Scale mixtures of Kotz–Dirichlet distributions

N. Balakrishnan a, E. Hashorva b,*

a Department of Mathematics and Statistics, McMaster University, 1280 Main Street West Hamilton, Ontario, Canada L8S 4K1
b Department of Actuarial Science, Faculty of Business and Economics, University of Lausanne Bâtiment Extraneuf, UNIL-Dorigny, 1015 Lausanne, Switzerland

A R T I C L E   I N F O

Article history:
Available online 23 August 2011

AMS subject classifications:
primary 60F05
secondary 60G70

Keywords:
Pearson–Kotz Dirichlet distribution
Dirichlet distribution
Kotz type distribution
Elliptical distribution
t-distribution
Conditional limiting theorem
Conditional excess distribution
Coefficient of tail dependence
Random scaling

A B S T R A C T

In this paper, we first show that a \(k\)-dimensional Dirichlet random vector has independent components if and only if it is a Kotz Type I Dirichlet random vector. We then consider in detail the class of \(k\)-dimensional scale mixtures of Kotz–Dirichlet random vectors, which is a natural extension of the class of Kotz Type I random vectors. An interesting member of the Kotz–Dirichlet class of multivariate distributions is the family of Pearson–Kotz Dirichlet distributions, for which we present a new distributional property. In an asymptotic framework, we show that the Kotz Type I Dirichlet distributions approximate the conditional distributions of scale mixtures of Kotz–Dirichlet random vectors. Furthermore, we show that the tail indices of regularly varying Dirichlet random vectors can be expressed in terms of the Kotz Type I Dirichlet random vectors.

© 2011 Elsevier Inc. All rights reserved.

1. Introduction

Let \(X_1\) and \(X_2\) be two independent random variables that are symmetric about zero satisfying further \(|X_i|^p \sim \text{gamma}(\alpha_i, 1/p)\), with \(\alpha_1, \alpha_2, p\) as positive constants. By \(\text{gamma}(a, b)\), we mean the gamma distribution function (df) with positive parameters \(a, b\), and \(Y \sim \text{gamma}(a, b)\) means that the random variable \(Y\) has df \(\text{gamma}(a, b)\); we then say \(X = (X_1, X_2)^\top\) is a Kotz Type I Dirichlet random vector. In the simple case when \(\alpha_1 = \alpha_2 = 1/p = 1/2\), \(X\) is bivariate Gaussian with independent components (\(^\top\) denotes the transpose sign). By the properties of generalized symmetrized Dirichlet random vectors (simply written hereafter as Dirichlet random vectors) introduced in Fang and Fang [13], we have the following equality of distribution functions (the corresponding notation is \(\overset{d}{=}\))

\[X \overset{d}{=} (RU_1, RU_2)^\top,\]  

(1)

with \(|U_1|^p + |U_2|^p = 1\) almost surely, \(R > 0\) independently of \(U = (U_1, U_2)^\top\), and \(R^p \sim \text{gamma}(\alpha_1 + \alpha_2, 1/p)\). Each component \(U_i\) (\(i = 1, 2\)) is symmetric about 0 and \(|U_i|^p\) is beta distributed with parameters \(\alpha_1, \alpha_2\). If \(R\) in (1) is positive with some df \(F\), then \(X\) is a Dirichlet random vector; see, for example, [13,17].

It is well-known that Gaussian random vectors possess some key distributional and asymptotic properties in the class of elliptically symmetric random vectors; interested readers may refer to [7,6,23]. As shall be shown in the present work, a
similar role is played by the Kotz Type I Dirichlet random vectors. Specifically, we prove that a Dirichlet random vector has independent components if and only if it is a Kotz Type I Dirichlet random vector.

The Kotz Type I Dirichlet random vectors are quite useful for deriving conditional distributional results, which are often of interest in a number of applications; see e.g., [29,8,18]. In fact, if $X$ is such a random vector, then for any $x \in \mathbb{R}$, the conditional random variable $Z_x \overset{d}{=} X | (X_1 = x)$ has the same df as $X_2$, i.e., $Z_x \overset{d}{=} X_2$ since both $X_1$ and $X_2$ are independent. In general, for a Dirichlet random vector $X$ with stochastic representation (1), the df of $Z_x$ depends on $x$. Surprisingly, as $x \to \infty$, we can approximate the df of $Z_x$ by some symmetrized gamma distribution, provided that $F$ is in the Gumbel max-domain of attraction (MDA). The importance of such approximations, referred to as Kotz approximations, relates to the fact that even if we do not know the df of some Dirichlet random vector, since $F$ will be usually unknown, we still can approximate the conditional distribution by the Kotz Type I distribution.

A similar role in the approximation of the conditional distributions of regularly varying Dirichlet random vectors is played by the Pearson–Kotz Dirichlet distribution which is defined via (1), where $R^d$ has the beta distribution of second kind with parameters $\alpha_1 + \alpha_2$ and $\theta > 0$; see [4]. The Pearson–Kotz Dirichlet random vectors are another important subclass of the Dirichlet class including two prominent cases: the Student $t$ random vectors (see e.g., [24] or [19] and the Pearson Type VII $L_p$ random vectors introduced by Gupta and Song [14]. Interestingly, the Pearson–Kotz Dirichlet random vectors are a special case of the scale mixtures of the Kotz–Dirichlet random vectors.

Given the prominent role that the Kotz–Dirichlet random vectors play in the literature, we study here the distributional as well as the asymptotic properties of the scale mixtures of Kotz–Dirichlet random vectors. After presenting the preliminary details in Section 2, we highlight in Section 3 the central distributional role of the Kotz Type I random vectors among the Dirichlet random vectors. Next, in Section 4, we discuss the Kotz approximation followed by Section 5, whereas we show that Kotz Type I Dirichlet random vectors are useful in the calculation of tail indices of regularly varying Dirichlet random vectors. Then, in Section 6, we obtain a distributional result for Pearson–Kotz Dirichlet random vectors. The proofs are relegated to Section 7 followed by an Appendix where we present an additional useful lemma.

2. Preliminaries

In this section, we describe our notation first and then present some definitions and results needed thereafter. We shall denote by $\text{beta}(a, b)$ and $\text{gamma}(a, b)$ the df of a beta and a gamma random variable with positive parameters $a, b$ with probability density functions

$$
\frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} x^{a-1}(1-x)^{b-1} \quad \text{for } x \in (0, 1), \quad \text{and} \quad \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx) \quad \text{for } x \in (0, \infty),
$$

respectively, where $\Gamma(\cdot)$ is the Euler gamma function. The beta distribution of second kind $\text{beta}_{\theta}(a_1, a_2)$ with positive parameters $a_1, a_2$ is the df of $Y_1/Y_2$, where $Y_i \sim \text{gamma}(a_i, 1)$, $i = 1, 2$, are independent; see [21]. Its probability density function (pdf) is

$$
\frac{\Gamma(a_1 + a_2)}{\Gamma(a_1) \Gamma(a_2)} x^{a_1-1}(1+x)^{-a_1-a_2} \quad \text{for } x \in (0, \infty).
$$

For any $k$-dimensional vector $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$, $k \geq 2$, we define its subvector with respect to some non-empty index subset $I \subset \{1, \ldots, k\}$ by $x_I := (x_i, i \in I)^\top$. Similarly, if $A$ is a real $k \times k$ matrix, we define its submatrices $A_{ij}, A_{ij}, A_{ij}$ for some non-empty index set $J \subset \{1, \ldots, k\} \setminus I$. Throughout the paper, the operations with vectors are meant componentwise. For instance, $x y$ is the vector $(x_1 y_1, \ldots, x_k y_k)^\top$, and we write simply $cx$, $c \in \mathbb{R}$, instead of $(cx_1, \ldots, cx_k)^\top$.

In what follows, $\alpha \in (0, \infty)^k$ and $p > 0$ are fixed parameters, and

$$
\alpha := \sum_{j=1}^k \alpha_j, \quad 1 := (1, \ldots, 1)^\top \in \mathbb{R}^k, \quad \|x\|_p := \left(\sum_{i=1}^k |x_i|^p\right)^{1/p}, \quad \text{for } x \in \mathbb{R}^k.
$$

Note in passing that $\| \cdot \|_p$ is a norm for $p \in [1, \infty)$.

If $X$ is a $k$-dimensional random vector with df $F$, we write this as $X \sim F$, and denote $\overline{F} = 1 - F$. When $k = 1$, then $\overline{F}$ is the usual survival function; we say that $\overline{F}$ is regularly varying at infinity, with some index $-\gamma \leq 0$, if

$$
\lim_{x \to \infty} \frac{\overline{F}(tx)}{\overline{F}(x)} = t^{-\gamma}, \quad \forall t > 0.
$$

One may refer to [3,11,20,28] for elaborate details on regular variation.

From extreme value theory (see e.g., [30] or [12]), it is well-known that the df $F$ with upper endpoint $x_F$ is in the Gumbel MDA if

$$
\lim_{u \to x_F} \frac{F(x + s/u(x))}{F(x)} = \exp(-s), \quad \text{for all } s \in [0, \infty),
$$

where $u(x)$ is some positive scaling function. It is well-known (see Eq. (9.2.26) of [27]) that for some $x_0 < x_F$

$$
F(x) = c(x) \exp \left(-\int_{x_0}^x u(t) \, dt\right), \quad x_0 < x < x_F,
$$
where \( c(x) \) is a measurable function such that \( \lim_{x \to 0^+} c(x) = c > 0 \), and \( w(x) \) is the scaling function in (3) such that \( a(x) = 1/w(x) \) has a positive Lebesgue density \( a' \) and \( \lim_{x \to 0^+} a'(x) = 0 \).

By this representation, for all \( u \) in a left-neighbourhood of \( x_F \) and \( \epsilon > 0 \) (see Proposition 1.1 of [10], and Eq. (6.31) of [16]),

\[
\frac{F(u + s/w(u))}{F(u)} \leq (1 + \epsilon)(1 + \epsilon s)^{-1/\epsilon}, \quad \forall s \geq 0.
\] (5)

Consequently, for any \( \gamma \geq 1 \),

\[
I_F(u) := \frac{(w(u))^{\gamma-1}}{F(u)} \frac{1}{\Gamma(\gamma)} \int_{u}^{\infty} (y-u)^{-1} dF(y) \to 1, \quad u \uparrow x_F.
\] (6)

In view of (3) for any \( \gamma > 0 \) Fatou’s Lemma implies \( \liminf_{u \uparrow x_F} I_F(u) \geq 1 \). Note that \( I_F(u), u \geq 0 \), is finite a.e. for any \( \gamma \in (0, 1) \). If \( I_F(u), u \geq 0 \), is ultimately monotone, or \( F \) is absolutely continuous with pdf \( f \) in a left-neighbourhood of \( x_F \) such that the following von Mises condition (see e.g., Proposition 1.17 in [30])

\[
\lim_{u \uparrow x_F} \frac{f(u)}{f(s)} = 1
\] (7)

holds, then the convergence in (6) is valid also for any \( \gamma \in (0, 1) \); see Lemma 6 in Appendix. The asymptotic expansion of \( I_F(u) \) implied by (6) is the key to the Kotz approximation discussed in Section 4. When \( \gamma \in (0, 1) \), the limit relation (6) might not hold in general. So, we introduce some restrictions on \( F \) which enable us to examine the asymptotic behaviour of \( I_F(u + z/w(u))/I_F(u), z \in \mathbb{R} \) as \( u \uparrow x_F \). Specifically, for the Kotz approximation discussed below, we shall further require that \( F \) possesses a pdf on a left-neighbourhood of \( x_F \) such that

\[
\lim_{u \uparrow x_F} \frac{f(u + s/w(u))}{f(u)} = \exp(-s), \quad \text{for all } s \in [0, \infty),
\] (8)

which is indeed weaker than condition (7).

Interesting tractable univariate distributions in the Gumbel MDA are the so-called Weibullian tail distributions; see [1]. Specifically, the df \( F \) has Weibullian tail asymptotics if

\[
F(x) = (1 + b(x))c_1x^2 \exp(-c_2x^2), \quad x \in \mathbb{R},
\] (9)

with \( c_i (i \leq 4) \) some positive constants and \( a(\cdot) \) such that \( \lim_{x \to -\infty} a(x) = 0 \).

3. Scale mixtures of Kotz-Type I random vectors

In \( k \geq 2 \) dimensions, for given \( \alpha \in (0, \infty)^k, p > 0 \), we can extend the definition of the Kotz Type I Dirichlet random vectors by simply considering the \( k \)-dimensional random vector \( X = (X_1, \ldots, X_k)^T \) with independent components that are symmetric about 0, such that

\[
|X_i|^p \sim \text{gamma}(\alpha_i, 1/p), \quad i = 1, \ldots, k.
\]

In order to avoid repetition, for such \( X \), we shall use the abbreviation \( X \sim \mathcal{K}_{\alpha, p} \) emphasizing its dependence on both \( \alpha \) and \( p \). In the special case when \( \alpha = 1/2 \) and \( p = 2 \), the random vector \( X \) is Gaussian with independent components.

From [13,17], it is known that \( X \) has a radial representation similar to (1) given by

\[
X \overset{d}{=} R U,
\] (10)

with \( \sum_{i=1}^{k} |U_i|^p = 1 \) almost surely, \( R > 0 \) independently of \( U \), and \( R^p \sim \text{gamma}(\alpha, 1/p) \). The random vector \( U \) has components symmetric about 0, and moreover

\[
|U_i|^p \overset{d}{=} \frac{|X_i|^p}{\sum_{j=1}^{k} |X_j|^p} \overset{d}{=} \frac{Y_i}{Y_i + Z_i}, \quad i = 1, \ldots, k,
\] (11)

with \( Y_i, Z_i (i \leq k) \) being independent with

\[
Y_i \sim \text{gamma}(\alpha_i, 1/p), \quad Z_i \sim \text{gamma}(\alpha - \alpha_i, 1/p).
\]

For such \( U \), we write \( U \sim \mathcal{D}(\alpha, p) \). When \( \alpha = 1/2 \) and \( p = 2 \), then \( X \) with the stochastic representation (10) is a spherically symmetric random vector. It is well-known (see [7]) that a spherically symmetric random vector \( X \) that possesses a positive pdf has independent components if and only if \( X \) is Gaussian with independent components. Similarly, Fang and Fang [13] showed that (for the case \( p = 2 \)) a Dirichlet random vector has independent components if and only if \( X \) is a Kotz Type I random vector, provided that the \( F \) of \( R \) possesses a positive pdf. Hashorva et al. [17] provided a minor extension of this result to the case of \( p \in (0, \infty) \). In the following theorem, we remove the restriction that \( F \) possesses a pdf.
Theorem 1. Let \( \mathbf{X} \) be a \( k \)-dimensional Dirichlet random vector with stochastic representation (10), with \( R \sim F \) being a positive random variable that is not constant. Then, the following statements are equivalent:

1. \( \mathbf{X} \) possesses independent components;
2. For any \( I \subset \{ 1, \ldots, k \} \), there exists some \( r > 0 \) such that \( \mathbf{X}_i/r \sim \mathcal{K}_{\alpha_i, \lambda} \);
3. There exist disjoint non-empty index sets \( I \) and \( J \) such that \( \mathbf{X}_i \) is independent of \( \mathbf{X}_j \);
4. There exist disjoint non-empty index sets \( I \) and \( J \) such that \( \mathbf{X}_i | \mathbf{X}_j \) is independent of \( \mathbf{X}_j \);
5. For any \( I \subset \{ 1, \ldots, k \} \) with at least one element, we have \( \sum_{i \in I} |X_i|^p \sim \text{gamma} \left( \sum_{i \in I} \alpha_i, r \right) \) for some positive constant \( r \).

Remarks. (a) In an analogous manner, we can define the \( k \)-dimensional Kotz Type I Liouville random vector \( \mathbf{X} \) by assuming that \( \mathbf{X} \) has independent components symmetric about 0 satisfying \( |X_i|^p \sim \text{gamma} \left( \alpha_i, \lambda \right) \) for some \( p_i > 0, i \leq k \). It follows that \( \mathbf{X} \) has the stochastic representation

\[
\mathbf{X} \overset{d}{=} (R^{1/p_1} \mathbf{V}_1 \cdots R^{1/p_k} \mathbf{V}_k)^\top, \quad p := \sum_{j=1}^k p_j,
\]

with \( R > 0 \) such that \( R \sim \text{gamma}(\alpha, \lambda) \) independent of \( \mathbf{V} = (\mathbf{V}_1 \ldots \mathbf{V}_k)^\top \), with \( \sum_{j=1}^k |V_j|^p = 1 \) almost surely. A result similar to Theorem 1 can be presented to display the role of Kotz Type I Liouville random vector \( \mathbf{X} \) among the Liouville random vectors with representation (12), where \( R \) has some df \( F \).

(b) If \( \mathbf{X} \) is a spherically symmetric random vector in \( \mathbb{R}^k, k \geq 3 \), then by Theorem 4.2.3 of [6] \( \mathbf{X} \) has independent components if and only if \( \mathbf{X} \) is Gaussian. In Theorem 1, we do not suppose that \( k \geq 3 \).

The class of Kotz Type I random vectors can be naturally enlarged by considering scale mixtures. So, let us define a scale mixture Kotz Type I Dirichlet (for short \( \delta \mathcal{K} \) Dirichlet) random vector \( \mathbf{X} \) by the stochastic representation

\[
\mathbf{X} \overset{d}{=} \mathbf{A} \mathbf{S} \mathbf{Y}, \quad \mathbf{Y} \sim \mathcal{K}_{\alpha, \lambda},
\]

where \( \mathbf{S} \sim H \) is some positive scaling random variable independent of \( \mathbf{Y} \), and \( \mathbf{A} \in \mathbb{R}^{k \times k} \) is a given matrix. When \( \mathbf{X} \) has representation (13), we denote by \( \mathbf{X} \sim \delta \mathcal{K}_{\alpha, \lambda}; H \). From (10) and (13), we obtain the alternative representation

\[
\mathbf{X} \overset{d}{=} \mathbf{A} \mathbf{R} \mathbf{U}, \quad \mathbf{R} := \mathbf{RS},
\]

where \( R > 0 \) is independent of \( \mathbf{S} \), and \( \mathbf{R}^p \sim \text{gamma}(\alpha, 1/p) \). Thus, the distributional properties of \( \mathbf{X} \sim \delta \mathcal{K}_{\alpha, \lambda}; H \) can be easily derived from those of the Dirichlet random vectors.

In view of Theorem 1, \( \delta \mathcal{K} \) Dirichlet random vectors have in general dependent components, but they have asymptotically independent components; see the derivation of (26) detailed in the Appendix.

Example 1. Let \( p = 1, \alpha \in (0, \infty), k \geq 2 \), and let \( \mathbf{S} \sim \text{beta}(\alpha, \alpha - c) \) with \( c \in (0, \alpha) \). If \( \mathbf{X} \sim \mathcal{K}_{\alpha, 1} \) is independent of \( \mathbf{S} \), then the new random vector \( \mathbf{Y} = \mathbf{S} \mathbf{X} \) is a scale mixture Kotz–Dirichlet random vector. By the independence of \( \mathbf{S} \) and \( \mathbf{X} \), we have

\[
\mathbf{Y} \overset{d}{=} \mathbf{R} \mathbf{U},
\]

where \( \mathbf{R} \sim \text{gamma}(c, 1) \) is independent of \( \mathbf{U} \sim \delta \mathcal{D} (\alpha, 1) \). In the limiting case \( c = \alpha := \sum_{j=1}^k \alpha_j \), Theorem 1 then implies that \( \mathbf{Y} \) has independent components. However, \( \mathbf{Y} \) has dependent components for any \( c \in (0, \alpha) \).

4. The Kotz Approximation

In this section, we are concerned with the approximation of conditional distributions of the \( \delta \mathcal{K} \) Dirichlet random vectors. Crucial for our approximation is the tractable form of the conditional distributions of such random vectors for specific forms of the square matrix \( \mathbf{A} \). In the sequel, we assume that the \( \delta \mathcal{K} \) Dirichlet random vector \( \mathbf{X} \) has stochastic representation (13), and we fix \( I, J \) as partitions of \( \{ 1, \ldots, k \} \). One issue of interest is about the approximation of the conditional distribution of \( \mathbf{X} \), given \( \mathbf{X}_I = \mathbf{x}_I \), where \( \mathbf{x}_I \) is such that one of its components tends to infinity. Suppose first that \( S = 1 \) almost surely. If \( \mathbf{A} \) is non-singular and \( A_{ij} \) has all its elements as 0, then we have the stochastic representation (see [4])

\[
\mathbf{X}_I | (\mathbf{X}_I = \mathbf{x}_I) - A_{ij} (A_{ij})^{-1} \mathbf{x}_j \overset{d}{=} A_{ij} \mathbf{Y}_I, \quad \mathbf{Y}_I \sim \mathcal{K}_{\alpha_i, \lambda}.
\]
If the scaling random variable \( S \) in the definition of the \( \Delta \mathcal{K} \) Dirichlet random vectors is bounded, say by 1, we expect that this random scaling does not influence the asymptotics, and therefore we shall try to derive the same approximation as in the aforementioned paper. If \( S \) is unbounded, large values of \( X_j \) are influenced by large values of \( S \). We therefore need to assume a tractable asymptotic tail behaviour of the random scaling \( S \), say that it has a Weibull tail asymptotics given by (9). For simplicity, we consider below only the case \( c_3 = c_4 = 1 \); the more general asymptotic tail behaviour given by (9) can be dealt with in a similar manner by borrowing the ideas of [25].

We now state the main result of this section.

**Theorem 2.** Let \( I, J \) be partitions of \( \{1, \ldots, k\} \), and let \( X \) be a \( \Delta \mathcal{K} \) Dirichlet random vector with stochastic representation (13) and \( S \sim H \) a positive random variable. Suppose that the square matrix \( A \) is non-singular, and \( A_f \) has all its elements to be 0. For given constants \( u_i \in \mathbb{R}^d \), define \( X_{i,n} \) as a sequence of random vectors in the same probability space, and set \( r_n := \| (u_i)_j (A_f)_j^{-1} \|_p, \ n \geq 1. \) Assume further that \( \lim_{n \to \infty} r_n = \infty \), and let \( Y \) be a k-dimensional random vector such that \( Y \sim \Delta_{\mathcal{K}} \).

(i) If \( H \) has upper endpoint 1, then we have the convergence in distribution

\[
X_{i,n} - \mu_{n,j} \xrightarrow{d} Y_i, \quad \text{as} \ n \to \infty, \tag{15}
\]

with \( \mu_{n,j} := \sum_j (\Sigma_j^{-1}(u_n)_j) \) and \( \Sigma : = AA^\top \);

(ii) If \( H \) satisfies (9) with \( c_3 = c_4 = 1 \), and \( c_1, c_2 \in (0, \infty) \) and \( b(x) = 0, x > 0 \) when \( \sum_i \alpha_i \in (0, 1) \), then we have the convergence in distribution

\[
r_n^{-p/(1+p)}(X_{i,n} - \mu_{n,j}) \xrightarrow{d} Y_i, \quad \text{as} \ n \to \infty. \tag{16}
\]

**Remarks.**

(a) When \( S \) is bounded, then the Kotz approximation substitutes the stochastic representation in (14), whereas if \( S \) is unbounded as in statement (ii) of Theorem 2, then we still can derive the Kotz approximation by utilizing a scaling sequence which converges to 0; see (16).

(b) The assumption that \( b(x) = 0, x > 0 \) imposed in the special case \( \sum_i \alpha_i \in (0, 1) \). The assumption that (8) holds, which is implied for random radius with von Mises distribution functions, ensures the validity of the Kotz approximation of general Dirichlet random vectors. In the Appendix, we present some sufficient conditions for (6) to hold.

(c) If \( p = 2 \) and \( \alpha = 1/2 \), then \( X \) in Theorem 2 is an elliptically symmetric random vector. Its df does not depend on \( A \), but instead on \( \Sigma = AA^\top \). Furthermore, we do not need to assume that \( A_f \) has all its elements to be 0.

5. Approximation of tail indices by Kotz Type I distributions

Investigation of the tail asymptotic behaviour of random vectors is of interest in diverse applications and theoretical problems. If \( X \sim \mathcal{G}(A; \alpha; p; \Gamma) \) is a given k-dimensional Dirichlet random vector with \( A \) non-singular, then the tail asymptotic behaviour of \( X \) is known when the survival function \( \tilde{F} \) is regularly varying at infinity. Specifically, by Theorem 11 in [17], the survival function \( \tilde{F} \) satisfies (2) with some index \(-\gamma < 0 \) if and only if some component of \( X \) (or \( X \) itself) is regularly varying with the same index \( \gamma \). Moreover, for any Borel set \( B \) of \( \mathbb{R}^k \) away from the origin, we have

\[
\lim_{t \to \infty} \frac{P[X/t \in B]}{\tilde{F}(t)} = \gamma \int_0^\infty P[rA \cup B]r^{-\gamma-1} \, dr \in (0, \infty), \quad B \sim \mathcal{G}(\alpha, p), \tag{17}
\]

which was initially shown in Theorem 12.6.1 of [2] for a k-dimensional spherically symmetric random vector. One may refer to [20,31] for details on regular variation of random vectors.

Relation (17) can be utilized for instance to calculate the tail index \( \tau_{\mathcal{K,L}} \) defined by

\[
\tau_{\mathcal{K,L}} := \lim_{u \to 1} P[X_i > G_i^{-1}(u), \ i \in K \mid \{X_i > G_i^{-1}(u), \ i \in L\}],
\]

with \( K, L \) being two non-empty subsets of \( \{1, \ldots, k\} \) and \( G_i^{-1} \) being the inverse of the df \( G_i \) of \( X_i \), \( i \leq k \). Note in passing that df \( G_i \) is absolutely continuous.

By assumption (2) and the fact that \( \max_{1 \leq i \leq k} \tilde{W}_i \) with \( W_{i,+} := \max((A \cup B)_i) \) is a bounded random variable, by applying Breiman’s Lemma (see e.g., [9]) for any \( i \leq k \) we obtain

\[
\lim_{t \to \infty} \frac{P[X_i > t]}{P[R > t]} = \lim_{t \to \infty} \frac{P[W_{i,+} > t]}{P[R > t]} = E[W_{i,+}^\gamma] \in (0, \infty),
\]

and consequently

\[
\lim_{u \to 1} \frac{G_i^{-1}(u)}{G_j^{-1}(u)} = \left( \frac{E[W_{i,+}^\gamma]}{E[W_{j,+}^\gamma]} \right)^{1/\gamma} := c_{ij} \in (0, \infty), \quad i, j \leq k. \tag{18}
\]
Hence, an expression for $\tau_{K,L}$ can be easily derived upon combining (17) and (18). In the bivariate setup, the index $\tau_{K,L}$ (denoted below by $\lambda$) is the well-known upper coefficient of tail dependence; see, for example, [26,22,5].

Quite surprisingly, the Kotz Type I Dirichlet random vectors can be utilized to calculate $\tau_{K,L}$ as shown in the next result.

**Theorem 3.** Let $X \sim g.D(A; \alpha, p; F)$ with $A \in \mathbb{R}^{k \times k}$, $k \geq 2$, being a non-singular matrix. If $\bar{F}$ is regularly varying at infinity with index $-\gamma < 0$, then we have

(a) For any Borel set $B \subset \mathbb{R}^k$ away from the origin,

$$
\lim_{t \to \infty} \frac{P\{X/t \in B\}}{\bar{F}(t)} = \gamma \int_0^\infty P\{|rAZ \in B\}r^{-\gamma-1} \, dr, \quad Z \sim \mathcal{K}_{\alpha,p},
$$

(b) For any two non-empty subsets $K, L$ of $\{1, \ldots, k\}$,

$$
\tau_{K,L} = \frac{\left(\min_{i \in K,j \in L} \tilde{W}_{i,j}/c_{i,j}\right)^{\gamma}}{\min_{i \in L} E[\tilde{W}_{i,i}]} \left(\frac{E[\tilde{W}_{j,j}]}{E[\tilde{W}_{j,j}]}\right)^{1/\gamma}, \quad i, j \leq k,
$$

is valid for some $j \leq k$, with $\tilde{W}_i := \max((AZ)_i, 0)$ and $c_{i,j} \in (0, \infty)$ as defined in (18).

**Remark.** The constants $c_{i,j}$ in (18), also appearing in (20), can be calculated by the alternate expression

$$
c_{i,j} = \left(\frac{E[\tilde{W}_{i,i}^{\gamma}]}{E[\tilde{W}_{j,j}^{\gamma}]}\right)^{1/\gamma}, \quad i, j \leq k,
$$

which is defined in terms of the Kotz Type I random vector $Z$.

**Example 2.** Let $X = (X_1, X_2)^\top$ be a Dirichlet bivariate random vector as in Theorem 3, where the elements of the matrix $A \in \mathbb{R}^{k \times k}$ are

$a_{11} = \sigma \in (0, 1), \quad a_{12} = \rho \in (-1, 1), \quad a_{21} = 0, \quad a_{22} = 1.$

In view of (18) and Theorem 3, if $X_i \sim G_i$, $i = 1, 2$, then we have

$$
\lambda := \lim_{u \uparrow 1} P\{X_1 > G_1^{-1}(u) \mid X_2 > G_2^{-1}(u)\}
$$

$$
= \lim_{u \uparrow 1} \frac{P\{X_1 > c_{1,2}G_2^{-1}(u), X_2 > G_2^{-1}(u)\}}{P\{X_2 > G_2^{-1}(u)\}} = \frac{E[\min((\sigma U + \rho V)_{+}, (\sigma U + \rho V)_{+})]}{E[\tilde{V}_2^+]}.
$$

where $(U, V)^\top \sim \mathcal{K}_{\alpha,p}$ and $(\cdot)_+ := \max(\cdot, 0)$.

6. Pearson–Kotz Dirichlet distributions

If $\mathcal{U}$ is a $k$-dimensional random vector with stochastic representation (11), then the Pearson–Kotz Dirichlet random vector $X$ considered in Balakrishnan and Hashorva [4] has the stochastic representation

$$
X \buildrel {d} \over \equiv AR\mathcal{U}, \quad R^p \sim \lambda beta_2(\alpha, \theta),
$$

where $A \in \mathbb{R}^{k \times k}$ is a given matrix, $R > 0$ almost surely and independent of $\mathcal{U}$, and $\alpha, \theta$ are two positive constants. Such an $X$ is a $g.D$ Dirichlet random vector since

$$
X \buildrel {d} \over \equiv ASZ, \quad Z \sim \mathcal{K}_{\alpha,p},
$$

where the positive scaling random variable $S$ is such that $S^p \buildrel {d} \over \equiv \lambda/W, \quad W \sim \text{gamma}(\theta, 1/p)$.

The Pearson–Kotz Dirichlet random vectors are interesting from both distributional as well as asymptotic point of view. When $\alpha = 1/2$ and $p = 2$, then $X$ in (21) is a (Student) $t$-random vector. Indeed, the $t$-distributions are both tractable and flexible forming a prominent subclass of the elliptically symmetric distributions; see [24]. The same role is played by Pearson–Kotz Dirichlet distributions within the class of the Dirichlet distributions; see [4]. For simplicity, we discuss below the case when $A = I_k$, with $I_k$ being the identity matrix in $\mathbb{R}^{k \times k}$. For a Pearson–Kotz Dirichlet random vector $X$ defined in (21) (with $A = I_k$), we use the notation $X \sim P.K.D(\alpha, p, \lambda, \theta)$. Complementing the findings of the aforementioned paper, we now establish a characterization result for the Pearson–Kotz Dirichlet random vectors.
Theorem 4. Let \( X = RU \) be a \( k \)-dimensional Dirichlet random vector, and let \( \lambda, \theta \) be two positive constants.

(i) If \( X_1 \sim \mathcal{P} \mathcal{K} \mathcal{D}(\alpha_i, p, \lambda, \theta) \) for some \( i \leq k \), and when \( k = 2 \), \( R \) possesses a positive pdf, then we have \( X \sim \mathcal{P} \mathcal{K} \mathcal{D}(\alpha, p, \lambda, \theta) \);

(ii) If for \( I, J \) as partitions of \( \{1, \ldots, k\} \) and some \( \mathbf{x} \in \mathbb{R}^k \) such that \( y := \|\mathbf{x}\|_{p}^{\lambda} = \sum_{i \in J} |x_i|^\lambda > 0 \) the condition

\[
X_i | (X_J = \mathbf{x}_J) \sim \mathcal{P} \mathcal{K} \mathcal{D}(\alpha_i, p, \lambda + y, \theta + \alpha_j)
\]

holds, and when \( I \cup J \) has \( k \) elements \( R \) possesses a positive pdf, then \( X \sim \mathcal{P} \mathcal{K} \mathcal{D}(\alpha, p, \lambda, \theta) \).

We note in passing that statement (i) is related to the fact that if we know the df of one component of the Dirichlet random vector \( X \), then we can determine the joint df of \( X \); see Section 7 for further details.

It follows that Pearson–Kotz Dirichlet random vectors are regularly varying, which is not the case for the Kotz Type I random vectors. Balakrishnan and Hashorva [4] proved the conditional approximation of regularly varying Dirichlet random vectors by the Pearson–Kotz Dirichlet random vectors, and so we do not focus on this topic here.

7. Proofs

For completeness, we present next a lemma given in [4], and then proceed with the proofs.

Lemma 5. Let \( R \) be a positive random variable independently of \( X \sim \beta(c, d) \). If \( Y \sim \beta(c, b) \) is independent of \( R \) and \( X \) is such that \( Y \overset{d}{=} RX \), then \( R \sim \beta(c + d, b) \).

Proof of Theorem 1. For \( p = 2 \) and \( F \) with some positive pdf \( f \), the proof follows from Theorem 4.3 of [13]. Theorem 6 in [17] extends Theorem 4.3 of [13] for any positive \( p \). In order to establish the proof, it suffices to show that we can remove the condition that \( F \) possesses a positive pdf \( f \). We show therefore that statement (1) implies \( X \) to be a Kotz Type I Dirichlet random vector. Since any subvector of \( X \) is again a Dirichlet random vector, we assume without loss of generality that \( k = 2 \). By the definition of \( X \), we have

\[
|X_1|^p = R^p |U_1|^p := R_1 W, \quad |X_2|^p = R^p |U_2|^p := R_2 (1 - W),
\]

with \( R_1 \) and \( W \) being independent. Next, if \( X_1 \) and \( X_2 \) are assumed to be independent, then clearly \( |X_1|^p \) and \( |X_2|^p \) are also independent. Applying Theorem 2.2 of [33], we then obtain

\[
R_1 \sim \text{gamma}(c_1 + c_2, \lambda) \quad \text{and} \quad W \sim \beta(c_1, c_2)
\]

for some positive constants \( \lambda, c_1, c_2 \). Since \( X \) is a Dirichlet random vector with parameters \( (\alpha_1, \alpha_2) \), then \( W \sim \beta(\alpha_1, \alpha_2) \). Consequently, \( c_1 = \alpha_1, c_2 = \alpha_2, \) and \( \lambda = 1/p \), thus establishing the proof. \( \square \)

Proof of Theorem 2. Let \( m \) be the number of elements of the index set \( I \), and let \( F^* \) denote the df of \( RS \). Set \( u_{n,j} := (u_n)_j, n \geq 1 \). Since \( A_{ij} \) has all its elements to be 0, the choice of \( u_n \) implies (see [4])

\[
X_i | (X_J = u_{n,j}) - A_{ij} (A_{ij})^{-1} u_{n,j} \overset{d}{=} A_{ij} R_n U_j, \quad U_j \sim \mathcal{D}(\alpha_i, p),
\]

with \( U_j \) being independent of the positive random variable \( R_n \), having df \( Q_n \) given by

\[
Q_n(x) = 1 - \frac{\int_{[0, +\infty]} (t^p - r_n)^{\alpha_i - 1} \frac{1}{t} \cdot f^*(t) dt}{\int_{[0, +\infty]} (t^p - r_n)^{\alpha_i - 1} \cdot f^*(t) dt}, \quad \forall x \in (0, \infty),
\]

where \( r_n := \| (u_n)_j (A_{ij})^{-1} \|_p, n \geq 1 \). By the assumption on the matrix \( A \), it follows further that

\[
X_i | (X_J = u_{n,j}) - \Sigma_J (\Sigma_J)^{-1} u_{n,j} \overset{d}{=} A_{ij} R_n U_j, \quad \Sigma := AA^T.
\]

Since \( R \) possesses the pdf \( f \), then the independence of \( R \) and \( S \) implies that the df \( F^* \) possesses a pdf \( f^* \) as well. Moreover, for any \( s > 0 \),

\[
f^*(s) = \int_{0}^{\infty} f(s/y) \frac{1}{y} dH(y),
\]

with \( H \) being the df of the random scaling \( S \). Consequently,

\[
Q_n(x) = 1 - \frac{\int_{[0, +\infty]} (t^p - r_n)^{\alpha_i - 1} \frac{1}{t} \cdot f^*(t) dt}{\int_{[0, +\infty]} (t^p - r_n)^{\alpha_i - 1} \cdot f^*(t) dt}, \quad \forall x \in (0, \infty),
\]

When \( p = 1 \), we have

\[
\lim_{n \to \infty} \frac{f(t + r_n)}{f(t)} = \lim_{n \to \infty} \frac{f^*(t + r_n)}{f^*(t)} = \exp(-t), \quad t > 0,
\]
and hence by applying Lemma 5.1 of [32], we obtain the convergence in distribution

\[ R_n \xrightarrow{d} R \sim \text{gamma} \left( \sum_{i \in I} \alpha_i, 1 \right), \quad n \to \infty. \]

If \( p \neq 1 \), by adopting a similar argument, we once again obtain the above convergence with \( R^p \sim \text{gamma} \left( \sum_{i \in I} \alpha_i, 1/p \right) \).

Since \( R_n \), \( n \geq 1 \) is independent of \( U_i \), the first claim of the theorem follows.

(ii) When \( \sum_{i \in I} \alpha_i \geq 1 \), the proof follows if we show that the df \( F^* \) of \( RS \) is in the Gumbel MDA. When the mentioned criteria are not fulfilled, then we need to show further that \( RS \) has a pdf that satisfies (6). This is easily shown when \( b(x) = 0 \) for \( x > 0 \). So, we prove next only the case that \( F^* \) is in the Gumbel MDA. Borrowing some techniques from [25], since \( R^p \sim \text{gamma} \left( \sum_{i \in I} \alpha_i, 1/p \right) \) and \( c_3 = c_4 = 1 \), it follows that \( F^* \) is in the Gumbel MDA with the scaling function \( u(x) = x^{1/(1+p)} \). We then obtain

\[ h_n \left( X_i | (X_j = u_{n,j}) - \Sigma_{ij} (\Sigma_y)^{-1} u_{n,j} \right) \xrightarrow{d} A \mathcal{R} U_i, \quad n \to \infty, \]

with \( R^p \) independently of \( U_i \) and

\[ h_n = \left( \frac{u(r_n)}{r_n} \right)^{1/p} = r_n^{-1/(p(1+p))} = r_n^{-p/(1+p)}, \]

which completes the proof. \( \Box \)

**Proof of Theorem 3.** The first part follows easily, and is therefore omitted. Next, we show (20) by a direct argument. Let \( c_{i,j}, i, j \leq k \), be as in (18), and let \( M = K \cup L \). Clearly, we have \( c_{i,j} \in (0, \infty) \). By the definition of \( \tau_{K,L} \) and Breiman’s Lemma, we may write

\[ \tau_{K,L} = \lim_{u \uparrow 1} \frac{\mathbb{P} \{ X_i > G_i^{-1}(u), i \in M \}}{\mathbb{P} \{ X_i > G_i^{-1}(u), i \in L \}} \]

\[ = \lim_{u \uparrow 1} \frac{\mathbb{P} \{ \min \max ((A U)_i, 0) > c_{i,j} G_i^{-1}(u) \}}{\mathbb{P} \{ \min \max ((A U)_i, 0) > c_{i,j} G_i^{-1}(u) \}} \]

\[ = \mathbb{E} \left\{ \min \max ((A U)_i/c_{i,j}, 0)^\gamma \right\} \]

\[ = \mathbb{E} \left\{ \min \max ((A U)_i/c_{i,j}, 0)^\gamma \right\}, \]

where \( \mathcal{U} \sim \Delta \mathcal{D} (\alpha, p) \). Note that \( \tau_{K,L} \in (0, \infty) \) follows from the first claim. For \( Z = \tilde{K} U \sim \mathcal{K}_w, p \), we have \( \tilde{R}^p \sim \text{gamma}(\alpha, 1/p) \), such that \( \mathbb{E} \{ R^\gamma \} < \infty \), and consequently

\[ \tau_{K,L} = \mathbb{E} \{ R^\gamma \} \mathbb{E} \left\{ \min \max ((A Z)_i/c_{i,j}, 0)^\gamma \right\} \]

\[ = \mathbb{E} \{ R^\gamma \} \mathbb{E} \left\{ \min \max ((A Z)_i/c_{i,j}, 0)^\gamma \right\} \]

\[ = \mathbb{E} \left\{ \min \max ((A Z)_i/c_{i,j}, 0)^\gamma \right\}, \]

which completes the proof. \( \Box \)

**Proof of Theorem 4.** (i) First, we mention the beta-independent splitting property of Dirichlet random vectors. Let \( \mathcal{V} \) be a \( k \)-dimensional random vector with independent components \( \mathcal{V}_i \sim \Delta \mathcal{D} (\alpha_i, 1) \) and \( \mathcal{V}_j \sim \Delta \mathcal{D} (\alpha_j, 1) \). If \( X = R \mathcal{U} \), where \( \mathcal{U} \sim \Delta \mathcal{D} (\alpha, p) \) is a \( k \)-dimensional Dirichlet random vector, then for any partitions \( I, J \) of \( \{ 1, \ldots, k \} \), the stochastic representation

\[ X_i \overset{d}{=} RW^{1/p} \mathcal{V}_i, \quad X_j \overset{d}{=} R (1 - W)^{1/p} \mathcal{V}_j \]

(25)
is valid with \( R > 0, W \sim \text{beta} \left( \sum_{i \in I} \alpha_i, \sum_{j \in J} \alpha_i \right) \) and \( R, \mathcal{V}_i, \mathcal{V}_j \) all being mutually independent.

The proof of the first statement follows from (25) and by Lemma 5.
(ii) Since any subvector of \( X \) is a Dirichlet random vector, and in addition it possesses a positive pdf, we can assume that \( X \) itself possesses a positive pdf, which is equivalent to the fact that \( R \) possesses a positive pdf \( f \). Next, we define some function \( g \), known as the generator of \( X \) (see [13, 4]) by

\[
f(r) = 2 \left( \frac{2}{p} \right)^{k-1} \prod_{i=1}^{k} \frac{\Gamma(a_i)}{\Gamma(\alpha)} g(r^p) r^{p\alpha-1}, \quad \forall r \in (0, \infty).
\]

From (23) and (25), we also have \( X_j \) and \( X_j | \{ X_j = x_j \} \) to be Dirichlet random vectors with some positive generators, denoted below by \( g_j \) and \( g_{ij} \), respectively. Setting now \( z := ||x_j||_p^p \), we find that the generator \( g_{ij} \) is related to both \( g \) and \( g_j \), and is given by

\[
g_{ij}(s) = \frac{g(s+z)}{g_j(z)}, \quad \forall s \in (0, \infty).
\]

Hence, for any positive \( t \), we have

\[
g(t) = g_j(t) g_{ij}(t-z) \\
= g_j(t) a_1(1 + (t-z)/(\lambda + z))^{-N} \\
= a_2(1 + t/\lambda)^{-N}, \quad N := \alpha + \theta,
\]

with \( a_1 \) and \( a_2 \) being two known positive constants. By Example 5 of [17], the random vector \( X \) has the pdf

\[
q(x) = \left( \frac{p}{2} \right)^k \frac{\Gamma(N)}{\Gamma(N-H)} \frac{\lambda^{-N}}{\prod_{i=1}^{k} \Gamma(a_i)} \left[ 1 + \sum_{i=1}^{k} |x_i|^p / \lambda \right]^{-N} \prod_{i=1}^{k} |x_i|^{p\alpha_i-1}, \quad \forall x \in \mathbb{R}^k,
\]

and hence the result. \( \square \)

Acknowledgements

We are grateful to the referees for making several comments and suggestions on an earlier draft of this paper. Enkelejd Hashorva gratefully acknowledges the support from Swiss National Science Foundation Project 200021–134785. Sincere thanks are due to Qihe Tang and Yang Yang for pointing out a gap in the proof of Lemma 5.1 of [15]. Finally, N. Balakrishnan expresses thanks to National Sciences and Engineering Council of Canada for funding this research.

Appendix

First, we establish the convergence in (6) under diverse conditions. Next, for a univariate df \( F \), define for any \( \gamma \in (0, \infty) \) the function \( q_\gamma \) by

\[
q_\gamma(x) = \frac{x^{-\gamma}}{\Gamma(1-\gamma) \Gamma(\gamma)} \int_{x}^{\infty} (y-x)^{\gamma-1} dF(y), \quad x > 0.
\]

If follows that \( q_\gamma \) is a pdf for \( \gamma \in (0, 1) \) and \( q_\gamma(u) \) is finite for all \( u \) large when \( F \) satisfies (5) and \( \gamma \in (1, \infty) \). In the case of \( \gamma \in (0, 1) \), we write \( Q_\gamma \) for the df corresponding to \( q_\gamma \).

Lemma 6. Let \( F \) be a univariate df such that (3) holds. If furthermore, when \( \gamma \in (0, 1) \), any of the following conditions

(a) \( q_\gamma \) is ultimately monotone,

(b) \( F \) possesses a pdf \( f \) in a left-neighbourhood of \( x_F \) satisfying (8),

(c) for all large \( u \) we have \( F(u+x/w(u))/F(u) \geq 1 - c \gamma^2 \) for all \( x \in [0, \varepsilon] \), \( \varepsilon > 0 \) and \( c \varepsilon > 0 \), \( \tau > \gamma \) some given constants, are satisfied, then the convergence in (6) holds for any \( \gamma \in (0, 1) \) under conditions (a) and (b), and for any \( \gamma \in (0, \tau) \) under condition (c).

Proof of Lemma 6. First note that when \( \gamma \in [1, \infty) \) then (5) implies (6) as shown in Section 2. Using this fact, for \( \gamma \in (0, 1) \) it follows that \( Q_\gamma \) is also in the Gumbel MDA with the scaling function \( w \). Hence, since \( q_\gamma \) is ultimately monotone, we have \( Q_\gamma \) to be a von Mises distribution function. The asymptotics for \( q_\gamma \) and the fact that the scaling function \( w \) is self-neglecting, i.e.,

\[
w(u+x/w(u))/w(u) \to 1, \quad u \uparrow x_F
\]

locally uniformly in \( \mathbb{R} \) (see e.g., [11]) imply then \( \lim_{u \uparrow x_F} I_\gamma(u) = 1 \).
Next, assume that condition (8) holds. Since $w$ is self-neglecting and (5) holds locally uniformly, it follows that
\[ q_{y}(u, y) = \frac{f(u + y/u(u))}{u(u)F(u)} \rightarrow \exp(-y), \quad u \uparrow x \]
locally uniformly for $y \in \mathbb{R}$, and therefore uniformly for $y \in [0, M]$ with $M > 0$ some constant. Consequently,
\[ \lim_{u \uparrow x} \int_{0}^{M} y^{\gamma-1} q_{y}(u, y) \, dy = \int_{0}^{M} y^{\gamma-1} \exp(-y) \, dy. \]
Furthermore, since $q_{y}$ is a pdf and $y^{\gamma-1} < M^{\gamma-1}, \forall y > M$ by Scheffe’s Lemma
\[ \lim_{u \uparrow x} \int_{M}^{\infty} y^{\gamma-1} q_{y}(u, y) \, dy = \int_{M}^{\infty} y^{\gamma-1} \exp(-y) \, dy \]
thus establishing the claim under condition (b). The last claim follows by a simple domination argument and is therefore omitted. Note in passing that under condition (b), condition (c) holds with $\tau = 1$, and hence the claim holds for any $\gamma \in (0, 1)$.

We remark that the case $\gamma \in (0, 1)$ in the above lemma corresponds to the case $\alpha \in (-1, 0)$ in Lemma 5.1 of [15]. It seems that the arguments of the aforementioned lemma might not be sufficient, and so we provide above additional conditions which along with the Gumbel MDA are sufficient for (6). Consequently, Theorem 3.1 therein holds for $\alpha \geq 0$, and if $\alpha \in (-1, 0)$, when additionally $F$ satisfies the conditions of Lemma 6.

To this end, we now show the asymptotic independence of $\mathcal{S} \mathcal{K}$ Dirichlet random vectors.

Let $X_{i} \sim \mathcal{S} \mathcal{K}_{k|a,p,H}$, and assume for simplicity that $p = 1$ and that $H$ has an upper endpoint 1. If $Y_{i}$ and $Y_{j}$ are two independent random variables symmetric about 0 such that
\[ |Y_{i}| \sim \text{gamma}(\alpha_{i}, 1), \quad |Y_{j}| \sim \text{gamma}(\alpha_{j}, 1), \quad i, j \leq k, \]
then $|X_{i}| = S|Y_{i}|$, $i \leq k$, and for any $x > 0$ we have
\[
\frac{P(|X_{i}| > x, |X_{j}| > x)}{P(|X_{i}| > x)} = \frac{P(S|Y_{i}| > x, S|Y_{j}| > x)}{P(S|Y_{i}| > x)} \\
\leq \frac{P(S|Y_{i}| + |Y_{j}| > 2x)}{P(S|Y_{i}| > x)} \\
\leq \frac{P(|Y_{i}| + |Y_{j}| > 2x)}{P(S > c)P(|Y_{i}| > x/c)}
\]
for some $c \in (0, 1)$. Choosing $c$ close to 1, we may further write
\[ 0 \leq \lim_{x \rightarrow \infty} \frac{P(|X_{i}| > x, |X_{j}| > x)}{P(|X_{i}| > x)} \leq \lim_{x \rightarrow \infty} \frac{P(|Y_{i}| + |Y_{j}| > 2x)}{P(S > c)P(|Y_{i}| > x/c)} = 0; \quad (26) \]
so, from the symmetry of $X_{i}$ and $X_{j}$ about 0, we obtain
\[ \lim_{x \rightarrow \infty} \frac{P(|X_{i}| > x, |X_{j}| > x)}{P(|X_{i}| > x)} = 0 \]
implicating readily that $X_{i}$ and $X_{j}$ are asymptotically independent.

References