), this operation is defined only for top entries of the columns of \(t\). Let \(a_0\) be such an entry, the tableau resulting from the evacuation of \(a_0\) is denoted \(ev(t, a_0)\) and is built as follows:

1. if \(a_0\) is a single-boxed column, then just delete this column and, if this is necessary, shift one component of the remaining tableau to connect it with the other one (e.g. of line 3 in Table 1);
2. otherwise, the box containing \(a_0\) is emptied and one compares the entry \(a_1\) that was just below \(a_0\) with the entry \(a_2\) on top of the column just to the right if any. If \(a_2 < a_1\) then this terminates the evacuation process (e.g. of line 4 in Table 1). Otherwise, move \(a_2\) and put it on top of \(a_1\), creating a new empty box in the tableau. If the new empty box is a single-boxed column, then this terminates the evacuation process (e.g. of line 7, step 4, in Table 1), otherwise, iteratively repeat the process with the entries just below and to the right of this new empty box.

Let \(t\) be a tableau of size \(n\) and shape \(u\), if one successively evacuates the entries \(n, (n-1), \ldots, 1\) from \(t\), labeling the boxes of \(u\) according to the positions of the empty cells at the end of the evacuation of entries, one gets a path tableau denoted \(ev(t)\). Recall that a path tableau is the canonical labeling of a saturated chain of YFL.
Remark 3.3. $ev(t)$ is the same tableau as the one described by Killpatrick [8], with Young–Fibonacci tableaux defined as in Definition 2.2.

Lemma 3.4. Let $w$ be a word with no letter repeated, let $a_0$ be one of its letters appearing as a top element in a column of $P(w)$, and let $w_0$ be the word obtained from $w$ by deleting the only occurrence of $a_0$, then $ev(P(w), a_0) = P(w_0)$.

Proof. Follows the description of the evacuation and the description of the insertion algorithm. □

We give a simpler proof of the following theorem by Killpatrick, relating $ev(P(\sigma))$ with $\widehat{P}$. Indeed, using the canonical labeling, Roby has converted the path $\widehat{P}$ into a path tableau $\widehat{P}(\sigma)$ and

Theorem 3.5. (See [8].) $ev(P(\sigma)) = \widehat{P}(\sigma)$.

Proof. Follows from Lemma 3.4 and the remark that any snakeshape $\widehat{P}_k$ appearing in $\widehat{P}$ is the shape of the tableau $P(\sigma/[1..k])$ where $\sigma/[1..k]$ is the restriction of $\sigma$ to the interval $[1..k]$. □

4. Fibonacci numbers and a statistic on Young–Fibonacci tableaux

In this section we point out a property of Young–Fibonacci numbers, defined as an analogue of Kostka numbers. Recall that the usual Kostka numbers $K_{\lambda, \mu}$ are defined for two partitions $\lambda$ and $\mu$ of the same integer $n$ and they appear when expressing Schur functions $s_{\lambda}$ in terms of the monomial symmetric functions $m_{\mu}$, and in the expression of the complete symmetric functions $h_{\mu}$ in terms of Schur functions $s_{\lambda}$,

$$s_{\lambda} = \sum_{\mu} K_{\lambda, \mu} m_{\mu}; \quad h_{\mu} = \sum_{\lambda} K_{\lambda, \mu} s_{\lambda}. \quad (4.1)$$

We are not motivated by the algebraic interpretation of the $K_{\lambda, \mu}$ but rather by their combinatorial interpretation in terms of tableaux. Indeed, $K_{\lambda, \mu}$ counts the number of distinct semi-standard Young tableaux of shape $\lambda$ and content $\mu$, that is to say with $\mu_i$ entries $i$ for $i = 1..\ell(\mu)$.

It is then natural to introduce an analogue definition for Young–Fibonacci tableaux.

Definition 4.1. A semi-standard Young–Fibonacci tableau is a numbering of a snakeshape with non-negative integers, not necessarily distinct, preserving the conditions stated in Definition 2.2.

Definition 4.2. Let $u$ and $v$ be two snakeshapes of size $n$, the Young–Fibonacci number associated to $u$ and $v$, denoted $N_{u,v}$ is the number of distinct semi-standard Young–Fibonacci tableaux of shape $u$ and valuation $v$, that is to say with $v_i$ entries $i$ for $i = 1..\ell(v)$.

For example, for $u = 221$ and $v = 1211$, there are 4 distinct semi-standard Young–Fibonacci tableaux of shape $u$ and valuation $v$. So $N_{221, 1211} = 4$. 

$$\begin{array}{ccc} 
 4 & 3 & 1 \\
 2 & 2 & 2 \\
\end{array} \quad \begin{array}{ccc} 
 4 & 3 & 2 \\
 2 & 1 & 2 \\
\end{array} \quad \begin{array}{ccc} 
 4 & 3 & 1 \\
 2 & 2 & 1 \\
\end{array} \quad \begin{array}{ccc} 
 4 & 2 & 1 \\
 3 & 1 & 2 \\
\end{array}$$
Table 2
Matrix of Young–Fibonacci numbers for \(n = 6\)

<table>
<thead>
<tr>
<th>(v)</th>
<th>222</th>
<th>2211</th>
<th>2121</th>
<th>2112</th>
<th>21111</th>
<th>1221</th>
<th>1212</th>
<th>12111</th>
<th>1122</th>
<th>11211</th>
<th>11121</th>
<th>11112</th>
<th>111111</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u = 222)</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>7</td>
<td>8</td>
<td>12</td>
<td>15</td>
</tr>
<tr>
<td>(2211)</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>(2121)</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>(2112)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>(21111)</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>(1221)</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>(1212)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>(12111)</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1122)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(11211)</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(11121)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(11112)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(111111)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The following matrices give Young–Fibonacci numbers \(N_{u,v}\) where \(u\) and \(v\) run over the snake-shapes of size 2, 3, 4 and 5.

\[
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 2 & 3 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
2 & 3 & 4 & 3 & 4 & 4 & 6 & 8 \\
1 & 1 & 1 & 2 & 3 & 3 & 4 & 4 \\
2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 \\
1 & 1 & 1 & 1 & 2 & 3 & 3 & 3 \\
1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 \\
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

**Proposition 4.3.** Young–Fibonacci numbers are generated by the recurrence formulas below:

\[
\begin{cases}
N_{\emptyset,\emptyset} = 1; & N_{2,2} = 0; \\
N_{1u,v1} = N_{u,v}; & N_{1u,v2} = N_{u,v1}; \\
N_{2u,v1} = \sum_{w \in v(-1)} N_{u,w}; & N_{2u,v2} = \sum_{w \in v(-1)} N_{u,w};
\end{cases}
\]

where \(v(-1)\) denotes the multiset of snake-shapes obtained from \(v\) either by deleting a single occurrence of 1, or by decreasing a single entry not equal to 1, for example \(2112(-1) = [1112, 212, 212, 2111]\).

**Proof.** Easily from the definition of Young–Fibonacci tableaux and Young–Fibonacci numbers. \(\square\)

**Theorem 4.4.** Let \(n \geq 2\) be a positive integer, then the number of couples \((u, v)\) of snake-shapes of size \(n\) such that \(N_{u,v} = 0\) is the \((n - 2)\)th Fibonacci number.

**Proof.** The proof is done by induction on \(n\). Indeed using Eq. (4.2) it is easy to see that \(N_{u,v} \neq 0\) whenever \(u \neq 1^{n-2}2\). So the problem is equivalent to counting the number of snake-shapes \(v\) such that \(N_{1^{n-2}2,v} = 0\). But \(N_{1^{n-2}2,v} = 0\) if and only if there exists a snake-shape \(w\) such that \(v = 2w\). Then the problem is finally equivalent to counting the snake-shapes of size \((n - 2)\), and hence the result. \(\square\)

5. A weak order on Young–Fibonacci tableaux

In what follows, we introduce a partial and graded order denoted \(\preceq\) on the set \(\text{YFT}_n\) of Young–Fibonacci tableaux of size \(n\). We will see (Theorem 5.8) that this partial ordering on \(\text{YFT}_n\) is such that the map from the weak order of the symmetric group \(\mathfrak{S}_n\) which sends each permutation \(\sigma\) onto its Young–Fibonacci insertion tableau \(P(\sigma)\) is order-preserving. Moreover, standard Young–Fibonacci classes of \(\mathfrak{S}_n\) are intervals of the weak order on \(\mathfrak{S}_n\).
An inversion of a permutation $\sigma$ is a couple $(j, i)$, $1 \leq i < j \leq n$ such that $\sigma^{-1}(i) > \sigma^{-1}(j)$, that is to say $j$ appears to the left of $i$ in $\sigma$. The set of inversions of a permutation $\sigma$ will be denoted $\text{inv}(\sigma)$, and the number of inversions denoted $\#\text{inv}(\sigma)$.

The weak order (also called right permutohedron order) on permutations of $\mathfrak{S}_n$ is the transitive closure of the relation $\sigma \leq_{\text{weak}} \tau$ if $\tau$ has more inversion than $\sigma$ and $\tau = \sigma \delta_i$ for some $i$, where $\delta_i$ is the adjacent transposition exchanging the numbers at places $i$ and $i+1$.

We will also make use of an analogous notion of non-inversion or rise of a permutation $\sigma$ which is a couple $(i, j)$, $1 \leq i < j \leq n$ such that $\sigma^{-1}(i) < \sigma^{-1}(j)$, that is to say $i$ appears to the left of $j$ in $\sigma$. The set of non-inversions of a permutation $\sigma$ will be denoted $\text{ord}(\sigma)$.

Before we introduce the partial order on Young–Fibonacci tableaux, let us first define the shifting operation on tableaux.

**Definition 5.1.** The shifting operation is defined on a Young–Fibonacci tableau $t$ as follows.

Let $a$ be the bottom entry of any column of $t$, and $c$ the bottom entry of the column just to the left of the column containing $a$. Let $b$ be the entry on top of $a$, if any.

1. If the columns containing $a$ and $c$ are single-boxed and $a < c$, then $a$ may bump up $c$.

\[
\begin{array}{cccc}
5 & 4 & 3 & 1 \\
2 & 4 & 3 & 1
\end{array}
\xrightarrow{\text{shift 1}}
\begin{array}{cccc}
5 & 4 & 3 & 1 \\
2 & 4 & 3 & 1
\end{array}
\]

2. If the column containing $a$ is single-boxed and the one containing $c$ is double-boxed with $a < c$, then one might exchange $a$ and $c$.

\[
\begin{array}{cccc}
5 & 4 & 3 & 1 \\
2 & 3 & 1
\end{array}
\xrightarrow{\text{shift 1}}
\begin{array}{cccc}
5 & 4 & 3 & 1 \\
2 & 1 & 3
\end{array}
\]

3. If the column containing $a$ is double-boxed and the one containing $c$ is single-boxed with $a < c$, then $a$ may bump up $c$ while $b$ just falls down.

\[
\begin{array}{cccc}
4 & 2 & 3 & 1 \\
5 & 2 & 3 & 1
\end{array}
\xrightarrow{\text{shift 2}}
\begin{array}{cccc}
5 & 2 & 4 & 3 \\
5 & 3 & 1
\end{array}
\]

\[
\begin{array}{cccc}
4 & 2 & 3 & 1 \\
5 & 2 & 3 & 1
\end{array}
\xrightarrow{\text{shift 3}}
\begin{array}{cccc}
5 & 2 & 3 & 1 \\
5 & 3 & 4 & 1
\end{array}
\]

---

**Fig. 4.** Hasse diagram of the permutohedron order on $\mathfrak{S}_4$. 

---

(4) If the columns containing \( a \) and \( c \) are double-boxed and \( a < c \), then \( a \) may replace \( c \) which on its turn is shifted to the right in such a way that

(i) if \( c < b \) then \( c \) will just replace \( a \)

\[
\begin{array}{c|c|c}
5 & 4 & 1 \\
2 & 1 & 3 \\
\end{array} \quad \text{shift 1} \quad \begin{array}{c|c|c}
5 & 4 & 1 \\
2 & 1 & 3 \\
\end{array}
\]

(ii) otherwise \( c \) creates a new single-boxed column just to the right, while \( b \) just falls down.

\[
\begin{array}{c|c|c}
5 & 3 & 1 \\
4 & 2 & 1 \\
\end{array} \quad \text{shift 2} \quad \begin{array}{c|c|c}
5 & 4 & 3 \\
2 & 4 & 1 \\
\end{array}
\]

Remark 5.2. It easily follows from the definition that shifting an entry in a tableau always produces a legitimate tableau of the same size.

In an analogous way, given a tableau \( t \), one defines the reverse operation of finding all the tableaux \( t' \) such that shifting an entry in \( t' \) gives back \( t \). For example,

\[
\begin{array}{c|c|c}
5 & 2 & 1 \\
3 & 4 & 1 \\
\end{array}
\]

is obtained from \( \begin{array}{c|c|c}
4 & 2 & 1 \\
5 & 3 & 1 \\
\end{array} \), \( \begin{array}{c|c|c}
5 & 2 & 1 \\
4 & 3 & 1 \\
\end{array} \) and \( \begin{array}{c|c|c}
5 & 3 & 1 \\
4 & 2 & 1 \\
\end{array} \) by shifting 3 or 1.

Finally it is clear that this operation is antisymmetric, that is to say if \( t' \) is obtained from \( t \) by shifting a given entry, then \( t \) cannot be obtained from \( t' \) by shifting another entry. This observation is enforced by the following lemma which also defines the graduation of the poset \((\mathbb{YFT}_n, \leq)\).

Lemma 5.3. Let \( t_2 \) be a tableau obtained by shifting an entry in a tableau \( t_1 \), and let \( \sigma_1 \) (resp. \( \sigma_2 \)) be the minimal permutation canonically associated to \( t_1 \) (resp. \( t_2 \)) as stated in Lemma 2.11, then the inversion numbers of \( \sigma_1 \) and \( \sigma_2 \) are related by the relation 

\[
# \text{inv}(\sigma_2) = # \text{inv}(\sigma_1) + 1.
\]

Proof. The proof takes into account all the configurations outlined in Definition 5.1.

(1) \( t_1 = T_2 \begin{array}{c|c}
\hline
\alpha & c \\
\hline
\end{array} T_1 \) and \( t_2 = T_2 \begin{array}{c|c}
\hline
\alpha & a \\
\hline
\end{array} T_1 \) with \( a < c \)

where \( T_1 \) and \( T_2 \) are partial \( \mathbb{YFT} \) having minimal canonical words \( w_1 \) and \( w_2 \). The minimal permutations are \( \sigma_1 = w_1 \cdot ac \cdot w_2 \) and \( \sigma_2 = w_1 \cdot ca \cdot w_2 \) respectively, and clearly \( \sigma_2 \) has one more inversion than \( \sigma_1 \).

(2) \( t_1 = T_2 \begin{array}{c|c}
\hline
\beta & c \\
\hline
\end{array} T_1 \) and \( t_2 = T_2 \begin{array}{c|c}
\hline
\alpha & c \\
\hline
\end{array} T_1 \) with \( a < c \)

the minimal permutations associated to \( t_1 \) and \( t_2 \) are \( \sigma_1 = w_1 \cdot adc \cdot w_2 \) and \( \sigma_2 = w_1 \cdot cda \cdot w_2 \).

(3) \( t_1 = T_2 \begin{array}{c|c}
\hline
\beta & c \\
\hline
\end{array} T_1 \) and \( t_2 = T_2 \begin{array}{c|c}
\hline
\alpha & b \\
\hline
\end{array} T_1 \), with \( a < c \)

the minimal permutations are \( \sigma_1 = w_1 \cdot bac \cdot w_2 \) and \( \sigma_2 = w_1 \cdot bca \cdot w_2 \), and clearly \( \sigma_2 \) has one more inversion than \( \sigma_1 \).

(4) \( t_1 = T_2 \begin{array}{c|c}
\hline
\beta & c \\
\hline
\end{array} T_1 \) and \( t_2 = T_2 \begin{array}{c|c}
\hline
\alpha & c \\
\hline
\end{array} T_1 \) with \( a < c < b \)

the minimal permutations are \( \sigma_1 = w_1 \cdot bdc \cdot w_2 \) and \( \sigma_2 = w_1 \cdot bda \cdot w_2 \).

\( t_1 = T_2 \begin{array}{c|c}
\hline
\beta & c \\
\hline
\end{array} T_1 \) and \( t_2 = T_2 \begin{array}{c|c}
\hline
\beta & c \\
\hline
\end{array} T_1 \) with \( a < c \) and \( c > b \)

the minimal permutations are \( \sigma_1 = w_1 \cdot bdc \cdot w_2 \) and \( \sigma_2 = w_1 \cdot bda \cdot w_2 \).
In (2) and (4), the inversion \((dc)\) appears in \(\sigma_1\) but not in \(\sigma_2\), whereas the inversions \((da)\) and \((ca)\) appear in \(\sigma_2\) but not in \(\sigma_1\); so \(\sigma_2\) has one more inversion. □

We are now in a position to provide \(\mathrm{YFT}_n\) with a structure of poset.

**Definition 5.4** (Weak order on \(\mathrm{YFT}_n\)). Let \(t\) and \(t'\) be two tableaux of size \(n\), then \(t\) is said being smaller than \(t'\) and we write \(t \preceq t'\) if one can find a sequence \(t_0 = t, t_1, \ldots, t_k = t'\) of tableaux of size \(n\) such that \(t_{i+1}\) be obtained from \(t_i\) by shifting an entry, for \(i\) from 0 to \(k - 1\).

**Proposition 5.5.** \((\mathrm{YFT}_n, \preceq)\) is a graded poset, the rank of a Young–Fibonacci tableau being the number of inversions of its minimal canonical permutation.

**Proof.** Follows from Lemma 5.3. □

**Remark 5.6.** This property of graduation of the poset of Young–Fibonacci tableaux of size \(n\) does not apply to the similar poset \(\mathrm{YT}_n\) of standard Young tableaux of size \(n\). The interested reader may refer to [9] where Taskin studied many nice properties of four partial orders on \(\mathrm{YT}_n\).
Remark 5.7. \((\mathcal{YFT}_n, \preceq)\) is in general not a lattice. For example, if \(n = 5\), then
\[
\begin{array}{c}
3 \\
5 & 4 & 2 & 1
\end{array}
\quad \text{and} \quad
\begin{array}{c}
5 \\
3 & 4 & 2 & 1
\end{array}
\]
do not have a least upper bound.

Proposition 5.8. Let \(t_1\) and \(t_2\) be two tableaux of size \(n\), then \(t_2\) covers \(t_1\) in \((\mathcal{YFT}_n, \preceq)\) if and only if there exist two permutations \(\sigma_1\) and \(\sigma_2\) such that \(P(\sigma_1) = t_1\), \(P(\sigma_2) = t_2\) and \(\sigma_1 \preceq_{\text{weak}} \sigma_2\).

Proof. Similar to the proof of Lemma 5.3. \(\Box\)

We will now study the structure of a Young–Fibonacci class, that is the set of permutations of prescribed insertion tableau.

Definition 5.9. Let \(t\) be a standard Young–Fibonacci tableau of size \(n\), its canonical poset \(\mathbb{P}_t\) is the poset defined on the set \(\{1, 2, \ldots, n\}\) with the cover relations as described below:

1. the right-to-left reading word of the bottom row of \(t\) forms a chain in the poset;
2. each entry on top in a two-boxed column of \(t\) is covered by the corresponding bottom entry.

Remark 5.10. Any permutation \(\sigma\) is a totally ordered set with cover relations defined by \(\sigma(i) \preceq_{\sigma} \sigma(j)\) whenever \(i < j\), that is to say \(x \preceq_\sigma y\) if \(x\) appears to the left of \(y\) in \(\sigma\).

Let \(\mathbb{P}\) be a poset and \(\sigma\) a permutation, \(\sigma\) is said to be a linear extension of \(\mathbb{P}\) if its relations preserve the relations in \(\mathbb{P}\), that is to say if \(x \preceq_\mathbb{P} y\) then \(x \preceq_\sigma y\). The set of linear extensions of a poset \(\mathbb{P}\) will be denoted \(\text{Ext}(\mathbb{P})\).

Theorem 5.11. Let \(t\) be a standard Young–Fibonacci tableau, then \(\mathcal{YFC}(t) = \text{Ext}(\mathbb{P}_t)\).

Proof. That any permutation \(\sigma\) having \(P(\sigma) = t\) is a linear extension of \(\mathbb{P}_t\) is clear from Definitions 2.5 and 5.9. Conversely, if \(\sigma\) is a linear extension of \(\mathbb{P}_t\), then \(P(\sigma) = t\) is naturally built reading \(\sigma\) from right to left following Definition 2.5. At each new step the first letter one reads is the maximal one (for \(\preceq_{\mathbb{P}_t}\)) not yet read in the chain described in rule (1) of Definition 5.9. \(\square\)

Theorem 5.12. Let \(t\) be a standard Young–Fibonacci tableau of size \(n\), then \(\text{Ext}(\mathbb{P}_t)\) is the interval \([w_{\text{min}}^t, w_{\text{max}}^t]\) of \((\mathfrak{S}_n, \preceq_{\text{weak}})\).

Lemma 5.13. Let \(\sigma\) and \(\tau\) be two permutations of \(\mathfrak{S}_n\), then the three properties below are equivalent.

1. \(\sigma \preceq_{\text{weak}} \tau\);
2. \(\text{ord}(\tau) \subseteq \text{ord}(\sigma)\);
3. \(\text{inv}(\sigma) \subseteq \text{inv}(\tau)\).

Proof of Theorem 5.12. It easily follows from the definition that \(\mathbb{P}_t\) can be partitioned into an antichain \(A = (y_1, y_2, \ldots, y_\ell)\) and a chain \(C = (x_1 <_{\mathbb{P}_t} x_2 <_{\mathbb{P}_t} \cdots <_{\mathbb{P}_t} x_k)\) such that for \(i = 1, \ldots, \ell\) there exists \(j(i) \leq k\) such that \(y_i <_{\mathbb{P}_t} x_{j(i)}\), and additionally for \(i_1 < i_2\) one has \(y_{i_1} < y_{i_2}\) and \(x_{j(i_1)} <_{\mathbb{P}_t} x_{j(i_2)}\). For illustrations, we use the following example.
\( A = (3, 6, 7) \)  
\( C = (2 <_{P_t} 5 <_{P_t} 1 <_{P_t} 4) \)

\[
\begin{array}{ccc}
7 & 6 & 3 \\
4 & 1 & 5 & 2
\end{array}
\]

a tableau \( t \) of shape \( u = 2212 \)

For \( \sigma \in \text{Ext}(P_t) \), \( \text{inv}(\sigma) \) includes at least the set

\[
I = \left\{ (y_i, x_{j(i)}), i = 1, \ldots, \ell \\
(\sigma_i, x_{j(i)}) / x_{j(i)} \geq x_{r_i} \text{ and } x_{j(i)} <_{\leq} x_{r_i} \\
(x_{i}, x_{j}) / x_{i} > x_{j} \text{ and } x_{i} < x_{j}
\right\}
\]

which is nothing but \( \text{inv}(w_{\text{min}}^t) \); so by Lemma 5.13(3), \( w_{\text{min}}^t \preceq \text{weak} \sigma \). Moreover, \( \text{ord}(\sigma) \) includes at least the set \( O = \{(y_i, x_{r_i}) / x_{r_i} <_{P_t} x_{r_i} \} \cup \{(x_{i}, x_{j}) / x_{i} < x_{j} \text{ and } x_{i} < x_{j} <_{P_t} x_{j}\} \), which is nothing but \( \text{ord}(w_{\text{max}}^t) \).

Conversely, for \( \sigma \in [w_{\text{min}}^t, w_{\text{max}}^t] \), applying Lemma 5.13 to \( w_{\text{min}}^t \), \( \sigma \) and \( w_{\text{max}}^t \) it appears that \( \sigma \) has the inversions \( y_i \preceq x_{j(i)} \) for \( i = 1, \ldots, \ell \), and the relations \( x_1 \prec x_2 \prec \cdots \prec x_k \). So \( P(\sigma) = t \) and hence \( \sigma \in \text{Ext}(P_t) \). \( \square \)

**Corollary 5.14.** Let \( t \) be a standard Young–Fibonacci tableau of size \( n \), then \( \mathbb{YFC}(t) \) is an interval of the weak order \( (\mathfrak{S}_n, \preceq_{\text{weak}}) \), more over \( \mathbb{YFC}(t) = [w_{\text{min}}^t, w_{\text{max}}^t] \).

**Definition 5.15.** Let \( u \) be a snakeshape of size \( n \), the row canonical tableau of shape \( u \) is denoted \( \text{rowTab}(u) \) and defined as follows:

1. top cells of \( \text{rowTab}(u) \) are labeled with entries \( n, n - 1, \ldots, \), from left to right;
2. bottom cells in two-boxed columns are labeled with entries \( 1, 2, \ldots, \), from left to right.

The column canonical tableau \( \text{columnTab}(u) \) is built by labeling the cells of \( u \) from right to left and bottom to top, with the entries \( 1, 2, \ldots, n \).

**Remark 5.16.** Note that by top cell we also mean the unique cell of a single-boxed column.

For example, for \( u = 2212 \), we have

\[
\text{rowTab}(2212) = \begin{array}{ccc}
7 & 6 & 4 \\
1 & 2 & 5 & 3
\end{array}
\quad \text{and} \quad \text{columnTab}(2212) = \begin{array}{ccc}
7 & 5 & 2 \\
6 & 4 & 3 & 1
\end{array}
\]

**Lemma 5.17.** Let \( u \) be a snakeshape of size \( n \), then \( \text{columnTab}(u) \) (resp. \( \text{rowTab}(u) \)) is the unique tableau of shape \( u \) having minimal rank \( \rho_{\text{min}}^u \) (resp. maximal rank \( \rho_{\text{max}}^u \)) in the poset \( \mathbb{YFT}_n, \preceq \). For any snakeshape \( u \), \( \rho_{\text{min}}^u \) is the number of double-boxed columns of \( u \) and \( \rho_{\text{max}}^u \) is obtained as follows. Label each bottom cell with the number of double-boxed columns on its left and do the same but add 1 for each top cell of double-boxed columns of \( u \). \( \rho_{\text{max}}^u \) is the sum of the labels obtained.

**Proof.** Easily from the definitions. \( \square \)
Fig. 6. Induced permutohedron order on Young–Fibonacci classes of $S_5$. 
For example,
\[
\rho \left( \begin{array}{ccc}
7 & 6 & 4 \\
1 & 2 & 5 & 3
\end{array} \right) = 11 \quad \text{and} \quad \rho \left( \begin{array}{ccc}
7 & 5 & 2 \\
6 & 4 & 3 & 1
\end{array} \right) = 3
\]

We will now relate \((\mathcal{YFT}, \preceq)\) to a transition matrix in Okada’s algebra associated to \(\mathcal{YFL}\).

6. A connection with Okada’s algebra associated to the Young–Fibonacci lattice

6.1. Okada’s algebra associated to the Young–Fibonacci lattice

We recall some basics facts about Okada’s algebras. The interested reader may refer to [14] for more details. Okada’s analogue of the symmetric group \(S_n\) is the associative algebra \(\mathcal{F}_n\) over a field of characteristic 0, defined by the generators \(e_1, e_2, \ldots, e_{n-1}\) and the following relations:

\[
\begin{align*}
\tag{6.1}
e_i^2 &= x_i e_i, \\
e_i e_{i-1} e_i &= y_i e_i, \quad i \geq 2, \\
e_i e_j &= e_j e_i, \quad |i - j| \geq 2,
\end{align*}
\]

where \(x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_{n-2}\) are generic parameters.

\(\mathcal{F}_n\) is semisimple of dimension \(n!\) and its irreducible representations are indexed by the snake-shapes of size \(n\). Let \(V^v_{\mathcal{F}_n}\) be the irreducible \(\mathcal{F}_n\)-module corresponding to \(v\), then the branching rule for the restriction to the subalgebra \(\mathcal{F}_{n-1} = \langle e_1, e_2, \ldots, e_{n-2} \rangle\) is described in the same way as in the case of the Young’s lattice, where \(u\) runs over the snake-shapes covered by \(v\) in \(\mathcal{YFL}\).

\[
V^v_{\mathcal{F}_n} \downarrow_{\mathcal{F}_{n-1}} = \bigoplus_u V^u_{\mathcal{F}_{n-1}} \tag{6.2}
\]

Okada’s algebra \(\mathcal{YFSym}\) is an algebra which is isomorphic to the algebra of polynomials in two noncommutative variables \(X\) and \(Y\), with \(d^X(X) = 1\) and \(d^X(Y) = 2\). The basis of \(\mathcal{YFSym}_n\) is also indexed by the snake-shapes of size \(n\). Okada has shown that \(\mathcal{YFSym}\) is a clone of the algebra of symmetric functions (\(\mathcal{Sym}\)) and the relations between \(\mathcal{YFSym}_n\) and \(\mathcal{F}_n\) are similar to the relations between \(\mathcal{Sym}_n\) and \(\mathcal{S}_n\).

In Okada’s algebra, the analogue of complete symmetric functions are the \(\{h_u\}\), obtained by identifying \(h_1\) (resp. \(h_2\)) with the monomial \(X\) (resp. \(Y\)), and by setting the multiplication rule:

\[
h_u h_v = h_{vu}.
\]

For example, if \(u = 1^k_0 2^1 2^1 \ldots 2^1 k_{e-1} 2^1 k_i\), then \(h_u \simeq X^{k_1} Y X^{k_{e-1}} Y \ldots X^{k_1} Y X^{k_0}\).

In the course of his study of the irreducible representations of \(\mathcal{F}_n\), Okada defined a new basis \(\{s_u\}\) of \(\mathcal{YFSym}\), and he showed that the \(\{s_u\}\) are the irreducible characters of \(\mathcal{F}_n\). These functions are called Fibonacci Schur functions. They are related to Okada’s analogues of complete symmetric functions by the following relation:

\[
h_v = \sum_u K_{u,v} s_u. \tag{6.3}
\]

The numbers \(K_{u,v}\) are called Kostka–Fibonacci numbers; they were characterized by Okada with the following recurrence relation, where \(\triangleright\) denotes the cover relation in \(\mathcal{YFL}\):

\[
\begin{align*}
K_{1u,1v} &= K_{u,v} \quad (r_1), \\
K_{2u,2v} &= K_{u,v} \quad (r_2), \\
K_{1u,2v} &= 0 \quad (r_3), \\
K_{2u,1v} &= \sum_{w \triangleright u} K_{w,v} \quad (r_4).
\end{align*} \tag{6.4}
\]
Bellow are the Kostka–Fibonacci matrices of order 2, 3, 4 and 5.

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & 1 & 2 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & 1 & 1 & 2 & 3 \\
0 & 1 & 1 & 1 & 3 \\
0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \quad \begin{pmatrix}
1 & 1 & 2 & 1 & 2 & 3 & 4 & 8 \\
0 & 1 & 1 & 1 & 1 & 3 & 4 \\
0 & 0 & 1 & 0 & 1 & 1 & 4 \\
0 & 0 & 0 & 1 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Rows and columns are indexed by snakeshapes listed in the dominance order. The dominance order is defined on snakeshapes the same way as it is defined on partitions of integers. So if \( u \) and \( v \) are two snakeshapes of size \( n \) such that \( \ell(u) \leq \ell(v) \), then \( u \) is smaller than \( v \) in the dominance order if \( u_1 + u_2 + \cdots + u_k \leq v_1 + v_2 + \cdots + v_k \), for each \( k = 1, \ldots, \ell(u) \).

As stated below, the hook-length formula for binary trees illustrated in Example 2.4 is a formula for computing \( K_{u,v} = F_u \), the dimension of the corresponding irreducible representation of Okada’s clone of the symmetric group.

**Proposition 6.1.** Let \( u \) be a snakeshape of size \( n \), then \( \dim(V^u_{\emptyset^n}) = F_u \).

**Proof.** \( \dim(V^u_{\emptyset^n}) \) is the number of saturated chains from \( \emptyset \) to \( u \) in \( \mathbb{YFT}_n \), hence the result. \( \square \)

The main theorem of this paper gives a combinatorial interpretation of the Kostka–Fibonacci numbers \( K_{u,v} \), using the Young–Fibonacci tableauhedron \( (\mathbb{YFT}_n, \prec) \).

**Theorem 6.2.** Let \( u \) and \( v \) be two snakeshapes of size \( n \), and let \( \hat{1} \) be the maximal tableau in \( (\mathbb{YFT}_n, \prec) \), then \( K_{u,v} \) is the number of tableaux of shape \( u \) in the interval \([\text{rowTab}(-v), \hat{1}]\).

**Example 6.3.** \( K_{u,1121} \) is the number of standard Young–Fibonacci tableaux of shape \( u \) in the interval \([\text{rowTab}(1121), \hat{1}]\) (see Fig. 7).

**Remark 6.4.** The tableau \( \hat{1} \) is the row canonical tableau of shape \( 2n/2 \) if \( n \) is odd, otherwise \( \epsilon \) is the empty word.

**Proof of Theorem 6.2.** A proof consists in showing that for any couple \((a, b)\) of snakeshapes appearing in the left-hand side of Eq. (6.4), there is a one-to-one correspondence between tableaux satisfying the conditions of the theorem for \((a, b)\) and those satisfying the conditions of the theorem for the couples of snakeshapes in the corresponding right-hand side of the relation.

For \((r_1)\), given a tableau \( t \) of shape \( u \) such that \( \text{rowTab}(v) \prec t \), \( t \) is mapped onto the tableau \( t' \) of shape \( 1u \) obtained from \( t \) by attaching a cell labeled \( n+1 \) to its left, and \( \text{rowTab}(1v) \prec t' \).

For \((r_2)\), one attaches a two-boxed column to the left of \( t \), with 1 as bottom entry and \( n+2 \) as top entry, in addition one standardizes \( t \) by increasing all its entries. Then \( t' \) is of shape \( 2u \) and \( \text{rowTab}(2v) \prec t' \).

For \((r_3)\) it easily follows from the definition of the operation of shifting an entry in a tableau that there is no tableaux \( t_1 \) and \( t_2 \) of shape \( 1u \) and \( 2v \) respectively, such that \( t_2 \prec t_1 \).

For \((r_4)\), let \( t \) be a tableau of shape \( 2u \) such that \( \text{rowTab}(1v) \prec t \), then \( t \) is mapped onto the tableau \( t' = ev(t,n) \), that is the tableau obtained from \( t \) by evacuation of its maximal letter (the evacuation process originally due to Killpatrick [8] is described in Section 3.2). Indeed, let \( w \) be the shape of \( t' \), then \( w \) covers \( u \) in \( \mathbb{YFL} \) and \( \text{rowTab}(v) \approx t' \). \( \square \)

**7. Young tableauhedron, Littlewood Richardson rule and Kostka numbers**

The Young–Fibonacci tableauhedron \( (\mathbb{YFT}_n, \prec) \) we defined in Section 5 is an analogue of one among four partial orders on the set \( \mathbb{YT}_n \) of standard Young tableaux of size \( n \).
We assume the reader is familiar with Young tableaux and the Robinson–Schensted correspondence. We are using French notation, so Young tableaux have increasing entries along rows from left to right, and strictly increasing entries along columns from bottom to top.

Let $t$ be a standard Young tableau, the Knuth class of $t$ also called plactic class is the set of permutations $\sigma$ such that $P(\sigma) = t$, where $P(\sigma)$ denotes the Schensted insertion tableau of $\sigma$.

The Young tableau order $(\mathcal{Y}T_n, \leq_{\text{weak}})$ is the order induced (by transitive closure) on Knuth classes by the weak order of the symmetric group $S_n$. Initially introduced under the name induced Duflo order by Melnikov [1], it has also been studied by Taskin [9].

**Definition 7.1.** Let $R$ and $S$ be two standard Young tableaux, then $R$ is said smaller than $S$ and we will write $R \leq_{\text{weak}} S$, if and only if one can find a sequence $t_0 = R, t_1, \ldots, t_k = S$ of Young tableaux such that for $i = 0, \ldots, (k - 1)$ there exists a permutation $\tau_{1i}$ in the Knuth class of $t_i$ and a permutation $\tau_{2i}$ in the Knuth class of $t_{i+1}$ such that $\tau_{1i} \leq_{\text{weak}} \tau_{2i}$.

Let $\lambda$ and $\mu$ be two partitions such that $\ell(\lambda) \leq \ell(\mu)$, then $\lambda$ is smaller than $\mu$ in the dominance order, and we write $\lambda \leq_{\text{dom}} \mu$, if $\lambda_1 + \lambda_2 + \cdots + \lambda_k \leq \mu_1 + \mu_2 + \cdots + \mu_k$, for each $k = 1, \ldots, \ell(\lambda)$.

Let $t$ be a standard Young tableau of size $n$, and $1 \leq i \leq j \leq n$, denote $\lambda(t[i:j])$ the shape of the tableau obtained from $t$ by restricting $t$ to the segment $[i, j]$, then lowering all entries by $i - 1$, and finally sliding the skew tableau obtained into normal shape by jeu-de-taquin. The dominance order $\leq_{\text{dom}}$ on standard Young tableaux is defined as follows.
Definition 7.2. (See [9].) Let $t$ and $t'$ be two standard Young tableaux of size $n$, then $t \preceq_{\text{dom}} t'$ if and only if for each $1 \leq i \leq j \leq n$, $\lambda(t'_{[i,j]}) \preceq_{\text{dom}} \lambda(t_{[i,j]})$.

The reader interested may refer to [9] for the definition of the two other orders, as well as for the properties of those posets. The four posets happen to coincide up to rank $n = 5$.

The dominance order on standard Young tableaux is naturally extended to semi-standard tableaux. On the left-hand side of Fig. 9 is the dominance order induced on semi-standard tableaux of content $\mu = 221$. On the right-hand side is the order on the standardized tableaux.

Definition 7.3. Let $\mu$ be a partition of the integer $n$, the row canonical (or hyperstandard) tableau of shape $\mu$ is denoted $\text{rowTab}(\mu)$ and obtained by filling the Ferrers diagram $F^\mu$ from left to right and bottom to top, with the entries $1, 2, \ldots, n$.

Below is a Young tableaux analogue of Theorem 6.2.
The Kostka numbers $K_{\lambda, 221}$ counts the number of standard Young tableaux of shape $\lambda$ in the interval $[\hat{0}, \text{rowTab}(221)]$.

Theorem 7.4. Let $\lambda, \mu$ be two partitions of the integer $n$, and let $\hat{0}$ be the standard Young tableau of shape $(n)$. The Kostka number $K_{\lambda, \mu}$ is the number of standard Young tableaux of shape $\lambda$ in the interval $[\hat{0}, \text{rowTab}(\mu)]$, for any one of the posets studied by Melnikov [1] and Taskin [9].

Proof. Let $\text{Tab}(\mu)$ denotes the set of all semi-standard Young tableaux of content $\mu$, that is each tableau in $\text{Tab}(\mu)$ has $\mu_i$ entries $i$ for $i = 1, \ldots, \ell(\mu)$. The natural standardization process is a canonical bijection mapping $(\text{Tab}(\mu), \preceq_{\text{dom}})$ onto $([\hat{0}, \text{rowTab}(\mu)], \preceq_{\text{dom}})$ and this map is order preserving. So Theorem 7.4 holds for the dominance order on tableaux. From [9, Theorem 1.1] and the remark that $[\hat{0}, \text{rowTab}(\mu)] = \text{rowTab}(\mu_1) \times \text{rowTab}(\mu_2) \times \cdots \times \text{rowTab}(\mu_{\ell(\mu)})$, it follows that the set of tableaux in $[\hat{0}, \text{rowTab}(\mu)]$ does not depend on the choice of the partial order. □

Acknowledgments

The author is grateful to F. Hivert for his helpful comments and suggestions. Experimentations and computations for this work were made with MuPAD-Combinat [7]. Figures were generated using MuPAD-Combinat [7], together with dot [5].
Appendix A. Young–Fibonacci tableauhedron of order $n = 6$

Fig. 10. Young–Fibonacci tableauhedron of order $n = 6$. 
Appendix B. Kostka–Fibonacci and Young–Fibonacci matrices of order $n = 6, 7$

Fig. 11. Kostka–Fibonacci matrix of order $n = 6$.

Fig. 12. Kostka–Fibonacci matrix of order $n = 7$.

Fig. 13. Young–Fibonacci matrix of order $n = 7$. 
References