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Induced fractional zero-point angular momentum for charged particles of the Bohm–Aharonov system by means of a “spectator” magnetic field

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ABSTRACT

An induced fractional zero-point angular momentum of charged particles by the Bohm–Aharonov (BA) vector potential is realized via a modified combined trap. It explores a “spectator” mechanism in this type of quantum effects: In the limit of the kinetic energy approaching one of its eigenvalues the BA vector potential alone cannot induce a fractional zero-point angular momentum at quantum mechanical level in the BA magnetic field-free region; But when there is a “spectator” magnetic field the BA vector potential induces a fractional zero-point angular momentum. The “spectator” does not contribute to such a fractional angular momentum, but plays essential role in guaranteeing non-trivial dynamics at quantum mechanical level in the required limit. This “spectator” mechanism is significant in investigating the BA effects and related topics in both aspects of theory and experiment.

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As is well known, quantum states of charged particles can be influenced by electromagnetic effects even if those particles are in a region of vanishing field strength [1,2]. As predicted by Bohm and Aharonov (BA) [2], experiments [3] showed that in a multiply connected region where field strength is zero everywhere the interference spectrum suffered a shift according to the amount of the loop integral of magnetic vector potential around an unshrinkable loop. Wu and Yang [4] pointed out that the BA effects is due to the non-trivial topology of the space where the magnetic field strength is vanishing. The BA effect is purely quantum mechanical one which explores far-reaching consequences of vector potential in quantum theory. This effect has been received much attention for years [5–7]. Recently investigations in this topic concentrated on revealing new types of quantum phases: The Aharonov–Casher effect [8], the He–McKellar–Wilkins phase [9] and the Anandan phase [10].

In another aspect a fractional angular momentum originated from the Poynting vector produced by crossing the Coulomb field of a charged particle with an external magnetic field has been predicted by Peshkin, Talmi and Tassie for years [6,11]. There are lots of works concerning fractional angular momentum in BA dynamics and their “fractional” statistics (see the reviews [12–17] and references therein). Spatial noncommutativity also leads to fractional angular momentum [18,19].

Recently Kastrup [20] considered the question of how to quantize a classical system of the canonically conjugate pair angle and orbital angular momentum. This has been a controversial issue since the founding days of quantum mechanics [21]. The problem is that the angle is a multivalued or discontinuous variable on the corresponding phase space. A crucial point is that the irreducible unitary representations of the euclidean group $E(2)$ or of its covering groups allow for orbital angular momentum $l = \hbar(n + \delta)$ where $n = 0, \pm 1, \pm 2, \dots$, and $0 \leq \delta < 1$. The case $\delta \neq 0$ corresponds to fractional zero-point angular momentum. Kastrup investigated the physical possibility of fractional orbital angular momentum in connection with the quantum optics of Laguerre–Gaussian laser modes in external magnetic fields, and pointed out that if implementable this would lead to a wealth of new theoretical, experimental and even technological possibilities.

In this Letter the induced fractional zero-point angular momentum of charged particles by the BA vector potential is realized via a modified combined trap. It explores a “spectator” mechanism in this type of quantum effects: In the limit of the kinetic energy approaching one of its eigenvalues the BA vector potential alone cannot induce a fractional zero-point angular momentum of charged particles at quantum mechanical level in a region of vanishing BA field strength; But when there is a “spectator” magnetic field the BA vector potential induces a fractional zero-point angular momentum in the same region. The “spectator” does not contribute to such a fractional angular momentum, but plays essential role in guaranteeing non-trivial dynamics at quantum mechanical level in the required limit. This type of quantum effects is so remarkable

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that in quantum mechanics the vector potential itself has physical significant meaning and becomes effectively measurable not only in shifts of interference spectra originated from quantum phases but also in physical observables.

1. Dynamics in a modified combined trap

We consider ions constrained in a modified combined trap including the BA type magnetic field. The Paul, Penning, and combined traps share the same electrode structure [22]. A combined trap operates in all of the fields of the Paul and Penning traps being applied simultaneously. The trapping mechanism in a Paul trap involves an oscillating axially symmetric electric potential $\tilde{U}(\rho, \phi, z, t) = U(\rho, \phi, z) \cos \tilde{\Omega}t$ with $U(\rho, \phi, z) = V(z^2 - \rho^2/2)/2d^2$ where ρ , ϕ and z are cylindrical coordinates, V and d are, respectively, characteristic voltage and length, and $\tilde{\Omega}$ is a large radio-frequency. The dominant effect of the oscillating potential is to add an oscillating phase factor to the wave function. Rapidly varying terms of time in Schrödinger equation can be replaced by their average values. Thus for $\tilde{\Omega} \gg \Omega \equiv (\sqrt{2}q|V|/\mu d^2)^{1/2}$ we obtain a time-independent effective electric potential [23] $V_{\text{eff}} = q^2 \nabla U \cdot \nabla U / 4\mu \tilde{\Omega}^2 = \mu \omega_p^2 (\rho^2 + 4z^2)/2$ where μ and $q(>0)$ are, respectively, the mass and charge of the trapped ion, and $\omega_p = \Omega^2/4\tilde{\Omega}$. A modified combined trap combines the above electrostatic potential and two magnetic fields¹: a homogeneous magnetic field \mathbf{B}_c aligned along the z -axis in a normal combined trap and a BA type magnetic field \mathbf{B}_0 produced by, for example, an infinitely long solenoid with radius $\rho = (x_1^2 + x_2^2)^{1/2} = a$. Inside the solenoid ($\rho < a$) $\mathbf{B}_{0,\text{in}} = (0, 0, B_0)$ is homogeneous along the z -axis, and outside the solenoid ($\rho > a$) $\mathbf{B}_{0,\text{out}} = 0$. The vector potential \mathbf{A}_c of \mathbf{B}_c is chosen as (Henceforth the summation convention is used) $A_{c,i} = -B_c \epsilon_{ij} x_j / 2$, $A_{c,z} = 0$ ($i, j = 1, 2$). The BA vector potential \mathbf{A}_0 is: Inside the solenoid $A_{0,i} = A_{\text{in},i} = -B_0 \epsilon_{ij} x_j / 2$, $A_{\text{in},z} = 0$; Outside the solenoid $A_{0,i} = A_{\text{out},i} = -B_0 a^2 \epsilon_{ij} x_j / 2x_k x_k$, $A_{\text{out},z} = 0$ ($i, j, k = 1, 2$). At $\rho = a$ the potential \mathbf{A}_{in} passes continuously over into \mathbf{A}_{out} . The Hamiltonian of the modified combined trap is $H = (p_i - qA_{c,i}/c - qA_{0,i}/c)^2/2\mu + p_z^2/2\mu + \mu\omega_p^2(x_1^2 + 4z^2)/2$. This Hamiltonian can be decomposed into a one-dimensional harmonic Hamiltonian $H_z(z)$ along the z -axis with the axial frequency $\omega_z = 2\omega_p$ and a two-dimensional Hamiltonian $H_{\perp}(x_1, x_2)$, $H = H_z(z) + H_{\perp}(x_1, x_2)$. Inside the solenoid the ion's motion is the same as the one with a total magnetic field $\mathbf{B}_c + \mathbf{B}_{0,\text{in}}$.

In the following we consider the motion outside the solenoid. The two-dimensional Hamiltonian outside the solenoid is [22,23]

$$H_{\perp}(x_1, x_2) = \frac{1}{2\mu} \left(p_i + \frac{1}{2} \mu \omega_c \epsilon_{ij} x_j + \mu \omega_0 a^2 \frac{\epsilon_{ij} x_j}{2x_k x_k} \right)^2 + \frac{1}{2} \mu \omega_p^2 x_i^2, \quad (1)$$

where $\omega_c = qB_c/\mu c$ and $\omega_0 = qB_0/\mu c$ are the cyclotron frequencies corresponding to, respectively, the magnetic fields \mathbf{B}_c and $\mathbf{B}_{0,\text{in}}$. The Hamiltonian H_{\perp} possess a rotational symmetry in (x_1, x_2) -plane. The z -component of the orbital angular momentum $J_z = \epsilon_{ij} x_i p_j$ commutes with H_{\perp} . They have common eigenstates.

¹ The derivation of a static effective potential V_{eff} from a rapidly oscillating one in Ref. [23] does not include a magnetic field or an associated vector potential. That derivation remains valid in the presence of magnetic fields, see Ref. [22] of the combined trap. A modified combined trap includes a BA type magnetic field. The BA effect consists essentially in a shift of the phase of the original wave function. One can adjust the radio-frequency $\tilde{\Omega}$ to compensate the phase shift, therefore for a modified combined trap the derivation in Ref. [23] also remains valid. The modification of $\tilde{\Omega}$ leads to the corresponding modification of the effective frequency $\omega_p = \Omega^2/4\tilde{\Omega}$ of V_{eff} . In Eq. (1) the ω_p means the modified effective frequency.

1.1. Dynamics in the limit of the kinetic energy approaching its lowest eigenvalue

In this limit the kinetic energy is $E_k = \mu \dot{x}_i \dot{x}_i / 2 = (K_1^2 + K_2^2)/2\mu$ where

$$K_i \equiv p_i + \frac{1}{2} \mu \omega_c \epsilon_{ij} x_j + \mu \omega_0 a^2 \frac{\epsilon_{ij} x_j}{2x_k x_k}, \quad [K_i, K_j] = i\hbar \mu \omega_c \epsilon_{ij}. \quad (2)$$

Here K_i is the mechanical momenta corresponding to the vector potentials $A_{c,i}$ and $A_{0,i}$. It is worth noting that the BA vector potential $A_{0,i}$ does not contribute to the commutator $[K_i, K_j]$. The canonical momenta p_i are quantized, $p_i = -i\hbar \partial/\partial x_i$. They commute each other $[p_i, p_j] = 0$. We define canonical variables $Q = K_1/\mu\omega_c$ and $\Pi = K_2$ which satisfy $[Q, \Pi] = i\hbar \delta_{ij}$. The kinetic energy E_k is rewritten as the Hamiltonian of a harmonic oscillator $E_k = \Pi^2/2\mu + \mu\omega_c^2 Q^2/2$. The lowest eigenvalue \mathcal{E}_{k0} of the kinetic energy E_k is $\mathcal{E}_{k0} = \hbar\omega_c/2$.

In a laser trapping field, using a number of laser beams and exploiting Zeeman tuning, the speed of atoms can be slowed to the extent of 1 ms^{-1} , see [24]. Ions are the common object in cooling and trapping. In order to experimentally realizing the limit of $E_k \rightarrow \mathcal{E}_{k0}$ through laser cooling in a trap ions are used.

In the limit of the kinetic energy approaching its lowest eigenvalue the Hamiltonian H_{\perp} in Eq. (1) has non-trivial dynamics [19,25,26]. The Lagrangian corresponding to H_{\perp} is

$$L = \frac{1}{2} \mu \dot{x}_i \dot{x}_i - \frac{1}{2} \mu \omega_c \epsilon_{ij} \dot{x}_i x_j - \mu \omega_0 a^2 \frac{\epsilon_{ij} \dot{x}_i x_j}{2x_k x_k} - \frac{1}{2} \mu \omega_p^2 x_i x_i. \quad (3)$$

In the limit of $E_k \rightarrow \mathcal{E}_{k0}$, the Hamiltonian H_{\perp} reduces to $H_0 = \hbar\omega_c/2 + \mu\omega_p^2 x_i x_i/2$. The Lagrangian corresponds to H_0 is

$$L_0 = -\frac{1}{2} \mu \omega_c \epsilon_{ij} \dot{x}_i x_j - \mu \omega_0 a^2 \frac{\epsilon_{ij} \dot{x}_i x_j}{2x_k x_k} - \frac{1}{2} \mu \omega_p^2 x_i x_i - \frac{1}{2} \hbar \omega_c. \quad (4)$$

1.2. Constraints

For the reduced system (H_0, L_0) the canonical momenta are

$$p_i = \frac{\partial L_0}{\partial \dot{x}_i} = -\frac{1}{2} \mu \omega_c \epsilon_{ij} x_j - \mu \omega_0 a^2 \frac{\epsilon_{ij} x_j}{2x_k x_k}. \quad (5)$$

Eq. (5) does not determine velocities \dot{x}_i as functions of p_i and x_j , but gives relations among p_i and x_j , that is, such relations are the primary constraints [19,26,27]

$$\varphi_i(x, p) = p_i + \frac{1}{2} \mu \omega_c \epsilon_{ij} x_j + \mu \omega_0 a^2 \frac{\epsilon_{ij} x_j}{2x_k x_k} = 0. \quad (6)$$

The physical meaning of Eq. (6) is that it expresses the dependence of degrees of freedom among p_i and x_j . The constraints (6) should be carefully treated.³ The subject can be treated simply by

² M. Peshkin pointed out that in the simplest case of the flux line and no other fields, there can be no wave function whose kinetic energy expectation is zero, hence no zero eigenvalue of the kinetic energy (a private communication).

³ The momentum p_i of Eq. (5) defined from the Lagrangian L_0 of Eq. (4) cannot determine the velocity \dot{x}_i as a function of p_i and x_j . This shows that L_0 is singular. The physical meaning of Eq. (5) is that the corresponding Eq. (6), $\varphi_i(x, p) = 0$, is a primary constraint which expresses the dependence of degrees of freedom among p_i and x_j . The Hamiltonian equations of such a constrained system are not unique. The standard way of deriving them is as follows.

From the Hamiltonian $H_0 = p_i \dot{x}_i - L_0$, using $p_i = \partial L_0/\partial \dot{x}_i$ and the Lagrangian equation $\dot{p}_i = \partial L_0/\partial x_i$, it follows that

$$\delta H_0 = \dot{x}_i \delta p_i - \dot{p}_i \delta x_i.$$

It indicates that H_0 can be expressed as a function of x_i and p_i . Thus we obtain

$$\delta H_0(x, p) = \frac{\partial H_0}{\partial x_i} \delta x_i + \frac{\partial H_0}{\partial p_i} \delta p_i.$$

Because of the constraints $\varphi_i(x, p) = 0$ of Eq. (6), H_0 plus any linear combination of φ_i is also a Hamiltonian of the system, i.e., the H_0 can be replaced by

the symplectic method in [28,29]. In this Letter we work in the Dirac formalism. The Poisson brackets of the constraints (6) are

$$C_{ij} = \{\varphi_i, \varphi_j\} = \mu\omega_c \epsilon_{ij}. \quad (7)$$

From Eq. (7), $\{\varphi_i, \varphi_j\} \neq 0$, it follows that the conditions of the constraints φ_i holding at all times do not lead to secondary constraints.

C_{ij} defined in Eq. (7) are elements of the constraint matrix C . Elements of its inverse matrix C^{-1} are $(C^{-1})_{ij} = -\epsilon_{ij}/\mu\omega_c$. The corresponding Dirac brackets of $\{\varphi_i, x_j\}_D$, $\{\varphi_i, p_j\}_D$, $\{x_i, x_j\}_D$, $\{p_i, p_j\}_D$ and $\{x_i, p_j\}_D$ can be defined. The Dirac brackets of φ_i with any variables x_i and p_j are zero so that the constraints (6) are strong conditions. It can be used to eliminate dependent variables. If we select x_1 and x_2 as the independent variables, from the constraints (6) the variables p_1 and p_2 can be represented by, respectively, the independent variables x_2 and x_1 as

$$p_1 = -\frac{1}{2}\mu\omega_c x_2 - \mu\omega_0 a^2 \frac{x_2}{2x_k x_k},$$

$$p_2 = \frac{1}{2}\mu\omega_c x_1 + \mu\omega_0 a^2 \frac{x_1}{2x_k x_k}. \quad (8)$$

The Dirac brackets of x_1 and x_2 is

$$\{x_1, x_2\}_D = \frac{1}{\mu\omega_c}. \quad (9)$$

We introduce new canonical variables $x = x_1$ and $p = \mu\omega_c x_2$. Their Dirac bracket is $\{x, p\}_D = 1$. According to Dirac's formalism of quantizing constrained systems the corresponding quantum commutation relation is $[x, p] = i\hbar$.

1.3. Quantum behavior of the reduced system

Now we consider quantum behavior of the reduced system (H_0, L_0) . By defining the following effective mass and frequency, $\mu^* \equiv \mu\omega_c^2/\omega_p^2$, $\omega^* \equiv \omega_p^2/\omega_c$, the Hamiltonian H_0 is represented as $H_0 = p^2/2\mu^* + \mu^*\omega^{*2}x^2/2 + \hbar\omega_c/2$. We introduce an annihilation operator $A = \sqrt{\mu^*\omega^*/2\hbar}x + i\sqrt{1/2\hbar\mu^*\omega^*}p$ and its conjugate one A^\dagger . The operators A and A^\dagger satisfies $[A, A^\dagger] = 1$. The eigenvalues of the number operator $N = A^\dagger A$ is $n = 0, 1, 2, \dots$. Using A and A^\dagger , the reduced Hamiltonian H_0 is rewritten as $H_0 = \hbar\omega^*(A^\dagger A + 1/2) + \hbar\omega_c/2$.

Now we consider the angular momentum of the ion. Using Eq. (8) to replace p_1 and p_2 by, respectively, the independent variables x_2 and x_1 , the orbital angular momentum $J_z = \epsilon_{ij}x_i p_j$ is rewritten as

$$J_z = \frac{q}{2\pi c}\Phi_0 + \frac{1}{2}\mu\omega_c(x_1^2 + x_2^2), \quad (10)$$

where $\Phi_0 = \pi a^2 B_0$ is the total flux of the magnetic field B_0 inside the solenoid. Similarly, using A and A^\dagger to rewrite J_z , we obtain

$H_0(x, p) + \lambda_i(x, p)\varphi_i(x, p)$. From the above two equations, including the contributions of $\delta(\lambda_i(x, p)\varphi_i(x, p))$, it follows that the Hamiltonian equations read

$$\dot{p}_i = -\frac{\partial H_0}{\partial x_i} - \lambda_k \frac{\partial \varphi_k}{\partial x_i}, \quad \dot{x}_i = \frac{\partial H_0}{\partial p_i} + \lambda_k \frac{\partial \varphi_k}{\partial p_i}.$$

Eq. (6) gives $\partial \varphi_k / \partial p_i = \delta_{ki}$. From the reduced Hamiltonian H_0 obtained from L_0 in Eq. (4) it follows that $\partial H_0 / \partial p_i = 0$. Thus the second equation reduces to

$$\dot{x}_i = \lambda_i.$$

In this example the Lagrange multiplier λ_i is just the velocity \dot{x}_i .

$J_z = q\Phi_0/2\pi c + \hbar(A^\dagger A + 1/2)$. The zero-point angular momentum of J_z is $\mathcal{J}_0 = \hbar/2 + q\Phi_0/2\pi c$. In the above the term⁴

$$\mathcal{J}_{AB} = \frac{q}{2\pi c}\Phi_0 \quad (11)$$

is the zero-point angular momentum induced by the AB vector potential. \mathcal{J}_{AB} takes fractional values. It is related to the region where the magnetic field $\mathbf{B}_{0,\text{out}} = 0$ but the corresponding vector potential $\mathbf{A}_{\text{out}} \neq 0$.

2. Dynamics in the case of $\mathbf{B}_c = 0$

It is worth noting that here \mathbf{B}_c , like a "spectator", does not contribute to \mathcal{J}_{AB} . In order to clarify the role played by \mathbf{B}_c , we consider the case of $\mathbf{B}_c = 0$. In this case the modified combined trap is as stable as a Paul trap. The corresponding kinetic energy reduces to $\tilde{E}_k = \mu\dot{x}_i\dot{x}_i/2 = (\tilde{K}_1^2 + \tilde{K}_2^2)/2\mu$ where

$$\tilde{K}_i \equiv p_i + \mu\omega_0 a^2 \frac{\epsilon_{ij}x_j}{2x_k x_k}, \quad [\tilde{K}_i, \tilde{K}_j] = 0. \quad (12)$$

In the above \tilde{K}_i is the mechanical momenta corresponding to the BA vector potential $A_{\text{out},i}$. Unlike the ordinary vector potential, the special feature of the BA vector potential is that it does not contribute to the commutator $[\tilde{K}_i, \tilde{K}_j]$. Because \tilde{K}_i are commuting, behavior of \tilde{E}_k is similar to a Hamiltonian of a free particle. Its spectrum is a continuous one. When \tilde{E}_k approaching some constant $\tilde{E}_k(\neq 0)$ the Hamiltonian H_\perp reduces to $\tilde{H}_0 = \tilde{E}_k + \mu\omega_p^2 x_i x_i/2$. The Lagrangian corresponding to \tilde{H}_0 is

$$\tilde{L}_0 = -\mu\omega_0 a^2 \frac{\epsilon_{ij}\dot{x}_i\dot{x}_j}{2x_k x_k} - \frac{1}{2}\mu\omega_p^2 x_i x_i - \tilde{E}_k. \quad (13)$$

From \tilde{L}_0 we obtain the canonical momenta

$$\tilde{p}_i = \frac{\partial \tilde{L}_0}{\partial \dot{x}_i} = -\mu\omega_0 a^2 \frac{\epsilon_{ij}x_j}{2x_k x_k}. \quad (14)$$

Now we clarify that the case $\tilde{E}_k = 0$ should be excluded. The limit of the kinetic energy $E_k = \mu\dot{x}_i\dot{x}_i/2 \rightarrow 0$ corresponds two possibilities: $\dot{x}_i = 0$ or $\mu \rightarrow 0$. In the case $\dot{x}_i = 0$ the Lagrangian L in Eq. (3) reduces to $\tilde{L}'_0 = -\mu\omega_p^2 x_i x_i/2$. The corresponding canonical momenta $\tilde{p}_i = \partial \tilde{L}'_0 / \partial \dot{x}_i = 0$. Therefore there is no dynamics. According to the definition of the frequency Ω the other possibility $\mu \rightarrow 0$ is forbidden.

Eq. (14) gives the reduced primary constraints

$$\tilde{\varphi}_i = \tilde{p}_{0i} + \mu\omega_0 a^2 \frac{\epsilon_{ij}x_j}{2x_k x_k} = 0. \quad (15)$$

Here the special feature is that the corresponding Poisson brackets are zero,

$$\tilde{C}_{ij} = \{\tilde{\varphi}_i, \tilde{\varphi}_j\} \equiv 0. \quad (16)$$

⁴ Ref. [11] investigated the angular momentum \mathbf{J} originated from the Poynting vector produced by crossing the Coulomb field \mathbf{E} of a charged particle with an external magnetic field \mathbf{B} ,

$$\mathbf{J} = \frac{1}{4\pi c} \int \mathbf{r} \times [\mathbf{E} \times \mathbf{B}(\mathbf{r})] d^3r.$$

In cases where the magnetic field is only in the z-direction, this angular momentum reduces to

$$J_z = -\frac{q\phi}{2\pi c},$$

where $\phi = \int \int B_z(x_1, x_2) dx_1 dx_2$ is the total magnetic flux. J_z is the angular momentum of the electromagnetic fields. In cases where the magnetic field B_z is produced by an infinitely long solenoid, this angular momentum exists only inside the solenoid. J_z should be distinguished from \mathcal{J}_{AB} of Eq. (11). \mathcal{J}_{AB} is the angular momentum of the charged particle. It is worth noting that \mathcal{J}_{AB} is induced by the BA vector potential outside the solenoid and does not exist inside the solenoid.

From Eq. (16), $\{\tilde{\varphi}_i, \tilde{\varphi}_i\} \equiv 0$, it follows that the conditions of the constraints $\tilde{\varphi}_i$ holding at all times lead to secondary constraints $\tilde{\varphi}_i^{(2)} = -\mu\omega_p^2 x_i$. The Poisson brackets $\{\tilde{\varphi}_i^{(2)}, \tilde{\varphi}_j\} = 0$, $\{\tilde{\varphi}_i^{(2)}, \tilde{\varphi}_j^{(2)}\} = 0$, and $\{\tilde{\varphi}_i^{(2)}, \tilde{H}_0\} = 0$, so that persistence of the secondary constraints $\tilde{\varphi}_i^{(2)}$ in course of time does not lead to further secondary constraints $\tilde{\varphi}_i^{(3)}$.

Because of $\tilde{C}_{ij} \equiv 0$, the inverse matrix \tilde{C}^{-1} does not exist. The Dirac brackets $\{\tilde{\varphi}_i, x_j\}_D$, $\{\tilde{\varphi}_i, p_j\}_D$, $\{\tilde{\varphi}_i^{(2)}, x_j\}_D$, $\{\tilde{\varphi}_i^{(2)}, p_j\}_D$, $\{x_i, x_j\}_D$, $\{p_i, p_j\}_D$, and $\{x_i, p_j\}_D$ cannot be defined. According to Dirac's formalism of quantizing constrained systems, there is no way to establish dynamics at quantum mechanical level. This means that the BA vector potential alone cannot lead to non-trivial dynamics at quantum mechanical level in the required limit, thus does not contribute to the energy spectrum and angular momentum at all.

It is clear that though the vector potential $A_{c,i}$ of the “spectator” magnetic field \mathbf{B}_c does not contribute to \mathcal{J}_{AB} , it plays essential role in guaranteeing non-trivial dynamics at quantum mechanical level in the limit of the kinetic energy approaching one of its eigenvalues. This example reveals that, unlike ordinary vector potential, the physical role played by the BA vector potential is subtle. This needs to be carefully analyzed at quantum mechanical level.

3. Dynamics in the case of $\mathbf{B}_0 = 0$

In order to further clarify the essential difference between \mathbf{A}_0 and \mathbf{A}_c in the region of $\mathbf{B}_{0,\text{out}} = 0$ we consider the case of $\mathbf{B}_0 = 0$. In this case the modified combined trap reduces to a combined trap. The Hamiltonian $H_\perp(x_1, x_2)$ in Eq. (1) reduces to $\hat{H}_\perp(x_1, x_2) = (p_i + \mu\omega_c\epsilon_{ij}x_j/2)^2/2\mu + \mu\omega_p^2 x_i^2/2$. Its kinetic energy is $\hat{E}_k = (\hat{K}_1^2 + \hat{K}_2^2)/2\mu$ where

$$\hat{K}_i \equiv p_i + \mu\omega_c\epsilon_{ij}x_j/2, \quad [\hat{K}_i, \hat{K}_j] = i\hbar\mu\omega_c\epsilon_{ij}. \quad (17)$$

In Eq. (17) \hat{K}_i is the mechanical momenta corresponding to the vector potentials $A_{c,i}$. The commutation relations between \hat{K}_i 's are the same as the ones between K_i 's in Eq. (2). The eigenvalues of \hat{E}_k is $\hat{E}_{kn} = \hbar\omega_c(n + 1/2)$, which are just the Landau levels of charged particles in an external magnetic field.

In the following we consider the limit of \hat{E}_k approaching the lowest eigenvalue $\hat{E}_{k0} = \hbar\omega_c/2$. The Lagrangian corresponding to \hat{H}_\perp is

$$\hat{L} = \mu\dot{x}_i\dot{x}_i/2 - \mu\omega_c\epsilon_{ij}\dot{x}_i x_j/2 - \mu\omega_p^2 x_i x_i/2. \quad (18)$$

In the limit of $\hat{E}_k \rightarrow \hat{E}_{k0}$, the Hamiltonian \hat{H}_\perp reduces to $\hat{H}_0 = \hbar\omega_c/2 + \mu\omega_p^2 x_i x_i/2$ which is the same as H_0 . The Lagrangian corresponds to \hat{H}_0 is

$$\hat{L}_0 = -\mu\omega_c\epsilon_{ij}\dot{x}_i x_j/2 - \mu\omega_p^2 x_i x_i/2 - \hbar\omega_c/2. \quad (19)$$

For the reduced system (\hat{H}_0, \hat{L}_0) the canonical momenta are $\hat{p}_i = \partial\hat{L}_0/\partial\dot{x}_i = -\mu\omega_c\epsilon_{ij}x_j/2$. It leads to the following constraints

$$\hat{\varphi}_i = p_i + \mu\omega_c\epsilon_{ij}x_j/2 = 0. \quad (20)$$

The Poisson brackets of $\hat{\varphi}_i$ are the same as ones of the constraints φ_i in Eq. (7):

$$\hat{C}_{ij} = \{\hat{\varphi}_i, \hat{\varphi}_j\} = \mu\omega_c\epsilon_{ij}. \quad (21)$$

From Eq. (21), $\{\hat{\varphi}_i, \hat{\varphi}_j\} \neq 0$, it follows that the conditions of the constraints $\hat{\varphi}_i$ holding at all times do not lead to secondary constraints.

By the similar procedure of treating the constraints (6), we find that the reduced system (\hat{H}_0, \hat{L}_0) has non-trivial dynamics at quantum mechanical level in the limit of $\hat{E}_k \rightarrow \hat{E}_{k0}$. The constraints (20) are strong conditions which can be used to eliminate dependent variables. We select x_1 and x_2 as the independent variables.

The variables p_1 and p_2 can be represented by, respectively, x_2 and x_1 as $p_1 = -\mu\omega_c x_2/2$, $p_2 = \mu\omega_c x_1/2$. The Dirac brackets of x_1 and x_2 is $\{x_1, x_2\}_D = 1/\mu\omega_c$. We introduce new canonical variables $x = x_1$ and $p = \mu\omega_c x_2$. Their Dirac bracket is $\{x, p\}_D = 1$. The corresponding quantum commutation relation is $[x, p] = i\hbar$. Using these results the orbital angular momentum $J_z = \epsilon_{ij}x_i p_j$ can be represented by the canonical variables x and p as $\hat{J}_z = (p^2/2\mu + \mu\omega_c^2 x^2/2)/\omega_c$. The zero-point angular momentum can be read out from this harmonic-like “Hamiltonian”, $\hat{J}_0 = \hbar/2$. We note that in this case there is no fractional zero-point angular momentum.

The above results elucidate that \mathbf{A}_c are essentially different from \mathbf{A}_0 : the \mathbf{A}_c alone can lead to non-trivial dynamics at quantum mechanical level in the limit of the kinetic energy approaching its lowest eigenvalue.

4. Gauge transformation

As is well known, we can perform a gauge transformation χ so that the resulting vector potential $\mathbf{A}'_{\text{out}} = \mathbf{A}_{\text{out}} + \nabla\chi = 0$. A suitable gauge function⁵ is $\chi = -B_0 a^2 \tan^{-1}(x_2/x_1)/2$. In the Schrödinger equation the corresponding gauge transformation is $\mathcal{G} = \exp(iq\chi/\hbar)$. Under this gauge transformation the Hamiltonian $H_\perp(x_1, x_2)$ in Eq. (1) is transformed into $H_\perp \rightarrow \mathcal{G}H_\perp\mathcal{G}^{-1} = H'_\perp = (p_i + \mu\omega_c\epsilon_{ij}x_j/2)^2/2\mu + \mu\omega_p^2 x_i^2/2$. Here H'_\perp is the same \hat{H}_\perp .

In the limit of the kinetic energy approaching its lowest eigenvalue the corresponding reduced constraints are the same $\hat{\varphi}_i$ in Eq. (20). Under the gauge transformation \mathcal{G} the angular momentum $J_z = \epsilon_{ij}x_i p_j$ is transformed into $J_z \rightarrow \mathcal{G}J_z\mathcal{G}^{-1} = J'_z = x_1 p_2 - x_2 p_1 + q\Phi_0/2\pi c$. Using the constraints $\hat{\varphi}_i$ in Eq. (20) to represent p_1 and p_2 by, respectively, the independent variables x_2 and x_1 , the first term in J'_z reads $x_1 p_2 - x_2 p_1 = \mu\omega_c(x_1^2 + x_2^2)/2$. Thus we obtain

$$J'_z = \frac{q}{2\pi c}\Phi_0 + \frac{1}{2}\mu\omega_c(x_1^2 + x_2^2). \quad (22)$$

J'_z is the same J_z in Eq. (10). This result shows that the fractional zero-point angular momentum induced by the BA vector potential is a real physical observable which cannot be gauged away by a gauge transformation.

In summary, this Letter explores a “spectator” mechanism in BA effects. It is clarified that the BA vector potential alone cannot lead to non-trivial dynamics at quantum mechanical level in the limit of the kinetic energy approaching one of its eigenvalues. In such a limit the BA vector potential alone cannot induce a fractional zero-point angular momentum. When there is a “spectator” magnetic field the BA vector potential induces a fractional zero-point angular momentum. The induced effect essentially depends upon the participation of a “spectator” magnetic field. The “spectator” vector potential does not contribute to the fractional angular momentum, but plays essential role in guaranteeing non-trivial dynamics at quantum mechanical level in the required limit. The “spectator” mechanism is significant in both aspects of theory and experiment. In the theoretical aspect, it is revealed that, unlike ordinary vector potentials, the physical role played by the BA vector potential is subtle. This needs to be carefully analyzed at quantum mechanical level. In the experimental aspect, existence of a “spectator” magnetic field is necessary for inducing the fractional angular momentum by the BA vector potential. As an example, the modified combined trap provides a realistic way to realize this “spectator” mechanism.

⁵ This gauge function is singular at $x_1 = 0$. The values of the polar angle ϕ on both sides of $x_2 = 0$, $x_1 < 0$ differ by 2π . There is a cut along the negative semi-axis of x_1 , the so-called Dirac string. However, there is no need to perform a singular gauge transformation. See, for example, Ref. [27].

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