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On a Nonconvolution Volterra Resolvent

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Under fairly weak assumptions, the solutions of the system of Volterra equations $x(t) = \int_0^t a(t, s)\, x(s)\, ds + f(t)$, $t > 0$, can be written in the form $x(t) = f(t) + \int_0^t r(t, s) f(s)\, ds$, $t > 0$, where r is the resolvent of a, i.e., the solution of the equation $r(t, s) = a(t, s) + \int_0^t a(t, v)\, r(v, s)\, dv$, $0 < s < t$. Conditions on a are given which imply that the resolvent operator $f \mapsto \int_0^t r(t, s) f(s)\, ds$ maps a weighted L^1 space continuously into another weighted L^1 space, and a weighted L^∞ space into another weighted L^∞ space. Our main theorem is used to study the asymptotic behavior of two differential delay equations.

1. INTRODUCTION

Consider the system of Volterra equations

$$x(t) = \int_0^t a(t, s)\, x(s)\, ds + f(t), \qquad t > 0.$$

Here x is the unknown solution with values in $\mathbf{R}^n$, a is a given kernel with n by n matrix values, and f is a given $\mathbf{R}^n$-valued forcing function. As is well known, under fairly weak assumptions, the unique solution of (1.1) is given by "the variation of constants formula"

$$x(t) = f(t) + \int_0^t r(t, s) f(s)\, ds, \qquad t > 0,$$

where r is the so-called resolvent of a. This reolvent satisfies two equations, namely, "the resolvent equation"

$$r(t, s) = a(t, s) + \int_s^t a(t, v)\, r(v, s)\, dv, \qquad 0 < s < t,$$

and "the adjoint resolvent equation"

$$r(t, s) = a(t, s) + \int_s^t r(t, v)\, a(v, s)\, dv, \qquad 0 < s < t.$$

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Because of the variation of constants formula, the more detailed information one can get on the resolvent, the better one can understand the behavior of the Volterra equation (1.1).

Here we shall concentrate on one particular property of r. We want to know whether the resolvent operator $f \mapsto \int_0^t r(t, s) f(s)\, ds$ maps some weighted L^1 space continuously into another weighted L^1 space, and/or some weighted L^∞ space continuously into another weighted L^∞ space. If it does so, then one can use standard perturbation techniques to study, e.g., the nonlinear equation

$$x(t) + \int_0^t a(t, s)[x(s) + g(s, x(s))]\, ds = f(t), \qquad t > 0,$$

in an L^p setting. For example, the stability analysis in [7] is based on this method (no weights are used in [7]).

A nonweighted version of the L^∞ case of our main Theorem 3.3 is proved by Gustaf Gripenberg in [5, Theorems 1 and 3]. In [9], the author uses the same technique as Gripenberg to prove the nonweighted L^1 version of Theorem 3.3. In [9], it is also shown how one can interpolate between the extreme cases $p = 1$ and $p = \infty$ to get an L^p result which applies to intermediate values of p, $1 < p < \infty$. A different L^p result is given by Richard Miller in his book [6, pp. 193–201]. For a discussion on how the interpolated L^p result relates to Miller's L^p result, see [9]. Also see [9] for a discussion on how the nonconvolution results given here relate to known results for convolution-type kernels.

In Section 2 we define our weighted L^1 and L^∞ spaces. The main results are given in Section 3, and are proved in Sections 4 and 5. Section 6 contains two applications to differential delay equations.

2. Operators on Weighted L^p Spaces

We begin by defining the weighted L^p spaces that we need, and describe a class of operators mapping one weighted L^p space into another.

Let (S, T) be an interval, $-\infty \leqslant S < T \leqslant \infty$. Let κ be a continuous, strictly positive function on (S, T). We define the weighted L^1 space $L^1(S, T; \kappa)$ to be the set of measurable, $\mathbf{R}^n$-valued functions x on (S, T) satisfying

$$\int_S^T |x(t)|/\kappa(t)\, dt < \infty.$$

Here $|x(t)|$ is the norm of $x(t)$ in $\mathbf{R}^n$. Analogously, we define $L^\infty(S, T; \kappa)$ to be the set of measurable, $\mathbf{R}^n$-valued functions x on (S, T) satisfying

$$\operatorname*{ess\,sup}_{S<t<T} |x(t)|/\kappa(t) < \infty.$$

Note that if κ is bounded from above and away from zero at the endpoints S and T, then these weighted spaces are isomorphic to the standard non-weighted L^1 and L^∞ spaces over (S, T).

In the sequel we shall use two different weight functions κ and λ. They are both supposed to be continuous and strictly positive on the interval (S, T).

LEMMA 2.1. *Let a be a measurable, n by n matrix valued function of (s, t), $S<s<t<T$, satisfying*

$$\operatorname*{ess\,sup}_{S<t<T}[\lambda(t)]^{-1} \int_S^t |a(t, s)|\, \kappa(s)\, ds < \infty. \tag{2.1}$$

Then the operator $x \to \int_S^t a(t, s)\, x(s)\, ds$ is continuous from $L^\infty(S, T; \kappa)$ into $L^\infty(S, T; \lambda)$. If a satisfies

$$\operatorname*{ess\,sup}_{S<s<T} \kappa(s) \int_s^T [\lambda(t)]^{-1}\, |a(t, s)|\, dt < \infty \tag{2.2}$$

instead of (2.1), *then the same operator is continuous from $L^1(S, T; \kappa)$ into $L^1(S, T; \lambda)$.*

Here $|a(t, s)|$ stands for a matrix norm of $a(t, s)$ compatible with the norm used in $\mathbb{R}^n$. The proof of Lemma 2.1 is a direct application of the version of Fubini's theorem which is given in [8, Theorem 7.12]. In [3, Theorem 2.6.1] it is shown that the condition (2.1) is necessary as well as sufficient in the L^∞ case.

On several occasions we shall also need "the adjoint operator" induced by a, namely, the operator $y \mapsto \int_s^T y(t)\, a(t, s)\, dt$. Here y is a row vector, whereas x was a column vector. It is easy to deduce from Lemma 2.1 that the adjoint operator is continuous from $L^1(S, T; \lambda^{-1})$ into $L^1(S, T; \kappa^{-1})$ if (2.1) holds, and from $L^\infty(S, T; \lambda^{-1})$ into $L^\infty(S, T; \kappa^{-1})$ if (2.2) holds.

3. THE VOLTERRA EQUATION

As in the previous section, let (S, T) be an interval, $-\infty \leqslant S < T \leqslant \infty$. We are interested in the solutions of the Volterra equation

$$x(t) = \int_S^t a(t, s)\, x(s)\, ds + f(t), \qquad S<t<T. \tag{3.1}$$

It turns out that it is convenient to also introduce the corresponding adjoint Volterra equation, i.e., the equation

$$y(s) = \int_s^T y(t)\, a(t, s)\, dt + g(s), \qquad S < s < T. \tag{3.2}$$

Here, and throughout in the sequel, a is an n by n matrix valued, measurable function of (s, t), $S < s < t < T$, x and f are measurable, $\mathbf{R}^n$ column vector valued functions on $S < t < T$, and y and g are measurable, $\mathbf{R}^n$ row vector valued functions on $S < s < T$.

We ask the following question: If the forcing functions f and g in (3.1) and (3.2) belong to a weighted L^1 or L^∞ space, is it then true that the solutions x and y belong to another weighted L^1 or L^∞ space? Before we give an answer to this question we want to discuss a local, nonweighted result.

THEOREM 3.1. (i) *Let a satisfy* (2.1) *with $\kappa \equiv \lambda \equiv 1$, let $-\infty < S < T < \infty$, and suppose that there exist constants ε and γ such that*

$$\operatorname*{ess\,sup}_{S' < t < T'} \int_{T'}^t |a(t, s)|\, ds \leqslant \gamma < 1, \qquad S' - T' < \varepsilon,\ S < S' < T' < T. \tag{3.3}$$

Then there is a unique function $r(t, s)$ satisfying

$$\operatorname*{ess\,sup}_{S < t < T} \int_S^t |r(t, s)|\, ds < \infty, \tag{3.4}$$

and for almost all (s, t), $S < s < t < T$,

$$r(t, s) = a(t, s) + \int_s^t a(t, v)\, r(v, s)\, dv, \tag{3.5}$$

and

$$r(t, s) = a(t, s) + \int_s^t r(t, v)\, a(v, s)\, dv. \tag{3.6}$$

Moreover, for each $f \in L^\infty(S, T)$ there is a unique solution $x \in L^\infty(S, T)$ of (3.1), *namely,*

$$x(t) = f(t) + \int_S^t r(t, s)\, f(s)\, ds, \tag{3.7}$$

and for each $g \in L^1(S, T)$ there is a unique solution $x \in L^1(S, T)$ of (3.2), *namely,*

$$y(s) = g(s) + \int_s^T g(t)\, r(t, s)\, dt. \tag{3.8}$$

(ii) *Let a satisfy* (2.2) *with* $\kappa \equiv \lambda \equiv 1$, *let* $-\infty < S < T < \infty$, *and suppose that there exist constants* ε *and* γ *such that*

$$\operatorname*{ess\,sup}_{S' < s < T'} \int_s^{T'} |a(t, s)| \, dt \leqslant \gamma < 1, \qquad T' - S' < \varepsilon, \; S < S' < T' < T. \tag{3.9}$$

Then there is a unique function $r(t, s)$ *satisfying*

$$\operatorname*{ess\,sup}_{S < s < T} \int_s^{T} |r(t, s)| \, dt < \infty, \tag{3.10}$$

and also (3.5) *and* (3.6) *for almost all* (s, t). *Moreover, for each* $f \in L^1(S, T)$ *there is a unique solution* $x \in L^1(S, T)$ *of* (3.1), *namely*, (3.7), *and for each* $g \in L^\infty(S, T)$ *there is a unique solution* $y \in L^\infty(S, T)$ *of* (3.2), *namely*, (3.8).

Gustaf Gripenberg proves the existence of a resolvent operator r satisfying (3.4), (3.5), and (3.6) in [5, Theorem 1]. Once the existence of such a resolvent operator is known, the proof of the rest of Theorem 3.1(i) is straightforward. Part (ii) of the theorem can be reduced to part (i) by a simple change of variables (see Section 4). By and large, Theorem 3.1 can be considered as an essentially known result. Still, for the convenience of the reader we have included a proof of Theorem 3.1 in Section 4 (a reader familiar with [5] may skip the proof of Lemma 4.1 below).

One version of Theorem 3.1 is also true when $S = -\infty$ and/or $T = \infty$. One simply adds smallness assumptions similar to (3.9) at plus and minus infinity. To get this version, take $\kappa \equiv \lambda \equiv 1$ in Theorem 3.3 below. The version one gets by taking $T = \infty$ in part (i) was discovered by Gripenberg [5, Theorem 3]. For still earlier related versions, see [4, Theorem 3, Part 1] and [10, p. 573].

We want to prove a global, weighted version of Theorem 3.1. As our weight functions are bounded from above and away from zero as long as one stays away from the endpoints S and T, in the interior of the interval (S, T) the weighted result is more or less equivalent to the nonweighted result. We shall therefore assume throughout that the hypothesis of Theorem 3.1 holds locally, i.e., that it holds whenever the interval (S, T) is replaced by an interval (S', T'), satisfying $S < S' < T' < T$.

To get an estimate on the growth rate of the resolvent close to the endpoints S and T we use the following lemma:

LEMMA 3.2. (i) *Let the assumption of Theorem* 3.1(i) *hold locally on* (S, T), *and suppose that for some* $\varepsilon > 0$,

$$\operatorname*{ess\,sup}_{S < t < T} \int_S^t \left[\frac{\lambda(s)}{\lambda(t)} + \varepsilon \frac{\kappa(s)}{\lambda(t)} \right] |a(t, s)| \, ds \leqslant 1. \tag{3.11}$$

Then the resolvent in Theorem 3.1 *satisfies*

$$\operatorname*{ess\,sup}_{S<t<T}[\lambda(t)]^{-1}\int_S^t |r(t,s)|\,\kappa(s)\,ds \leqslant 1/\varepsilon. \tag{3.12}$$

(ii) *Let the assumption of Theorem* 3.1(ii) *hold locally on* (S, T), *and suppose that for some* $\varepsilon > 0$,

$$\operatorname*{ess\,sup}_{S<s<T}\int_s^T \left[\varepsilon\frac{\kappa(s)}{\lambda(t)}+\frac{\kappa(s)}{\kappa(t)}\right]|a(t,s)|\,dt \leqslant 1. \tag{3.13}$$

Then the resolvent in Theorem 3.1 *satisfies*

$$\operatorname*{ess\,sup}_{S<s<T}\kappa(s)\int_s^T [\lambda(t)]^{-1}\,|r(t,s)|\,dt \leqslant 1/\varepsilon. \tag{3.14}$$

The proof of Lemma 3.2, given in Section 5, has been modeled after the proof of Theorem 2.1 in [2].

The following theorem is our main result:

THEOREM 3.3. (i) *Let the hypothesis of Theorem* 3.1(i) *hold locally on* (S, T). *Suppose that there exist constants* S' *and* T', $S < S' < T' < T$, *such that* (3.11) *holds with* T *replaced by* S', *and with* S *replaced by* T'. *In addition, suppose that for each* V, $S < V < T$,

$$\operatorname*{ess\,sup}_{V<t<T}\left[\frac{1}{\kappa(t)}+\frac{1}{\lambda(t)}\right]\int_S^V |a(t,s)|\,[\kappa(s)+\lambda(s)]\,ds < \infty. \tag{3.15}$$

Then the resolvent r *in Theorem* 3.1 *satisfies*

$$\operatorname*{ess\,sup}_{S<t<T}[\lambda(t)]^{-1}\int_S^t |r(t,s)|\,\kappa(s)\,ds < \infty. \tag{3.16}$$

Moreover, for each $f \in L^\infty(S, T; \kappa)$, *formula* (3.7) *defines a solution* $x \in L^\infty(S, T; \lambda)$ *of* (3.1), *and for each* $g \in L^1(S, T; \lambda^{-1})$ *formula* (3.8) *defines a solution* $y \in L^1(S, T; \kappa^{-1})$ *of* (3.2).

(ii) *Let the hypothesis of Theorem* 3.1(ii) *hold locally on* (S, T), *and suppose that there exist constants* S' *and* T', $S < S' < T' < T$, *such that* (3.13) *holds with* T *replaced by* S', *and with* S *replaced by* T'. *In addition, suppose that for each* V, $S < V < T$,

$$\operatorname*{ess\,sup}_{S<s<V}[\kappa(s)+\lambda(s)]\int_V^T \left[\frac{1}{\kappa(t)}+\frac{1}{\lambda(t)}\right]|a(t,s)|\,dt < \infty. \tag{3.17}$$

Then the resolvent r in Theorem 3.1 *satisfies*

$$\operatorname*{ess\,sup}_{S<s<T} \kappa(s) \int_s^T [\lambda(t)]^{-1} |r(t,s)| \, dt < \infty. \tag{3.18}$$

Moreover, for each $f \in L^1(S, T; \kappa)$, *formula* (3.7) *defines a solution* $x \in L^1(S, T; \lambda)$ *of* (3.1), *and for each* $g \in L^\infty(S, T; \lambda^{-1})$ *formula* (3.8) *defines a solution* $y \in L^\infty(S, T; \kappa^{-1})$ *of* (3.2).

(iii) *In both* (i) *and* (ii), *if* $\kappa(s)/\lambda(s)$ *is bounded away from zero as* $s \to S+$, *then the solution x of* (3.1) *is unique in* $L^\infty(S, T; \lambda)$ *or* $L^1(S, T; \lambda)$, *and if* $\kappa(s)/\lambda(s)$ *is bounded from above as* $s \to T-$, *then the solution y of* (3.2) *is unique in* $L^1(S, T; \kappa^{-1})$ *or* $L^\infty(S, T; \kappa^{-1})$.

4. Proof of Theorem 3.1

The proof of Theorem 3.1 is based on two lemmas. Our Lemma 4.1 is essentially extracted from the proof of [5, Theorem 1], but our Lemma 4.2 differs from Gripenberg's corresponding argument.

Lemma 4.1. (i) *Suppose that*

$$\operatorname*{ess\,sup}_{S<t<T} \int_S^t |a(t,s)| \, ds \leqslant \gamma < 1. \tag{4.1}$$

Then there exists a resolvent r satisfying (3.4), *and for almost all* (s, t), *also* (3.5) *and* (3.6).

(ii) *If instead,*

$$\operatorname*{ess\,sup}_{S<s<T} \int_s^T |a(t,s)| \, dt \leqslant \gamma < 1, \tag{4.2}$$

then there exists a resolvent r satisfying (3.10), *and for almost all* (s, t), *also* (3.5) *and* (3.6).

Lemma 4.1 is also true with $S = -\infty$ and/or $T = \infty$.

Before we prove Lemma 4.1, let us make a remark on the nature of the integrals in (3.5) and (3.6). For simplicity we discuss only (3.5), with one type of bounds on a and r, but the same discussion applies to all the different cases. We claim that if a satisfies (2.1) with $\kappa \equiv \lambda \equiv 1$, and r satisfies (3.4), then the integral $\int_S^t a(t,v)\, r(v,s) \, dv$ is measurable in (s, t), and it satisfies the same type of bound as a and r do. That this is true one can see in the following way. The functions in this integral are jointly measurable in the three variables s, t, and v. So is the characteristic function of the set

$s < v < t$. The product of these three functions is integrable. Apply Fubini's theorem [8, Theorem 7.12], grouping (s, t) together to one variable to get the measurability of the integral. One more application of Fubini's theorem gives the desired bound.

Let us also note that it suffices to prove only part (i) of Lemma 4.1, as part (ii) can be reduced to part (i) as follows. Define $b(t, s)$ to be the transpose of $a(-s, -t)$, and apply part (i) with a replaced by b, and the interval (S, T) by the interval $(-T, -S)$. This gives a function r, satisfying (3.4), (3.5), and (3.6) in the interval $(-T, -S)$. The desired solution in part (ii) is the transpose of $r(-s, -t)$.

The same transformation can be applied to reduce parts (ii) of all of our Lemmas and Theorems to the corresponding parts (i). Therefore throughout we omit the proofs of the second parts of the Lemmas and Theorems. Observe that this transformation interchanges (3.1) and (3.2) with each other, (3.5) and (3.6) with each other, and κ and λ^{-1} with each other. (This is our main reason for studying (3.2) in addition to (3.1).)

Proof of Lemma 4.1. Define

$$r_1(t, s) = a(t, s), \tag{4.3}$$

$$r_{n+1}(t, s) = \int_s^t a(t, v)\, r_n(v, s), \qquad n \geq 1, \tag{4.4}$$

for almost all (t, s), $S < s < t < T$. It follows from (4.1), (4.3), (4.4), Fubini's theorem, and an induction argument that the functions $r_n(t, s)$ are measurable in (s, t), $S < s < t < T$, and that

$$\operatorname*{ess\,sup}_{S<t<T} \int_S^t |r_n(t, s)|\, ds \leq \gamma^n. \tag{4.5}$$

Formally, we can define

$$r(t, s) = \sum_{n=1}^{\infty} r_n(t, s), \qquad S < s < t < T. \tag{4.6}$$

By (4.1), (4.3), (4.4), (4.5), and Lebesgue's dominated convergence theorem, $r(t, s)$, as defined in (4.6), exists for almost all (s, t), $S < s < t < T$, it is measurable, it satisfies (3.4), and (3.5) holds.

We claim that each r_n in addition to (4.4) satisfies also

$$r_{n+1}(t, s) = \int_s^t r_n(t, v)\, a(v, s)\, dv, \qquad n \geq 1, \tag{4.7}$$

for almost all (s, t), $S < s < t < T$. To prove this claim, use (4.3), (4.4), Fubini's theorem, and an induction argument. Now (4.4), (4.5), (4.6), (4.7), and Lebesgue's dominated convergence theorem give

$$\int_s^t r(t, v)\, a(v, s)\, dv = \int_s^t a(t, v)\, r(v, s)\, dv$$

for almost all (s, t), $S < s < t < T$. Together with (3.5) this implies (3.6), and completes the proof of Lemma 4.1. ▮

LEMMA 4.2. (i) *Let a satisfy* (2.1) *with* $\kappa \equiv \lambda \equiv 1$. *Let* r_1 *be an almost everywhere solution of* (3.5) *and* (3.6) *on the interval* (S, V), *satisfying* (3.4) *with* T *replaced by* V, *and let* r_2 *be an almost everywhere solution of* (3.5), (3.6) *on the interval* (V, T), *satisfying* (3.4) *with* S *replaced by* V. *Define* $r(t, s)$ *almost everywhere by* $r(t, s) = r_1(t, s)$, $S < s < t < V$,

$$r(t, s) = a(t, s) + \int_s^V a(t, u)\, r_1(u, s)\, du + \int_V^t r_2(t, v)\, a(v, s)\, dv$$

$$+ \int_V^t r_2(t, v)\, dv \int_s^V a(v, u)\, r_1(u, s)\, du, \qquad S < s < V < t < T, \quad (4.8)$$

and $r(t, s) = r_2(t, s)$, $V < s < t < T$. *Then r satisfies* (3.4), *and for almost all* (s, t), *also* (3.5) *and* (3.6).

(ii) *Let a satisfy* (2.2) *with* $\kappa \equiv \lambda \equiv 1$. *Let* r_1 *be an almost everywhere solution of* (3.5) *and* (3.6) *on the interval* (S, V), *satisfying* (3.10) *with* T *replaced by* V, *and let* r_2 *be an almost everywhere solution of* (3.5), (3.6) *on the interval* (V, T), *satisfying* (3.10) *with* S *replaced by* V. *Define* $r(t, s)$ *in the same way as in part* (i). *Then r satisfies* (3.10), *and for almost all* (s, t), *also* (3.5) *and* (3.6).

Formally, one can derive (4.8), e.g., by writing (3.5) in the form

$$r(t, s) = a(t, s) + \int_s^V a(t, u)\, r(u, s)\, du + \int_V^t a(t, u)\, r(u, s)\, du,$$

and solving this equation by using the variation of constants formula (3.7), with S replaced by V.

The proof of Lemma 4.2 is rather long, but it is quite straightforward. We therefore leave it to the reader. That r satisfies (3.4) or (3.10) is proved in the same way as in the proof of Lemma 5.1 below (take $\kappa \equiv \lambda \equiv 1$ in the proof of Lemma 5.1). To prove that r satisfies (3.5) and (3.6) one simply multiplies (4.3) by a from the left and from the right, integrates, and uses Fubini's theorem and the fact that r_1 and r_2 satisfy (3.5) and (3.6) on their respective intervals of definition.

Proof of Theorem 3.1. Divide the interval (S, T) into a finite number of subintervals, each with length at most ε. Then by (3.3) and Lemma 4.1, in each subinterval we have a solution of (3.5), (3.6) satisfying (3.4). Applying Lemma 4.2 repeatedly to two successive intervals at a time we conclude that there exists a function r, satisfying (3.4), (3.5), and (3.6) in the whole interval (S, T).

To prove the uniqueness of r, suppose that r_1 and r_2 satisfy (3.4), (3.5), and (3.6). Multiply (3.6) with r replaced by r_2 from the right by r_1, integrate, and use Fubini's theorem and the fact that r_1 satisfies (3.5) to show that for almost all (s, t),

$$\int_s^t a(t, v)\, r_2(v, s)\, dv = \int_s^t r_1(t, v)\, a(v, s)\, dv.$$

This together with (3.5) and (3.6) imply that $r_1(t, s) = r_2(t, s)$ for almost all (s, t).

Let $f \in L^\infty(S, T)$, and define x by (3.7). Then, by Lemma 2.1 with $\kappa \equiv \lambda \equiv 1$, $x \in L^\infty(S, T)$. Multiply (3.7) from the left by a, integrate, and use (3.5) to show that for almost all t,

$$\int_S^t a(t, s)\, x(s)\, ds = \int_S^t r(t, s)\, f(s)\, ds. \tag{4.9}$$

Together with (3.7) this implies that x is a solution of (3.1). To prove that (3.7) is the unique solution of (3.1), multiply (3.1) from the left by r, integrate, and use (3.6) to get (4.9). Substitute (4.9) into (3.1) to get (3.7).

The proof of the claim concerning the solution (3.8) of the adjoint equation (3.2) is completely analogous, so we leave it to the reader. ∎

5. Proof of Theorem 3.3

We begin the proof of Theorem 3.3 by proving Lemma 3.2. We remind the reader of the argument in Section 4, which shows that part (ii) of all our Lemmas and Theorems can be reduced to the corresponding parts (i). Therefore, we only prove parts (i) below.

Proof of Lemma 3.2. Define $\varphi(s) = \lambda(s) + \varepsilon\kappa(s)$. Multiply (3.6) by $\varphi(s)$, integrate, and use (3.11) to get

$$\begin{aligned} &\operatorname*{ess\,sup}_{S<t<T} [\lambda(t)]^{-1} \int_S^t |r(t, s)|\, \varphi(s)\, ds \\ &\quad \leqslant \operatorname*{ess\,sup}_{S<t<T} [\lambda(t)]^{-1} \int_S^t |a(t, s)|\, \varphi(s)\, ds \end{aligned}$$

$$+ \operatorname*{ess\,sup}_{S<t<T} [\lambda(t)]^{-1} \int_S^t |r(t, v)| \, dv \int_S^v |a(v, s)| \, \varphi(s) \, ds$$

$$\leqslant 1 + \operatorname*{ess\,sup}_{S<t<T} [\lambda(t)]^{-1} \int_S^t |r(t, v)| \, \lambda(v) \, dv.$$

If the very last term is finite, then we can move it over to the far left side, and get (3.12). If it is not, then we replace the interval (S, T) by an interval (S', T'), with $S < S' < T' < T$. In this new interval, because of Theorem 3.1 and the fact that λ is bounded from above and away from zero, the last term is finite, and we get (3.12) with (S, T) replaced by (S', T'). Letting $S' \to S+$ and $T' \to T-$ we finally get (3.12). ∎

In our proof of Theorem 3.3 we also need the following modified version of Lemma 4.2.

LEMMA 5.1. (i) *Let $S < V < T$, and let the hypothesis of Theorem* 3.1(i) *hold locally on (S, T). Let the solution r of* (3.5) *and* (3.6) *satisfy* (3.16) *with T replaced by V, and with S replaced by V. Also suppose that* (3.15) *holds. Then r satisfies* (3.16).

(ii) *Let $S < V < T$, and let the hypothesis of Theorem* 3.1(ii) *hold locally on (S, T). Let the solution r of* (3.5) *and* (3.6) *satisfy* (3.18) *with T replaced by V, and with S replaced by V. Also suppose that* (3.17) *holds. Then r satisfies* (3.18).

Proof of Lemma 5.1. Clearly, as we assume that (3.16) holds with T replaced by V, and with S replaced by V, to prove (3.16) it suffices to show that

$$\operatorname*{ess\,sup}_{V<t<T} [\lambda(t)]^{-1} \int_S^V |r(t, s)| \, \kappa(s) \, ds < \infty.$$

It follows from (4.8) and the uniqueness of the resolvent operator that for almost all (s, t), $S < s < V < t < T$,

$$r(t, s) = k(t, s) + \int_s^V k(t, v) \, r(v, s) \, dv, \tag{5.1}$$

where $k(t, s)$ is defined for almost all (s, t), $S < s < V < t < T$, by

$$k(t, s) = a(t, s) + \int_V^t r(t, v) \, a(v, s) \, dv.$$

Now, for almost all t, $V<t<T$,

$$
\begin{aligned}
&[\lambda(t)]^{-1}\int_S^V |k(t,s)|[\kappa(s)+\lambda(s)]\,ds \\
&\quad\leqslant [\lambda(t)]^{-1}\int_S^V |a(t,s)|[\kappa(s)+\lambda(s)]\,ds \\
&\qquad + [\lambda(t)]^{-1}\int_V^t |r(t,v)|\,dv \int_S^V |a(v,s)|[\kappa(s)+\lambda(s)]\,ds,
\end{aligned}
$$

so by (3.16) with S replaced by V, and by (3.15),

$$
\operatorname*{ess\,sup}_{V<t<T}[\lambda(t)]^{-1}\int_v^T |k(t,s)|[\kappa(s)+\lambda(s)]\,ds<\infty. \tag{5.2}
$$

Use (5.1), (5.2) and the fact that (3.16) holds with T replaced by V to get

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{V<t<T}[\lambda(t)]^{-1}\int_S^V |r(t,s)|\,\kappa(s)\,ds \\
&\quad\leqslant \operatorname*{ess\,sup}_{V<t<T}[\lambda(t)]^{-1}\int_S^V |k(t,s)|\,\kappa(s)\,ds \\
&\qquad + \operatorname*{ess\,sup}_{V<t<T}[\lambda(t)]^{-1}\int_S^V |k(t,v)|\,dv\int_S^v |r(v,s)|\,\kappa(s)\,ds<\infty.
\end{aligned}
$$

This completes the proof of Lemma 5.1. ∎

Proof of Theorem 3.3. Divide the interval (S,T) into three intervals (S,S'), (S',T'), and (T',T). By Lemma 3.2, in the first and the last interval (3.16) is satisfied, i.e., (3.16) holds with T replaced by S', and with S replaced by T'. It is also satisfied in the middle interval, because κ and λ are bounded from above and away from zero in (S',T'), and (3.4) holds with S replaced by S' and T replaced by T'. Applying Lemma 5.1 two times we get (3.16).

To prove that (3.7) is a solution of (3.1), observe that our hypothesis implies

$$
\operatorname*{ess\,sup}_{S<t<T}[\lambda(t)]^{-1}\int_S^t |a(t,s)|[\kappa(s)+\lambda(s)]\,ds<\infty,
$$

and argue exactly in the same way as in the proof of the corresponding claim of Theorem 3.1.

The uniqueness of the solution of (3.1) in part (iii) follows from the fact that if $\kappa(s)/\lambda(s)$ is bounded away from zero as $s \to S+$, then for each T', $S < T' < T$, (3.16) implies that

$$\operatorname*{ess\,sup}_{S<t<T'} [\lambda(t)]^{-1} \int_S^t |r(t,s)| [\kappa(s) + \lambda(s)] \, ds < \infty.$$

When this condition is true, we can use the same uniqueness computation as in the proof of Theorem 3.1 to get uniqueness in the interval (S, T'). Letting $T' \to T$ we get uniqueness in the whole interval (S, T).

The uniqueness of the solution of the adjoint equation is proved in the same way. ∎

6. Two Examples

Let us illustrate what type of results one can get from Theorem 3.3 by applying it to two examples, namely, to the equations

$$x(t) = \int_{t-r}^t a(s)\, x(s)\, ds + f(t), \qquad 0 < t < \infty, \tag{6.1}$$

and

$$x(t) = a(t) \int_{t-r}^t x(s)\, ds + f(t), \qquad 0 < t < \infty. \tag{6.2}$$

In both equations we give an initial condition

$$x(t) = \varphi(t), \qquad -r < t < 0. \tag{6.3}$$

These two equations are closely related to certain differential delay equations. Differentiating (6.1) one gets

$$x'(t) = a(t)\, x(t) - a(t-r)\, x(t-r) + f'(t), \qquad 0 < t < \infty,$$

and (6.2) is the equation satisfied by the derivative of the solution of

$$x'(t) = a(t)[x(t) - x(t-r)] + f(t), \qquad 0 < t < \infty. \tag{6.4}$$

Also note that if x is a solution of (6.2), then $\int_{t-r}^t x(s)\, ds$ is a solution of (6.1), with f replaced by $\int_{t-r}^t f(s)\, ds$, plus a correction term supported on $[0, r]$, originating from the initial condition (6.3).

To apply Theorem 3.3, we have to write (6.1) and (6.2) in the standard form (3.1). Define

$$\begin{aligned} b(t,s) &= a(s), \qquad t-r<s<t,\ s>0, \\ &= 0, \qquad \text{otherwise}, \end{aligned} \tag{6.5}$$

$$\begin{aligned} c(t,s) &= a(t), \qquad 0<s<t<s+r, \\ &= 0, \qquad \text{otherwise}. \end{aligned} \tag{6.6}$$

Then (6.1) becomes

$$x(t)=\int_0^t b(t,s)\,x(s)\,ds+g(t), \qquad 0<t<\infty, \tag{6.7}$$

where

$$\begin{aligned} g(t) &= f(t)+\int_{t-r}^{0} a(s)\,\varphi(s)\,ds, \qquad 0<t<r, \\ &= f(t), \qquad r<t<\infty, \end{aligned}$$

and (6.2) becomes

$$x(t)=\int_0^t c(t,s)\,x(s)\,ds+h(t), \qquad 0<t<\infty, \tag{6.8}$$

where

$$\begin{aligned} h(t) &= f(t)+a(t)\int_{t-r}^{0} \varphi(s)\,ds, \qquad 0<t<r, \\ &= f(t), \qquad r<t<\infty. \end{aligned}$$

We suppose that a is locally integrable, that κ and λ are strictly positive functions on $[0,\infty)$, that $g\in L^\infty(0,\infty;\kappa)$, and that $h\in L^1(0,\infty;\kappa)$. We want to apply Theorem 3.3(i) to (6.7), and Theorem 3.3(ii) to (6.8). Because of (6.5), (6.6) and the local integrability of a, on each interval $(0,T)$ with $0<T<\infty$, the kernel b satisfies (2.1) with $\kappa\equiv\lambda\equiv 1$, and also (3.3). Likewise, on each interval $(0,T)$, $0<T<\infty$, the kernel c satisfies (2.2) with $\kappa\equiv\lambda\equiv 1$, and also (3.9). It is also easy to show that (3.15) and (3.17) hold (as $b(t,s)=c(t,s)=0$ for $t-s>r$, one can restrict t to the interval $V<t<V+r$, and s to the interval $V<s<V+r$; observe that (3.15) and (3.17) need not hold uniformly in V). In this case, as κ and λ are continuous and strictly positive at zero, the conditions (3.11) and (3.13)

will automatically be true on an interval $(0, T)$, for T sufficiently small. The only conditions which are not more or less automatically satisfied are (3.11) and (3.13) with $T=\infty$, and with S to be chosen freely, $0<S<\infty$. For instance, if we take $\kappa(t)=(1+t)^{-1}$ and $\lambda(t)\equiv 1$, then (3.11) is satisfied with a replaced by b if it is true that

$$\int_{t-r}^{t} |a(s)|\, ds \leqslant 1-K/t, \qquad S<t<\infty, \tag{6.9}$$

for some positive constants K and S. With the same choice of κ and λ, we have (3.13) satisfied with a replaced by c if (6.9) holds for some $S>0$ and $K>r$ (note that the condition $K>0$ is not sufficient in this case).

We have obtained the following result: If a is locally integrable on $[0, \infty)$, if (6.9) holds for some positive constants K and S, and if g is measurable and satisfies

$$\operatorname*{ess\,sup}_{0<t<\infty}(1+t)\,|g(t)|<\infty,$$

then the solution x of (6.7) satisfies

$$\operatorname*{ess\,sup}_{0<t<\infty} |x(t)|<\infty.$$

If in addition (6.9) is true for some constant $K>r$, and if h is measurable and satisfies

$$\int_0^\infty (1+t)\,|h(t)|\, dt<\infty,$$

then the solution x of (6.8) satisfies

$$\int_0^\infty |x(t)|\, dt<\infty.$$

As we mentioned earlier, the derivative of the solution of (6.4) satisfies (6.2). Equation (6.4) has been studied, e.g., in [1] and [2]. The condition (6.9) with $K>r$ is the same as the condition Atkinson and Haddock get in [1, Theorem 3.2]. This is no coincidence. Recall that condition (6.9) with $K>r$ comes from Lemma 3.2(ii), and that the proof of Lemma 3.2(ii) has been modeled after the proof of Theorem 2.1 in [2].

If a satisfies

$$\liminf_{t\to\infty} \int_{t-r}^{t} |a(s)|\, ds<1$$

rather than (6.9), then one can choose κ and λ to be decaying exponentials, and one gets exponential convergence of the solutions of (6.7) and (6.8) to zero (cf. [2, Corollary 1 and Theorem 2.2]).

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