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Applications of braid group techniques to the decomposition of moduli spaces, new examples [☆]

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Abstract

Every smooth minimal complex algebraic surface of general type, X, may be mapped into a moduli space, $\mathcal{M}_{c_1^2(X),c_2(X)}$, of minimal surfaces of general type, all of which have the same Chern numbers. Using the braid group and braid monodromy, we construct infinitely many new examples of pairs of minimal surfaces of general type which have the same Chern numbers and nonisomorphic fundamental groups. Unlike previous examples, our results include X for which $|\pi_1(X)|$ is arbitrarily large. Moreover, the surfaces are of positive signature. This supports our goal of using the braid group and fundamental groups to decompose $\mathcal{M}_{c_1^2(X),c_2(X)}$ into connected components. © 1997 Elsevier Science B.V.

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0. Introduction

It was proven by Gieseker that there exists a quasi-projective coarse moduli space, $\mathcal{M}$, of minimal surfaces of general type. This space is a union of components, $\mathcal{M}_{c_1^2(X),c_2(X)}$, in which all members have the same Chern numbers. A major problem in the theory of surfaces is the search for discrete invariants which characterize the connected components of $\mathcal{M}_{c_1^2,c_2}$ [3,16].

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The Chern numbers are, of course, topological invariants; $c_2(X)$ is the topological Euler characteristic, and $c_1^2(X)$ may be computed from $c_2(X)$ and the signature of X. Hence, this may also be regarded as a problem in four-dimensional topology.

A discrete invariant which has already been used successfully to distinguish connected components of $\mathcal{M}_{c_1^2,c_2}$ is the divisibility index, $r(X)$, of a surface, X. If K_X is the canonical divisor of X, then $r(X)$ is the largest positive integer such that K_X is linearly equivalent to rD for some divisor D of X. The divisibility index is a deformation invariant. Catanese and Manetti have each produced examples of minimal surfaces of general type with the same Chern numbers and different divisibilities [2]. Later Catanese and Manetti produced examples of homeomorphic minimal surfaces of general type with the same divisibility which are not deformations of each other, i.e., they are in different connected components of moduli spaces (see [4,9,10]). All of their surfaces are simply-connected.

Another discrete invariant is the fundamental group. There exist pairs of surfaces which have the same Chern numbers and nonisomorphic fundamental groups. For example, the Godeaux construction can be used to produce surfaces, X, such that $c_1^2(X) = 2$, $c_2(X) = 10$, and $\pi_1(X)$ is one of the following: $\bigoplus_3 \mathbb{Z}_2$; $\mathbb{Z}_2 \oplus \mathbb{Z}_4$; $\mathbb{Z}_8$; the multiplicative group $\{\pm 1, \pm i \pm j \pm k\} \subset \mathbb{H}$ [1]. However, there are few examples of such pairs of surfaces and the possible fundamental groups are all relatively small.

In this article, we use new results on Galois covers of Hirzebruch surfaces [15] to construct infinitely many new examples of pairs of minimal surfaces of general type, X and Y, such that

$$c_1^2(X) = c_1^2(Y), \qquad c_2(X) = c_2(Y), \qquad \pi_1(X) \ncong \pi_1(Y).$$

In all of our examples, X and Y have positive signature and $\pi_1(Y) = \{0\}$. The fundamental group of X is finite but may be arbitrarily large. Indeed, for every $n \in \mathbb{Z}$, there exists a pair X, Y, such that $|\pi_1(X)| > n$.

This article is organized as follows. Section 1 is an introduction to the braid monodromy map associated to an algebraic curve. Section 2 describes the Galois cover of an algebraic surface (our examples are Galois covers of Hirzebruch surfaces). Section 3 describes how to use braid monodromy theory to calculate the fundamental group of a Galois cover. Section 4 contains our examples of pairs of surfaces with the same Chern numbers and nonisomorphic fundamental groups. Section 5 gives more examples, which are based on a work in preparation by the authors. It also describes how more examples might be obtained using Galois covers of K3 surfaces.

1. The braid group techniques

We introduce braid monodromy, which is used to compute fundamental groups related to surfaces of general type: fundamental groups of the complement of a branch curve and fundamental groups of Galois covers.

We consider the following situation:

- S is a curve in $\mathbb{C}^2$, $p = \deg S$.

- $\pi:\mathbb{C}^2 \to \mathbb{C}$ a projection on the first coordinate.
- $N = \{x \mid \#\pi^{-1}(x) \cap S \lneqq p\}$.
- Let E be a closed disk on x-axis such that $N \subset \mathrm{Int}(E)$. We choose $u \in \partial E$.
- Let D be a closed disk on the y-axis such that $\pi^{-1}(E) \cap S \subset E \times D$.
- $K = \{y \mid (u,y) \in S\} = \{q_1, \dots, q_p\}$.

In such a situation, we are going to introduce "braid monodromy".

Definition (*Braid monodromy of S w.r.t. $E \times D, \pi, u$*). Every loop in $E - N$ starting at u has liftings to a system of p paths in $(E - N) \times D$ starting at $q_1, \dots, q_p$. Projecting them to D we get p paths in D defining a motion $\{q_1(t), \dots, q_p(t)\}$ of p points in D starting and ending at K.

This motion defines a braid in $B_p[D,K]$, as explained in [14, Chapter III]. Thus we get a map $\varphi: \pi_1(E - N, u) \to B_p[D,K]$. This map is evidently a group homomorphism, and it is the braid monodromy of S w.r.t. $E \times D, \pi, u$. We sometimes denote φ by φ_u.

Definition (*Braid monodromy of S w.r.t. π, u*). When considering the braid induced from the previous motion as an element of the group $B_p[\mathbb{C}_u, K]$ we get the homomorphism $\varphi: \pi_1(E - N, u) \to B_p[\mathbb{C}_u, K]$ which is called the braid monodromy of S w.r.t. π, u. We sometimes denote φ by φ_u.

Proposition 1.1 (Example). *Let*

$$E = \{x \in \mathbb{C} \mid |x| \leqslant 1\}, \qquad D = \{y \in \mathbb{C} \mid y \leqslant R\},$$

$R \gg 1$, S is the curve $y^2 = x^\nu$, $u = 1$. Clearly, here $n = 2$, $N = \{0\}$, $K = \{-1, +1\}$ and $\pi_1(E - N, 1)$ is generated by $\Gamma = \partial E$ (positive orientation). Denote by $\varphi: \pi_1(E - N, 1) \to B_2[D,K]$ the braid monodromy of S w.r.t. $E \times D, \pi, u$.

Then $\varphi(\Gamma) = H^\nu$, where H is the positive half-twist defined by $[-1,1]$ ("positive generator" of $B_2[D,K]$).

Proof. We can write $\Gamma = \{e^{2\pi i t} \mid t \in [0,1]\}$. Lifting Γ to S we get two paths:

$$\delta_1(t) = \left(e^{2\pi i t}, e^{2\pi i \nu t/2}\right), \qquad \delta_2(t) = \left(e^{2\pi i t}, -e^{2\pi i \nu t/2}\right).$$

Projecting $\delta_1(t)$, $\delta_2(t)$ to D we get two paths:

$$a_1(t) = e^{\pi i t \cdot \nu}, \quad a_2(t) = -e^{\pi i t \cdot \nu}, \qquad 0 \leqslant t \leqslant 1.$$

This gives a motion of $\{1, -1\}$ in D. This motion is the νth power of the motion $\mathcal{M}$:

$$b_1(t) = e^{\pi i t}, \quad b_2(t) = -e^{\pi i t}, \qquad 0 \leqslant t \leqslant 1.$$

The braid of $B_2\left[D, \{1, -1\}\right]$ induced by $\mathcal{M}$ coincides with the half-twist H corresponding to $[-1,1] \subset D$. Thus $\varphi(\Gamma) = H^\nu$. □

Proposition 1.2 (Example). *Let S be a union of p lines, meeting in one point s_0, $s_0 = (x(s_0), y(s_0))$. Let D, E, u, K be as before. Let φ be the braid monodromy of S w.r.t.*

$E \times D, \pi, u$. Clearly, here N = single point $x(s_0)$ and $\pi_1(E - N, u)$ is generated by $\Gamma = \partial E$. Then $\varphi(\Gamma) = \Delta_p^2 = \Delta_p^2\left[D, K(u)\right]$.

Proof. By a continuous change of s_0 and the n lines passing through s_0 (and by uniqueness of Δ_p^2) we can reduce the proof to the following case: $S = UL_k$, L_k: $y = j_k x$, $j_k = \mathrm{e}^{2\pi \mathrm{i} k/p}$, $k = 0, \ldots, p-1$. Then $N = \{0\}$. We can take $E = \{c \mid |x| \leqslant 1\}$, $u = 1$, $\Gamma = \partial E = \{x = \mathrm{e}^{2\pi \mathrm{i} t} \mid t \in [0,1]\}$. Lifting ∂E to S and then project it to D we get n loops:

$$a_k(t) = \mathrm{e}^{2\pi \mathrm{i}(t+k/p)}, \qquad k = 0, \ldots, p-1, \quad t \in [0,1].$$

Thus the motion of $a_k(0)$ represented by $a_k(t)$ is a full twist which defines the braid $\Delta_p^2\left[D, \{a_k(0)\}\right] = \Delta_p^2\left[D, K(1)\right]$. (To check the last fact, see the corresponding actions in $\pi_1(D - K, u)$.) □

Let S be a curve in $\mathbb{C}^2$, $p = \deg S$, $\mathbb{C}_u = \{(u, y)\}$. There exists an epimorphism $\pi_1(\mathbb{C}_u - S, u_0) \to \pi_1(\mathbb{C}^2 - S, u_0)$, so a set of generators for $\pi_1(\mathbb{C}_u - S, u_0)$ determines a set of generators for $\pi_1(\mathbb{C}^2 - S, u_0)$.

There is a classical theorem of Van Kampen from the 30's [19,20], which states that all relations in $\pi_1(\mathbb{C}^2 - S, u_0)$ come from the braid group B_p via the braid monodromy φ_u of S. We shall formulate it precisely.

Choose L, a line in infinity transverse to S. Let $\mathbb{C}^2 = \mathbb{CP}^2 - L$.

Choose coordinates x, y on $\mathbb{C}^2$. Let $\varphi_u : \pi_1(\mathbb{C} - N, u) \to B_p$ the braid monodromy of S with respect to π, u.

The group $\pi_1(\mathbb{C}_u - S, u_0)$ is a free group.

Van Kampen Theorem. $\pi_1\left(\mathbb{C}^2 - S, u_0\right) \simeq \pi_1(\mathbb{C}_u - S, u_0)/\{\beta(V) = V \mid \beta \in \operatorname{Im} \varphi_u, V \in \pi_1(\mathbb{C}_u - S, u_0)\}$.

The above formulation of van Kampen is not very practical because the group presentation is not finite. It is possible to simplify the presentation so that it is finite. See, for example, [8]. Moreover, since we consider branch curves which are cuspidal, one can formulate van Kampen's theorem with relations of types AB, $ABA^{-1}B^{-1}$ and $ABAB^{-1}A^{-1}B^{-1}$. Even this presentation is rather long and complicated, and in order to apply it we have to find symmetries in the braid monodromy factorizations.

2. Galois covers and their Chern numbers

We use the Galois cover construction of Miyaoka to construct our examples.

Definition (*Galois cover*). For X a surface in $\mathbb{CP}^N$ and $f : X \to \mathbb{CP}^2$ a generic projection, we define the Galois cover of X and f w.r.t. the full symmetric group as:

$$\widetilde{X} = X_{\mathrm{Gal}} = \overline{\underbrace{(X_{\mathbb{CP}^2} \times \cdots \times X_{\mathbb{CP}^2}) - \Delta}_{n \text{ times}}},$$

where Δ is the set of n-tuple $(a_1,\ldots,a_n)$,$a_i = a_j$ for some $i \neq j$. Let $\tilde{f}:\widetilde{X} \to \mathbb{CP}^2$ denote the natural projection.

Let $S \subset \mathbb{CP}^2$ denote the branch curve of f. The curve S is singular, with ordinary singularities—nodes and cusps.

The surface $\widetilde{X}$ is smooth. If $\deg(S) > 6$, then $\widetilde{X}$ is minimal and of general type [12].

Lemma 2.1. *Let $n = \deg(X)$ and $m = \deg(S)$. Let d and ρ denote the respective numbers of nodes and cusps of S. Then*

$$c_1^2(\widetilde{X}) = \frac{n!}{4}(m-6)^2, \qquad c_2(\widetilde{X}) = n!\left(\frac{1}{2}m(m-3)+3-\frac{3d}{4}-\frac{4\rho}{3}\right).$$

Proof. See 7.1.1 of [15]. □

Lemma 2.2. *Let E and K denote respective hyperplane and canonical divisors of X. Then the Chern numbers of $\widetilde{X}$ are functions of $c_1^2(X)$, $c_2(X)$, $\deg(X)$, and $E\cdot K$.*

Proof. Let g denote the genus of an algebraic curve and let e denote the topological Euler characteristic of of a space.

Let $R \subset X$ denote the ramification locus of f. The curve R is a nonsingular model of S. By the Riemann–Hurwitz formula, $R = K + 3E$. Thus

$$e(R) = -R\cdot(R+K) = -(K+3E)\cdot(2K+3E).$$

It follows that $g(R)$ is determined by $n = E^2$, $c_1^2(X) = K^2$, and $K\cdot E$. Similarly, $e(E)$ is determined by these quantities. Because $m = \deg(S) = \deg(R) = E\cdot(K+3E)$, we have that m is determined by these quantities.

Let S^* denote the dual curve to S and let $\mu = \deg(S^*)$. By the preceding section, and by Lemma 2.1, it suffices to show that μ, d, and ρ are determined by m, n, $e(E)$, $c_2(X)$, and $g(R)$. We show this by presenting three linearly independent formulae:

$$m(m-1) = \mu + 2d + 3\rho,$$

$$g(R) = \frac{(m-1)(m-2)}{2} - d - \rho,$$

$$c_2(X) + n = 2e(E) + \mu.$$

The first two are classical Plucker formulae. For the third, we may find a Lefschetz pencil of hyperplane sections of X whose union is X. Thus,

$$e(X) + n = e(\mathbb{CP}^1)\cdot e(E) + (\text{number of singular curves in the pencil}),$$

where e is the topological Euler characteristic. The number of singular curves is equal to μ. □

Remark 2.3. Lemma 2.1 can easily be modified to give explicit formulae for $c_1^2(\widetilde{X})$ and $c_2(\widetilde{X})$ in terms of $c_1^2(X)$, $c_2(X)$, n, and $E\cdot K$. However, such formulae are not necessary for our result.

3. Fundamental groups of Galois cover

Let X be a surface in $\mathbb{CP}^N$. Let f be a generic projection $f: X \to \mathbb{CP}^2$. Let X_{Gal} be the Galois cover of X and f w.r.t. the full symmetric group.

Consider the natural homomorphism $\pi_1(\mathbb{C}^2 - S, u_0) \xrightarrow{\psi} S_n$ for S the branch curve of f and u_0 any point not in S. In fact, lifting a loop at u_0 to n paths in X $(n = \deg f)$, induces a permutation of $f^{-1}(u_0)$. Since $\# f^{-1}(u_0) = n$ we thus get an element of S_n. Because f is a generic projection, we have

$$1 \to \ker\psi \to \pi_1(\mathbb{C}^2 - S, u_0) \to S_n \to 1.$$

To obtain an isomorphic form of $\pi_1(X_{\text{Gal}})$ related to the braid monodromy we have to choose a certain system of generators for $\pi(\mathbb{C}^2 - S, u_0)$.

Let $\pi : \mathbb{C}^2 - \mathbb{C}$ be the projection on the first coordinate. Let $M' \subseteq S \subseteq \mathbb{C}^2$ be the points of S where $\pi|_S$ is not etale. Let $M = \pi(M')$. The set M is finite. Let $u \in \mathbb{C}_M$. $(\pi|_S)^{-1}(u)$ is a "good" fibre. Let us take u real, "far enough" from the "bad" points. Let u_0 be a point in $\mathbb{C}_u = \pi^{\prime}(u)$, $u_0 \notin S$.

Let $S \cap \mathbb{C}_u = \{q_1, \ldots, q_p\}$. Let γ_j be paths from u_0 to q_j, such that the γ_j's do not meet each other in any point except u_0. Let η_j be a small circle around q_j. Let γ_j' be the part of γ_j outside η_j. Take $\Gamma_j = \gamma_j' g_j (\gamma_j')^{-1}$. The set $\{\Gamma_j\}$ freely generates $\pi_1(\mathbb{C}_u - S, u_0)$. $\{\Gamma_j\}_{j=1}^p$ is called a good system of generators for $\pi_1(\mathbb{C}_u - S, u_0)$.

We have a surjection $\pi_1(\mathbb{C}_u - S, u_0) \xrightarrow{\nu} \pi_1(\mathbb{C}^2 - S, u_0) \to 0$. The set $\{\nu(\Gamma_j)\}$ generates $\pi_1(\mathbb{C}^2 - S, u_0)$. By abuse of notation, we shall denote $\nu(\Gamma_j)$ by Γ_j.

Since f is stable, Γ_j induces a transposition in S_n. So $\Gamma_j^2 \in \ker\psi$. Let $\langle \Gamma_j^2 \rangle$ be the normal subgroup generated by Γ_j^2. Then $\langle \Gamma_j^2 \rangle \subseteq \ker\psi$. By the standard isomorphism theorems, we have:

$$1 \to \ker\psi / \langle \Gamma_j^2 \rangle \to \pi_1(\mathbb{C}^2 - S, u_0)/\langle \Gamma_j^2 \rangle \xrightarrow{\psi} S_n \to 1.$$

It is convenient to replace $\mathbb{CP}^2$ by its "generic" affine part. Let S be now the branch curve of f in $\mathbb{C}^2$. Let $X_{\text{Gal}}^{\text{Aff}}$ be the part of X_{Gal} lying over $\mathbb{C}^2$ ($\subseteq \mathbb{CP}^2$). It is evident that $X_{\text{Gal}}^{\text{Aff}} \to X_{\text{Gal}}$ is surjective. We consider first $\pi_1(X_{\text{Gal}}^{\text{Aff}})$.

Proposition 3.1. *$\pi_1(X_{\text{Gal}}^{\text{Aff}})$ is isomorphic to* $\ker\psi/\langle \Gamma_j^2 \rangle$.

Proof. In [11]. □

Proposition 3.1 reduces the problem of computing $\pi_1(X_{\text{Gal}}^{\text{Aff}})$ to the computation of a subgroup of $\pi_1(\mathbb{C}^2 - S, u_0)/\langle \Gamma_j^2 \rangle$. Using the braid monodromy we get via the Van Kampen method (see Section 1) a finite presentation of $\pi_1(\mathbb{C}^2 - S, u_0)/\langle \Gamma_j^2 \rangle$. From this presentation via the Reidemeister–Schreier method we get a finite presentation of its subgroup $\ker\psi/\langle \Gamma_j^2 \rangle$ which is $\pi_1(X_{\text{Gal}}^{\text{Aff}})$. Passing to the projective case means adding one relation which is $\prod_j \Gamma_j = 1$.

4. The new examples

Let F_k denote the Hirzebruch (rational ruled) surface of order k. The Picard group of F_k is generated by divisors C and E_0, where $C \subset F_k$ is a fiber and $E_0 \subset F_k$ is a zero section. We have $E_0^2 = k$, $E_0 \cdot C = 1$, and $C^2 = 0$. Let K_k denote the canonical divisor of F_k. It is well known that for any k,

$$K_k = -2E_0 + (k-2)C, \qquad c_1^2(F_k) = 8, \qquad c_2(F_k) = 4.$$

Let $F_{k,(a,b)}$ denote the image of F_k under the embedding induced by $aC+bE_0$ $(a, b \geqslant 1)$. It is elementary that

(a) $\deg(F_{k,(a,b)}) = (aC + bE_0)^2 = 2ab + b^2k$,

(b) $K_k \cdot (aC + bE_0) = \big(-2E_0 + (k-2)C\big)(aC + bE_0) = -2a - 2b - bk.$

For every pair of positive integers s, t, define

$$X_{s,t} = \widetilde{F}_{0,(s+t,2t)}, \qquad Y_{s,t} = \widetilde{F}_{1,(s,2t)}.$$

Proposition 4.1. *For any s, t,*

$$c_1^2(X_{s,t}) = c_1^2(Y_{s,t}), \qquad c_2(X_{s,t}) = c_2(Y_{s,t}).$$

Proof. Since $X_{s,t}$ and $Y_{s,t}$ are Galois covers we can prove the proposition by using Lemma 2.2. Because all rational ruled surfaces have the same Chern numbers, then in order to apply Lemma 2.2 it suffices to prove that the degrees and the intersection of the hyperplane divisor with the canonical divisor, are the same for $F_{0,(s+t,2t)}$ and $F_{1,(s,2t)}$. This is easily done with the above formulae (a) and (b). □

Theorem 4.2. *Let s, t be odd integers such that* $\gcd(s,t) = 1$. *Let*

$$n(s,t) = \deg(F_{0,(s+t,2t)}) = 4st + 4t^2.$$

Then

$$\pi_1(X_{s,t}) \cong \bigoplus_{n(s,t)-2} \mathbb{Z}/2\mathbb{Z}, \qquad \pi_1(Y_{s,t}) \cong \{0\}.$$

Proof. The case of $X_{s,t}$ follows directly from [13, Theorem 10.2]. This states that $\pi_1(\widetilde{F}_{0,(a,b)}) \cong \bigoplus_{n(a,b)-2} \mathbb{Z}/c\mathbb{Z}$ where $c = \gcd(a,b)$. The case of $Y_{s,t}$ follows directly from [15, Theorem 0.1]. This states that for any k, $\pi_1(\widetilde{F}_{k,(a,b)})$ is trivial when $\gcd(a,b) = 1$. □

Theorem 4.3. *$X_{s,t}$ and $Y_{s,t}$ are minimal surfaces of general type which are 4-manifolds with positive signature.*

Proof. In [15].

By combining Proposition 4.1, Theorems 4.2 and 4.3, we obtain our examples. Each example is a pair of 2 smooth minimal surfaces of general type with positive signature which have the same Chern numbers and nonisomorphic fundamental groups. One fundamental group is trivial and the other one is a commutative finite group. The examples include X for which $|\pi_1(X)|$ is arbitrarily large.

5. Other examples

It is possible to use Galois covers of other embeddings of Hirzebruch surfaces. For example, we obtain nonisomorphic fundamental groups if s and t are both odd but not necessarily relatively prime. Nonisomorphic fundamental groups are also obtained if $s = t$.

However, these cases require Theorem 5.1, whose proof is in [6]. This theorem is a generalization of the results of [13] and [15]. Its proof uses the topological and group-theoretic techniques of these articles.

Theorem 5.1. *Let $\widetilde{F}_{k,(a,b)}$ denote the Galois cover of $F_{k,(a,b)}$. Let $n(a,b) = \deg(F_{k,(a,b)})$ and $c = \gcd(a,b)$ then*

$$\pi_1(\widetilde{F}_{k,(a,b)}) \cong \bigoplus_{n(a,b)-2} \mathbb{Z}/c\mathbb{Z}.$$

It is likely that the Galois construction can be used to produce many more examples of pairs of surfaces with the same Chern numbers and nonisomorphic fundamental groups. To illustrate this, we will consider covers of K3 surfaces.

An analog of Theorem 5.1 is Conjecture 5.2.

Conjecture 5.2. *Let $X \subset \mathbb{CP}^N$ be an embedded* K3 *surface, with hyperplane section E and degree n. Let $D \subset X$ be the hyperplane section of a embedding of X of minimal degree, and assume that $E \equiv sD$. Then*

$$\pi_1(\widetilde{X}) \cong \bigoplus_{n-2} \mathbb{Z}_s.$$

Conjecture 5.2 can be proven in the cases for which $D^2 = 4, 6, 8$ (in these cases, X is a complete intersection). It should be possible to combine results on the degenerations of K3 surfaces of Ciloberto, Lopez, and Miranda [5] with braid monodromy techniques [11,13,15] to prove the other cases. It should be mentioned that similar results exist for other surfaces [18].

Let $a, b \in \mathbb{Z}^+$ be distinct even square integers. Let X and Y be embedded K3 surfaces, each of degree a^2b^2. Assume that the smallest possible degrees of embeddings of X and Y are a^2 and b^2 respectively. It is elementary that the K3 surfaces X and Y above satisfy the hypotheses of Proposition 4.1. Hence, $\widetilde{X}$ and $\widetilde{Y}$ have the same Chern numbers. Conjecture 5.2 would imply that $\pi_1(\widetilde{X}) \cong \bigoplus_{n-2} \mathbb{Z}_b$ and $\pi_1(\widetilde{X}) \cong \bigoplus_{n-2} \mathbb{Z}_a$, which would imply the existence of many more families of examples.

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