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ON PROPERTY $B(s)$, II

H.L. ABBOTT

Mathematics Department, University of Alberta, Edmonton, Alberta, T6G 2G1, Canada

A.C. LIU

Mathematics Department, University of Regina, Regina, Saskatchewan, S4S 0A2, Canada

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Denote by $m(n, s)$ the size of a smallest family $\mathscr{F}$ of n-element sets with the property that if $|S \cap F| \geq 1$ for all $F \in \mathscr{F}$, then $|S \cap F| \geq s$ for some $F \in \mathscr{F}$. We obtain some bounds for $m(n, s)$.

1. Introduction

By an n-graph, $n \geq 2$, is meant a family $\mathscr{F}$ of n-sets called edges. Elements of $\bigcup \mathscr{F}$ are called vertices. Let $2 \leq s \leq n$. An n-graph $\mathscr{F}$ is said to have property $B(s)$ if there exists a subset S of $\bigcup \mathscr{F}$ such that $1 \leq |S \cap F| \leq s-1$ for all $F \in \mathscr{F}$. Such an S is called a $B(s)$-set for $\mathscr{F}$. Denote by $m(n, s)$ the least integer k for which there exists an n-graph $\mathscr{F}$ with k edges which does not have property $B(s)$. Such an $\mathscr{F}$ will be called an (n, s)-graph.

It is known that $m(2k, 2) = 3$ and $m(2k+1, 2) = 4$ for all positive integers k [6]. The value of $m(n, 3)$ is not known for all values of n. The following summarizes the available information. (See [1] for details.)

$$\begin{aligned} & m(n,3)=7, \quad \text{whenever } n \text{ is a multiple of 3 or 4;} \\ & 9 \leq m(5,3) \leq 10; \\ & 8 \leq m(11,3) \leq 10; \\ & 8 \leq m(n,3) \leq 9, \quad \text{for all other values of } n. \end{aligned} \tag{1}$$

In the case where $n = s$ the principal results are (denoting $m(n, n)$ by $m(n)$):

$$m(n) < n^2 2^{n+1} \tag{2}$$

and

$$m(n) > n^{1-\varepsilon} 2^n \tag{3}$$

for every $\varepsilon > 0$, $n \geq n_0(\varepsilon)$. The upper bound is due to Erdös [6] and the lower bound to Beck [3]. The value of $m(4)$ is not known. Seymour [9] and Toft [10] independently showed that $m(4) \leq 23$ and Seltridge and Aizley [8] have announced that $m(4) \geq 19$.

Erdös [5] remarked that the family of n-subsets of a set of $n+s-1$ elements does not have property $B(s)$ and hence that

$$m(n, s) \leq \binom{n+s-1}{n}.$$

In [5], Erdös stated that he can prove there exist constants c_1 and c_2, $1 < c_1 < c_2$, such that

$$c_1^s < m(n, s) < c_2^s \tag{4}$$

but did not give explicit values for c_1 and c_2. Everts [7] proved that if $0 < c < 1$ and $s = [cn]$, then

$$m(n, s) > 2^n \left((2-c)^n + \left(\frac{2-c}{1-c}\right)^{n(1-c)} \right)^{-1}. \tag{5}$$

Since $2 > ((2-c)/(1-c))^{1-c} > 2-c$ we obtain from (5), on putting $f(c) = 2^{1/c}((1-c)/(2-c))^{(1-c)/c}$ and on noting that $f(c) > 1$ that

$$m(n, s) > \tfrac{1}{2}(f(c))^s. \tag{6}$$

We note that f is increasing on $(0, 1)$ and that $\lim_{c \to 1} f(c) = 2$ and

$$\lim_{c \to 0} f(c) = \frac{2}{\sqrt{e}} = 1.213 \ldots.$$

In Section 2 of this paper we prove a number of recurrence inequalities for $m(n, s)$. These are contained in the following theorem.

Theorem 1. *The following recurrence inequalities hold:*

$$m(n, s) \leq s m(n-1, s-1) + T(n, n-s, n-s+1) + 1, \tag{7}$$

$$m(n, s) \leq (s-1) m(n-1, s-1) + \binom{n}{s-1} + 1, \tag{8}$$

$$m(n, s) \leq s m(n-t, s) + 1 \quad \textit{provided } 0 < t \leq n-s \quad \textit{and} \quad n \geq ts, \tag{9}$$

$$m(kl, uv) \leq m(k, u) m(l, v)^k \quad \textit{provided } k \geq u, \quad l \geq v. \tag{10}$$

Here, for $2 \leq k \leq b \leq n$, $T(n, k, b)$ denotes the well-known Turán number, namely, the least number of edges a k-graph $\mathcal{T}$ on n vertices can have so that every b-subset of $\bigcup \mathcal{T}$ contains at least one edge of $\mathcal{T}$.

We deduce from some of these inequalities that the following upper bounds for $m(n, s)$ hold.

Theorem 2.

$$m(n, s) \leq s^3 2^{s+1} \quad \textit{for all } n \geq s(s-1) \tag{11}$$

and

$$m(n, s) \leq s^3 2^{2s+1} \quad \textit{for all } n \geq s. \tag{12}$$

We also record in Section 2 the best upper bounds for $m(n, 4)$ that we have been able to obtain for $5 \leq n \leq 10$.

In Section 3 we obtain, by a modification of Everts' argument, a lower bound for $m(n, s)$ which is stronger than the one given by (6) for a certain range of values of n and s. We prove the following theorem.

Theorem 3. *Let* λ *and* ε *be positive numbers satisfying*

$$\lambda + 1 + \varepsilon + \log \lambda < 0. \tag{13}$$

Then, for $s_0 \leq s \leq \delta n$

$$m(n, s) \geq \tfrac{1}{2} e^{\lambda s} \tag{14}$$

where $\delta = 2\varepsilon/(\lambda(\lambda + 2\varepsilon))$ *and* s_0 *is the least integer such that* $e^{s_0(\lambda+1+\varepsilon+\log\lambda)} < 1 - \lambda$.

Theorem 3 gives an improvement over (6) in the case where $s = [cn]$ and c is small. For example, if we choose $\lambda = 0.27$ and $\varepsilon = 0.03$ we find that (13) is satisfied. We also find from the argument leading to (14) that one may then choose $s_0 = 34$ and $\delta = \frac{1}{4}$, so that

$$m(n, s) > \tfrac{1}{2}(1.30)^s \quad \textit{for } 34 \leq s \leq \tfrac{1}{4}n$$

while with $s = [\frac{1}{4}n]$, the lower bound given by (6) is

$$m(n, s) > \tfrac{1}{2}(1.25)^s.$$

In fact, there is a number c_0 (=0.45 approximately) such that if $0 < c < c_0$ and $s = [cn]$, the lower bound given by (14) is sharper than that given by (6), for all sufficiently large s. However, for $c_0 \leq c < 1$, (6) is stronger.

2. Upper bounds for $m(n, s)$

First we record three simple inequalities for $m(n, s)$, the first two of which are obvious; the third is proved in [1].

$$m(n, s) \leq m(n, s+1), \tag{15}$$

$$m(n, s) \leq m(n+1, s+1), \tag{16}$$

$$m(kn, s) \leq m(n, s). \tag{17}$$

Proof of Theorem 1. Let $\mathcal{T}$ be an $(n-s)$-graph on n vertices such that every $(n-s+1)$-subset of $V = \bigcup \mathcal{T}$ contains an edge of $\mathcal{T}$. We suppose that $\mathcal{T}$ is minimal and thus has $T(n, n-s, n-s+1)$ edges. Let $\mathcal{G}$ be an $(n-1, s-1)$-graph. We suppose that the vertex sets of $\mathcal{G}$ and $\mathcal{T}$ are disjoint. Let $A = \{a_1, a_2, \ldots, a_s\}$ be a set which is disjoint from $\bigcup \mathcal{T}$ and $\bigcup \mathcal{G}$. Let $\mathcal{F}$ be the n-graph consisting of the following edges:

(1) all sets of the form $G\cup\{a_i\}$, $G\in\mathscr{G}$, $i=1,2,\ldots,s$;

(2) all sets of the form $T\cup A$, $T\in\mathscr{T}$;

(3) the set V.

Then $\mathscr{F}$ clearly has $sm(n-1,s-1)+T(n,n-s,n-s+1)+1$ edges. Suppose $\mathscr{F}$ has a $B(s)$-set S. Then $S\not\supset A$. Consider first the case where $S\cap A\neq\emptyset$. We may suppose, without loss of generality, that $a_1\in S$ and $a_2\notin S$. Then if $|S\cap G|\geq 1$ for all $G\in\mathscr{G}$, we must have $|S\cap G_1|\geq s-1$ for some $G_1\in\mathscr{G}$. Then $|S\cap(G_1\cup\{a_1\})|\geq s$, a contradiction. Hence $S\cap G_2=\emptyset$ for some $G_2\in\mathscr{G}$ but then $S\cap(G_2\cup\{a_2\})=\emptyset$, another contradiction. Secondly, consider the case where $S\cap A=\emptyset$. Now $|S\cap V|\leq s-1$. Hence $|\bar{S}\cap V|\geq n-s+1$, and thus, by the definition of $\mathscr{T}$, $\bar{S}\supset T$ for some $T\in\mathscr{T}$. Thus $S\cap T=\emptyset$ and consequently $S\cap(T\cup A)=\emptyset$, a contradiction. It follows that $\mathscr{F}$ does not have property $B(s)$. This proves (7).

In order to prove (8), let $\mathscr{G}$ be an $(n-1,s-1)$-graph and let $\mathscr{H}$ be the complete $(n-s+1)$-graph on n vertices. Let $A=\{a_1,a_2,\ldots,a_{s-1}\}$. We suppose that A, $\bigcup\mathscr{G}$ and $\bigcup\mathscr{H}$ are pairwise disjoint. Let $\mathscr{F}$ be the n-graph consisting of the following edges:

(1) all sets of the form $G\cup\{a_i\}$, $G\in\mathscr{G}$, $i=1,2,\ldots,s-1$;

(2) all set of the form $H\cup A$, $H\in\mathscr{H}$;

(3) the set $\bigcup\mathscr{H}$.

Then $\mathscr{F}$ has $(s-1)m(n-1,s-1)+\binom{n}{s-1}+1$ edges. We leave to the reader the verification that $\mathscr{F}$ does not have property $B(s)$.

We now construct the graph that leads to (9). Let $\mathscr{G}$ be an $(n-t,s)$-graph. Let $A=\{a_1,a_2,\ldots,a_n\}$ be disjoint from $\bigcup\mathscr{G}$. For $i=1,2,\ldots,s$ let $A_i=\{a_k:(i-1)t+1\leq k\leq it\}$. Let $\mathscr{F}$ be the n-graph whose edges are all sets of the form $G\cup A_i$, $G\in\mathscr{G}$, $i=1,2,\ldots,s$ together with the set A. Then $\mathscr{F}$ has $sm(n-t,s)+1$ edges. We leave to the reader the verification that $\mathscr{F}$ does not have property $B(s)$.

We do not present the proof of (10) since the result is a generalization of the inequality $m(ab)\leq m(a)m(b)^a$ given in [2]. The proof there carries over with little change. This completes the proof of Theorem 1.

Proof of Theorem 2. It is now a simple matter to prove Theorem 2. Write $n=qs+t$, $0\leq t\leq s-1$. If $t=0$ we have, by (17) and (2), $m(n,s)=m(qs,s)\leq m(s)<s^2 2^{s+1}$. Hence we may suppose $t>0$. Then it is easy to check that the inequality $n\geq s(s-1)$ implies that the conditions required by (9) are satisfied. Thus by (9), (17) and (2) we get

$$m(n,s)\leq sm(qs,s)+1\leq sm(s)+1\leq s^3 2^{s+1}.$$

This proves (11).

In order to prove (12) write $n=qs+r$, $0<r<s$. (Again, the case where $r=0$ is covered by (17).) Let k be defined by $(k-1)q<r\leq kq$ and let $t=n-q(s+k-1)$. Then one may check that $0<t\leq n-s$ and $n\geq ts$ so that (9) may be used. Note

also that $k \leq s-1$. We have

$$\begin{aligned} m(n,s) &\leq sm(n-t,s)+1 && \text{(by (9))} \\ &= sm(q(s+k-1),s)+1 \\ &\leq sm(s+k-1,s)+1 && \text{(by (17))} \\ &\leq sm(s+k-1,s+k-1)+1 && \text{(by (15))} \\ &\leq sm(2s-2)+1 && \text{(by (16) and } k \leq s-1) \\ &< s^3 2^{2s-1} && \text{(by (2)).} \end{aligned}$$

This completes the proof of Theorem 2.

It would be of interest to know whether for every $\varepsilon > 0$, $m(n,s) < (2+\varepsilon)^s$ when $n \geq s \geq s(\varepsilon)$. It is not difficult to show, using the results of Theorem 1, that if $m(n,s) < c^s$ for $s \leq n < 2s$ and if $s \geq s(\varepsilon)$, then $m(n,s) < (c+\varepsilon)^s$ for $s \leq n < s(s-1)$. Thus the difficulty lies in getting a good upper estimate for $m(n,s)$ when $s \leq n \leq 2s$. We remark that it follows easily from (10), by an argument which parallels closely one used in [2], that if $0 < c < 1$ and $s = [cn]$, then $\lim_{s\to\infty} m(n,s)^{1/s}$ exists.

We conclude this section by listing the best upper bounds we have been able to obtain for $m(n,4)$ for $5 \leq n \leq 10$. For the values of the Turán numbers quoted in Table 1, see [4].

Table 1

n	$m(n,4) \leq$	*Results used*
5	32	(8), and $m(4,3) = 7$
6	48	(10), $m(2) = 3$ and $m(3,2) = 4$
7	41	(7), $m(6,3) = 7$ and $T(7,3,4) = 12$
8	23	(17), and $m(4) \leq 23$
9	50	(7), $m(8,3) = 7$ and $T(9,5,6) = 21$
10	48	(10), $m(2) = 3$ and $m(5,2) = 4$

3. Lower bounds for $m(n,s)$

In this section we prove Theorem 3. The argument is probabilistic in nature and the underlying ideas are similar to those used by Everts [7].

Let $t \geq n$ and let $V = \{1, 2, \ldots, t\}$. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$ be a family of n-subsets of V. Let $X_1, X_2, \ldots, X_t$ be independent random variables with values 0 and 1 such that for each i,

$$\mathbb{P}_r(X_i = 1) = \frac{\lambda s}{n} \quad \text{and} \quad \mathbb{P}_r(X_i = 0) = 1 - \frac{\lambda s}{n}.$$

For each of the 2^t sequences $(X_1, X_2, \ldots, X_t) = x$, let $S_x = \{i : X_i = 1\}$. S_x is thus a random subset of V whose expected size is $t\lambda s/n$. For each $F \in \mathcal{F}$ and each k,

$0 \leqslant k \leqslant n$,

$$\mathbf{P}_r(|S_x \cap F| = k) = \binom{n}{k}\left(\frac{\lambda s}{n}\right)^k \left(1 - \frac{\lambda s}{n}\right)^{n-k}.$$

The expected number of sets in $\mathcal{F}$ which are either disjoint from S_x or meet S_x is s or more places is thus

$$E = r\left(\left(1 - \frac{\lambda s}{n}\right)^n + \sum_{k=s}^{n} \binom{n}{k}\left(\frac{\lambda s}{n}\right)^k \left(1 - \frac{\lambda s}{n}\right)^{n-k}\right).$$

If we use the inequality $(1 - h/n)^n < e^{-h}$ we get

$$E < re^{-\lambda s}\left(1 + \sum_{k=s}^{n} \binom{n}{k}\left(\frac{\lambda s}{n}\right)^k \left(1 - \frac{\lambda s}{n}\right)^{-k}\right) = re^{-\lambda s}\{1 + H\}, \quad \text{say.}$$

Now

$$\begin{aligned} -n \log\left(1 - \frac{\lambda s}{n}\right) &= n \sum_{l=1}^{\infty} \frac{1}{l}\left(\frac{\lambda s}{n}\right)^l \\ &< \lambda s + n \sum_{l=2}^{\infty} \left(\frac{\lambda s}{n}\right)^l \\ &= \lambda s + \frac{\lambda^2 s^2}{n - \lambda s} \\ &< (\lambda + \varepsilon)s \text{ provided } s < \frac{2\varepsilon n}{\lambda(\lambda + z\varepsilon)} = \delta_0 n, \end{aligned}$$

so that

$$\left(1 - \frac{\lambda s}{n}\right)^{-k} \leqslant \left(1 - \frac{\lambda s}{n}\right)^{-n} < e^{(\lambda + \varepsilon)s}.$$

Thus

$$\begin{aligned} H &\leqslant e^{(\lambda+\varepsilon)s} \sum_{k=s}^{n} \frac{(\lambda s)^k}{k!} \\ &\leqslant e^{(\lambda+\varepsilon)s} \frac{(\lambda s)^s}{s!}\left\{1 + \sum_{k=1}^{\infty} \frac{(\lambda s)^k}{(s+1)(s+2)\cdots(s+k)}\right\} \\ &\leqslant e^{(\lambda+\varepsilon)s} \frac{(\lambda s)^s}{s!(1-\lambda)} < e^{(\lambda+\varepsilon)s} \frac{(\lambda s)^s}{\left(\frac{s}{e}\right)^s (1-\lambda)} \\ &= \frac{1}{1-\lambda} e^{s(\lambda + 1 + \log \lambda)} \\ &< 1 \quad \text{provided } s \geqslant s_0(\varepsilon, \lambda) \qquad \text{(by (13).} \end{aligned}$$

It follows that $E < 2re^{-\lambda s}$ so that $E < 1$ provided $r < \frac{1}{2}e^{\lambda s}$. Hence there exists an S_x which satisfies $1 \leqslant |S_x \cap F| \leqslant s - 1$ for all $F \in \mathcal{F}$ and this S_x is a $B(s)$-set for $\mathcal{F}$. Hence $m(n, s) \geqslant \frac{1}{2}e^{\lambda s}$. This completes the proof of Theorem 3.

We conclude by mentioning the following interesting question which was brought to our attention some time ago by P. Erdös: Is it true that $m(n, s) \geqslant m(s)$ for all $n \geqslant s$? If the answer to this question is yes, and if one could answer affirmatively the question raised near the end of Section 2, it would follow that $m(n, s)$ behave essentially like 2^s for all $n \geqslant s$.

References

[1] H.L. Abbott and A. Liu, On property *B(s)*, Ars Combin. 7 (1979) 255–260.
[2] H.L. Abbott and L. Moser, On a combinatorial problem of Erdös and Hajnal, Canad. Math. Bull. 7 (1964) 177–182.
[3] J. Beck, On 3-chromatic graphs, Discrete Math. 24 (1978) 127–137.
[4] V. Chvátal, Hypergraphs and Ramseyian theorems, Ph.D. Thesis, University of Waterloo (1970).
[5] P. Erdös, On a combinatorial problem, Nordisk Mat. Tidskr. 11 (1963) 5–10.
[6] P. Erdös, On a combinatorial problem II, Acta Math. Acad. Sci. Hungar. 15 (1964) 445–447.
[7] F. Everts, Colorings of sets, Ph.D. Thesis, University of Colorado (1977).
[8] J.E. Selfridge and P. Aizley, Notices Amer. Math. Soc. 24 (1977) A-452.
[9] P. Seymour, A note on a combinatorial problem of Erdös and Hajnal, J. London Math. Soc. 8(2) (1974) 681–682.
[10] B. Toft, On color critical hypergraphs, in: A. Hajnal et al., eds., Infinite and Finite Sets (North-Holland, Amsterdam, 1975) 1445–1457.