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Orthogonal and symplectic Yangians and Yang–Baxter R-operators

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Abstract

Yang–Baxter R operators symmetric with respect to the orthogonal and symplectic algebras are considered in an uniform way. Explicit forms for the spinorial and metaplectic R operators are obtained. L operators, obeying the RLL relation with the orthogonal or symplectic fundamental R matrix, are considered in the interesting cases, where their expansion in inverse powers of the spectral parameter is truncated. Unlike the case of special linear algebra symmetry the truncation results in additional conditions on the Lie algebra generators of which the L operators is built and which can be fulfilled in distinguished representations only. Further, generalized L operators, obeying the modified RLL relation with the fundamental R matrix replaced by the spinorial or metaplectic one, are considered in the particular case of linear dependence on the spectral parameter. It is shown how by fusion with respect to the spinorial or metaplectic representation these first order spinorial L operators reproduce the ordinary L operators with second order truncation.

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1. Introduction

Let $\mathcal{G}$ be a Lie algebra of a Lie group G and V_j be spaces of representations ρ_j of $\mathcal{G}$ and G. We consider the Yang–Baxter (YB) relations in the general form

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \in \mathrm{End}(V_1 \otimes V_2 \otimes V_3)\,, \tag{1.1}$$

where the operator R_{ij} acts nontrivially only in the spaces V_i and V_j and u, v are spectral parameters. It is well known that (1.1) is the basic relation in the treatment of integrable quantum systems and is considered as an analog of the Jacobi identities in the formulation of the related algebras [1–7].

A solution $R_{ij}(u)$ of the YB relation (1.1) is called symmetric with respect to the group G or the algebra $\mathcal{G}$ if the action of $R_{ij}(u)$ on $V_i \otimes V_j$ commutes with the action of the group G (or its Lie algebra $\mathcal{G}$) in the representation $\rho_i \otimes \rho_j$:

$$\begin{gathered}[\rho_i(g) \otimes \rho_j(g)\,,\; R_{ij}(u)] = 0 \quad (\forall g \in G) \qquad \Leftrightarrow \\ [\rho_i(A) \otimes 1_j + 1_i \otimes \rho_j(A)\,,\; R_{ij}(u)] = 0 \;\; (\forall A \in \mathcal{G}).\end{gathered}$$

The present paper is concerned with the specific features of the YB relations and the involved R operators in the cases of symmetry with respect to orthogonal (*so*) and symplectic (*sp*) algebra actions. The less trivial representation theories in those algebras compared to the special linear ($s\ell$) ones imply more involved structures in the Yang–Baxter R operators. A distinguishing feature of *so* and *sp* algebras compared to the $s\ell$ ones is the presence of an invariant metric – the second rank tensor ε, determining the scalar product in the defining (fundamental) representation. It is symmetric, $\varepsilon^T = \varepsilon$, in the *so* case and anti-symmetric, $\varepsilon^T = -\varepsilon$, in the *sp* case. In particular this results in the analogy between the *so* and *sp* cases connected with the interchange of symmetrization with anti-symmetrization and gives us the possibility to treat both cases simultaneously.

The *so* (or *sp*) symmetric matrix $R_{ij}(u)$ obeying the Yang–Baxter (YB) relation (1.1), where $V_1 = V_2 = V_3$ are spaces of a defining (fundamental) representation, is not a linear function in the spectral parameter u as it is for the $s\ell$ symmetric fundamental R-matrix. The explicit form of the fundamental *so* (and *sp*) symmetric R-matrices were found first in [7,8,11].

The generic YB relation (1.1) specifies to the *RLL* relation if two of the three spaces $V_1 = V_2 = V_f$ carry the fundamental representation ρ_f while the third space $V_3 = V$ is the space of any representation ρ of $\mathcal{G}$. In this case the R-operator acting on the product $V_f \otimes V$ is called L-operator (or L matrix). For the *so* and *sp* cases the *RLL* version of the YB relation involving the fundamental R matrices [7,8,11] together with the L operators of the form

$$L(u) = u\mathbf{1} + \frac{1}{2}\rho_f(G^a_{\ b})\,\rho(G^b_{\ a})\,, \tag{1.2}$$

does not hold for an arbitrary representation ρ of the generators $G^b_{\ a}$. The spinor representation ρ_s of the orthogonal algebra with $\rho_s(G_{ab}) = M_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, where γ_a are Dirac gamma-matrices, is a distinguished case, where the *RLL* relation is obeyed with L of first order in the spectral parameter u (see (1.2)). Also the spinorial R matrix $R_{ss}(u)$, intertwining two spinor representations ρ_s, is known [9,10]. This and other representations of the orthogonal algebra distinguished in this sense as well as the corresponding R operators (including spinorial R operator) have been recently considered and analyzed in detail in [14,15].

As we mentioned above we rely on the known similarity of the *so* and *sp* algebras and treat the related R operators in a uniform way. In Section 2 we recall the fundamental R matrices and present them for both cases uniformly. Further we identify the symplectic counterpart of the spinor representation.

It is known that the $\mathcal{G}$-symmetric *RLL* relations are defining relations for the infinite dimensional algebra called Yangian $Y(\mathcal{G})$ of the type $\mathcal{G}$. This concept was introduced by Drinfeld in [6] and provides the appropriate general viewpoint onto the known examples of simple forms of operators $L(u)$ mentioned above. We use this concept to answer the question what are the general conditions for such simple solutions to exist. In general $L(u)$ obeying the *RLL* relation expands in inverse powers of the spectral parameter u with infinitely many terms. The truncation of the expansion of $L(u)$ at the first non-trivial term (1.2) is known to be consistent for arbitrary representations ρ of the $s\ell$ algebras (the evaluation representation of $Y(s\ell(n))$). However, in the cases of *so* and *sp* symmetric *RLL* relations the truncation of the type (1.2) results in additional conditions, which can be fulfilled only for distinguished representations ρ.

We shall investigate the additional conditions arising from the truncation at the first and second order. The additional conditions appear as characteristic equations in the matrix of generators $\rho_f(G^a_{\ b})\, G^b_{\ a}$ which enters the definition of the L-operator. We stress here that a number of such examples has been considered in [11].

In Section 5 the Yang–Baxter R operator intertwining two spinorial representations and two metaplectic representations is obtained in both orthogonal and symplectic cases in the uniform way.

It can be checked that the fundamental R matrix (quadratic in u) can be reproduced by fusion including projection from the product of the spinorial L with its conjugate. By fusion of R operators acting in the tensor product of the spinorial and Jordan–Schwinger type representation spaces we obtain the L operator of second order in u acting in $V_f \otimes V$ with V carrying a representation of Jordan–Schwinger type, bosonic in the *so* case and fermionic in the *sp* case.

2. Fundamental Yang–Baxter R-matrices

Let V_f be the space of the defining (fundamental) representation of the Lie algebra $\mathcal{G}$ and its group G. Let V_f be the n-dimensional vector space with the basis vectors $\vec{e}_a \in V_f$ $(a = 1, \ldots, n)$. Introduce the operator $R(u)$ which acts in the space $V_f \otimes V_f$

$$R(u) \cdot (\vec{e}_{a_1} \otimes \vec{e}_{a_2}) = (\vec{e}_{b_1} \otimes \vec{e}_{b_2})\, R^{b_1 b_2}_{a_1 a_2}(u)\, , \tag{2.3}$$

and depends on the spectral parameter u. The matrix with elements $R^{b_1 b_2}_{a_1 a_2}(u)$ is called fundamental Yang–Baxter (YB) R-matrix if it satisfies the Yang–Baxter equation of the form

$$\begin{gathered} R^{a_1 a_2}_{b_1 b_2}(u) R^{b_1 a_3}_{c_1 b_3}(u+v) R^{b_2 b_3}_{c_2 c_3}(v) = R^{a_2 a_3}_{b_2 b_3}(v) R^{a_1 b_3}_{b_1 c_3}(u+v) R^{b_1 b_2}_{c_1 c_2}(u) \;\Rightarrow \\ R_{12}(u) R_{13}(u+v) R_{23}(v) = R_{23}(v) R_{13}(u+v) R_{12}(u)\, . \end{gathered} \tag{2.4}$$

This equation is understood as a relation of operators acting in $V_f \otimes V_f \otimes V_f$ and $R_{13}(u)$ or $R_{23}(u)$, etc., denotes that the R-operator (2.3) acts nontrivially only in first and third, or in second and third, etc., factors of $V_f \otimes V_f \otimes V_f$.

The simplest solution to Yang–Baxter equation (2.4) is the Yang matrix

$$R^{a_1 a_2}_{b_1 b_2}(u) = u \delta^{a_1}_{b_1} \delta^{a_2}_{b_2} + \delta^{a_1}_{b_2} \delta^{a_2}_{b_1} = (uI + P)^{a_1 a_2}_{b_1 b_2}\, , \tag{2.5}$$

Here I and P denote the unit and the permutation operators. This YB R matrix is $g\ell(n)$ symmetric and acts in the tensor product of two fundamental representations. The hierarchy of solutions of the Yang–Baxter equations, corresponding to higher representations can be obtained by the fusion method.

In the orthogonal and symplectic cases the analoga of the matrix (2.5) and the hierarchy of fusion solutions of Yang–Baxter equations look more complicated and do not realize in the simplest way. To explain this we recall that operators A acting in the n-dimensional vector space V_f are elements of the algebra $so(n)$, or $sp(2m)$ $(2m = n)$, if the matrices $||A^a{}_b||_{a,b=1,\dots,n}$ of the operators A in the basis $\vec{e}_a \in V_f$: $A \cdot \vec{e}_a = \vec{e}_b\, A^b{}_a$ satisfy the conditions

$$A^d{}_a\, \varepsilon_{db} + \varepsilon_{ad}\, A^d{}_b = 0\,, \tag{2.6}$$

where ε_{ab} is a non-degenerate invariant metric in V_f

$$\varepsilon_{ab} = \epsilon\, \varepsilon_{ba}\,, \quad \varepsilon_{ab}\varepsilon^{bd} = \delta^d_a\,, \tag{2.7}$$

which is symmetric $\epsilon = +1$ for $SO(n)$ case and skew-symmetric $\epsilon = -1$ for $Sp(n)$ case. We denote by ε^{bd} (with upper indices) the elements of the inverse matrix ε^{-1}. Namely the existence of the invariant tensor ε_{ab} leads to the above mentioned complications in $SO(n)$ and $Sp(n)$ cases, e.g., it causes a third term in the corresponding expressions of the R-matrices and leads to the dependence on the spectral parameter of second power.

The well known R-matrices [7–9,11] for the $SO(n)$ and $Sp(n)$ $(n = 2m)$ cases can be written in a unified form for arbitrary metrics ε_{ab} (2.7) as follows (see, e.g., [12])

$$R^{a_1a_2}_{b_1b_2}(u) = u(u + \frac{n}{2} - \varepsilon) I^{a_1a_2}_{b_1b_2} + (u + \frac{n}{2} - \varepsilon)\, P^{a_1a_2}_{b_1b_2} - \epsilon\, u\, K^{a_1a_2}_{b_1b_2}\,, \tag{2.8}$$

where

$$I^{a_1a_2}_{b_1b_2} = \delta^{a_1}_{b_1}\delta^{a_2}_{b_2}\,, \quad P^{a_1a_2}_{b_1b_2} = \delta^{a_1}_{b_2}\delta^{a_2}_{b_1}\,, \quad K^{a_1a_2}_{b_1b_2} = \varepsilon^{a_1a_2}\,\varepsilon_{b_1b_2}\,, \tag{2.9}$$

and the choices $\epsilon = +1$ and $\epsilon = -1$ correspond to the $SO(n)$ and $Sp(n)$ cases respectively. We note that the R-matrix (2.8) is invariant under the adjoint action of any real form (related to the metric ε^{ab}) of the complex groups $SO(n, \mathbb{C})$ and $Sp(n, \mathbb{C})$.

Let the index range be $a_1, a_2, \dots = 1, \dots, n$ for the $SO(n)$ case and $a_1, a_2, \dots = -m, \dots, -1, 1, \dots, m$ – for the $Sp(n)$ $(n = 2m)$ case. For the choice $\varepsilon^{a_1a_2} = \delta^{a_1a_2}$ in the $SO(n)$ case we have

$$R^{a_1a_2}_{b_1b_2}(u) = u(u+\beta)\delta^{a_1}_{b_1}\delta^{a_2}_{b_2} + (u+\beta)\delta^{a_1}_{b_2}\delta^{a_2}_{b_1} - u\delta^{a_1a_2}\delta_{b_1b_2}\,, \quad \beta = (n/2 - 1)\,, \tag{2.10}$$

and for the choice $\varepsilon^{a_1a_2} = \varepsilon_{a_2}\delta^{a_1,-a_2}$ (here $\varepsilon_a = sign(a)$ and $\varepsilon^{ab} = -\varepsilon_{ab}$) in the $Sp(2m)$ case we have

$$R^{a_1a_2}_{b_1b_2}(u) = u(u+\beta)\delta^{a_1}_{b_1}\delta^{a_2}_{b_2} + (u+\beta)\delta^{a_1}_{b_2}\delta^{a_2}_{b_1} - u\varepsilon_{a_2}\varepsilon_{b_2}\delta^{a_1,-a_2}\delta^{b_1,-b_2}\,, \quad \beta = (m+1)\,. \tag{2.11}$$

3. Yangians of *so* and *sp* types

Let $\mathcal{G}$ be the Lie algebra $so(n)$ or $sp(2m)$ $(2m = n)$. The Yangian $Y(\mathcal{G})$ of $\mathcal{G}$-type is defined [6] as an associative algebra with the infinite number of generators $(L^{(k)})^a_b$ arranged as $(n \times n)$ matrices $||(L^{(k)})^a_b||_{a,b=1,\dots,n}$ $(k = 0, 1, 2, \dots)$ such that $(L^{(0)})^a_b = I\delta^a_b$, where I is an unit element in $Y(\mathcal{G})$, and $(L^{(k)})^a_b$ for $k > 0$ satisfy the quadratic defining relations which we shall describe now. The generators $(L^{(k)})^a_b \in Y(\mathcal{G})$ are considered as coefficients in the expansion of

$$L^a_b(u) = \sum_{k=0}^{\infty} \frac{(L^{(k)})^a_b}{u^k}\,, \quad L^{(0)} = I\,, \tag{3.12}$$

where u is called spectral parameter. The function $L(u)$ is called L-operator and the defining relations of $Y(\mathcal{G})$ are represented [6] as RLL-relations

$$\begin{aligned} R^{a_1a_2}_{b_1b_2}(u-v)L^{b_1}_{c_1}(u)L^{b_2}_{c_2}(v) = L^{a_2}_{b_2}(v)L^{a_1}_{b_1}(u)R^{b_1b_2}_{c_1c_2}(u-v) \;\Leftrightarrow \\ R_{12}(u-v)\,L_1(u)\,L_2(v) = L_2(v)\,L_1(u)\,R_{12}(u-v)\,. \end{aligned} \tag{3.13}$$

Here $R^{a_1a_2}_{b_1b_2}(u-v)$ is the Yang–Baxter R-matrix (2.8) and in the second line of (3.13) we use the standard matrix notations of [13]. Recall that in this notations the matrices $R_{12}(u)$ (2.8) and P_{12}, K_{12} (2.9) are operators in $V_f \otimes V_f$, where V_f denotes a n-dimensional vector space, while L_1 and L_2 are matrices $||L^a{}_b||$ which act nontrivially only in the first and in the second factors of $V_f \otimes V_f$, respectively, and have algebra valued matrix elements.

The defining relations (3.13) are homogeneous in L and one can do in (3.12), (3.13) the redefinition $L(u) \to f(u)\,L(u+b_0)$ with any scalar function $f(u) = 1 + b_1/u + b_2/u^2 + \dots$, where b_i are parameters. Now it is clear that the Yangian (2.8), (3.13) possesses the set of automorphisms

$$L(u) \to \frac{(u-a)^k}{u^k}\,L(u)\,, \qquad (k = 1, 2, \dots)\,,$$

where a is a constant (in general a is a central element in $Y(\mathcal{G})$). In particular for $k=1$ we obtain that the generators $L^{(j)}$ are transforming as

$$L^{(1)} \to L^{(1)} - a I_n\,, \quad L^{(2)} \to L^{(2)} - a\,L^{(1)}\,, \quad L^{(3)} \to L^{(3)} - a L^{(2)}\,, \quad \dots, \tag{3.14}$$

Taking $a = \frac{1}{n}\mathrm{Tr}(L^{(1)})$ one can fix $L^{(1)}$ such that $\mathrm{Tr}(L^{(1)}) = 0$ (below we show that $\mathrm{Tr}(L^{(1)})$ is central element in $Y(\mathcal{G})$).

We represent the fundamental R-matrix (2.8) in the concise form

$$R_{12}(u) = u(u+\beta)\,I + (u+\beta)\,P_{12} - \epsilon\,u\,K_{12}\,, \tag{3.15}$$

where $\beta = (\frac{n}{2} - \epsilon)$. Further, after the shift of the spectral parameter $u \to u - v$, we write it as

$$\tfrac{1}{u^2v^2}R(u-v) = (\tfrac{1}{v} - \tfrac{1}{u})(\tfrac{1}{v} - \tfrac{1}{u} + \tfrac{\beta}{uv}) - (\tfrac{1}{uv^2} - \tfrac{1}{u^2v} + \tfrac{\beta}{u^2v^2})P - \epsilon(\tfrac{1}{uv^2} - \tfrac{1}{u^2v})K. \tag{3.16}$$

Then we substitute (3.16) and (3.12) into (3.13) and obtain, as a coefficient at $u^{-k}v^{-j}$, the explicit quadratic relations for the generators $(L^{(k)})^a_b$ of the Yangians $Y(\mathcal{G})$

$$\begin{gathered} [L_1^{(k)},\, L_2^{(j-2)}] - 2[L_1^{(k-1)},\, L_2^{(j-1)}] + [L_1^{(k-2)},\, L_2^{(j)}] + \\ + \beta([L_1^{(k-1)},\, L_2^{(j-2)}] - [L_1^{(k-2)},\, L_2^{(j-1)}]) + \\ + P\Big(L_1^{(k-1)}\,L_2^{(j-2)} - L_1^{(k-2)}\,L_2^{(j-1)} + \beta L_1^{(k-2)}\,L_2^{(j-2)}\Big) - \\ - \Big(L_2^{(j-2)}L_1^{(k-1)} - L_2^{(j-1)}L_1^{(k-2)} + \beta L_2^{(j-2)}\,L_1^{(k-2)}\Big)P + \\ + \epsilon\Big(K\,(L_1^{(k-2)}\,L_2^{(j-1)} - L_1^{(k-1)}\,L_2^{(j-2)}) - (L_2^{(j-1)}L_1^{(k-2)} - L_2^{(j-2)}L_1^{(k-1)})\,K\Big) = 0\,, \end{gathered} \tag{3.17}$$

where the operators K, P are given in (2.9), $\epsilon = +1$ for $\mathcal{G} = so(n)$ and $\epsilon = -1$ for $\mathcal{G} = sp(2m)$. For the special value $k = 1$ we obtain from (3.17) the set of relations

$$[L_1^{(1)},\ L_2^{(j-2)}] = -\Big[(P_{12} - \epsilon K_{12}),\ L_2^{(j-2)}\Big]\ , \quad (\forall j)\ , \tag{3.18}$$

which in particular lead to the statement that $\mathrm{Tr}(L^{(1)})$ is a central element in $Y(\mathcal{G})$: $[\mathrm{Tr}(L^{(1)}),\ (L^{(j)})^a{}_b] = 0$ $(\forall j)$. For $j = 3$ we deduce from (3.18) the defining relations for the Lie algebra generators $G^a{}_b \equiv -(L^{(1)})^a{}_b$:

$$[G_1,\ G_2] = [(P_{12} - \epsilon K_{12}),\ G_2]\ . \tag{3.19}$$

The permutation of the indices $1 \leftrightarrow 2$ in this equation gives the consistency conditions and the same conditions are obtained from (3.17) directly.

$$K_{12}\,(G_1 + G_2) = (G_1 + G_2)\,K_{12}\ .$$

Acting on this equation by K_{12} from the left (or by K_{12} from the right) we write it as

$$K_{12}\,(G_1 + G_2) = \frac{2}{n}\,\mathrm{Tr}(G)\,K_{12} = (G_1 + G_2)\,K_{12}\ , \tag{3.20}$$

where we have used

$$K_{12}^2 = \epsilon\, n\, K_{12}\ , \quad K_{12}\,N_1\,K_{12} = K_{12}\,N_2\,K_{12} = \epsilon\,\mathrm{Tr}(N)\,K_{12}\ . \tag{3.21}$$

Here N is any $n \times n$ matrix.

Then, according to the automorphism (3.14) we redefine the elements $G \to G - \frac{1}{n}\,\mathrm{Tr}(G)$ in such a way that for the new generators we have $\mathrm{Tr}(G) = 0$ This leads to the (anti)symmetry conditions for the generators (cf. (2.6))

$$\begin{gathered} K_{12}\,(G_1 + G_2) = 0 = (G_1 + G_2)\,K_{12} \quad \Rightarrow \\ G^d{}_a\,\varepsilon_{db} + \varepsilon_{ad}\,G^d{}_b = 0\ . \end{gathered} \tag{3.22}$$

The equations (3.19) and (3.22) for $\epsilon = +1$ and $\epsilon = -1$ define the Lie algebra $\mathcal{G} = so(n)$ and $\mathcal{G} = sp(2m)$ $(2m = n)$, respectively. The defining relations (3.19) and (anti)symmetry condition (3.22) for the generators $G_{ab} = \varepsilon_{ad}\,G^d{}_b$ can be written in the familiar form

$$[G_{ab},\ G_{cd}] = \varepsilon_{cb}G_{ad} + \varepsilon_{db}G_{ca} + \varepsilon_{ca}G_{db} + \varepsilon_{da}G_{bc}\ , \quad G_{ab} = -\epsilon\, G_{ba}\ . \tag{3.23}$$

This means (see [6]) that an enveloping algebra $\mathcal{U}(\mathcal{G})$ of the Lie algebra $\mathcal{G} = so(n), sp(2m)$ is always a subalgebra in the Yangian $Y(\mathcal{G})$.

4. *L* operators

Now we consider two reductions of the Yangian $Y(\mathcal{G})$ (3.17) which we call *linear and quadratic* evaluations, i.e. the two cases where $L(u)$ is represented by a linear or a quadratic polynomial in u.

1. Linear evaluation of $Y(\mathcal{G})$.

We put equal to zero all generators $L^{(k)} \in Y(\mathcal{G})$ with $k > 1$. In this case the L-operator (3.12), after the multiplication by u, is represented as

$$L^a{}_b(u) = u\delta^a_b - G^a{}_b\ , \tag{4.24}$$

where for the Lie algebra generators we again use the notation $G^a{}_b = -(L^{(1)})^a{}_b$. It happens that for the choice of the L-operator in the form (4.24) the *RLL* relations (3.13) in addition to the Lie algebra defining relations (3.19) and (3.20) lead to further constraints on the generators $G^a{}_b \in \mathcal{G}$.

Proposition 1. *For $so(n)$ and $sp(n)$ type R-matrices (3.15) the L-operator (4.24) is a solution of (3.13) iff the elements $G^a{}_b$ satisfy (3.19), (3.20) and in addition obey the quadratic characteristic identity*

$$G^2 - \left(\beta + \tfrac{2}{n}\mathrm{Tr}G\right) G - \tfrac{z}{n}\, I_n = 0\,,$$
$$z \equiv \mathsf{C}^{(2)} - \left(\beta + \tfrac{2}{n}\mathrm{Tr}(G)\right)\mathrm{Tr}(G)\,, \tag{4.25}$$

where $\beta = \frac{n}{2} - \epsilon$. The quadratic Casimir operator $\mathsf{C}^{(2)} = \mathrm{Tr}(G^2) = G^a{}_b\, G^b{}_a$ is the central element in $\mathcal{U}(\mathcal{G})$ and I_n is $n \times n$ unit matrix.

Proof. Substitute the R-matrix (3.15) and the L-operator (4.24) into (3.13). After a straightforward calculation we obtain that the L-operator (4.24) is a solution of the equation (3.13) iff $G^a{}_b$ satisfy the equations (3.19), (3.20) and

$$K_{12}\,(G_1 \cdot G_2 + \beta\, G_2) = (G_2 \cdot G_1 + \beta\, G_2)\, K_{12}\,, \tag{4.26}$$

where $\beta = (\frac{n}{2} - \epsilon)$. Note that the same condition (4.26) can be obtained directly from the defining relations (3.17) of the Yangian if we put there $L^{(k)} = 0$, for $\forall k > 1$, and fix $j = 2, k = 3$ (or $j = 3, k = 2$). Taking into account (3.20) we write (4.26) in the form

$$\left[K_{12}\,,\; G_2^2 - \beta'\, G_2\right] = 0\,, \tag{4.27}$$

where $\beta' = \left(\beta + \frac{2}{n}\mathrm{Tr}(G)\right)$. We act on the left hand side of (4.27) by K_{12} from the right and use formulas (3.21) which lead to the identities

$$K_{12}\, G_2\, K_{12} = \epsilon \mathrm{Tr}(G_2)\, K_{12}\,, \quad K_{12}\,(G_2)^2\, K_{12} = \epsilon\, \mathsf{C}^{(2)}\, K_{12}\,,$$

where $\mathsf{C}^{(2)} = \mathrm{Tr}(G_2)^2 = G^a{}_b\, G^b{}_a$. As a result we have

$$K_{12}\,(G_2^2 - \beta' G_2)\, K_{12} - \epsilon\, n\,(G_2^2 - \beta'\, G_2)\, K_{12} =$$
$$= \epsilon\Big(\mathsf{C}^{(2)} - \beta'\,\mathrm{Tr}(G) - n\,(G_2^2 - \beta'\, G_2)\Big)K_{12} = 0\,, \tag{4.28}$$

which is equivalent to (4.25). □

Remark 1. Since $\mathrm{Tr}(G)$ is a central element for the algebra (3.19), one can shift the spectral parameter $u \to u + \frac{1}{n}\mathrm{Tr}(G)$ in (3.13), (4.24) and fix the generators $G^a{}_b$ such that $\mathrm{Tr}(G) = G^a{}_a = 0$. In this case, $G^a{}_b$ are generators of the Lie algebras $\mathcal{G} = so(n), sp(n)$ which satisfy (3.19), (3.22). In this case the condition (4.25) is simplified to

$$G^2 - \left(\tfrac{n}{2} - \epsilon\right) G - \tfrac{1}{n}\, \mathsf{C}^{(2)}\, I_n = 0, \tag{4.29}$$

that can be written also in the form

$$G^a{}_d\, G^d{}_b - \frac{1}{n}\, \delta^a_b \left(G^e{}_d\, G^d{}_e\right) = \beta\, G^a{}_b\,.$$

The left hand side is the traceless quadratic combination of the matrices G and the right hand side is proportional to the traceless matrix of generators $G^a{}_b \in \mathcal{G} = so(n), sp(2m)$ $(2m = n)$.

Writing $(G^2)^a_b$ as a sum of commutator and anti-commutator we obtain

$$(G^2 - \beta G)^a_b = \frac{1}{2}[G^a_c, G^c_b]_+.$$

This leads still to another form of the additional condition (4.29),

$$[G^a_c, G^c_b]_+ = \frac{1}{n}\mathsf{C}^{(2)}\delta^a_b.$$

We stress that (4.29) does not hold as an identity in the enveloping algebras $\mathcal{U}(so(n))$, or $\mathcal{U}(sp(n))$. This condition can be fulfilled only when the generators $G^a{}_d$ are taken in some special representations of $so(n)$, or $sp(n)$. Thus, for the ansatz of the L-operator (4.24) we find that (3.13) is valid only for a restricted class of representations of the Lie algebras $so(n)$ and $sp(2m)$. For the *so* case this fact has already been noticed in [11,14,15] and for the *sp* case it was discussed in [11]. We note also that the quadratic condition (4.29) in the universal form (when the quadratic Casimir operator is not fixed) has been discussed for the *so* case in [15] and in the context of some special representations of *so* and *sp* in [11]. Below we give examples of special $so(n)$ and $sp(n)$ representations which fulfill the condition (4.29).

Remark 2. The definition of the of Yangian $Y(\mathcal{G})$ of $\mathcal{G}$-type applies of course to case of $g\ell_n$. We take the L-operator and the *RLL* relations in the same form as in (3.12), (3.13) and use in (3.13) Yang's R-matrix (2.5). Thus, all relations for the Yangian $Y(g\ell_n)$ can be deduced from (3.15), (3.17) and (3.19) if we put everywhere $K = 0$ and $\beta = 0$. For the $g\ell_n$ case the analog of the Proposition 1 states that the L-operator (4.24) constructed from the set of elements $G^b{}_c$ obeying the *RLL* relation (3.13) with the fundamental Yang R-matrix (2.5) iff elements $G^b{}_c$ satisfy the defining relations for generators of $g\ell_n$ (cf. (3.19) for $K = 0$)

$$[G^{a_1}_{b_1},\, G^{a_2}_{b_2}] = \delta^{a_2}_{b_1} G^{a_1}_{b_2} - \delta^{a_1}_{b_2} G^{a_2}_{b_1}\,,$$

and there do not arise any additional constraints (like (4.29)) on the generators $G^b{}_c \in g\ell_n$. It means that for $Y(g\ell_n)$ we have a homomorphic map of the Yangian $Y(g\ell_n)$ into the enveloping algebra $\mathcal{U}(g\ell_n)$ such that

$$L^{(1)} \to -G\,, \quad L^{(k)} \to 0 \quad \forall k > 1\,.$$

This map is called *evaluation representation* of the Yangian $Y(g\ell_n)$.

Now we construct a representation of the Lie algebra with the defining relations (3.19), (3.22) which fulfill the condition (4.29). This distinguished representation is realized in terms of fermionic and bosonic oscillators.

First we introduce an algebra $\mathcal{A}$ of fermionic or bosonic oscillators with generators c^a ($a = 1, \dots, n$) and the defining relations

$$[c^a, c^b]_\epsilon \equiv c^a c^b + \epsilon\, c^b c^a = \varepsilon^{ab}\,. \tag{4.30}$$

Here for the $SO(n)$ case ($\epsilon = +1$) the elements c^a are fermionic oscillators and for the $Sp(n)$ ($n = 2m$) case ($\epsilon = -1$) the elements c^a are bosonic oscillators. The algebra $\mathcal{A}$ with the defining relations (4.30) is covariant under the action of $SO(n)$ or $Sp(n)$ group $c^a \to U^a_b\, c^b$, where $U \in SO(n)$ or $U \in Sp(n)$. For the set of dual oscillators $c_a = \varepsilon_{ab}\, c^b$ we obtain from (4.30) the following relations (cf. (4.30))

$$[c_a, c_b]_\epsilon \equiv c_a c_b + \epsilon\, c_b c_a = \varepsilon_{ba} = \epsilon\, \varepsilon_{ab} \quad \Leftrightarrow \quad c_a c^b + \epsilon\, c^b c_a = \delta^b_a\,. \tag{4.31}$$

In particular we have

$$c_a c^a = \epsilon\, c^a c_a = \frac{1}{2}\varepsilon^{ab}(c_a c_b + \epsilon\, c_b c_a) = \frac{n}{2}\,.$$

Proposition 2. *Let c_a and c^b be the generators of the oscillator algebra $\mathcal{A}$ with the defining relations (4.30), (4.31). The operators*

$$\rho(G^a_{\ b}) = F^a_b = (c^a\, c_b - \tfrac{\epsilon}{2}\delta^a_b) = \tfrac{1}{2}(c^a\, c_b - \epsilon\, c_b\, c^a) = \varepsilon^{ad}\, c_{[d}c_{b)} = \varepsilon_{bd}\, c^{[a}c^{d)}\,,$$
$$\rho(G^a_{\ a}) = F^a_a = \mathrm{Tr}(F)\,, \tag{4.32}$$

satisfy the relations (3.19) and (3.22), i.e., the operators $F^a_b = \rho(G^a_{\ b})$ define representations ρ of $so(n)$ and $sp(n)$ generators. Moreover, the generators (4.32) obey the quadratic characteristic identity

$$F^a_{\ d}\, F^d_{\ b} - \beta\, F^a_{\ b} = \frac{1}{4}\,(n\epsilon - 1)\,\delta^a_b\,, \tag{4.33}$$

which is nothing but the conditions (4.29), where the quadratic Casimir operator is fixed as $\mathsf{C}^{(2)} = \frac{1}{4}n(n\epsilon - 1)I$. Therefore the L-operator (4.24) in the representation (4.32)

$$L^a_{\ b}(u) = u\delta^a_b - \varepsilon^{ad}\, c_{[d}c_{b)} = u\delta^a_b - \varepsilon_{bd}\, c^{[a}c^{d)}\,, \tag{4.34}$$

where $c_{[d}c_{b)} = \frac{1}{2}(c_d\, c_b - \epsilon\, c_b\, c_d)$ is the symmetrized product of c_d and c_b, solves the RLL relations (3.13).

Proof. First, we check that the operators (4.32) satisfy the symmetry condition (3.22)

$$\Big(K_{12}(F_1 + F_2)\Big)^{a_1a_2}_{d_1b_2} = K^{a_1a_2}_{d_1b_2}\, F^{d_1}_{b_1} + K^{a_1a_2}_{b_1d_2}\, F^{d_2}_{b_2} =$$
$$= \varepsilon^{a_1a_2}\Big((\epsilon c_{b_2}c_{b_1} + c_{b_1}c_{b_2}) - \epsilon\, \varepsilon_{b_1b_2}\Big) = 0\,.$$

Then after the substitution of (4.32) into the l.h.s. of (3.19) we obtain

$$\Big[F^{a_1}_{b_1},\, F^{a_2}_{b_2}\Big] = \big[c^{a_1}\, c_{\,b_1},\, c^{a_2}\, c_{\,b_2}\big] = \big[c^{a_1}\, c_{\,b_1},\, c^{a_2}\big]\, c_{\,b_2} + c^{a_2}\, \big[c^{a_1}\, c_{\,b_1},\, c_{\,b_2}\big] =$$
$$= \Big(c^{a_1}\, \big[c_{b_1},\, c^{a_2}\big]_\epsilon - \epsilon\, \big[c^{a_1},\, c^{a_2}\big]_\epsilon\, c_{b_1}\Big)c_{b_2} + c^{a_2}\Big(c^{a_1}\, \big[c_{b_1},\, c_{b_2}\big]_\epsilon - \epsilon\, \big[c^{a_1},\, c_{b_2}\big]_\epsilon\, c_{b_1}\Big) =$$
$$= \delta^{a_2}_{b_1}c^{a_1}c_{b_2} - \delta^{a_1}_{b_2}c^{a_2}c_{b_1} + \epsilon\big(c^{a_2}c^{a_1}\varepsilon_{b_1b_2} - \varepsilon^{a_1a_2}c_{b_1}c_{b_2}\big) =$$
$$= \delta^{a_2}_{b_1}\, F^{a_1}_{b_2} - \delta^{a_1}_{b_2}\, F^{a_2}_{b_1} + \epsilon\Big(F^{a_2}_{d_2}\, K^{a_1d_2}_{b_1b_2} - K^{a_1a_2}_{b_1d_2}\, F^{d_2}_{b_2}\Big)$$

which is equivalent to (3.19). Finally we have

$$F^a_{\ d}\, F^d_{\ b} - \beta\, F^a_{\ b} = (c^a\, c_d - \tfrac{\epsilon}{2}\delta^a_d)(c^d\, c_b - \tfrac{\epsilon}{2}\delta^d_b) - \beta(c^a\, c_b - \tfrac{\epsilon}{2}\delta^a_b) =$$
$$= (\tfrac{n}{2} - \epsilon - \beta)c^a\, c_b + \tfrac{1}{4}\,\delta^a_b(1 + 2\beta\epsilon) = \tfrac{1}{4}\,\delta^a_b(n\epsilon - 1)\,,$$

and the condition (4.29) is fulfilled for the value $\rho(\mathsf{C}^{(2)}) = \frac{1}{4}n(n\epsilon - 1)I$ of the quadratic Casimir operator. $\square$

Remark 3. In the *so* case the considered representation is formulated in terms of fermionic oscillators. Their anti-commutation relation can be read also as the defining relation of the Clifford algebra and one can identify the oscillator generators with the Dirac gamma matrices.

$$\gamma^a = \sqrt{2}c^a\,,\quad [\gamma^a, \gamma^b]_+ = 2\varepsilon^{ab}.$$

Therefore the representation is called also spinorial. The bosonic oscillator representation is the appropriate counterpart in the *sp* case and we shall call it metaplectic or also (symplectic) spinorial representation.

$$\Gamma^a = \sqrt{2}c^a, \quad [\Gamma^a, \Gamma^b]_- = 2\varepsilon^{ab}.$$

The oscillator algebra and the spinor representation generators can be regarded as appearing from the restriction of a Jordan–Schwinger type representation of a general linear algebra [17].

2. Quadratic evaluation of $Y(\mathcal{G})$.

Now we put all generators $L^{(k)} \in Y(\mathcal{G})$ with $k > 2$ equal to zero. In this case the L-operator (3.12), after a multiplication by u^2, can be written in the form

$$L(u) = u(u + a) - u\, G + N \,, \tag{4.35}$$

where we introduce

$$L^{(1)} = a - G \,, \quad L^{(2)} = N \,, \tag{4.36}$$

and a is a constant.

We present examples of distinguished representations allowing for a quadratic evaluation of the Yangian of *so* or *sp* type and consider the characteristic identities. The investigation of the additional condition in the general case is subject of further study.

Now we show that the fundamental representations $T = \rho_f$ of *so* and *sp* is just an example of the quadratic evaluation of the Yangian.

Proposition 3. *The set of n^2 matrices $T(G^a_{\ b})$ (here indices a and b enumerate matrices) with elements*

$$T^c_{\ d}(G^a_{\ b}) = -(P - \epsilon K)^{ac}_{bd} \equiv G^{ac}_{bd} \,, \tag{4.37}$$

define the fundamental representation T of generators $G^a_{\ b}$ for $SO(n)$ $\epsilon = +1$ and $Sp(n)$ $\epsilon = -1$ cases. Generators (4.37) satisfy the cubic characteristic identity

$$G^3 + (1 - n\epsilon)G^2 - G = (1 - n\epsilon)I \,. \tag{4.38}$$

The corresponding L-operator which solves the RLL equation (3.13) has the form (cf. (4.35))

$$L(u) = I + \frac{1}{u}(\beta I - G) + \frac{1}{2u^2}\Big(G^2 - 2\beta\, G - I\Big) \,, \tag{4.39}$$

where $\beta = n/2 - \epsilon$.

Proof. It is not hard to check that the generators (4.37) satisfy the conditions (3.22) and (3.19) which in our notations (4.37) can be written in the form

$$K_{12}\,(G_{13} + G_{23}) = 0 = (G_{13} + G_{23})\,K_{12} \,,$$
$$[G_{13},\, G_{23}] + [G_{12},\, G_{23}] = 0 \,.$$

Finally for the matrix (4.37) we have the relations

$$G^2 = I + (n\epsilon - 2)\,K \,, \quad G^3 = G + (n - 2\epsilon)\,(n\epsilon - 1)K \,, \tag{4.40}$$

and the characteristic identity (4.38) follows immediately from (4.40).

We search the solution of (3.13) as an L-operator acting in the space $V \otimes V$, where V – the space of fundamental representation of *so* (or *sp*). In view of the Yang–Baxter equation (2.4), it is clear that this L-operator is given by the R-matrix (2.8) and can be represented in the form

$$L(u) = \frac{1}{u^2}\, R(u) = I + \frac{(\beta I - G)}{u} + \frac{\beta P}{u^2} = I + \frac{(\beta I - G)}{u} + \frac{1}{2u^2}\Big(G^2 - 2\beta\, G - I\Big)\,,$$

where we have used the definition of G (4.37) (written as $P = \epsilon K - G$) and the identities (4.40). □

Remark. The quadratic Casimir operator in the representation (4.37) is

$$\frac{1}{2}\, T^c_{\ d}(\mathrm{Tr}(G^2)) = \frac{1}{2}\,(P - \epsilon K)^{ac}_{br}(P - \epsilon K)^{br}_{ad} = (n - \epsilon)\delta^c_{\ d}\,. \tag{4.41}$$

As a further example, consider the monodromy built as the product of the above spinor L operators (4.34) $L_{12}(u) = L_1(u - \mu_1)L_2(u - \mu_2)$ with multiplication in the fundamental representation space V_f and tensor product of two copies of the spinor space, i.e. it is acting in $V_f \otimes V_{12}$, $V_{12} = V_s \otimes V_s$. The monodromy obeys the *RLL* relation and in this way we obtain an obvious example of the second order evaluation of $Y(\mathcal{G})$. Examples of higher order evaluations are given by monodromies with more factors. In Sect. 6 we shall consider the fusion procedure with projection from the tensor product of spinor representations to the fundamental one resulting in the fundamental R matrix.

As the third example we consider the representations of *so* and *sp* of Jordan–Schwinger (JS) type where the generators are built from Heisenberg pairs in the form (cf. (4.32))

$$M_{ab} = \epsilon(x_a \partial_b - \epsilon\, x_b\, \partial_a) = (\epsilon\, x_a\, \partial_b - x_b\, \partial_a)\,, \tag{4.42}$$

where

$$\partial_a\, x_b - \epsilon\, x_b\, \partial_a = \varepsilon_{ab}\,, \quad \partial_a\, \partial_b = \epsilon\, \partial_b\, \partial_a\,, \quad x_a\, x_b = \epsilon\, x_b\, x_a\,, \tag{4.43}$$

and for *so* and *sp* we have $\epsilon = +1$ and $\epsilon = -1$ respectively. Contrary to the realization (4.30), (4.31), for the *so* case $\{x_a, \partial_b\}$ are bosonic and for the *sp* case $\{x_a, \partial_b\}$ are fermionic. JS type representations have been considered for the $s\ell$ algebras e.g. in [16] and for the *so* and *sp* algebras in [17]. The defining relations and the (anti)symmetry condition are the same as in (3.23)

$$[M_{ab},\ M_{cd}] = \varepsilon_{cb} M_{ad} + \varepsilon_{db} M_{ca} + \varepsilon_{ca} M_{db} + \varepsilon_{ad} M_{bc}\,, \qquad M_{ab} = -\epsilon M_{ba}\,. \tag{4.44}$$

Let us introduce the operator

$$H = (\varepsilon^{bc} x_b\, \partial_c) = x_b\, \partial^b = \epsilon(\varepsilon^{bc}\partial_c x_b - n)\,,$$

which has the properties

$$H\, x_a = x_a\,(H + 1)\,, \quad H\, \partial_a = \partial_a\,(H - 1)\,.$$

Using (4.42) we find

$$\begin{aligned}
(M^2)_{ad} &= M_{ab} M^b_{\ d} = \varepsilon^{bc} M_{ab} M_{cd} = \varepsilon^{bc}(\epsilon\, x_a\, \partial_b - x_b\, \partial_a)(\epsilon\, x_c\, \partial_d - x_d\, \partial_c) = \\
&= (\epsilon n - 2)\, x_a\, \partial_d + \varepsilon_{ad}\, H + (H - 1)(x_a\, \partial_d + \epsilon x_d\, \partial_a) - x_a x_d \partial_b \partial^b - x_b x^b \partial_a \partial_d = \\
&= (\epsilon n + 2H - 4)\, x_a\, \partial_d + \varepsilon_{ad}\, H - \epsilon(H - 1) M_{ad} - x_a x_d \partial_b \partial^b - x_b x^b \partial_a \partial_d\,,
\end{aligned}$$

and for *so* and *sp* Lie algebras in the considered representations (4.42)) we have the following explicit form of the quadratic Casimir operator

$$\mathrm{Tr}(M^2) = M^d_{\ b} M^b_{\ d} = \varepsilon^{da}\,(M^2)_{ad} = 2(n - 2\epsilon)\, H + 2\epsilon H^2 - 2\epsilon x^2 \partial^2\,, \tag{4.45}$$

where $x^2 = x_b x^b$, $\partial^2 = \partial_d \partial^d$.

In the *so* case we have here the finite dimensional representations of integer spins m and they are spanned by the homogeneous harmonic polynomials $P_m(x)$

$$\partial^2 P_m(x) = 0\,, \quad H\, P_m(x) = m\, P_m(x)\,.$$

Then by using (4.45) we obtain

$$\frac{1}{2}\mathrm{Tr}(M^2)\, P_m(x) = \Big((n-2\epsilon)\, m + \epsilon\, m^2\Big)\, P_m(x)\,. \tag{4.46}$$

In spin $m = 1$ case we obtain the eigenvalue of the Casimir operator as $(n - \epsilon)$ which coincides with the value for the fundamental representation (4.41) as expected.

Proposition 4. *In the JS type representation (4.42) the characteristic identity for the generators of so and sp algebras is*

$$\begin{gathered} M^3_{ad} = (n-\epsilon)M^2_{ad} + (2-\epsilon n)M_{ad} - \tfrac{1}{2}\mathrm{Tr}(M^2)(\varepsilon_{ad} - \epsilon M_{ad})\,, \\ M^3 + (\epsilon - n)M^2 + (\epsilon n - 2)M + \tfrac{1}{2}\mathrm{Tr}(M^2)(I - \epsilon M) = 0\,. \end{gathered} \tag{4.47}$$

Proof. Multiply the matrix M^2 as written above by M. After straightforward calculations we get (4.47). □

Now we compare this formula (4.47) with the characteristic identity (4.38) found for the case of the fundamental (defining) representation (4.37). First we note that (4.38) is transformed to (4.47) if we redefine $G \to \epsilon M$. Then (4.47) gives (4.38) for the choice of the value of the Casimir operator

$$\frac{1}{2}\mathrm{Tr}(M^2) = (n - \epsilon)\,,$$

which is compatible with the spectrum (4.46) for $m = 1$ and with (4.41).

The L-operator can be written in the form

$$L(u) = (u-\lambda)(u-\mu)I + (u-\sigma)M + M^2 \tag{4.48}$$

In Section 6 we shall show that this form is obtained by fusion from YB operators acting in the product of the spinorial and the JS representation. It can be checked that the *RLL* relation with the fundamental R matrix (3.13) is fulfilled for particular values of the parameters.

5. Spinorial and metaplectic Yang–Baxter operator $\check{\mathfrak{R}}$

Let $\mathcal{A}$ be the algebra of (fermionic or bosonic) oscillators with the defining relations (4.30) and denote by $\mathcal{G}$ the Lie algebra $so(n)$ or $sp(n)$. Consider the L-operator in the general form

$$L(u) = u\, I + \frac{1}{2}\,\rho(G^a_{\ b}) \otimes G^b_{\ a} = u\, I - \frac{1}{2}\, c^{[a}\, c^{b)} \otimes G_{ab} \;\in\; \mathcal{A} \otimes \mathcal{U}_{\mathcal{G}}\,, \tag{5.1}$$

where u is the spectral parameter, $G^b_{\ a}$ are the generators of the Lie algebra $\mathcal{G}$, $\mathcal{U}_{\mathcal{G}}$ denotes the enveloping algebra of $\mathcal{G}$. $\rho(G^a_{\ b}) \in \mathcal{A}$ denote the image of the generators $G^b_{\ a} \in \mathcal{G}$ in the oscillator representation (4.32). Note that if we evaluate the second factor in (5.1) in the fundamental representation (4.37) then the L-operator (5.1) takes the form (4.34):

$$L^a_{\ b}(u) = u\,\delta^a_b + \frac{1}{2}\,\rho(G^c_{\ d})\,T^a_b(G^d_{\ c}) = u\,\delta^a_b + \frac{1}{2}\,\rho(G^c_{\ d})(\epsilon K^{da}_{\ cb} - P^{da}_{\ cb}) = u\,\delta^a_b - \rho(G^a_{\ b})\,,$$

where we have used the symmetry properties (3.23) of generators $G^c_{\ d}$.

Now we consider the new version of the *RLL* relation (cf. (3.13))

$$\check{\mathfrak{R}}_{12}(u)\,L_1(u+v)\,L_2(v) = L_1(v)\,L_2(u+v)\,\check{\mathfrak{R}}_{12}(u)\,, \tag{5.2}$$

different from (3.13) because $\check{\mathfrak{R}}(u)$ is not the fundamental R matrix, but rather an element of the algebra $\mathcal{A}\otimes\mathcal{A}$, $\check{\mathfrak{R}}(u) = \mathsf{P}\cdot\mathfrak{R}(u) \in \mathcal{A}\otimes\mathcal{A}$. Operator P permutes the factors in the tensor product $\mathcal{A}\otimes\mathcal{A}$ and the operators $\check{\mathfrak{R}}_{12}(u)$, L_1, L_2 are elements of $\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{U}_{\mathcal{G}}$: $\check{\mathfrak{R}}_{12}(u) = \check{\mathfrak{R}}(u)\otimes I_{\mathcal{U}_{\mathcal{G}}}$,

$$\begin{aligned} L_1(v) &= v\,I - \tfrac{1}{2}\,c^{[a}\,c^{b)} \otimes I_{\mathcal{A}} \otimes G_{ab} \equiv v\,I - \tfrac{1}{2}\,c_1^{[a}\,c_1^{b)}\,G_{ab}\,, \\ L_2(v) &= v\,I - \tfrac{1}{2}\,I_{\mathcal{A}} \otimes c^{[a}\,c^{b)} \otimes G_{ab} \equiv v\,I - \tfrac{1}{2}\,c_2^{[a}\,c_2^{b)}\,G_{ab}\,. \end{aligned} \tag{5.3}$$

Here $I_{\mathcal{A}}$ is the unit element in $\mathcal{A}$ and I is the unit element in $\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{U}_{\mathcal{G}}$. We introduce the short-hand notations c_1^a for oscillators c^a in the first factor of $\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{U}_{\mathcal{G}}$ and c_2^a in the second factor of $\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{U}_{\mathcal{G}}$.

Since the L-operator is fixed in (5.1), we interpret (5.2) as the defining equation for the operator $\check{\mathfrak{R}}_{12}(u)$.

Introduce the basis in the algebra $\mathcal{A}$ of fermionic or bosonic oscillators which is formed by the unit element $I_{\mathcal{A}}$ and the (anti-)symmetrized products

$$c^{[a_1}\cdots c^{a_k)} = \sum_{\sigma\in S_k} (-\epsilon)^{p(\sigma)}\,c^{a_{\sigma(1)}}\cdots c^{a_{\sigma(k)}} \quad (k=1,2,\dots)\,, \tag{5.4}$$

where S_k is a symmetric group, the sum is performed over all permutations $\sigma\in S_k$ of k indices $(1,2,\dots,k)$ and $p(\sigma)$ is the parity of σ. Under the transposition of any two indices a_i and a_j, the basis elements (5.4) have the (anti)symmetric property

$$c^{[a_1}\cdots c^{a_i}\cdots c^{a_j}\cdots c^{a_k)} = -\epsilon\,c^{[a_1}\cdots c^{a_j}\cdots c^{a_i}\cdots c^{a_k)}\,.$$

Preparing the proof of the following theorem where we encounter the products of elements of the (anti-)symmetrized basis we study the expansion of such products into the elements of this (anti-)symmetrized basis. This is conveniently done by using generating functions (see, e.g., [15]). We introduce the auxiliary (anti-)commuting variables $\kappa^a, \kappa'^a, \dots$ such that

$$\begin{gathered} \kappa_a = \varepsilon_{ab}\,\kappa^b\,, \quad \kappa^a\,\kappa^b = -\epsilon\kappa^b\,\kappa^a\,, \quad \kappa^a\,\kappa'^b = -\epsilon\kappa'^b\,\kappa^a\,, \\ \kappa^a\,c^b = -\epsilon c^b\,\kappa^a\,, \quad \kappa'^a\,c^b = -\epsilon\,c^b\,\kappa'^a\,, \end{gathered}$$

and define the operators $\partial^b = \partial/\partial\kappa_b$, $\partial'^b = \partial/\partial\kappa'_b$, with relations

$$\begin{gathered} [\partial^b,\,\kappa_a]_\epsilon = [\partial'^b,\,\kappa'_a]_\epsilon = \delta^b_a\,, \quad [\partial_b,\,\kappa^a]_\epsilon = [\partial'_b,\,\kappa'^a]_\epsilon = \epsilon\,\delta^a_b\,, \\ [\partial^b,\,\kappa^a]_\epsilon = [\partial'^b,\,\kappa'^a]_\epsilon = \varepsilon^{ab}\,, \quad [\partial'^b,\,\partial^a]_\epsilon = [\partial'^b,\,\kappa_a]_\epsilon = [\partial^b,\,\kappa'_a]_\epsilon = 0\,, \end{gathered} \tag{5.5}$$

where $[A,\,B]_\epsilon = A\,B + \epsilon\,B\,A$ and according to our agreement we have $(\partial_i)_b = \varepsilon_{ba}\,\partial_i^a = \epsilon\,\partial/\partial\kappa_i^b$. The scalar products of variables κ^b with themselves and with generators of $\mathcal{A}$ are

$$\begin{gathered} (\kappa\cdot\kappa') \equiv \kappa_a\kappa'^a = \varepsilon_{ab}\,\kappa^b\kappa'^a = \epsilon\,\kappa^a\kappa'_a = -\kappa'_a\kappa^a = -(\kappa'\cdot\kappa)\,, \\ (\kappa\cdot c) = \kappa_a c^a = -c_a\kappa^a = -(c\cdot\kappa)\,, \quad (\kappa\cdot\kappa) = 0\,. \end{gathered}$$

Thus, the variables κ^a, κ'^a have the same grading as c^a's, i.e., they are anti-commuting variables in the *so* case ($\epsilon = +1$) and commuting ones in the *sp* case ($\epsilon = -1$). Then derivatives of the expression $(\kappa \cdot c)^k$ with respect to $\partial^{a_k} \dots \partial^{a_1}$ will give the elements of the symmetrized basis of $\mathcal{A}$:

$$\partial^{a_1} \dots \partial^{a_k} (\kappa \cdot c)^k = k!\, c^{[a_1} \dots c^{a_k)} ,$$

and it also can be written in the form

$$c^{[a_1} \dots c^{a_k)} = \partial^{a_1} \dots \partial^{a_k} \; e^{(\kappa \cdot c)}|_{\kappa=0} .$$

Then we consider the product of two basis elements of algebra $\mathcal{A}$ (see (5.18))

$$c^{[a_1} \dots c^{a_k)} \cdot c^{[a} c^{b)} = \partial^{a_1} \dots \partial^{a_k} e^{(\kappa \cdot c)}\, \partial'^a \partial'^b e^{(\kappa' \cdot c)}\Big|_{\kappa=\kappa'=0} =$$
$$= \partial^{a_1 \dots a_k} \partial'^{ab} e^{(\kappa \cdot c)}\; e^{(\kappa' \cdot c)}\Big|_{\kappa=\kappa'=0} = \partial^{a_1 \dots a_k} \partial'^{ab}\, e^{((\kappa+\kappa') \cdot c)} e^{(\frac{\epsilon}{2}\kappa' \cdot \kappa)}\Big|_{\kappa=\kappa'=0} .$$

Here we denoted $\partial^{a_1 \dots a_k} \equiv \partial^{a_1} \dots \partial_i^{a_k}$ $(i = 1, 2)$, and used the Baker–Hausdorff formula

$$e^{(\kappa \cdot c)}\; e^{(\kappa' \cdot c)} = e^{(\kappa \cdot c) + (\kappa' \cdot c) + \frac{1}{2}[(\kappa \cdot c),\, (\kappa' \cdot c)]} ,$$
$$[(\kappa \cdot c), (\kappa' \cdot c)] = \kappa'_b \kappa_a [c^a, c^b]_\epsilon = \kappa'_b \kappa_a\, \varepsilon^{ab} = \epsilon\, (\kappa' \cdot \kappa) .$$

Then we change the variables $\{\kappa, \kappa'\}$ to $\{\bar{\kappa} = \kappa + \kappa', \kappa'\}$ and it leads to $(\kappa' \cdot \kappa) = (\kappa' \cdot \bar{\kappa})$ while the κ-derivatives are transformed as following

$$\partial \to \bar{\partial},\ \partial' \to \bar{\partial} + \partial' .$$

Finally this change of variables results in (for simplicity we remove bar for variables $\bar{\kappa}$, etc.)

$$c^{[a_1} \dots c^{a_k)} c^{[a} c^{b)} = \partial^{a_1 \dots a_k} (\partial^a + \partial'^a)(\partial^b + \partial'^b)\; e^{(\kappa \cdot c)} e^{\frac{\epsilon}{2}(\kappa' \cdot \kappa)}\Big|_{\kappa=\kappa'=0} =$$
$$= \partial^{a_1 \dots a_k} [(\partial^a \partial^b e^{(\kappa \cdot c)} - \epsilon\, \partial^b e^{(\kappa \cdot c)} \partial'^a + \partial^a e^{(\kappa \cdot c)} \partial'^b + e^{(\kappa \cdot c)} \partial'^a \partial'^b]\; e^{\frac{\epsilon}{2}(\kappa' \cdot \kappa)}\Big|_{\kappa=\kappa'=0} =$$
$$= \partial^{a_1 \dots a_k} \left[\partial^a \partial^b + \frac{1}{2}(\epsilon \kappa^a \partial^b - \kappa^b \partial^a) + \frac{1}{4}\kappa^a \kappa^b \right] e^{(\kappa \cdot c)}\Big|_{\kappa=0} . \tag{5.6}$$

In the same way we obtain

$$c^{[a} c^{b)} c^{[a_1} \dots c^{a_k)} = \partial'^a \partial'^b e^{(\kappa' \cdot c)}\, \partial^{a_1} \dots \partial^{a_k} e^{(\kappa \cdot c)}\Big|_{\kappa=\kappa'=0} =$$
$$= \partial^{a_1 \dots a_k} \partial'^{ab} e^{(\kappa' \cdot c)}\; e^{(\kappa \cdot c)}\Big|_{\kappa=\kappa'=0} = \partial^{a_1 \dots a_k} \partial'^{ab}\, e^{((\kappa+\kappa') \cdot c)} e^{-\frac{\epsilon}{2}(\kappa' \cdot \kappa)}\Big|_{\kappa=\kappa'=0} =$$
$$= \partial^{a_1 \dots a_k} [(\partial^a \partial^b e^{(\kappa \cdot c)} - \epsilon\, \partial^b e^{(\kappa \cdot c)} \partial'^a + \partial^a e^{(\kappa \cdot c)} \partial'^b + e^{(\kappa \cdot c)} \partial'^a \partial'^b]\; e^{-\frac{\epsilon}{2}(\kappa' \cdot \kappa)}\Big|_{\kappa=\kappa'=0} =$$
$$= \partial^{a_1 \dots a_k} \left[\partial^a \partial^b - \frac{1}{2}(\epsilon \kappa^a \partial^b - \kappa^b \partial^a) + \frac{1}{4}\kappa^a \kappa^b \right] e^{(\kappa \cdot c)}\Big|_{\kappa=0} . \tag{5.7}$$

The difference between expressions (5.6) and (5.7) appears only in the sign of terms which are linear in κ^d. So we denote

$$[\pm]_i^{ab} = \left[\partial_i^a \partial_i^b \pm \frac{1}{2}(\epsilon \kappa_i^a \partial_i^b - \kappa_i^b \partial_i^a) + \frac{1}{4}\kappa_i^a \kappa_i^b \right] , \tag{5.8}$$

where the index i refers the first or second factor in the tensor product $\mathcal{A} \otimes \mathcal{A}$.

Proposition 5. *The so or sp invariant R-operator $\check{\mathfrak{R}}_{12}(u)$ which satisfies the RLL-relations (5.2) with the L-operator given in (5.1) has the form*

$$\check{\mathfrak{R}} = \sum_k \frac{r_k(u)}{k!} \sum_{\vec{a},\vec{b}} \varepsilon_{a_1 b_1} \cdots \varepsilon_{a_k b_k} \, c^{[a_1} \ldots c^{a_k)} \otimes c^{[b_1} \ldots c^{b_k)} \,, \tag{5.9}$$

where ε_{ab} is the so or sp invariant metric and the coefficient functions $r_k(u)$ are written separately for even and odd k as

$$\begin{aligned} r_{2m}(u) &= \frac{2^{2m}\, \Gamma(m + \epsilon \frac{u}{2})}{\Gamma(m + 1 - \epsilon \frac{(u+n)}{2})} A_0(u) \,, \\ r_{2m+1}(u) &= \frac{2^{2m}\, \Gamma(m + \frac{1}{2} + \epsilon \frac{u}{2})}{\Gamma(m + \frac{1}{2} - \epsilon \frac{(u+n)}{2})} A_1(u) \,, \end{aligned} \tag{5.10}$$

Here $A_0(u)$, $A_1(u)$ are arbitrary functions and $\epsilon = +1$ for the so case and $\epsilon = -1$ for the sp case.

Proof.

1st part: Extracting the defining conditions from the RLL relation

After the substitution of (5.1) into (5.2) the quadratic parts in spectral parameters (proportional to v^2 and uv) in both sides of (5.2) are canceled. The linear in v term gives the symmetry condition

$$[(\rho(G^a_{\ b}) \otimes I_{\mathcal{A}} + I_{\mathcal{A}} \otimes \rho(G^a_{\ b})),\, \check{\mathfrak{R}}(u)] = 0 \quad \Leftrightarrow \quad [(c_1^{[a} c_1^{b)} + c_2^{[a} c_2^{b)}),\, \check{\mathfrak{R}}(u)] = 0 \,, \tag{5.11}$$

which indicate that $\check{\mathfrak{R}}(u)$ is invariant under the adjoint action of *so* and *sp* algebras. The terms in (5.2) which contains no v leads to the condition:

$$\begin{aligned} &u\Big(\check{\mathfrak{R}}_{12}(u)\, c_2^{[a} c_2^{b)} - c_1^{[a} c_1^{b)}\, \check{\mathfrak{R}}_{12}(u)\Big) G_{ab} - \\ -&\frac{1}{2}\Big(\check{\mathfrak{R}}_{12}(u)\, c_1^{[a} c_1^{e)}\, c_2^{[f} c_2^{b)} - c_1^{[a} c_1^{e)}\, c_2^{[f} c_2^{b)}\, \check{\mathfrak{R}}_{12}(u)\Big) G_{ae} G_{fb} = 0 \,. \end{aligned} \tag{5.12}$$

Now we substitute $G_{ae}G_{fb} = \frac{1}{2}\big([G_{ae},\, G_{fb}]_- + [G_{ae},\, G_{fb}]_+\big)$ and use commutation relations (3.23). Then the condition (5.12) is written in the form

$$\Big(u\big[\check{\mathfrak{R}}_{12}(u)\, c_2^{[ab)} - c_1^{[ab)}\, \check{\mathfrak{R}}_{12}(u)\big] - \varepsilon_{fe}\, X^{[ae)[fb)}\Big)\, G_{ab} = \tfrac{1}{4}\, X^{[ae)[fb)}\, [G_{ae},\, G_{fb}]_+ \,, \tag{5.13}$$

where we denote $c^{[ab)} = c^{[a} c^{b)}$ and

$$X^{[ae)[fb)} = \big(\check{\mathfrak{R}}_{12}(u)\, c_1^{[ae)}\, c_2^{[fb)} - c_1^{[ae)}\, c_2^{[fb)}\, \check{\mathfrak{R}}_{12}(u)\big)$$

Following [15] we search the solution of (5.13) as the solution of two equations

$$\Big(u\big[\check{\mathfrak{R}}_{12}(u)\, c_2^{[ab)} - c_1^{[ab)}\, \check{\mathfrak{R}}_{12}(u)\big] - \varepsilon_{fe}\, X^{[ae)[fb)}\Big)\, G_{ab} = 0 \,, \tag{5.14}$$

$$X^{[ae)[fb)}\, [G_{ae},\, G_{fb}]_+ = 0 \,. \tag{5.15}$$

Below it will be shown that equation (5.15) leads to the condition

$$[G_{[ae} G_{f)b}]_+ = 0 \,, \tag{5.16}$$

where $[aef)$ denotes the (anti)symmetrization over three indices. This condition is not valid for the enveloping algebra $\mathcal{U}_{\mathcal{G}}$. Therefore we need to restrict our consideration to appropriate representations of $\mathcal{U}_{\mathcal{G}}$ for which (5.16) is fulfilled. Note that the operators G_{ab} with symmetric property $G_{ab} = -\epsilon G_{ba}$ (cf. (3.23)) satisfy the identity $[G_{[ae}G_{f)b}]_+ = [G_{a[e}G_{fb)}]_+$ and thus (5.16) is valid for (anti)symmetrization of any three of four indices (a, e, f, b). For the *so* case the condition (5.16) was discussed in [15].

The Yangian-type condition (5.14) fixes the operator $\check{\mathfrak{R}}$ and we start to solve it now. We apply the method developed in [15] for *so* case. As we will see below this method works perfectly also for *sp* case.

2nd part: The symmetry condition and the generating function form

The symmetry condition (5.11) implies that the spinorial R-operator decomposes into the sum (5.9) over invariants which are tensor products of the basis elements (5.4) of the algebra $\mathcal{A}$. We write (5.9) in the concise form

$$\check{\mathfrak{R}} = \sum_k \frac{r_k(u)}{k!} \sum_{\vec{a},\vec{b}} \varepsilon_{\vec{a},\vec{b}}\; c_1^{[a_1\dots a_k)}\; c_2^{[b_1\dots b_k)}\,, \tag{5.17}$$

where we have denoted again the first and the second factors of tensor the product by subscripts 1 and 2 and have introduced the shorthand notations

$$\varepsilon_{\vec{a},\vec{b}} = \varepsilon_{a_1b_1}\cdots\varepsilon_{a_kb_k}\,, \quad c_1^{[a_1\dots a_k)} = c_1^{[a_1}\cdots c_1^{a_k)}\,, \quad c_2^{[a_1\dots a_k)} = c_2^{[a_1}\cdots c_2^{a_k)}\,.$$

After substituting (5.17) the Yangian-type condition (5.14) takes the form

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{r_k(u)}{k!} \sum_{\vec{a},\vec{b}} \varepsilon_{\vec{a},\vec{b}} \Big(u\left[c_1^{[a_1\dots a_k)} \cdot c_2^{[b_1\dots b_k)} c_2^{[ab)} - c_1^{[ab)} c_1^{[a_1\dots a_k)} \cdot c_2^{[b_1\dots b_k)} \right] - \\ &- \varepsilon_{fe} \left[c_1^{[a_1\dots a_k)} c_1^{[ae)} \cdot c_2^{[b_1\dots b_k)} c_2^{[fb)} - c_1^{[ae)} c_1^{[a_1\dots a_k)} \cdot c_2^{[fb)} c_2^{[b_1\dots b_k)} \right] \Big) = 0\,.\end{aligned} \tag{5.18}$$

Using the representations (5.6), (5.7) for the factors in the tensor products appearing in the equation (5.18) we write this equation as

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{r_k(u)}{k!}\, \partial_\lambda^k\, e^{\lambda(\partial_2\cdot\partial_1)} \Big(u \left([+]_2^{ab} - [-]_1^{ab} \right) - \\ &- \varepsilon_{fe} \left([+]_1^{ae}\cdot[+]_2^{fb} - [-]_1^{ae}\cdot[-]_2^{fb} \right) \Big) e^{(\kappa_1\cdot c_1)} e^{(\kappa_2\cdot c_2)} \Big|_{\lambda=\kappa_1=\kappa_2=0} = 0\,,\end{aligned} \tag{5.19}$$

where we have applied also the formula

$$\partial_\lambda^k\, e^{\lambda(\partial_2\cdot\partial_1)}\Big|_{\lambda=0} = (\varepsilon_{ab}\partial_2^b\,\partial_1^a)^k = \varepsilon_{a_1b_1}\dots\varepsilon_{a_kb_k}\partial_1^{a_1\dots a_k}\;\partial_2^{b_1\dots b_k}\,.$$

Our task is now to commute in (5.19) all derivatives ∂_i^a to the right and then put to zero all variables κ_i^a appearing on left of the derivatives ∂_i^a. Taking into account (5.5) we have the rules

$$e^{\lambda(\partial_2\cdot\partial_1)}\,\kappa_1^a = (\kappa_1^a + \lambda\,\partial_2^a)\, e^{\lambda(\partial_2\cdot\partial_1)}\,, \quad e^{\lambda(\partial_2\cdot\partial_1)}\,\kappa_2^a = (\kappa_2^a + \epsilon\lambda\,\partial_1^a)\, e^{\lambda(\partial_2\cdot\partial_1)}\,, \tag{5.20}$$

3rd part: Evaluating the Yangian-type condition

Applying these rules to the first term in (5.19) we obtain

$$e^{\lambda(\partial_2\cdot\partial_1)}\Big([+]_2^{ab}-[-]_1^{ab}\Big)|_{\kappa=0}=\Big(\Big[\partial_2^a\partial_2^b+\lambda\tfrac{1}{2}(\partial_1^a\partial_2^b-\epsilon\,\partial_1^b\partial_2^a)+\lambda^2\tfrac{1}{4}\partial_1^a\partial_1^b\Big]-$$
$$-\Big[\partial_1^a\partial_1^b-\lambda\tfrac{1}{2}(\epsilon\,\partial_2^a\partial_1^b-\partial_2^b\partial_1^a)+\lambda^2\tfrac{1}{4}\partial_2^a\partial_2^b\Big]\Big)e^{\lambda(\partial_2\cdot\partial_1)}=$$
$$=(\lambda^2\tfrac{1}{4}-1)\Big(\partial_1^a\partial_1^b-\partial_2^a\partial_2^b\Big)e^{\lambda(\partial_2\cdot\partial_1)}\,, \tag{5.21}$$

where after reordering we put to zero all variables κ_i^a appearing on left hand side with respect to derivatives ∂_i^a.

The second term in (5.19) contains the expression

$$Y^{[ae)[fb)}(\kappa_1,\kappa_2)\equiv\Big([+]_1^{ae}\cdot[+]_2^{fb}-[-]_1^{ae}\cdot[-]_2^{fb}\Big)=$$
$$=\left[\partial_1^a\partial_1^e+\frac{1}{4}\kappa_1^a\kappa_1^e\right](\epsilon\kappa_2^f\,\partial_2^b-\kappa_2^b\partial_2^f)+(\epsilon\kappa_1^a\partial_1^e-\kappa_1^e\partial_1^a)\left[\partial_2^f\partial_2^b+\frac{1}{4}\kappa_2^f\kappa_2^b\right],$$

where we substituted (5.8). Further we act to this expression by the operator $e^{\lambda(\partial_2\cdot\partial_1)}$ from the left and then move $e^{\lambda(\partial_2\cdot\partial_1)}$ to the right with the help of rules (5.20). The result is

$$e^{\lambda(\partial_2\cdot\partial_1)}\cdot Y^{[ae)[fb)}(\kappa_1^a,\kappa_2^a)=Y^{[ae)[fb)}(\nabla_2^a,\nabla_1^a)\cdot e^{\lambda(\partial_2\cdot\partial_1)}\,,$$

where $\nabla_1^a=(\kappa_2^a+\epsilon\lambda\,\partial_1^a)$, $\nabla_2^a=(\kappa_1^a+\lambda\,\partial_2^a)$. After reordering of variables κ_i^a and ∂_i^b and canceling all κ_i^a appearing at the left we deduce

$$Y^{[ae)[fb)}(\nabla_2^a,\nabla_1^a)\big|_{\kappa_i=0}=\Big[\big(\epsilon\nabla_1^f\partial_2^b-\nabla_1^b\partial_2^f\big)\partial_1^a\partial_1^e+\tfrac{1}{4}\nabla_2^a\nabla_2^e\big(\epsilon\nabla_1^f\partial_2^b-\nabla_1^b\partial_2^f\big)+$$
$$+(\epsilon\nabla_2^a\partial_1^e-\nabla_2^e\partial_1^a)\,\partial_2^f\partial_2^b+\tfrac{1}{4}\nabla_1^f\nabla_1^b\big(\epsilon\nabla_2^a\partial_1^e-\nabla_2^e\partial_1^a\big)\Big]\Big|_{\kappa_i=0}=$$
$$=\lambda\Big[\partial_1^{ae}+\tfrac{1}{4}\lambda^2\partial_2^{ae}\Big](\partial_1^f\partial_2^b-\epsilon\,\partial_1^b\partial_2^f)+\lambda(\epsilon\,\partial_2^a\partial_1^e-\partial_2^e\partial_1^a)\Big[\partial_2^{fb}+\tfrac{1}{4}\lambda^2\,\partial_1^{fb}\Big]+$$
$$+\tfrac{1}{4}\lambda^2\Big((\varepsilon^{ef}\,\partial_2^{ab}+\varepsilon^{af}\,\partial_2^{be}+\varepsilon^{eb}\,\partial_2^{fa}+\varepsilon^{ab}\,\partial_2^{ef})-(2\to1)\Big)\,.$$

Here the formulas $\partial_i^a\partial_i^e\kappa_i^f\Big|_{\kappa_i=0}=\varepsilon^{fe}\partial_i^a-\varepsilon^{af}\partial_i^e$ are helpful.

Note that we obtain the expression for $Y^{[ae)[fb)}(\nabla_2^a,\nabla_1^a)\big|_{\kappa_i=0}$ which is (anti-)symmetric under the permutation of any three indices out of four $\{aefb\}$. This implies that the same index symmetry holds for the operator $X^{[ae)[fb)}$ in (5.15). This fact proves that the condition (5.16) follows from (5.15).

Finally, after contraction with ε_{fe}, we obtain

$$\varepsilon_{fe}\,Y^{[ae)[fb)}(\nabla_2^a,\nabla_1^a)\Big|_{\kappa=0}=\left[\frac{\lambda^2}{4}(n-2\epsilon)-\epsilon\lambda\Big(1+\frac{\lambda^2}{4}\Big)(\partial_2\cdot\partial_1)\right](\partial_2^{ab}-\partial_1^{ab})\,,$$

and together with (5.21) we write the equation (5.19) as following

$$\sum_{k=0}^{\infty}\frac{\eta_k\,r_k(u)}{k!}\,W_k((\partial_2\cdot\partial_1))\,(\partial_2^{ab}-\partial_1^{ab})\,e^{(\kappa_1\cdot c_1+\kappa_2\cdot c_2)}\Big|_{\kappa_1=\kappa_2=0}=0\,, \tag{5.22}$$

where we have introduced the notation $W_k\big((\partial_2\cdot\partial_1)\big)$ for the operator

$$W_k\big((\partial_2\cdot\partial_1)\big)=\partial_\lambda^k\left(u\left(1-\frac{\lambda^2}{4}\right)-\frac{\lambda^2}{4}(n-2\epsilon)+\epsilon\lambda\Big(1+\frac{\lambda^2}{4}\Big)(\partial_2\cdot\partial_1)\right)\,e^{\lambda(\partial_2\cdot\partial_1)}\Big|_{\lambda=0}=$$
$$=\partial_\lambda^k\,\Big(u-\tfrac{\lambda^2}{4}(n-2\epsilon+u)+\epsilon\lambda(1+\tfrac{\lambda^2}{4})\partial_\lambda\Big)\,e^{\lambda(\partial_2\cdot\partial_1)}\Big|_{\lambda=0}=$$

$$= \left((u+\epsilon k)\partial_\lambda^k + \frac{k(k-1)}{4}(n-\epsilon k+u)\partial_\lambda^{k-2} \right) e^{\lambda(\partial_2\cdot\partial_1)}\Big|_{\lambda=0} . \tag{5.23}$$

By substitution of the operator (5.23) into (5.22) one obtains the recurrent relation for the coefficients $r_k(u)$:

$$r_{k+2}(u) = \frac{4\,(k+\epsilon\, u)}{((k+2)-\epsilon(u+n))}\, r_k(u) \, . \tag{5.24}$$

The solution of (5.24) separates for even and odd k and it is now not hard to check that it is given by the formulas (5.10). □

In the $so(n)$ case ($\epsilon = +1$) our results (e.g. relation (5.24)) coincide with the results of [15] after rescaling of generators $c^a \to \sqrt{2}c^a$ which gives the standard definition of the Clifford algebra $c^a c^b + c^b c^a = 2\,\epsilon^{ab}$.

We have encountered the additional condition (5.16). Representations for the generators of which it is fulfilled result in L operators linear in u obeying the RLL relation (5.2) with the spinorial YB operator $\mathfrak{R}$ (5.9).

Proposition 6. *The representations of Jordan–Schwinger type (4.42), (4.43)*

$$M_{ab} = \epsilon(x_a\partial_b - \epsilon\, x_b\, \partial_a) = (\epsilon\, x_a\, \partial_b - x_b\, \partial_a) \, ,$$

obey the additional condition (5.16), i.e.

$$[M_{[ae} M_{f)b}]_+ = 0 \, .$$

Thus the L operators

$$L(u) = uI - F_{ab}M^{ab}, \widetilde{L}(u) = uI + F^t_{ab}M^{ab}$$

built from the generators of the spinor representation F_{ab} and of the JS representation M^{ab} obey the RLL relation (5.2) with the spinorial YB operator $\mathfrak{R}$ (5.9). Here F_{ab} are understood as operators acting in the spinor space and the superscript t, F^t, means transposition.

Proof. The proof is done by straightforward calculations. □

In Section 4 we have seen that the representations of JS type obey the cubic characteristic identity for the matrix of its generators. We see now that this identity and the additional condition (5.16) are related.

Proposition 7. *If the generators $G^a{}_b$ of the algebra so, or sp, in a representation ρ' obey the additional condition (5.16), i.e.*

$$\rho'\Big([G_{[ae}\, , \, G_{f)b}]_+ \Big) = 0 \, , \tag{5.25}$$

then the matrix $G = ||G^a{}_b||$ (for simplicity here and below we omit the symbol of the representation ρ') obeys the cubic characteristic identity (cf. (4.47))

$$G^3 + (\epsilon - n)G^2 + (\epsilon n - 2)G + \frac{1}{2}\mathrm{Tr}(G^2)(I - \epsilon G) = 0 \, . \tag{5.26}$$

Proof. It is convenient to rewrite (5.25) using the commutation relation (3.23) as:

$$G_{a_2a_1}G_{c_1c_2} + G_{a_1c_1}G_{a_2c_2} + G_{c_1a_2}G_{a_1c_2} = \varepsilon_{c_2a_1}G_{c_1a_2} + \varepsilon_{c_2a_2}G_{a_1c_1} + \varepsilon_{c_2c_1}G_{a_2a_1}. \quad (5.27)$$

The multiplication by $G^{a_1a_2}$ and summation over a_1, a_2 leads to

$$\begin{aligned} &m_2G_{c_1c_2} - 2G_{c_1a_1}\varepsilon^{a_1a}G_{ab}\varepsilon^{ba_2}G_{a_2c_2} + 2(n-2\epsilon)G_{c_1b}\varepsilon^{ba_2}G_{a_2c_2} = \\ &\qquad = -2G_{c_2b}\varepsilon^{ba_2}G_{a_2c_1} + m_2\varepsilon_{c_2c_1} \, , \end{aligned} \quad (5.28)$$

where $m_2 = \mathrm{Tr}G^2$. From commutation relations (3.23) we also deduce

$$G_{c_2b}\varepsilon^{ba_2}G_{a_2c_1} = \epsilon[G_{c_1a}\varepsilon^{ab}G_{bc_2} - (n-2\epsilon)G_{c_1c_2}],$$

and applying this identity to the right hand side of (5.28) we obtain

$$\begin{aligned} &G_{c_1a_1}\varepsilon^{a_1a}G_{ab}\varepsilon^{ba_2}G_{a_2c_2} - (n-2\epsilon)G_{c_1b}\varepsilon^{ba_2}G_{a_2c_2} - \tfrac{1}{2}m_2G_{c_1c_2} = \\ &\qquad = \epsilon[G_{c_1a}\varepsilon^{ab}G_{bc_2} - (n-2\epsilon)G_{c_1c_2} - \tfrac{1}{2}m_2\varepsilon_{c_1c_2}], \end{aligned} \quad (5.29)$$

or simply for the matrix $G = ||G^a{}_b|| = ||\varepsilon^{ac}\, G_{cb}||$ we have the characteristic relation

$$G^3 - (n-\epsilon)\, G^2 + \epsilon \left(n - 2\epsilon - \frac{m_2}{2}\right) G + \epsilon\, \frac{m_2}{2} = 0\, , \quad (5.30)$$

which coincides with (5.26). □

Note, that by parameterizing the eigenvalue of the quadratic Casimir operator as in (4.46)

$$m_2 \equiv \mathrm{Tr}G^2 = 2\, m\, (m\epsilon + n - 2\epsilon),$$

one can rewrite the polynomial (5.26) in the factorized form

$$\begin{aligned} &G^3 + (\epsilon - n)G^2 + (\epsilon n - 2 - \epsilon\tfrac{m_2}{2})G + \tfrac{m_2}{2} = \\ &= (G + \epsilon m)(G - \epsilon m - n + 2\epsilon)(G - \epsilon) = 0\, . \end{aligned} \quad (5.31)$$

Note that this condition was presented in [11] (see eq. (3.9) there) for the case of $so(n)$ Lie algebra and for the generators $G_{ab} \in so(n)$ which coincides with ours (satisfying the commutation relations (3.23) and (4.44)) up to the redefinition $G_{ab} \to -G_{ab}$.

An obvious generalization of the Yangian concept discussed in Sect. 3 can be considered where the fundamental R matrix is replaced by the spinorial $\mathfrak{R}$. The spinorial L operators of the first order evaluation of the spinorial Yangian $Y_s(\mathcal{G})$,

$$L(u) = Iu - F_{ab}G^{ab}, \quad (5.32)$$

where $F^a_b = \frac{1}{2}\varepsilon_{bd}c^{(a}c^{d]}$ are the spinor representation generators and G^{ab} are generators acting in V and obeying the condition (5.16). Monodromies built as products with multiplication in the spinor space V_s but tensor product of copies of V result in examples of higher order evaluations of the spinorial Yangian. Below we shall consider instead the monodromy of two L factors of this form but with the roles of the representations V_s and V interchanged. The fusion procedure involving the projection of $V_s \otimes V_s$ onto the fundamental representation space V_f results in examples of the second order evaluation of the (fundamental) Yangian $Y(\mathcal{G})$.

6. Fusion operations

Recalling the known procedure we consider the YB relation (1.1) with the representation by operators in the space $V_1 \otimes V_2 \otimes V_3$. If it holds for the particular choices for $R_{i,3}$ as $L_{i,3}$ and $L_{i,\tilde{3}}$ then it will hold also for the choice

$$T_{i,3\tilde{3}}(u) = L_{i,3}(u-\lambda)L_{i,\tilde{3}}(u-\mu) \tag{6.33}$$

defined by the product of operators in the space V_i and as tensor product action in $V_3 \otimes V_{\tilde{3}}$. Consider then the projection Π on the invariant subspace $V_{\Pi 3} = \Pi \cdot V_3 \otimes V_{\tilde{3}}$ and the restriction of the operator $T_{i,3\tilde{3}}(u)$ to this subspace $L_{i,\Pi 3}$.

$$((V_3 \otimes V_{\tilde{3}}) \otimes (V_3 \otimes V_{\tilde{3}})^t \to V_{\Pi 3} \otimes V^t_{\Pi 3}.$$

The YB relation then holds for the substitution of $R_{i,3}$ by $L_{i,\Pi 3}$.

We consider first the case with the fundamental representation for both V_1 and V_2. V_3 and $V_{\tilde{3}}$ are both the spinorial spaces and

$$L_{i,3}(u) = uI - F, \quad L_{i,\tilde{3}}(u) = uI + F^t$$

The fermionic oscillators used above to express the spinorial generators are related to the conventional gamma matrices and the transposition t is defined in the matrix sense.

$$F^a_b = \frac{1}{2}\varepsilon_{bd}c^{(a}c^{d]} = \frac{1}{4}\varepsilon_{bd}\gamma^{ad}, \gamma^{ab} = \gamma^{(a}\gamma^{d]}, \gamma^a = \sqrt{2}c^a \tag{6.34}$$

The gamma matrices represent the intertwiner of the fundamental representation space V_f labeled by the index a and the corresponding invariant subspace in the tensor product of the spinor spaces $V_s \otimes V_s$ labeled by two matrix indices, i.e. they project $V_s \otimes V_s \to V_f$. For the reduction of the product of L matrices the projection $((V_s \otimes V_s) \otimes (V_s \otimes V_s)^t \to V_f \otimes V^t_f$ by contracting the spinor indices with $\gamma^{a_2\alpha_2}_{\alpha_1}\gamma^{\beta_1}_{b_1\beta_2}$ is to be done.

This operation is performed by calculating gamma matrix traces of products with 2, 4 and 6 γ, e.g.

$$\mathrm{tr}(\gamma^a\gamma^b) = C\ \varepsilon^{ab}\mathrm{tr}(\gamma^a\gamma^b\gamma^c\gamma^d) = C\ \left(\varepsilon^{ab}\varepsilon^{cd} - \varepsilon^{ac}\varepsilon^{bd} + \varepsilon^{ad}\varepsilon^{bc}\right)$$

which follows from the Clifford anti-commutation relation with C being the trace of the unit matrix in spinor space.

The symplectic counterpart of the spinor representation is infinite-dimensional. Nevertheless, the operators Γ_a can be interpreted as projectors of the tensor product to the fundamental representation labeled by a. The matrix elements can be calculated with respect to the standard basis of oscillator states or in coherent states. The infinite sum or integration over the basis states requires a regularization. Here we need only that this regularization can be done in such a way that the analogous relations for the traces hold. The following result relies on the relations

$$\mathrm{tr}(\gamma^a\gamma^b) = C\varepsilon^{ab}, \mathrm{tr}(\gamma^a\gamma^b\gamma^c\gamma^d) = C\left(\varepsilon^{ab}\varepsilon^{cd} - \epsilon\varepsilon^{ac}\varepsilon^{bd} + \varepsilon^{ad}\varepsilon^{bc}\right)$$

Proposition 8. *The fundamental R matrix with so or sp symmetry (2.8), (3.15) is reproduced by the fusion procedure applied to the product $L(u-\mu)\widetilde{L}(u-\lambda)$ of the spinor L matrix (4.34) and its transposition involving the projection of the tensor product of the two spinor representation spaces to the invariant subspace corresponding to the fundamental (spin 1 in so case) representation.*

Proof. Let us write the monodromy and the projection with explicit indices for the *so* case. Indices $a, b, ..$ label the basis of the fundamental representation space and $\alpha, \beta...$ the basis of the spinor spaces.

$$(L_{1,f}(u))^{a_1 a_2}_{b_1 b_2} = L^{a_1 \alpha_1}_{1,3\; c_1 \beta_1}(u-\lambda) L^{c\beta_2}_{1,\tilde{3} b_1 \alpha_2}(u-\mu) \times \gamma^{a_2 \alpha_2}_{\alpha_1} \gamma^{\beta_1}_{b_1 \beta_2}$$

We obtain

$$\begin{aligned}
&(u-\lambda)(u-\mu)\delta^{b_1}_{d_1}\mathrm{tr}(\gamma^{b_2}\gamma_{d_2}) - \frac{1}{2}(u-\mu)\mathrm{tr}(\gamma^{b_2}\gamma^{b_1}{}_{d_1}\gamma_{d_2}) + \frac{1}{2}(u-\lambda)\mathrm{tr}(\gamma^{b_1}{}_{d_1}\gamma^{b_2}\gamma_{d_2}) - \\
&\quad - \frac{1}{4}\mathrm{tr}(\gamma^{c_1}{}_{d_1}\gamma^{b_2}\gamma^{b_1}{}_{c_1}\gamma_{d_2}) \\
&\; = C \cdot [\{(u-\lambda)(u-\mu) - \frac{n-3}{4}\}\delta^{b_1}_{d_1}\delta^{b_2}_{d_2} + \{u - \frac{\lambda+\mu}{2} + \frac{n-2}{4}\}\delta^{b_1}_{d_2}\delta^{b_2}_{d_1} - \\
&\quad - \{u - \frac{\lambda+\mu}{2} - \frac{n-2}{4}\}\delta^{b_1 b_2}\delta_{d_1 d_2}].
\end{aligned}$$

In the symplectic case we have the analogous calculation and the result for both cases is proportional to

$$\{(u-\lambda)(u-\mu) - \frac{\varepsilon n-3}{4}\}I + \{u - \frac{\lambda+\mu}{2} + \frac{\varepsilon n-2}{4}\}P - \{u - \frac{\lambda+\mu}{2} - \frac{\varepsilon n-2}{4}\}\varepsilon K.$$

If we impose $\lambda + \mu = \frac{1}{2}(2-\varepsilon n)$ and $\lambda\mu = \frac{1}{4}(\varepsilon n - 3)$ we recognize the fundamental R-matrix (2.10). □

Proposition 9. *Consider the product $L(u-\mu)\widetilde{L}(u-\lambda)$ of the spinorial L operators (5.32)*

$$L(u) = uI - F_{ab}G^{ab}, \widetilde{L}(u) = uI + F^t_{ab}G^{ab}$$

where F^a_b are the spinor representation generators and G^{ab} are generators obeying the condition (5.16), i.e.

$$[G_{[ae}G_{f)b}]_+ = 0$$

The application of the fusion procedure involving the projection of the tensor product of the two spinor representation spaces to the invariant subspace corresponding to the fundamental representation results in the L operator of the form of the quadratic evaluation type, which at the shift parameter values

$$(\lambda-\mu)^2 = \frac{(n-4)^2}{4}. \;\; \lambda+\mu = 0, \tag{6.35}$$

reads

$$\begin{aligned}
&L(u) = u^2 I + uG + N, \\
&N = \frac{1}{2}(G^2 - \beta G) - \frac{1}{4}\beta^2 I - \frac{m_2}{8}, \;\; \beta = \frac{n}{2} - \varepsilon.
\end{aligned} \tag{6.36}$$

Proof. We may start from the YB relation with the JS representation in V_1, and the spinorial in V_2, V_3 or from the *RLL* relation with the JS representation in V_1, V_2 and the spinorial one in V_3. In the first case the fusion operation will lead in particular to $T_{1,3\tilde{3}}$ (6.33) with JS operator multiplication for the action in V_1 and in the tensor product of two copies of spinorial representations V_3. The projection of the latter to the fundamental representation is done by just the same calculation as above.

Let us write the monodromy $T_{1,3\tilde{3}}$ and the projection with explicit indices for the *so* case.

$$[(u-\lambda)\mathbb{I} - \frac{1}{4}\gamma^{ab}G_{ab}]^{\alpha}_{\beta}[(u-\mu)\mathbb{I} + \frac{1}{4}\gamma^{cd}G_{cd}]^{\gamma}_{\delta}.$$

The contraction with $(\gamma^e)^{\beta}_{\gamma}(\gamma^f)^{\delta}_{\alpha}$ leads to

$$\begin{aligned}\mathrm{L}(u) &= (u-\lambda)(u-\mu)\mathrm{tr}(\gamma^e\gamma^f) - \frac{1}{4}(u-\mu)\mathrm{tr}(\gamma^{ab}\gamma^e\gamma^f)G_{ab} + \\ &\quad + \frac{1}{4}(u-\lambda)\mathrm{tr}(\gamma^e\gamma^{cd}\gamma^f)G_{cd} - \frac{1}{16}\mathrm{tr}(\gamma^{ab}\gamma^e\gamma^{cd}\gamma^f)G_{ab}G_{cd} \\ &= \mathrm{tr}\mathbb{I}\cdot[(u-\lambda)(u-\mu)\delta^{ef} + \frac{1}{2}(u-\mu)G^{ef} + \frac{1}{2}(u-\lambda)G^{ef} + \\ &\quad + \frac{1}{4}(G^{eb}G^{bf} + G^{fb}G^{be} - \frac{1}{2}\delta^{ef}G^{cd}G_{dc})].\end{aligned}$$

By a shift $u \to u - \lambda$ this can be written as

$$\begin{aligned}&u^2\delta^{ef} + u(G^{ef} - (\mu+\lambda)\delta^{ef}) + N^{ef}, \\ N^{ef} &= \frac{1}{2}(F^2)^{ef} - \frac{1}{8}\delta^{ef}\mathrm{tr}G^2 + (\frac{n-2}{4} - \frac{\mu+\lambda}{2})G^{ef},\end{aligned} \tag{6.37}$$

Here we have used the Lie algebra relations to transform the commutator of generators G. □

By direct calculation it is checked that the obtained L operator at the particular parameter values (6.36) obeys the *RLL* relation (3.13) with the fundamental R matrix (3.15).

7. Summary

We have considered Yang–Baxter R operators symmetric with respect to the orthogonal and symplectic algebras. We have started from known examples illustrating how the more involved structure of these algebras compared to the one of the special linear type result in more involved features of the R operators. We have shown how both cases can be treated in a uniform way, which amounts in particular in the interchange of symmetrization with anti-symmetrization. It is known that this feature of analogy allows a supersymmetric formulation starting from the graded orthosymplectic algebra. We have preferred the more explicit parallel treatment of the two cases and decided not to add the supersymmetric formulation here.

The L operators obeying the *RLL* relation together with the *so* or *sp* symmetric fundamental R matrix define the corresponding Yangian algebra. Unlike the case of $s\ell$ symmetry the truncation of the expansion of $L(u)$ in inverse powers of the spectral parameter u results in constraints, which cannot be fulfilled in the enveloping algebra, but lead to the restriction to distinguished representations the generators of which can build such L operators.

The known example of truncation at the first order, the linear evaluation of the Yangian, is given by the spinor representation of the orthogonal algebra. It can be formulated on the basis of a fermionic oscillator or Clifford algebra. We have indicated its symplectic counterpart (metaplectic representation), which is formulated on the basis of a bosonic oscillator algebra. The constraint resulting form the first order truncation can be formulated as a characteristic identity of second order in terms of the matrix of generators.

The fundamental R matrix can be regarded as an example for the truncation at the second order, the quadratic evaluation of the Yangian algebra. The Jordan–Schwinger type representations provide more examples. The constraint resulting form the second order truncation can be formulated as a characteristic identity of third order in terms of the matrix of generators.

The YB relation involving the spinor and metaplectic representation L operators together with the particular R operator acting in the tensor product of two spinor and metaplectic representations has been studied. On its basis the explicit form of this spinorial and metaplectic R operator has been derived in an uniform treatment of both the orthogonal and symplectic cases. Further, we have studied a similar YB relation involving this spinorial R operator together with YB operators acting in a tensor product space of the spinor (and metaplectic) with another representation different from the fundamental one. The demanded YB relation results in a constraint on the generators of this representation. It is fulfilled by the fundamental representation, but also by Jordan–Schwinger type representations. For the constraint the latter have to be based on bosonic Heisenberg algebras in the orthogonal case and on fermionic Heisenberg algebras in the symplectic case. We have shown that the latter constraint is directly related to the third order characteristic identity of the quadratic Yangian evaluation.

We have studied fusion operations on products of YB operators acting on tensor products where one tensor factor is the spinor representation and the fusion involves the projection of the tensor product of two spinor representations onto the fundamental (vector) representation. In particular we have demonstrated how the fundamental R matrix is reproduced performing the fusion operation of the product of spinor L operators. The fusion operation with the same projection has been done also on the product of the R operators obeying the above YB relation with the spinorial R operator. These examples of fusion are chosen, because they result in the examples of L operators of the quadratic Yangian evaluation considered before, the fundamental R matrix and the L operator of the Jordan–Schwinger type. The explicit form of the latter is found in this way.

Yang–Baxter operators and in particular L operators of simple structure, which can be formulated explicitly, are of interest for integrable quantum systems. In particular the monodromy operators defined as products of L operators are applied in the investigation of integrable interaction models and in the construction of symmetric correlators and operators.

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