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The number of permutations containing exactly one increasing subsequence of length three

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Abstract

It is proved that the number of permuations on $\{1,2,\ldots,n\}$ with exactly one increasing subsequence of length 3 is $\frac{3}{n}\binom{2n}{n+3}$ [0, 0, 1, 6, 27, 110, 429, ... (Sloane A3517)].

Given a permutaion $\sigma \in S_n$, an *abc* subsequence is a set of three elements, $\sigma(i), \sigma(j), \sigma(k)$, with $\sigma(i) < \sigma(j) < \sigma(k)$ and $i < j < k$. It is known [2–4] that the number of permutations on $\{1,2,\ldots,n\}$ with no *abc* subsequences is given by the Catalan number $1/(n+1)\binom{2n}{n}$. A natural question is: is there a nice expression for the generating function $\sum_{r=0}^{\binom{n}{3}} B(n,r)q^r$, where $B(n,r)$ is the number of permutations on $\{1,2,\ldots,n\}$ with exactly r *abc*'s, for $1 \leqslant r \leqslant \binom{n}{3}$? Another question is, for a fixed r, what can one say about the sequence in n, $B(n,r)$? Doron Zeilberger conjectures that for any given r, the coefficients $B(n,r)$ of the generating function are P-recursive in n, i.e. they satisfy a linear recurrence with polynomial coefficients. This is supported by the fact that $B(n,0)$, being closed form, satisfies a *first-order* recurrence and hence is P-recursive.

It would be too much to hope for a closed form formula for $B(n,r)$ for general r, and a priori, there is no reason to hope that even $B(n,1)$ is closed form. To our surprise $B(n,1)$ did turn out to be closed form, and in this paper we present and prove such a formula. We hope to treat $B(n,r)$, for $r > 1$, in a subsequent paper.

Theorem. *The number of permutations on n objects that have exactly one abc subsequence is*

$$\frac{3}{n}\binom{2n}{n+3}. \tag{1}$$

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As is the case with many results, in order to prove this, we must first look at a more general result. For $\sigma \in S_n$, let $\phi_k(\sigma) = |\{(i,j): \sigma(i) < \sigma(j) = k \text{ and } i < j\}|$. Let $P_{(n,I)}$ denote the set of all $\sigma \in S_n$ with no *abc* subsequences and for which $\phi_j(\sigma) = 0$ for all $j \leqslant I$. Let $P(n,I) = |P_{(n,I)}|$. The result that the number of permutations on n elements with no *abc* subsequences is a Catalan number can be stated as $P(n,1) = 1/(n+1)\binom{2n}{n}$. Notice that $P(n,n) = 1$. Furthermore, from our definition of $P_{(n,I)}$ it follows that $P_{(n,0)} = P_{(n,1)}$.

Lemma 1.

$$P(n,I) = \binom{2n-I-1}{n-I} - \binom{2n-I-1}{n-I-2}. \tag{2}$$

These are the famous ballot numbers, and the proof below can be easily bijectified. Erikson and Linusson [1] had a similar result. We will show that both sides of (2) satisfy the same recursion;

$$F(n,I) = F(n-1,I-1) + F(n,I+1) \quad \text{for } n > 0 \text{ and } I > 0, \tag{2'}$$

with initial conditions

$$F(n,0) = F(n,1) \quad \text{for } n > 0 \tag{2''}$$

and

$$F(n,n) = 1 \quad \text{for } n > 0 \tag{2'''}$$

That the right-hand side of (2) satisfies (2′), (2″), and (2‴) is purely routine and is left to the reader. As a result of our definition, $P_{(n,0)} = P_{(n,1)}$. Furthermore, from the definition of $P_{(n,I)}$, $P_{(n,n)}$ is the set of permutations on $\{1,2,\ldots,n\}$ with no *abc* subsequences and no non-inversions. There is only one such permutation, namely $[n, n-1, \ldots, 2, 1]$, hence $P(n,n) = 1$.

Separate the set $P_{(n,I)}$ into two sets, K_1 and K_2. Let $K_1 := \{\sigma \in P_{(n,I)}: \phi_{I+1}(\sigma) = 0\}$ and $K_2 := \{\sigma \in P_{(n,I)}: \phi_{I+1}(\sigma) > 0\}$. The set K_1 is $P_{(n,I+1)}$.

Sublemma 1.1. *If $\sigma \in K_2$ then $\sigma(n) = 1$ or $\sigma(n) = I+1$.*

Proof. Let $\sigma \in K_2$. Assume $1 < \sigma(n) = j < I+1$. We must have $\sigma(i) = 1$ for some $i < n$. Thus $\phi_j(\sigma) > 0$, contradicting our construction of K_2. Assume $\sigma(n) > I+1$. By our construction of K_2, we know that $\phi_{I+1}(\sigma) > 0$. Let i and j be chosen so that $\sigma(i) < \sigma(j) = I+1$ and $i < j$. Then $\sigma(i) < \sigma(j) < \sigma(n)$ and $i < j < n$. Hence σ has an *abc* subsequence contradicting our consturction of K_2. □

Let $\sigma \in K_2$ and let $\sigma_1 \in S_{n-1}$ be defined by

$$\sigma_1(i) = \begin{cases} \sigma(i) - 1 & \text{if } \sigma(n) = 1, \\ \sigma(i) & \text{if } \sigma(n) = I + 1 \text{ and } \sigma(i) < I + 1, \\ \sigma(i) - 1 & \text{if } \sigma(n) = I + 1 \text{ and } \sigma(i) > I + 1. \end{cases}$$

Notice that σ_1 has no *abc* subsequences and $\phi_j(\sigma_1) = 0$ for $j \leqslant I - 1$. Let $\psi : K_2 \to P_{(n-1,I-1)}$ be defined by $\psi(\sigma) = \sigma_1$. We will show that ψ is a bijection between K_2 and $P_{(n-1,I-1)}$.

First we prove that ψ is one-to-one. Suppose $\sigma, \pi \in K_2$ such that $\psi(\sigma) = \psi(\pi)$ and $\sigma \neq \pi$. Then $\sigma(n) \neq \pi(n)$. From Sublemma 1.1, we must have $\sigma(n), \pi(n) \in \{1, I + 1\}$. Without loss of generality we may assume that $\sigma(n) = 1$ and $\pi(n) = I + 1$. Let $\sigma_1 = \psi(\sigma)$ and $\pi_1 = \psi(\pi)$. Since $\sigma \in K_2$, $\sigma(i) < \sigma(j) = I + 1$ for some $i < j$. It follows that $\sigma_1(i) < \sigma_1(j) = I$. Now $\pi_1 = \sigma_1$ and so $\pi_1(i) < \pi_1(j) = I$. It follows that $\pi(i) < \pi(j) = I < \pi(n) = I + 1$ and $i < j < n$, contradicting our assumption that π has no *abc* subsequences. Therefore, $\pi = \sigma$ and ψ is one-to-one.

Now we prove that ψ is onto. Suppose $\sigma_1 \in P_{(n-1,I-1)}$. If $\phi_I(\sigma_1) > 0$ then let σ be defined as

$$\sigma(i) = \begin{cases} \sigma_1(i) + 1 & \text{if } i \neq n, \\ 1 & \text{if } i = n. \end{cases}$$

If $\phi_I(\sigma_1) = 0$ then let σ defined as

$$\sigma(i) = \begin{cases} \sigma_1(i) & \text{if } \sigma_1(i) \leqslant I, \\ \sigma_1(i) + 1 & \text{if } \sigma_1(i) > I, \\ I + 1 & \text{if } i = n. \end{cases}$$

In both cases notice that σ has no *abc* subsequences and that $\phi_j(\sigma) = 0$ for $j \leqslant I$. So $\sigma \in K_2$ and $\psi(\sigma) = \sigma_1$. Therefore, ψ is onto and a bijection. We have $|P_{(n,I)}| = |P_{(n,I+1)}| + |P_{(n-1,I-1)}|$, so $P(n,I) = P(n,I+1) + P(n-1,I-1)$.

Notice that when $I = 1$, using (2) for $P(n, I)$, we rederive the above-mentioned result that the number of permutations with no *abc*'s is

$$P(n,1) = \binom{2n-2}{n-1} - \binom{2n-2}{n-3} = \frac{(2n-2)![n(n+1) - (n-2)(n-1)]}{(n-1)!(n+1)!}$$

$$= \frac{(2n)!}{n!(n+1)!} = C_n.$$

Let $P^{(1)}_{(n,I)} = \{\sigma \in S_n\colon \sigma \text{ has no } abc \text{ subsequences and } \phi_j(\sigma) = 0 \text{ for } j \leqslant I\}$. Let $P^{(1)}(n,I) = |P^{(1)}_{(n,I)}|$. Thus $P^{(1)}(n,1)$ is the number of permutations on $\{1, 2, \ldots, n\}$ with exactly one *abc* subsequence. Notice that $P^{(1)}(n, n-1) = 0$ for all n, and $P^{(1)}(3,1) = 1$.

Lemma 2.

$$P^{(1)}(n,I) = \binom{2n-I-1}{n} - \binom{2n-I-1}{n+3} + \binom{2n-2I-2}{n-I-4} - \binom{2n-2I-2}{n-I-1} + \binom{2n-2I-3}{n-I-4} - \binom{2n-2I-3}{n-I-2} \qquad (3)$$

To prove this, we prove that both sides of this equation satisfy the recursion

$$F(n,I) = F(n-1,I-1) + F(n,I+1) + P(n-I,2) \quad \text{for } n > 0 \text{ and } I > 0, \quad (3')$$

where $P(n-I,2)$ is as defined above and with the initial conditions

$$F(n,0) = F(n,1) \quad \text{for } n > 0 \qquad (3'')$$

and

$$F(n,n-2) = n-2 \quad \text{for } n > 0. \qquad (3''')$$

That the right-hand side of (3) satisfies (3′), (3″), and (3‴) is routine. As a result of our definition, $P^{(1)}_{(n,0)} = P^{(1)}_{(n,1)}$ and so $P^{(1)}(n,0) = P^{(1)}(n,1)$. We can easily compute $P^{(1)}(n,n-2)$. If $\sigma \in P^{(1)}_{(n,n-2)}$ then $\phi_j(\sigma) = 0$ for $j \leqslant n-2$ and σ has exactly 1 *abc* subsequence. Thus, σ is of the form $[n-2, n-1, n-3, \ldots, n-i, n, n-i-1, \ldots, 2, 1]$. There are exactly $n-2$ such permutations, hence $P^{(1)}(n,n-2) = n-2$. So we see that $P^{(1)}(n,I)$ satisfies (3″) and (3‴).

We prove that $P^{(1)}(n,I)$ satisfies (3′) by separating the set $P^{(1)}_{(n,I)}$ into three sets K_1, K_2, and K_3. Let $K_1 = \{\sigma \in P^{(1)}_{(n,I)}\colon \phi_{I+1}(\sigma) = 0\}$, $K_2 = \{\sigma \in P^{(1)}_{(n,I)}\colon \phi_{I+1}(\sigma) > 0$ and $\sigma(n)$ participates in the *abc* subsequence$\}$, and $K_3 = \{\sigma \in P^{(1)}_{(n,I)}\colon \phi_{I+1}(\sigma) > 0$ and $\sigma(n)$ does not participate in the *abc* subsequence$\}$. The first set is $P^{(1)}_{(n,I+1)}$.

We must show that $|K_2| = |P^{(1)}_{(n-1,I-1)}|$ and $|K_3| = |P_{(n-I,2)}|$.

Sublemma 2.1. *If* $\sigma \in K_2$ *then* $\sigma(n) \in \{1, I+1\}$.

Proof. Let $\sigma \in K_2$. If $1 < \sigma(n) = j < I+1$ then $\sigma(i) = 1$ for some $i < n$, but then $\phi_j(\sigma) > 0$ contradicting our construction of K_2. If $\sigma(n) > I+1$ then by our construction of K_2, we know that $\phi_{I+1}(\sigma) > 0$. Let i and j be chosen so that $\sigma(i) < \sigma(j) = I+1$ and $i < j$. Then $\sigma(i) < \sigma(j) < \sigma(n)$ and $i < j < n$. Hence $\sigma(n)$ participates in an *abc* subsequence which contradicts our construction of K_2. □

Let $\sigma \in K_2$ and let $\sigma_1 \in S_{n-1}$ be defined by

$$\sigma_1(i) = \begin{cases} \sigma(i) - 1 & \text{if } \sigma(n) = 1, \\ \sigma(i) & \text{if } \sigma(n) = I+1 \text{ and } \sigma(i) < I+1, \\ \sigma(i) - 1 & \text{if } \sigma(n) = I+1 \text{ and } \sigma(i) > I+1. \end{cases}$$

Notice that σ_1 has precisely one *abc* subsequence and $\phi_j(\sigma_1) = 0$ for $j \leqslant I - 1$. Let $\psi : K_2 \to P_{(n-1,I-1)}$ be defined by $\psi(\sigma) = \sigma_1$. First we prove that ψ is one-to-one. Suppose there exist σ and π in K_2 such that $\psi(\sigma) = \psi(\pi)$ and $\sigma \neq \pi$. Let $\sigma_1 = \psi(\sigma)$ and $\pi_1 = \psi(\pi)$. We must have $\sigma(n) \neq \pi(n)$. By Sublemma 2.1, $\sigma(n)$ and $\pi(n)$ are in $\{1, I+1\}$. Without loss of generality we may assume that $\sigma(n) = 1$ and $\pi(n) = I + 1$. If $\sigma \in K_2$ then $\sigma(i) < \sigma(j) = I + 1$ for some $i < j < n$. It follows that $\sigma_1(i) < \sigma_1(j) = I$. Thus $\pi_1(i) < \pi_1(j) = I$. But then $\pi(i) < \pi(j) < \pi(n) = I + 1$ which contradicts our construction of K_2. Therefore, ψ is one-to-one.

Now we prove that ψ is onto. Suppose $\sigma_1 \in P^{(1)}_{(n-1,I-1)}$. If $\phi_I(\sigma_1) > 0$ then $\sigma \in S_n$ be defined by

$$\sigma(i) = \begin{cases} \sigma_1(i) + 1 & \text{if } 1 \leqslant i < n, \\ 1 & \text{if } i = n. \end{cases}$$

If $\phi_I(\sigma_1) = 0$ then let $\sigma \in S_n$ defined by

$$\sigma(i) = \begin{cases} \sigma_1(i) & \text{if } \sigma_1(i) \leqslant I + 1, \\ \sigma_1(i) + 1 & \text{if } \sigma_1(i) > I + 1, \\ I + 1 & \text{if } i = n. \end{cases}$$

In either case, it follows that σ has exactly one *abc* subsequence, $\phi_{I+1}(\sigma) > 0$, and $\sigma(n)$ does not participate in the *abc* subsequence of σ. So $\sigma \in K_2$ and $\psi(\sigma) = \sigma_1$. Therefore, ψ is onto and a bijection and $|K_2| = |P^{(1)}_{(n-1,I-1)}|$.

Finally, we must construct a bijection between $P_{(n-I,2)}$ and K_3. Let $\sigma \in P_{(n-I,2)}$. Let k be chosen so that $\sigma(k) = 1$. If $\sigma(k-1) \neq 2$ then let $\sigma_1 \in S_n$ be defined by

$$\sigma_1 = \begin{cases} I + \sigma(i) & \text{if } i < k - 1 \text{ or } k < i < n - I, \\ I & \text{if } i = k - 1, \\ I + \sigma(i-1) & \text{if } i = k, \\ I + 1 & \text{if } i = n - I, \\ n - i & \text{if } n - I < i < n, \\ I + \sigma(n - I) & \text{if } i = n. \end{cases}$$

If $\sigma(k-1) = 2$ then let $\sigma_1 \in S_n$ be defined by

$$\sigma_1 = \begin{cases} I + \sigma(i) & \text{if } i < k - 1, \\ I & \text{if } i = k - 1, \\ I + \sigma(i+1) & \text{if } k - 1 < i < n - I, \\ I + 1 & \text{if } i = n - I, \\ n - i & \text{if } n - I < i < n, \\ I + 2 & \text{if } i = n. \end{cases}$$

Notice that if $\sigma \in P_{(n-I,2)}$ then $\sigma_1 \in K_3$. Indeed by the way we constructed it, $\phi_j(\sigma_1) = 0$ for $j \leqslant I$. Furthermore, $\phi_{I+1}(\sigma_1) > 0$, and σ_1 has exactly one *abc* subsequence, consisting of I, $I+1$, and the last element of $\varphi(\pi)$..

Let $\varphi : P_{(n-I,2)} \to K_3$ be define as $\varphi(\sigma) = \sigma_1$. We will prove that φ is a bijection.

First we prove φ is one-to-one. Suppose $\pi, \sigma \in P_{(n-I,2)}$ and $\varphi(\pi) = \varphi(\sigma)$. Let

$$\sigma = [\gamma_1, \gamma_2, \ldots, \gamma_{k_1}, \eta_1, 1, \eta_2, \eta_3, \ldots, \eta_{m_1}]$$

and

$$\pi = [\alpha_1, \alpha_2, \ldots, \alpha_{k_2}, \beta_1, 1, \beta_2, \beta_3, \ldots, \beta_{m_2}].$$

Then by the position of I in $\varphi(\sigma)$ and $\varphi(\pi)$, we may conclude that $m_1 = m_2$ and $k_1 = k_2$. Next we note that the last element of $\varphi(\sigma)$ must be the same as the last element of $\varphi(\pi)$, and so either $\beta_1 = \eta_1 = 2$ or $\beta_{m_2} = \eta_{m_1}$. Similarly, we may conclude that $\beta_i = \eta_i$ for $1 \leqslant i \leqslant m_1 = m_2$ and $\alpha_i = \gamma_i$ for $1 \leqslant i \leqslant k_1 = k_2$. Thus $\sigma = \pi$ and φ is one-to-one.

Now we prove that φ is onto. Suppose $\sigma = [I + \alpha_1, I + \alpha_2, \ldots, I + \alpha_k, I, I + \beta_1, I + \beta_2, \ldots, I + \beta_m, I + 1, I - 1, I - 2, \ldots, 2, 1, I + j]$, where $j \neq 2$. It is easy to see that $\sigma_1 = [\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, 1, \beta_2, \ldots, \beta_m, j] \in P_{(n-I,2)}$ and $\varphi(\sigma_1) = \sigma$. If $\sigma = [I + \alpha_1, I + \alpha_2, \ldots, I + \alpha_k, I, I + \beta_1, I + \beta_2, \ldots, I + \beta_m, I + 1, I - 1, I - 2, \ldots, 2, 1, I + 2]$ then $\sigma_1 = [\alpha_1, \alpha_2, \ldots, \alpha_k, 2, 1, \beta_1, \beta_2, \ldots, \beta_m,] \in P_{(n-I,2)}$ and $\varphi(\sigma_1) = \sigma$. Therefore, φ is onto and a bijection and $|K_3| = |P(n-I,2)|$.

We have $|P^{(1)}_{(n,I)}| = |P^{(1)}_{(n,I+1)}| + |K_2| + |K_3| = |P^{(1)}_{(n,I+1)}| + |P^{(1)}_{(n-1,I-1)}| + |P_{(n-I,2)}|$. Therefore, $P^{(1)}(n,I) = P^{(1)}(n,I+1) + P^{(1)}(n-1,I-1) + P(n-I,2)$. □

From the definition of $P^{(1)}(n,I)$, we see that $P^{(1)}(n,1)$ is the number of permutations on n objects with exactly one *abc* subsequence and no other restrictions. Using (3) with $I = 1$, we have

$$P^{(1)}(n,1) = \binom{2n-2}{n} - \binom{2n-2}{n+3} + \binom{2n-4}{n-5} - \binom{2n-4}{n-2} + \binom{2n-5}{n-5} - \binom{2n-5}{n-3} = \frac{3}{n}\binom{2n}{n+3}$$

We observe that $P^{(1)}(n,1) = P(n+2,5)$, so the number of permutations on $\{1,2,\ldots,n\}$ with exactly 1 *abc* equals the number of permutations σ on $\{1,2,\ldots,n+3\}$ with no *abc*'s and with $\phi_j(\sigma) = 0$ for $j \leqslant 6$. Doron Zeilberger offers 25 dollars for a *nice* bijective proof.

Note: A small Maple package accompanying this paper, labc.maple can be obtained by using your favorite world wide web browser at http://www.math.temple.edu/ ~ noonan or by anonymous ftp to ftp.math.temple.edu, directory /pub/noonan.

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