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Symmetric Lie Algebras

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Lie Rootsystems R are introduced, with axioms which reflect properties of the rootset of a Lie algebra L as structured by representations of compatible simple restricted rank 1 subquotients of L. The rank 1 Lie rootsystems and the rank 2 Lie rootsystems defined over $\mathbb{Z}_p$ are classified up to isomorphism. *Base*, *closure* and *core* are discussed. The rootsystems of *collapse* on passage from R to Core R are shown to be of type S_m.

Given any Lie rootsystem R, its independent root pairs are shown to fall into eleven classes. Where the eleventh (anomoly) pair T_2 never occurs, it is shown that R is contained in $R_0 + S$ (not always equal), where R_0 is a *Witt rootsystem* and S is a *classical rootsystem*. This result is of major importance to two papers (D. J. Winter, Generalized classical–Albert–Zassenhaus Lie algebras, to appear; Rootsystems of simple Lie algebras, to appear), since it implies that the rootsystems of the simple nonclassical Lie algebras considered there are Witt rootsystems.

Toral Lie algebras and *symmetric Lie algebras* are introduced and studied as generalizations of classical–Albert–Zassenhaus Lie algebras. It is shown that their rootsystems are Lie rootsystems. The cores of toral Lie algebras are shown to be classical–Albert–Zassenhaus Lie algebras.

These results form the basis for the abovementioned papers on rootsystems of simple Lie algebras and the classification of the rootsystems of two larger classes of Lie algebras, the *generalized classical–Albert–Zassenhaus Lie algebras* and the *classical–Albert–Zassenhaus–Kaplansky Lie algebras*.

Symmetric Lie algebras are introduced as generalizations of classical–Albert–Zassenhaus Lie algebras. It is shown that their rootsets R are Lie rootsystems. Consequently, symmetric Lie algebras can be studied locally using the classification of rank 2 Lie rootsystems. This is done in detail for *toral Lie algebras*.

INTRODUCTION

In Part I, *Lie rootsystems* R of characteristic $p > 3$ are introduced, with axioms corresponding to properties of the set R of roots (with 0) of a Lie algebra L. More specifically, axioms for R are chosen to reflect com-

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binatorial structure of R determined by representations of simple restricted rank 1 subquotients of L compatible with R. Accordingly, Lie rootsystems appear as the rootsets of the *symmetric Lie algebras* studied in Part II, Sections 7–12.

Since the simple restricted rank 1 Lie algebras are A_1 (classical) and W_1 (Witt), each nonzero root of a Lie rootsystem R is assumed to be either *classical*, $R \cap \mathbb{Z}a = \{0, \pm a\}$, or *Witt*, $\mathbb{Z}a \subset R$. It is shown in Theorem 2.5 that all roots are classical if and only if R is isomorphic as groupoid to a rootsystem in the sense of Bourbaki [1]. This theorem is proved for symmetrysets in Winter [7], except when the characteristic is 5 or 7, and the cases $p=5$ and $p=7$ are resolved by the development, given in Section 1, of an extension to symmetrysets of teh Mills [2] theory of classical Lie algebras of characterisics 5 and 7.

Rank 1 Lie rootsystems R are shown, in Theorem 2.6, to be either A_1 or a group, which corresponds to the classification, recently announced by Georgia Benkart and J. Marschall Osborn, of rank one simple Lie algebras.

The irreducible rank 2 Lie rootsystems defined over $\mathbb{Z}_p$ are classified in Section 3. This classification leads to the determination of all possible pairs of $\mathbb{Z}_p$-independent roots a, b of any Lie rootsystem R, given in Table 3.1 for a, b k-independent and by Theorem 2.6 for a, b k-dependent.

Base, *closure* and *core* are discussed for Lie rootsystem R, and it is shown that any *collapse* in the passage $R \to \text{Core } R = R_\infty$ is a Lie rootsystem of type S_m. This result, given in Theorem 5.3, is used in the passage from a Lie algebra $L = \sum_{a \in R} L_a$ to Core $L = L^\infty/\text{Nil } L^\infty = \sum_{a \in \text{Core } R} (\text{Core } L)_a$ for the identification of toral Lie algebras.

Whenever $R_b(a) = \{b - ra, ..., b + qa\}$ is bounded, the Cartan integer $a^*(b) = r - q$ may be used in place of the axiomatically specified $a^0(b) \in \mathbb{Z}_p$, as in the case of $b^*(a)$ with b in the set $R^{\cdot}$ of classical roots, in all cases except the anomaly, type T_2.

In Section 6 we consider *Lie rootsystems R excluding* T_2, that is, Lie rootsystems which involve only the first 10 of the 11 possible classes of roots. We then use the rank 2 classification of Lie rootsystems above and the closure and homomorphism theory of Winter [16] to prove the following theorem which reduces the problem of determining R to that of determining all *Witt rootsystems* R^0 (only Witt roots occur) and all Witt-classical *amalgamations* $R \subset R^0 + R^1$ with R^0, R^1 Witt and classical rootsystems, respectively:

$$R = \bigcup_{\substack{a \in R^0, b \in R^1 \\ a+b \in R}} a + R_b,$$

where $R_b = \{-b, 0, b\}$ of type A_1.

DECOMPOSITION THEOREM. *Let R be a Lie rootsystem excluding T_2 and let $\hat{a}_1, \ldots, \hat{a}_r$ be a base for the classical Lie rootsystem $\tilde{R}$. Then $R^1 = \{n_1 a_1 + \cdots + n_r a_r | n_1 \in \mathbb{Z},\ n_1 \hat{a}_1 + \cdots + n_r \hat{a}_r \in \hat{R}\}$ is a classical rootsystem isomorphic to $\hat{R}$, R^0 is a Witt rootsystem and $R \subset R^0 + R^1$.*

In Part II, Sections 7–12, Lie algebras $L = \sum_{a \in R} L_a$ of characteristic $p > 3$ are studied by studying their rootsystems and using the abstract theory of rootsystems developed in Part I. Structural preliminaries are given in Sections 7 and 8. *Toral Lie algebras* $L = \sum_{a \in R} L_a$ are discussed in Section 9. It is shown for $p > 5$ "toral Lie algebras are classified by classical–Albert–Zassenhaus Lie algebras up to isomorphism of rootsystems" by showing that the core Core $L = L^\infty / Nil\, L^\infty$ of a toral Lie algebra $L = \sum_{a \in R} L_a$ is a classical–Albert–Zassenhaus Lie algebra whose rootsystem is isomorphic to the rootsystem R of L. The core Core $R = R - Nil\, R \cup \{0\}$ is discussed in Section 10. In Section 11, *symmetric Lie algebras* $L = \sum_{a \in R} L_a$ are introduced for $p > 3$ as generalizations of classical–Albert–Zassenhaus Lie algebras. It is shown in Theorem 11.6 that the rootsystem R of a symmetric Lie algebra L is a Lie rootsystem in the sense of Part I. Consequences of *exclusion* of certain rank 1 and 2 subtypes are considered in Section 12.

The results form the basis for the papers [12, 15] on rootsystems of simple Lie algebras and classification of the roothsystems of two large classes of Lie algebras, the *generalized classical–Albert–Zassenhaus Lie algebras* and the *classical–Albert–Zassenhaus–Kaplansky Lie algebras.*

I. ROOTSYSTEMS

1. SYMMETRYSETS

Let G be an abelian group with additively written product $a + b$. For any finite subset S of G, we regard S as *groupoid*: $a + b \in S$ only for certain $a, b \in S$. For $b \in S$, $a \in G$, we let $S_b(a) = \{b - ra, \ldots, b + qa\}$ be the equivalence class of b in S determined by the equivalence relation generated by the relation $\{(c, c + a) | c, c + a \in S\}$. We call $S_b(a)$ the *a-orbit* of b and $q + r$ the *length* $L(S_b(a))$ of $S_b(a)$. If $b - (r - 1)\ a \notin S$ and $b + (q + 1)\ a \notin S$, we may say that $S_b(a)$ is a *bounded* a-orbit.

For a bounded a-orbit $S_b(a)$, we define the *Cartan integer* $a^*(b) = r - q$, and the *reflection* r_a reversing $S_b(a)$ by $r_a(b + ia) = b + (q - r - i)\ a = (b + ia) - a^*(b + ia)\ a (b + ia \in S_b(a))$.

For any finite subset R of G, an *automorphism* of R (as groupoid) is a bijection $r: R \to R$ such that $a + b \in R$ if and only if $r(a) + r(b) \in R$, in which case $r(a+b)=r(a)+r(b)$, for all $a, b \in R$. The group of such automorphisms of R is denoted Aut R. For $a \in R-\{0\}$, a *symmetry* of R at a is an automorphism s_a of R which stabilzes all a-orbits $R_b(a)$ ($b \in R$) such that $s_a(a)=-a$.

A *symmetryset* of G is a finite subset R of G containing 0 having a symmetry s_a of R at a for each $a \in R-\{0\}$. Since s_a stabilizes $R_b(a)$, we have $s_a(b)=b-a^0(b)\,a$; where $a^0(b) \in \mathbb{Z}_{|a|}$ (ring of integers modulo the order $|a|$ of a, set equal to 0 if a has infinite order).

Henceforth, in this section, we assume that R is a symmetryset of G which is *reduced*, that is, $2a \notin R$ for all $a \in R-\{0\}$. Since $R_0(a)=\{-ra,..., 0, a\}$ is s_a-stable, it is $\{-a, 0, a\}$. Note that $a^0(b+c)=a^0(b)+a^0(c)$ for all $b, c, b+c \in R$, $a \in R-\{0\}$, and that $s_a(b)=r_a(b)$ and $a^0(b)\,a=a^*(b)\,a$ if the a-orbit $R_b(a)$ is bounded.

For any abelian group A, we let $\mathrm{Hom}(R, A)$ be the set of *homomorphisms* f from R to A: $f(a+b)=f(a)+f(b)$ for all $a, b, a+b \in R$. Thus $a^0 \in \mathrm{Hom}(R, \mathbb{Z}_{|a|})$ for all $a \in R-\{0\}$. In this section, we prove, in the absence of certain torsion, that a^* lifts a^0 to $a^* \in \mathrm{Hom}(R, \mathbb{Z})$ for $a \in R-\{0\}$.

We begin by stating the following generalization of a decisive result of Seligman [8], proved along similar lines.

1.1. THEOREM (Winter [16]). *If G has no 2 torsion, then $L(R_b(a)) \leqslant 3$ and $|a^*(b)| \leqslant 3$ for all $a \in R-\{0\}$, $b \in R$. If G has no 2, 3, 5, 7 torsion, then $a^* \in \mathrm{Hom}(R, \mathbb{Z})$ for all $a \in R-\{0\}$.*

Henceforth, we assume that G has no 2 torsion, so that all orbits in R are bounded of length at most 3. It follows that $(-a)^*(b)=a^*(-b)=-a^*(b)$ ($a \in R-\{0\}$, $b \in R$). Note also that $a^*(a)=2$, since R is reduced. We now state generalizations of results of Mills [2], whose proofs carry over withou difficulty, except for Lemma 1.5, which is reformulated and proved in this paper (cf. Lemmas B, A, D, G of Mills [2] for 1.2, 1.3, 1.4, 1.5, below).

1.2. LEMMA (Seligman [2]). *Let $a, b \in R-\{0\}$ with $a \neq \pm b$. Then $0 \leqslant a^*(b)\, b^*(a) \leqslant 3$.*

1.3. LEMMA (Mills [2]). *Let $a, b \in R-\{0\}$. Then $a^*(b) \geqslant 2 \Rightarrow a+b \notin R$ and $a^*(b) \leqslant -2 \Rightarrow a-b \notin R$.*

1.4. THEOREM (Mills [2]). *Suppose that G has no torsion other than 7 torsion. Let $a, b, c \in R$ and $a+b+c=0$. Then $d^*(a)+d^*(b)+d^*(c)=0$ for $d \in R-\{0\}$.*

Finally, we turn to results of Mills [2] for characteristic 5. These results are proved using elements h_a in a Lie algebra L. We bypass use of the Lie algebra, in order to generalize and simplify the Mills [2] theory, by defining counterparts h_a, h_b, etc., in terms of R alone, namely: $h_a = a^0$, $h_b = -b^0/2$, $h_{a+b} = h_a + h_b$, $h_{a+2b} = h_a + 2h_b$, $h_{a+3b} = h_a + 3h_b$, $h_{2a+3b} = 2h_a + 3h_b$ in $\mathrm{Hom}(R, \mathbb{Z}_5)$.

1.5. LEMMA. *Suppose that G is a vector space over $\mathbb{Z}_5$. Let $a, b \in R - \{0\}$ such that $b^*(a) = -3$. Then*

(1) *$a+b$, $a+2b$, $a+3b$, $2a+3b$ are nonzero roots;*

(2) *the h_a, h_b, h_{a+b}, h_{a+2b}, h_{a+3b}, h_{2a+3b} defined above have the following properties: $h_a(a) = 2$, $h_b(b) = h_b(a) = h_a(b) = -1$, $h_{a+b}(a+b) = h_{a+2b}(a+2b) = -1$, $h_{a+3b}(a+3b) = h_{2a+3b}(2a+3b) = 2$.*

Proof. (1) is proved as in Mills [2]. For (2), we have $h_a(a) = a^0(a) = 2$, $h_b(b) = -\frac{1}{2}b^0(b) = -1$, $h_b(a) = -\frac{1}{2}b^0(a) \equiv -\frac{1}{2}b^*(a) \equiv -\frac{1}{2}(-3) \equiv -\frac{1}{2}(2) = a^0(b) = -1$, $h_a(b) = -1$ by Lemma 1.2, $h_{a+b}(a+b) = (h_a + h_b)(a+b) = h_a(a) + h_a(b) + h_b(a) + h_b(b) = 2-1-1-1 = -1$, $h_{a+2b}(a+2b) = (h_a + 2h_b)(a+2b) = h_a(a) + 2h_a(b) + 2h_b(a) + 4h_b(b) = 2-2-2-4 = -6 \equiv -1$, $h_{a+3b}(a+3b) = (h_a + 3h_b)(a+3b) = h_a(a) + 3h_a(b) + 3h_a(a) + 9h_6(b) = 2-3-3-9 \equiv 2$ and $h_{2a+3b}(2a+3b) = (2h_a + 3h_b)(2a+3b) = 4h_a(a) + 6h_a(b) + 6h_b(a) + 9h_b(b) = 8-6-6-9 \equiv 2$. ∎

Using the above result, the following theorem can now be proved along the same lines as its counterpart in Mills [2].

1.6. THEOREM. *Suppose that G is a vector space over $\mathbb{Z}_5$. Let $a, b, c \in R$ and $a+b+c=0$. Then $d^*(a) + d^*(b) + d^*(c) = 0$ for $d \in R - \{0\}$.*

Finally, we need the following Theorem 2.3 of Winter [10].

1.7. THEOREM (Winter [10]). *Let R be a symmetryset whose a-orbits $R_b(a)$ ($b \in R$) are bounded and whose Cartan functions a^* are in* $\mathrm{Hom}(R, \mathbb{Z})$ *($a \in R - \{0\}$). Then $\{a^* \mid a \in R - \{0\}\}$ separates R and R is a* $\mathbb{Z}$-rootsystem, *that is, R isomorphic (as groupoid) to a rootsystem in the sense of Bourbaki* [1] *with* 0 *added.*

We now can prove the following theorem needed for this paper.

1.8. THEOREM. *Let R be a reduced symmetryset of G and suppose that G has no 2, 3, 5, 7 torsion, or that G is a vector space over $\mathbb{Z}_5$ or $\mathbb{Z}_7$. Then R is a rootsystem.*

Proof. This follows from the above theorem, since $a^* \in \mathrm{Hom}(R, \mathbb{Z})$ for all $a \in R - \{0\}$ by Theorems 1.1, 1.4, 1.6. ▮

The following corollary to Theorem 1.1 is also needed.

1.9. COROLLARY. *Let R be a reduced symmetryset in $\mathbb{Z}_p$ whose symmetries are the reversals $r_a(b) = -b(a \in R - \{0\}, b \in R)$. Then $R = \{0\}$ or $R = \{-a, 0, a\}$.*

Proof. Suppose that $a \in R - \{0\}$ and choose k minimal with $2 \leqslant k \leqslant p-1$ and $ka \in R$. If $k = p-1$, then $R = \{-a, 0, a\}$. Otherwise, R has exactly two a-orbits $R = \{-a, 0, a\} \cup \{ka,..., (p-k)\,a\}$, since $R_{ka}(a) = \{ka,..., (p-k)\,a\}$ and, consequently $R_{ka}(a)$ contains every a-orbit except $\{-a, 0, a\}$ by the minimality of k. Repeat the above argument, replacing a by ka, and observe that $R = \{-ka, 0, ka\} \cup \{k(ka),..., (p-k)\,ka\}$, by counting. By Theorem 1.1, the length m of the second orbit is 0, 1, 2 or 3. Hence, its cardinality is 1, 2, 3 or 4 with 1, 3 eliminated since 0 does not occur. Thus, $m = 2$ or 4. It follows that $k = (p-1)/2$ or $k = (p-3)/2$. Consequently, $((p+1)/2)\; a \notin R$. But then $a = 2((p+1)/2)\, a) \notin R$, which is not possible since R is reduced. We conclude that R has only one a-orbit, namely $\{-a, 0, a\}$. ▮

2. ROOTSYSTEMS

2.1. DEFINITION. A *rootsystem* is a pair (V, R), where V is a vectorspace and R is a finite subset of V containing 0 which has a *symmetry* $r_a(v) = v - a^0(v)\, a$ $(v \in V)$ for each $a \in R - \{0\}$:

(1) $a^0 \in \mathrm{Hom}_k(V, k)$ and $a^0(a) = 2$;

(2) $r_a(R_b(a)) = R_b(a)$ for every bounded a-orbit $R_b(a)$ $(b \in R)$.

We also assume that R spans V.

The *rank* of (V, R) is the dimension of the span $V = kR$ of R. The *$\mathbb{Z}$-rank* of R is the rank of the groupoid dual $\mathrm{Hom}(R, \mathbb{Z})$.

We let $Ra = R \cap \mathbb{Z}a$ and define the set $R^{\cdot} = \{a \in R - \{0\} \mid Ra = \{-a, 0, a\}\}$ of *classical roots*, and the set $R^0 = \{a \in R \mid Ra \supset \mathbb{Z}a\}$ of *Witt roots*.

2.2. DEFINITION. A *Lie rootsystem* is a rootsystem (V, R) such that $R = R^0 \cup R^{\cdot}$ and $R_b(a)$ has 1 or $p-1$ or p elements for every $a \in R^0 - \{0\}$, $b \in R$.

We call the orbits $R_b(a)$ $(a \in R^{\cdot}\; b \in R)$ *classical orbits*, and the orbits $R_b(a)$ $(a \in R^0 - \{0\}, b \in R)$ *Witt orbits*. Accordingly, a rootsystem is a Lie rootsystem if all roots are either classical or Witt and every Witt orbit has 1 or $p-1$ or p elements.

In Winter [16, Sect. 4], it is noted that if $2a$, $2(b+a)$, $2(b+3a) \notin R$, then $R_b(a)$ does not contain a, $b+a$, $b+2a$, $b+3a$, $b+4a$ and therefore, is bounded of length at most 3. Thus, orbits $R_b(a)$ of length greater than three exist only when R^0 contains one of a, $b+a$, $b+3a$. It follows that if R^0 is a group, that is, $R^0+R^0=R^0$, then $R_b(a)$ with $b \in R^0$ has length greater than three only for $a \in R^0$.

2.3. Proposition. *Let R^0 be a group, $b \in R^0$ and $R_b(a) \neq \{b\}$. Then:*

(1) *for $p>5$, $a \in R^0$ if and only if $R_b(a)$ has $p-1$ or p elements;*

(2) *for $p=5$, $a \in R^0$ if $R_b(a)=\mathbb{Z}a+b$.*

By the same argument, it follos that $R=R^{\cdot} \cup \{0\}$ if and only if every orbit $R_b(a)$ ($a \in R-\{0\}$, $b \in R$) is bounded, in which case the Lie rootsystem R is a reduced symmetryset.

2.4. Definition. A *classical rootsystem* is a Lie rootsystem all of whose nonzero roots aere classical, that is, $R=R^{\cdot} \cup \{0\}$.

We can now restate Theorem 1.8 as follows.

2.5. Theorem. *A rootsystem (V, R) is classical if and only if R is isomorphic as groupoid to a reduced rootsystem in the sense of Bourbaki with 0 added.*

In the next section, we classify the rank two Lie rootsystems $Rab=R \cap (\mathbb{Z}a+\mathbb{Z}b)$ up to isomorphism. All turn out to be symmetry sets. At the same time, only two rank 1 Lie rootsystem are symmetrysets, namely A_1 and W_1. Accordingly, we refer to A_1 and W_1 as the Lie rootsystems of rank 1 defined over $\mathbb{Z}_p$. The general situation for rank one is as follows.

2.6. Theorem. *Let R be a rank one Lie rootsystem. Then either $R=R^0$ and R is a group, or $R=\{-a, 0, a\}$.*

Proof. Let $a \in R^0-\{0\}$, $b \in R-\{0\}$ and write $b=ma$, where $m \in k$ (which is possible since R is of rank 1). If $R_b(a)$ is bounded, then $a^0(b)=2m$ is in the prime field $\mathbb{Z}_p=\mathbb{Z}1$, so that $b=ma$ is in $\mathbb{Z}a=R_0(a)$. But then $R_b(a)=R_0(a)=\mathbb{Z}a$ and $R_b(a)$ is not bounded, a contradiction. Thus, $R_b(a)=\mathbb{Z}a+b$. Iterating, we have $R^0+R \subset R$, $R^0+R^0+R \subset R, \ldots$, $G=R^0+\cdots+R^0$ (n times) $\subset R$. Since R is finite, G is a group for some n. Since every subgroup of R is in R^0, $G=R^0$ and R^0 is a group. Next, let $a \in R^0-\{0\}$, $b \in R-\{0\}$ and consider $R_a(b)$. If it is unbonded, we have

$b \in R^0$, by Proposition 2.3. Suppose that it is bounded: $R_a(b) = \{a - rb, \ldots, a + qb\} = R_c(b) = \{c, \ldots, c + (r+q)\, b\}$. Then $b^0(c) = -(r+q) \in \mathbb{Z}_p$. We may write $c = sb$ with $s \in k$, by invoking "rank 1," so that $2s = r + q$ and $s \in \mathbb{Z}_p$. Then we have $b \in R^0$: $c = sb \in \mathbb{Z}_p b \Rightarrow a = c + rb \in \mathbb{Z}_p b \Rightarrow \mathbb{Z}_p b = \mathbb{Z}_p a \subset R^0 \Rightarrow b \in R^0$. Thus, an element $b \in R - \{0\}$ is in R^0 in all cases, so that $R = R^0$ if and only if R^0 is nonzero. Suppose, finally, that $R^0 = \{0\}$, so that $R = R^{\cdot} \cup \{0\}$. By the remarks preceding Definition 2.4, each orbit $R_b(a)$ is bounded: $R_b(a) = \{b - ra, \ldots, b + qa\}$ $(a \in R^{\cdot} = R - \{0\},\ b \in R)$. Then $b + qa = r_a(b - ra)$, $a^0(b) = r - q$ and $2s = a^0(sa) = a^0(b) = r - q$, where $b = sa$ $(s \in k)$, so that $s \in \mathbb{Z}p$ and $b \in R \cap \mathbb{Z}a = ra = \{-a, 0, a\}$. It follows that $R = \{-a, 0, a\}$. ∎

We construct Lie rootsystems of rank 1 and 2 as follows. Any finite subgroup G of k^+ determines the Lie rootsystem (k, G). For the others, define $S \vee T = \{(s, t) \in S \times T \mid s = 0 \text{ or } t = 0\}$ and $S \oplus T = \{(s, t) \mid s \in S,\ t \in T\}$, where S and T are sets with a distinguished element: $0 \in S$, $0 \in T$. Identify $s = (s, 0)$, $t = (0, t)$ and write $s + t = (s, t)$. Let $A = \{-a, 0, a\} \subset ka$ and $W = \{-a, 0, \ldots, p-2\} \subset ka$, the rank 1 Lie rootsystems defined over $\mathbb{Z}_p$. Next construct $A \vee A$, $A \vee W$, $W \vee W$, the *reducible* rank 2 Lie rootsystems defined over $\mathbb{Z}_p$.

Let $A_2 = \{(0, 0), \pm(1, 0), \pm(0, 1), \pm(1, 1)\}$, $B_2 = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,1), \pm(1,-1)\}$, $G_2 = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,1), \pm(1,2), \pm(2,1), \pm(1,-1)\}$, which are symmetrysets in $\mathbb{Z}_p^2$ whose groupoid reflections $r_a(b) = b - a^*(b)\, a$ determine $a^* \in \operatorname{Hom}(R, \mathbb{Z})$ and $a^0 \in \operatorname{Hom}(\mathbb{Z}_p^2, \mathbb{Z}_p)$ (a^* reduced modulo p) for $a \in R - \{0\}$ and $R = A_2, B_2, G_2$. These are the irreducible rank 2 *classical* rootsystems. Note that the *irreducible nonreduced classical symmetrysets* $2A = \{0, \pm a, \pm 2a\}$, $BC_2 = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,1), \pm(1,-1), \pm(0,2), \pm(2,0)\}$ are not Lie rootsystems, due to the occurence of a, $2a \in R$, $3a \notin R$.

Finally, construct $W_2 = W \oplus W$, $W \oplus A$, $S_2 = \{(i, j) \in \mathbb{Z}_p^2 \mid i + j \neq 0\} \cup \{(0,0)\}$, $T = T_2(n) = S_2 \cup \{\pm(n, -n)\} = S_2 \cup A = S_2 + A$, where $A = \{(0,0), \pm(n, -n)\}$ $(1 \leqslant n \leqslant p-1)$, the irreducible rank 2 *nonclassical* Lie rootsystems defined over $\mathbb{Z}_p$. To see that $W \oplus A = \mathbb{Z}a - b \cup \mathbb{Z}a + 0 \cup \mathbb{Z}a + b$, S_2 and $T_2(n)$ are Lie rootsystems, we define the symmetries $r_c(b) = b - c^0(b)\, c$ in the three cases $W \oplus A$. S_2, $T_2(n)$ by specifying the appropriate *Cartan functions* $c^0 \in \operatorname{Hom}(\mathbb{Z}_p^2, \mathbb{Z}_p)$:

$$(ia)^0(ia) = 2, \qquad (ia)^0(b) = 0$$

$$(\pm b + ja)^0(\pm b + ja) = 2, \qquad (\pm b + ja)^0(a) = 0$$

$$(i, j)^0(r, s) = 2\,\frac{r+s}{i+j} \qquad ((i, j) \in S_2,\ (r, s) \in \mathbb{Z}_p^2)$$

$$(n, -n)^0(n, -n) = 2, \qquad (n, -n)^0(1, 0) = \text{anything}.$$

2.7. DEFINITION. A rootsystem (V, R) is *defined over* $\mathbb{Z}_p$ if R is contained in some $\mathbb{Z}_p$-form $V_{\mathbb{Z}_p}$ of V: any $\mathbb{Z}_p$-basis for $V_{\mathbb{Z}_p}$ is a k-basis for V.

The following propositions are straightforward. In the first proposition, $Ra_1,..., a_n = R \cap (\mathbb{Z}a_1 + \cdots \mathbb{Z}a_n)$.

2.8. PROPOSITION. *A Lie rootsystem R of rank n is defined over $\mathbb{Z}_p$ if and only if $R = Ra_1 \cdots a_n$ for some $a_1,..., a_n \in R$.*

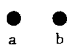
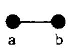
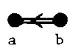
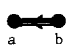
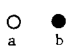
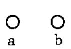
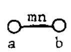
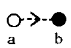
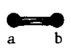
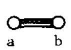
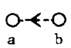
2.9. PROPOSTION. *For $a, b \in R$ and $R_b(a)$ bounded, $a^0(b)$ is in $\mathbb{Z}_p$.*

3. CLASSIFICATION OF LIE ROOTSYSTEMS OF RANK 2

We now determine Recognition Properties for all possible k-independent pairs a, b of roots in a Lie rootsystem R, and classify all corresponding Lie rootsystems Rab of rank 2 up to isomorphism. The results are given in Table I, the irreducible rootsystems of rank 2 defined over $\mathbb{Z}_p$ being A_2, B_2,

TABLE I

Possibilities for Pairs of Independent Roots a, b, Up to Change of Signs

No.	Diagram	Recognition conditions on a and b	Type of Rab	$a^0(b)$ $b^0(a)$
1.		$a, b \in R^{\cdot}, R_b(a) = \{b\}$	$A \vee A$	0 0
2.		$a, b \in R^{\cdot}, a^*(b)\, b^*(a) = 1$	A_2	-1 -1
3.		$a, b \in R^{\cdot}, a^*(b)\, b^*(a) = \frac{a^*(b)}{b^*(a)} = 2$	B_2	-2 -1
4.		$a, b \in R^{\cdot}, a^*(b)\, b^*(a) = \frac{a^*(b)}{b^*(a)} = 3$	G_2	-3 -1
5.		$a \in R^0, b \in R^{\cdot}, R_b(a) = \{b\}$	$W \vee A$	0 0
6.		$a, b \in R^0, R_b(a) = \{b\}$	$W \vee W$	0 0
7.		$a, b \in R^0, R_b(a) = \mathbb{Z}a + b$	W_2	$-m$ $-n$
8.		$a \in R^0, b \in R^{\cdot}, a + b \in R^{\cdot}$	$W \oplus A$	$-m$ 0
9.		$a, b \in R^{\cdot}, a^*(b)\, b^*(a) = 4, \frac{a^*(b)}{b^*(a)} = 1$	$W \oplus A$	-2 -2
10.		$a, b \in R^0, a^0(b)\, b^0(a) = 4$	S_2	$-m$ $-4/m$
11.		$a \in R^0, b \in R^{\cdot}, a + b \in R^0$	T_2	0 $-m$

G_2, W_2, $W \oplus A$, S_2, T_2. In this table, diagrams are introduced to represent each of the 11 classes of k-independent rootpairs a, b. Classical and Witt roots are denoted by black and white nodes, respectively. The *number of solid lines* is the product $a^0(b)\ b^0(a)$. *No lines* indicates that a and b are *orthogonal*: $a \pm b$ are not roots and $a^0(b)\, b^0(a) = 0$. A dotted line indicates that a and b are not orthogonal and $a^0(b)\ {}^0(a) = 0$, which occurs for types S_2, T_2, and for type W_2 if $m = 0$ or $n = 0$.

Orientation indicates which root is shorter, for types B_2, B_2. Orientation indicates that $a + b$ is classical or Witt, for types $W \oplus A$ and T_2, depending on whether the black or white node is "less" (which suffices to distinguish between types $W \oplus A$ and T_2). Actual values for m, n in 7, 8, 10, 11 are suppressed in the diagrams. Adjustments in a, b would lead to default values -1, -1 in 7, -1, 0 (or $-p$) in 8, -2, -2 in 10 and 0 (or $-p$), -1 in 11, which bring the use of orientation (or lack thereof) in these diagrams in line with its conventional use in the diagrams of the classical rootsystems 1, 2, 3, 4.

Here and in the sequel, we decompose a rootsystem into its irreducible components as follows. We say that $S \subset R$ is *closed* if $0 \in S$, $S = -S$ and $a + b \in S$ whenever $a, b \in S$ and $a + b \in R$. If $(R - S) \cup \{0\}$ is closed, we say that $S \subset R$ is *open*. Then $\{S - \{0\} \mid S$ is open and closed in $R\}$ is a topology for $R - \{0\}$, whose connected components $R_1 - \{0\}, \ldots, R_n - \{0\}$ determine the *irreducible components* $R_1, \ldots, R_n$ of R:

(1) $R = R_1 \cup \cdots \cup R_n$ with $R_i \cap R_j = \{0\}$ for $i = j$;

(2) $a = a_1 + \cdots + a_n \in R$ with $a_i \in R_i$ $(1 \leqslant i \leqslant n)$ implies $a = a_i \in R_i$ for some i.

We use the notation $R = S \uplus R = S \cup T$, $S \cap T = \{0\}$, where S, T are open and closed in R. Then R is *irreducible* if and only if $R = R_1$ if and only if $R = \{0\}$ is connect if and only if $R = S \uplus$ implies $R = S$ or $R = T$.

3.1. Proposition. *The irreducible components R_i of a Lie rootsystem are Lie rootsystems.*

We begin with the following theorem, which establishes Recognition Conditions 1, 5, 6. It is proved in the cases of $a \in R^0$, $b \in R$ needed for the ensuing rank 2 classification. The case $a \in R^{\cdot}$ then follows from the classification.

3.2. Theorem. *Let R be a Lie rootsystem and let $a, b \in R - \{0\}$. Then $Rab = Ra \cup Rb$ with $Ra \cap Rb = \{0\}$ if and only if $R_b(a) = \{b\}$.*

Proof. For one direction, note that $a \pm b \in R_b(a) \subset Ra \subset \mathbb{Z}a$, say, implies that $Rab = \{-a, 0, a\}$ or $\mathbb{Z}a$ and $R_b(a) = Rab$.

For the other direction, suppose first that $a \in R^0$, $b \in R^{\cdot}$ and $R_b(a) = \{b\}$. Thus, we know that $a \pm b \notin R$. We claim that $R = Ra \cup Rb$, $Ra \cap Rb = \{0\}$. Suppose, otherwise, that there exist $b' = ra + sb \in R$ with $r, s \neq 0$. We claim that $s \in \{(p-1)/2, \ (p+1)/2\}$, and that $R = s((Ra + Rb) - Rb) \cup Rb = s((W + A) - A) \cup A$, where $Ra = W = \mathbb{Z}a$, $Rb = A = \{-b, 0, b\}$. For this, note first that $r_a(b) = b$ and $a^0(b) = 0$, so $a^0(b') \neq 0$ and $r_a(b') \neq b'$. It follows that $R_{b'}(a) \neq \{b'\}$ and, therefore, has p or $p-1$ elements, by Definition 2.2, since $a \in R^0$. We refer to Table II in what follows.

Note that $R_{b'}(a)$ is contained in Column C_s, among the columns $C_0 =$ Column 0,..., $C_{-1} = C_{p-1} =$ Column $p-1$ of Table II, each of which has at most p-elements. Since $R \cap C_{\pm 1}$ exclude $\pm(b-a)$, $\pm(b+a)$, we must have $s \neq \pm 1$. Thus, since $b \in R^{\cdot}$, we have $sb \notin R$ and $sb \notin R_{b'}(a)$. These constraints force $R_{b'}(a) = \mathbb{Z}(a) + sb - \{sb\}$. Moreover, the constraint $s \neq \pm 1$ forces $R_{b'}(b)$ to have fewer than p-elements, so that $b'' =_{\text{def}} r_b(b') = ra - sb \in R$. Repeating the above argument, we have $R_{b''}(a) = \mathbb{Z}a - sb - \{-sb\}$. Thus, $R \supset ((\mathbb{Z}a + s\{-b, 0, b\}) - s\{-b, 0, b\}) \cup \{-b, 0, b\} = ((W + sA) - sA) \cup A = s((W + A) - A) \cup A$. It follows that $s(b-a) \in R$. Since $(b-a) \notin R$, s can take on only two values, by Proposition 2.3. It follows that $R = s((W + A) - A) \cup A$. Interchanging signs, if necessary, we have $2 \leqslant s \leqslant \cdots \leqslant (p-1) \ s \leqslant \cdots \leqslant p-2$. It follows that $R_{sb+a}(b) = \{sb + a, \ (s+1) \ b + a\}$ and, therefore, that $s + 1 = -s$ and $s = -\frac{1}{2}$, as asserted. Now compute $R_b(sb + a) = \{b, \frac{1}{2}b + a, \ 0 + 2a, \ -\frac{1}{2}b + 3a\}$. Then $(sb + a)^0(b) = -3$ and $(sb + a)^0(sb) = \frac{3}{2}$. Thus, $(sb + a)^0(a) = (sb + a)^0(sb + a) - (sb + a)^0(sb) = 2 - \frac{3}{2} = \frac{1}{2}$. But $R_a(sb + a) = \{a, sb + 2a\}$ implies, to the contrary, that $(sb + a)^0(a) = -1$. We must therefore, conclude that $R = Ra \ \dot{\cup} \ R_b$ for $b \in R^{\cdot}$ and $R_b(a) = \{b\}$.

Next, suppose that $b \in R^0$ and $R_b(a) = \{b\}$. We again claim that $Rab = Ra \cup Rb$, $Ra \cap Rb = \{0\}$. Note that $C_1 \cap R$ excludes $b \pm a$ and, therefore no

TABLE II

The Roots of $W_2 \supset T_2(n) \supset S_2$ as Successively Generated in a-orbits from the Nonroots $m(b-a)$ of S_2

	0	1	2	3		$p-1$
0	0	$\underline{b-a}$	$\underline{2b-2a}$	$\underline{3b-3a}$	— — — — — — —	$\underline{((p-1)\,b+a}$
1	a	b	$2b-a$	$3b-2a$	— — — — — — —	$(p-1)\,b+2a$
2	$2a$	$b+a$	$2b$	$3b-a$	— — — — — — —	$(p-1)\,b+3a$
3	$3a$	$b+2a$	$2b+a$	$3b$	╲	$(p-1)\,b+4a$
4	$4a$	$b+3a$	$2b+2a$	$3b+a$	╲	$(p-1)\,b+5a$
5	$5a$	$b+4a$	$2b+3a$	$3b+2a$	╲	$(p-1)\,b+6a$
6	$6a$	$b+5a$	$2b+4a$	$3b+3a$	╲	$(p-1)\,b+7a$
⋮	⋮	⋮	⋮	⋮	╲	⋮
$p-1$	$(p-1)\,a$	$b+(p-2)\,a$	$2b+(p-3a)$	$3b+(p-4)\,a$	— — — — —	$(p-1)\,b$

$b' = b + ra$ $(r = 0)$ is in R: $b' \in R \Rightarrow R_{b'}(a) = \{b'\} \Rightarrow a^*(b') = 0 \Rightarrow 0 = a^*(ra) = 2r \Rightarrow r = 0$. Similarly, no $a' = a + sb$ $(s \neq 0)$ is in R. But then every column C_s excludes $sb \pm a$, so that each $R_{b'}(a)$ is $\{b'\}$ for $b' \in C_s - \{sb\}$, which is impossible since $0 = a^*(b') = a^*(sb + ra) = 2r$ implies $r = 0$. Thus, $Rab = Ra \uplus Rb$.

The remaining case $a, b \in R^{\cdot}$ is not needed for the rank 2 classification given below. Applying this classification, we need only observe that one of the conditions $R_b(a) = \{b\}$ or $a^*(b) = 0$ is not met for each pair 2, 3, 4, 9, to complete the proof. ∎

3.3. THEOREM. *Let R be a Lie rootsystem and let $a \in R^0$, $b \in R^{\cdot}$ with Rab irreducible. Then $b - a \in R$ and $Rab \supset Ra + Rb = W \oplus A$.*

Proof. Suppose that $b - a \notin R$. Since Rab is irreducible, $b + a \in R$, by Theorem 3.2. Thus, $R_b(a) = b, \ldots, b + (p-2)\, a\}$, by Definition 2.2. It follows that $a^0(b) = 2 = a^0(a)$ and $a^0(b - a) = 0$. Thus, $a^0(2b - 4a) = 2a^0(b - a) - 2a^0(a) = -4$ and $r_a(2b - 4a) = 2b - 4a - a^0(2b - 4a)\, a = 2b \notin R$. It follows that $2b - 4a \notin R$. Since $2b$, $2b - 4a \notin R$ and $-2b$, $-2b + 4a \notin R$, R contains no root $\pm(2b + ra)$; otherwise $R_{2b+ra}(a)$, say, has fewer than $p - 1$ elements, so $R_{2b+ra}(a) = \{2b + ra\}$, which contradicts the irreducibility of Rab. It follows that $R_a(b) = \{a, a + b\}$, whereas $R_{2a}(b) = \{-b + 2a, 2a, b + 2a\}$. Thus, $-1 = b^0(a)$ and $-2 = b^0(-b + 2a) = -2 + 2b^0(a) = -4$ and $2 = 0$, a contradiction.

We conclude that $b - a \in R$ for all $b \in R^{\cdot}$. If $\mathbb{Z}a + b \subset Rab$, we are done. Otherwise, choose r such that $b' = b - ra \in R$ and $b' - a \notin R$. From our discussion above, we conclude that $b' \in R^0$. But then the classification below, which does not depend on this case, implies that Rab' is W_2, S_2 or T_2: Nos. 7, 10, 11 of Table I. But then $Rab' = T_1$, since $R^{\cdot}$ is empty for $R = W_2$ or S_2, in which case $b \in T_2^{\cdot}$ and $T_2 = R \supset Ra + Rb = W \oplus A$ as asserted. ∎

3.4. THEOREM. *Every nonclassical irreducible rank 1 or 2 Lie rootsystem defined over $\mathbb{Z}_p$ R is one of W, $W \oplus A$, W_2, S_2, T_2.*

Proof. Let R be nonclassical, irreducible and not one of W, $W \oplus A$, W_2. We claim that R is S_2 or T_2, as asserted. Since R is not classical it is not reduced, by Theorem 1.8, so that there exists $a \in R^0$ by Definition 2.2. We know that R is an irreducible rank 2 rootsystem defined over $\mathbb{Z}_p$. Letting V be the $\mathbb{Z}_p$-span of R, we have $R \subset \mathbb{Z}a + \mathbb{Z}b = V$ for any $b \in V - \mathbb{Z}a$.

Suppose first that $(b + \mathbb{Z}a) \cap R = \varnothing$ for some $b \in V - \mathbb{Z}a$. Then $c \in R - \mathbb{Z}a$ implies that $c \notin R^0$, so that $c \in R^{\cdot}$. But then it follows from the proven part of Theorem 3.3 that $c + \mathbb{Z}a \subseteq R$, so that $R \supseteq \mathbb{Z}a + \{-c, 0, c\} = W \oplus A$. Moreover, any element $d \in R - \mathbb{Z}a$ has the form $d = r(c + ta)$ with

$c+ta \in R^{\cdot}$ (just as in the above case $r=1$, $t=0$), so that $r=\pm 1$ and $d \in \pm c+\mathbb{Z}a$. It follows that $R=W\oplus A$.

Suppose next that $(b+\mathbb{Z}a)\cap R \neq \varnothing$ for every $b \in V-\mathbb{Z}a$. Then $(b+\mathbb{Z}a)\cap R$ has more than one element for every $b\in V-\mathbb{Z}a$, by Theorem 3.2 and the irreducibility of R. But then $(b+\mathbb{Z}a)\cap R$ has at least $p-1$ elements for every $b\in V-\mathbb{Z}a$, by Definition 2.2. Take $c\in V-\mathbb{Z}a$ with $\mathbb{Z}c\cap R$ as small as possible. Then $R\subseteq \mathbb{Z}a+\mathbb{Z}c$ and $\mathbb{Z}c\cap R$ has one or three or p elements. If $\mathbb{Z}c\cap R$ has p elements, then $R=W_2$. If $\mathbb{Z}c\cap R$ has only one element, namely 0, then each set $(\mathbb{Z}a+jc)\cap R$ $(j\neq 0)$ has exactly $p-1$ elements in it and $R=((\mathbb{Z}a+\mathbb{Z}c)-\mathbb{Z}c)\cup\{0\}=S_2$. In the remaining case, $\mathbb{Z}c\cap R$ has three elements, which we may take to be $\{-c, 0, c\}$ with no loss in generality. Then R contains $ia+jc$ when $i\neq 0$ and $j\neq \pm 1$. It follows that R contains four or more, hence all p, mulitples of every $b\in V-\mathbb{Z}c$. Thus, $R=(\mathbb{Z}a+\mathbb{Z}c-\mathbb{Z}c)\cup\{-c,0,c\}=T_2$.

4. Base and Closure

Let (V, R), (W, S) be rootsystems and consider $R\oplus S = \{a\oplus b \mid a\in R, b\in S\}\subset V\oplus W$. Introduce $r_{a\oplus b}(c\oplus d) = c\oplus d-(a\oplus b)^0(c\oplus d)(a\oplus b)$ by specifying $(a\oplus b)^0 \in \operatorname{Hom}(V\oplus W, \mathbb{Z}_p)$ as follows:

$$(a\oplus 0)^0(c\oplus d)=a^0(c)$$

$$(a\oplus b)^0(c\oplus d)=b^0(d) \qquad (b\neq 0).$$

Note that $R+S_{c\oplus d}(a\oplus 0) = R_c(a)\oplus d$ and $r_{a\oplus 0}(c\oplus d) = c\oplus d-a^0(c)(a\oplus 0) = r_a(c)\oplus d$. Next, suppose that R is a group, that is, $R+R=R$, and note that $R+S_{c\oplus d}(a\oplus b) = \{c\oplus d-r(a\oplus b),..., c\oplus d+q(a\oplus b)\}$, where $S_d(b) = \{d-rb,..., d+qb\}$, so that $d+qb = r_b(d-rb) = (d-rb)-b^0(d-rb)\,b$ implies $c\oplus d+q(a\oplus b) = r_{a\oplus b}(c\oplus d-r(a\oplus b))$. It follows that $V\oplus W, R\oplus S)$ is a rootsystem, provided that R is a group (Table III).

Next, let $V=\mathbb{Z}_p^n$ with basis $a_1,..., a_n$. Consider $S_n=\{0\}\cup\{r_1a_1+\cdots+r_na_n \mid r_1+\cdots+r_n\neq 0\}$ and note that $S_n=\{v\in V \mid f(v,v)\neq 0\}$, where f is the symmetric bilinear form $f(a_i, a_j)=1$ for all i, j. Then S_n is a Lie rootsystem

TABLE III

Rootsystems Constructed from a Given Rootsystem R

$G\oplus R$	(G finite subgroup of k^+)
$W_n\oplus R$	($W_n=\mathbb{Z}_p^n$)
$S_n+R=S_n\cup R=S_n(R)\subset \mathbb{Z}_p^n$	($n>\operatorname{rank} R$)

with the symmetries $r_a(b) = b - 2(f(a, b)/f(a, a))\, a = b - 2(\sum s_i/\sum r_i)\, a$ for $a = \sum r_i a_i$, $b = \sum s_i a_i$. The condition $0 = f(a, a) = (\sum r_i)^2$ defines the hyperplane $V - S_n = W$ of dimension $n-1$. Let (W, R) be any Lie rootsystem in W. Then $S_n + R = S_n \cup R$ is a Lie rootsystem:

(1) $(S_n + R)_b(a) = \mathbb{Z}a + b$ and $a^*(b) = 0$ for $a \in S_n - \{0\}$, $b \in R$;

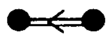
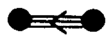
(2) $(S_n + R)_a(b) = a + \mathbb{Z}b$ for $a \in S_n - \{0\}$, $b \in R - \{0\}$;

(3) $(S_n + R)_c(b) = R_c(b)$ for $c \in R$, $b \in R - \{0\}$;

(4) $(S_n + R)_b(a) = (S_n)_b(a)$ or $b + \mathbb{Z}a$ for $a, b \in S_n - \{0\}$ (use the latter if $b + ia \in R$ for some i).

4.1. Definition. We define *base* for a Lie rootsystem (V, R) to be a basis $\pi = \{a_1, \ldots, a_n\}$ for V contained in R such that

(1) if Ra_ia_j is type A_2, B_2, or G_2, then the diagram for a_i, a_j is

,  or  $(i \neq j)$;

(2) $R = R^-(\pi) \cup \{0\} \cup R^+(\pi)$, where $R^-(\pi) = -R^+(\pi)$ and $R^+(\pi)$ is the set of those $a \in R$ for which there exist $a_{i_1}, \ldots, a_{i_s} \in \pi$ such that $\sum_{j=1}^r a_{i_j} \in R$ for $1 \leqslant r \leqslant s$ and $a = \sum_{j=1}^s a_{i_j}$.

In characteristic 0, this is the usual concept of base, since condition (1) implies that $(a_i, a_j) \leqslant 0$ for all $i \neq j$ and condition (1) implies condition (2).

If R has base $a_1, \ldots, a_n$, then $W_m \oplus R$ has base $b_1, \ldots, b_m, a_1, \ldots, a_n$, where $b_1, \ldots, b_m$ is a basis for W_m as a group. And $S_m + R$ $(m > n)$ has base $a_1, \ldots, a_n, a_{n+1}, \ldots, a_m$ obtained by taking any $a \in S_m$, forming the independent set $a + a_1, \ldots, a + a_n, a$, showing that it is part of a base $a + a_1, \ldots, a + a_n, a = a_{n+1}, \ldots, a_m$ for S_m and then returning to $a_1, \ldots, a_n, a_{n+1}, \ldots, a_m$.

4.2. Definition. A Lie rootsystem (V, R) which has a base is said to be *regular*.

A Lie rootsystem (V, R) need not be regular. In fact, a classical rootsystem (V, R) of type A_r, where $p \mid (r+1)$ need not be regular, since it is possible that $\dim V' < r$. We illustrate this by describing two rootsystems (V, R), (V', R') of type A_r $(p \mid (r+1))$ with (V, R) regular and (V', R') not regular. For this, let k be a field of characteristic $p > 0$, let $e_0, \ldots, e_r$ be the basis of k^{r+1} with coordinate conditions $(e_i)_j = \delta_{ij}$, let $R = \{e_i - e_j \mid i \neq j,\ 0 \leqslant i, j \leqslant r\}$, let $\pi = \{a_j \mid 1 \leqslant i \leqslant r\}$ with $a_j = e_{j-1} - t_j$ and let V be the k-span of R. Then (V, R) is a rootsystem of type A_r, π is a basis for (V, R) in the sense of Definition 4.1 and (V, R) is regular. Next, assuming that $p \mid r+1$, note that V contains $e_0 + \cdots + e_r$ and pass to quotients of V modulo $k(e_0 + \cdots + e_r)$. Let $f(v) = v + k(e_0 + \cdots + e_r)$ be the quotient map and define $v' = f(v)$, $V' = \{v' | v \in V\}$, $R' = \{a' | a \in R\}$. Then (V', R') is a rootsystem of type A_r and $f: R \to R'$ is an isomorphism of groupoids. Since a dimension is lost in passing from V to V', π' is not a basis for V' and (V', R') is not regular.

To bypass this pathology for classical rootsystems (V, R), we pass to their *k-closures* $(H^*, \bar{R})$, described below. This passage corresponds, for certain Lie algebras L, to passage to certain Lie algebras $H^* + \text{ad } L^\infty$ of derivations, $H^* = \text{Hom}(R, k)$, where $[h^*, x] = h^*(a)x$ for $a \in R$, $x \in L_a$, $h^* \in H^*$. When L is classical, this Lie algebra is Der $L = H^* + \text{ad } L$, which, since L is centerless and idempotent, is complete by Schenkman [7].

4.3. DEFINITION. $H = \text{Hom}(R, k) = \{f: R \to K | f(a+b) = f(a) + f(b)$ for all $a, b, a+b \in R\}$ is called the *Cartan space* of (V, R) and H_∞ denotes the k-span of its subset $\{a^0|_R \mid a \in R - \{0\}\}$. The groupoid homomorphism $R \to H^* = \text{Hom}_k(H, k)$ which sends $a \in R$ to $\bar{a} \in H^*$ defined by $\bar{a}(f) = f(a)$ is called the *k-closure homomorphism*, and $\bar{R} = \bar{a} | a \in R\}$ is the *set of roots* of the Cartan space H.

We identify a^0 and a^0/R. Then the k-closure $(H^*, \bar{R})$ of (V, R) is a rootsystem with Cartan functions $\bar{a}^0(\bar{b}) = a^0(b)$ and symmetries $r_{\bar{a}}(\bar{b}) = \overline{r_a(b)}$ $(a, b \in R, \bar{a} \neq \bar{0})$.

Using k-closures and *regular functions*, we show in Theorem 4.6 that we can pass from a classical rootsystem (V, R) that may not be regular to an isomorphic rootsystem $(H^*, \bar{R})$ which is regular.

If the k-closure homomorphism $R \to \bar{k}$ is an isomorphism of groupoids, and if it can be extended to an isomorphism of vector-spaces from V to H^*, then we say that (V, R) coincides, up to identification, with its k-closure.

4.4. DEFINITION. (V, R) is *k-closed* if (V, R) coincides, up to identifications, with its k-closure.

It is convenient to have passed from a rootsystem (V, R) to its k-closure $(H^*, \bar{R})$, in order to have realized all latent independence among roots, as in the case of passage from $A_r(k)$ $(p | r+1)$ covered by Theorem 4.6 below. Moreover, R is isomorphic to $\bar{R}$ in the absence of the rootsystem S_2, in which case we may simplify notation and work with the closed rootsystem (H^*, R) with $R \subset H^*$, $a^0 =_{\text{def}} h_a \in H_\infty$ and $a^0(b) = b(h_a) \in k$ for all $a \in R - \{0\}$. This is proved in Theorem 5.4.

We recall from Winter [16] and Theorem 1.8 the $\mathbb{Z}$-closure homomorphism $R \to \hat{R} = \{\hat{a} | a \in R\}$ from R into $H_{\mathbb{Z}} = \text{Hom}(R, \mathbb{Z}) \subset H_Q = \text{Hom}(R, Q) \subset H_{\mathbb{R}} = \text{Hom}(R, \mathbb{R})$, defined by $\hat{a}(f) = f(a)$, is an isomorphism of groupoids to a $\mathbb{Z}$-rootsystem, provided that R is reduced. Thus, there exists a *regular function* on R, that is, a function $f \in H_{\mathbb{R}}$ such that $f(a) \neq 0$ for all $a \in R - \{0\}$; and we then define $R^{\pm} = R^{\pm}(f) = \{a \in R \mid \pm f(a) > 0\}$ and $\pi^{\pm} = \pi^{\pm}(f) = \{a \in R^{\pm}(f) \mid a$ is not contained in $R^{\pm}(f) + R^{\pm}(f)\}$. Conversely, let there exist a regular function $f \in H_{\mathbb{R}}$ for R. Then $f|_{Rab}$ is a regular function on the rank 2 rootsystem Rab. A look at the possibilities for Rab, given in Section 3, show that Rab is classical. It follows that R is a classical rootsystem.

4.5. THEOREM. *A rootsystem (V, R) is classical if and only if there is a regular function $f \in H_{\mathbb{R}}$ on R. For any regular functon $F \in H_{\mathbb{R}}$, $\pi^+(f)$ is base for R if and only if $\pi^+(f)$ is linearly independent.*

We observe that the closure $(H^*, \bar{R})$ of a classical rootsystem R is a regular rootsystem $\bar{R}$ isomorphic to R as groupoid such that $\overline{\pi^+(f)} = \{\bar{a}_1,..., \bar{a}_r\}$ is a base for $\bar{R}$ for any regular function $f \in H_{\mathbb{R}}$ on R. For this, define $\hat{d}_i \in \mathrm{Hom}(\hat{R}, \mathbb{Z})$ such that $\hat{d}_i(\hat{a}_j) = \delta_{ij}$ $(1 \leqslant i, j \leqslant r)$. This is possible since $\pi^+(\hat{f}) = \{\hat{a}_1,..., \hat{a}_r\}$ is a base for $\hat{R}$, where $\hat{f} \in \mathrm{Hom}(\hat{R}, \mathbb{Z})$ is defined by $\hat{f}(\hat{a}) = f(a)$ $(a \in R)$. Then define $d_i: R \to K$ by taking $d_i(a)$ to be $\hat{d}_i(\hat{a})$ reduced modulo p. Then $d_i(\bar{a}_j) = \delta_{ij}$, so that $\overline{\pi^+(f)}$ is a basis for $\bar{R}$ and $\bar{R}$ is a regular rootsystem isomorphic to R.

4.6. THEOREM. *The closure $(H^*, \bar{R})$ of a classical rootsystem (V, R) is a regular rootsystem with R isomorphic to $\bar{R}$ as groupoid and base $\bar{\pi} = \overline{\pi^+(f)}$.*

5. Rigidity and Collapse under Core

In studying a rootsystem R of Lie algebra L, it is important to understand the passage from R to Core R and from L to Core $L = L^\infty/\mathrm{Nil}\, L^\infty$. For this, we further develop concepts introduced in Definition 4.3.

5.1. DEFINITION. The *core* of a rootsystem (V, R) is (H_∞^*, R_∞), where Core $R = R_\infty = \{a_\infty \mid a \in R\}$ and $a_\infty = \bar{a}|_{H_\infty}$. Here, $a \mapsto \bar{a}$ is the closure map and H_∞ is the k-span of $\{a^0 \mid a \in R - \{0\}\}$. We call $R \mapsto R_\infty$, sending a to a_∞ $(a \in R)$, the *core* map. If the core map is bijective, we say that R is *rigid.*

The following proposition is evident.

5.2. PROPOSITION. *If R is rigid, the closure mapping is an isomorphism.*

We say that a set $\{a_1,..., a_n\}$ $(n \geqslant 2)$ of n distinct roots *collapse* if $a_{1\infty} = \cdots = a_{n\infty}$. The following theorem on collapse shows that R is rigid (has no collapse) if R has no rootsystem Rab of type S_2.

5.3. THEOREM. *Let $\{a_1,..., a_n\}$ collapse. Then $Ra_1 \cdots a_n = R \cap (\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n)$ is a rootsystem of type S_m for some m.*

Proof. Without loss of generality, take $a_1,..., a_n$ linearly independent. First, take distinct elements $a, b \in \{a_1,..., a_n\}$.

Note that $b-a \notin R$: for otherwise $2=(b-a)^0(b-a)=(b-a)^0 (b_\infty - a_\infty)=0$. Thus, $R_b(a) = \{b,..., b+qa\}$, where $-q=a^0(b) = a^0(b_\infty) = a^0(a_\infty) = a^0(a)=2$, and $R_b(a)=\{b,..., b-2a\}$. Similarly, $R_a(b)=\{a,..., a-2b\}$. As in Section 3, we now proceed to cogenerate roots and *non-roots* of S_2, as in Table II. Note that no difference $mb-ma=m(b-a)$ is a root. For this, observe that $2b-a$, $a \in R$ with $2b-2a \notin R$: for otherwise $(2b-2a)^0(2b-2a)=(2b-2a)^0(2b_\infty - 2a_\infty)=0$. Similarly, $2a-b$, $b \in R$ with $2a-2b \notin R$. This generalizes easily to $mb-(m-1)a$, $a \in R$, with $m(b-a) \notin R$. Therefore, this cogeneration leads to $Rab=\{ra+sb \mid r+s \neq 0\} = S_2$. Next, suppose that we have $m \geqslant 2$ such that $a=\sum_1^m r_i a_i \in R$ for $\sum_1^m r_i \neq 0$. For any such a, consider $b=(\sum_1^m r_i)\ a_{m+1}$ and note that $b-a \notin R$; otherwise $2=(b-a)^0(b-a)=(b-a)^0(b_\infty - a_\infty) = \sum_1^m r_i(b-a)^0 (a_{m+1\infty} - a_{i\infty})=0$. Thus, $R_b(a)=\{b,..., b+qa\}$, where $-q=a^0(b)=\text{etc.}= a^0(a)=2$, and $R_b(a)=\{b,..., b-2a\}$. Similarly, $R_a(b)=\{a,..., a-2b\}$. It follows that $a=\sum_1^{m+1} r_i a_i \in R$, provided that $\sum_1^{m+1} r_i \neq 0$. By induction, therefore, R contains S_n. Finally, let $a=\sum_1^n r_i a_i \in R$ and suppose that $\sum_1^n r_i=0$. Then $a_\infty=0$, which is impossible: $2=a^0(a)=a^0(a_\infty)=0$. It follows that $Ra_1 \cdots a_n = S_n$. ∎

6. Lie Rootsystems Excluding T_2

Following Sections 1–5 and Winter [10], a Lie rootsystem R has a *closure* $\hat{R}=\{\hat{a} \mid a \in R\}$ over the field $\mathbb{R}$ of real numbers, and a *closure homomorphism* $R \to \hat{R}$ sending $a \in R$ to $\hat{a} \in H^* = \mathrm{Hom}_{\mathbb{R}}(H, \mathbb{R})$, where $\hat{a}$ is defined by $\hat{a}(f)=f(a)$ for $f \in H = \mathrm{Hom}_{\mathrm{Groupoid}}(R, \mathbb{R}^+)$, $\mathbb{R}^+ = (\mathbb{R}, +)$: $\widehat{a+b}=\hat{a}+\hat{b}$ for $a, b, a+b \in R$. We let $\mathbb{R}\hat{R}$ denote the $\mathbb{R}$-span of $\hat{R}$ in H^*. The closure $(\mathbb{R}\hat{R}, \hat{R})$ of (V, R) is a rootsystem over $\mathbb{R}$ in the sense of Winter [10]:

(1) each $\hat{a} \in \hat{R} - \{\hat{0}\}$ has an associated $\hat{a}^0 \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}\hat{R}, \mathbb{R})$, with $\hat{a}^0(\hat{a})=2$ defined by the condition $r_{\hat{a}}(\hat{b})=\hat{b}-\hat{a}^0(\hat{b})\,\hat{a}$, where $r_{\hat{a}}$ is defined as in (2) below;

(2) each $\hat{a} \in \hat{R} - \{\hat{0}\}$ has an associated symmetry $r_{\hat{a}} \in \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}\hat{R})$ defined as $r_{\hat{a}}=\hat{r}_a$, where $\hat{s} \in \mathrm{Aut}_{\mathbb{R}}\mathbb{R}\hat{R}$ is as defined below for $s \in \mathrm{Aut}\, R$.

Here, we define $\hat{s}=s^{**}|_{\mathbb{R}\hat{R}}$, where $s^* \in \mathrm{Aut}_{\mathbb{R}} H$ and $s^{**} \in \mathrm{Aut}_{\mathbb{R}} H^*$ are the *adjoints* of s, s^*: $s^*(f)=f \circ s (f \in h)$, $s^{**}(g)=g \circ s^*(g \in H^*)$, $\hat{s}(\hat{a}) = s^{**}(\hat{a}) = \hat{a} \circ s^*$, $\hat{s}(\hat{a})(f) = \hat{a} \circ s^*(f) = (s^*(f))(a) = (f \circ s)(a) = f(s(a)) = \widehat{s(a)}(f)$ $(a \in R,\ f \in H)$. Thus, $\hat{s}(\hat{a})=\widehat{s(a)}$ and $r_{\hat{a}}(\hat{b})=\widehat{r_a(b)}$. Note that

$\hat{a} \in \hat{R} - \{\hat{0}\}$ implies that $\hat{R}_{\hat{b}}(\hat{a})$ is bounded, thus that $R_b(a)$ is bounded and $a^0(b) = a^*(b) \bmod p$, where $a^*(b)$ is the Cartan integer in the sense of Winter [6, 8].

By Sections 1–5 and Winter [16], $(\mathbb{R}\hat{R}, \hat{R})$ is a rootsystem in the sense of Bourbaki [1] with 0 added.

The following theorem is needed for the proof of the Main Theorem. It is a variation of Theorem 2.4 of Winter [16] and is proved in the same way.

6.1. THEOREM. *Let R be a Lie rootsystem. Then*

(1) *$R_a(b)$ is bounded and $\hat{a}^*(\hat{b}) = a^*(b)$ for all $a, b \in R$, $\hat{a} \neq \hat{0}$;*

(2) *for any $a, b \in R$, $\hat{a} = \hat{0}$, there exists $c \in R$ such that the closure mapping maps $R_c(a)$ bijectively onto $R_{\hat{b}}(\hat{a})$;*

(3) *the closure mapping $R \to \hat{R}$ is an isomorphism (of groupoids) if and only if it is bijective.*

In Theorem 6.1, $R_a(b)$ is bounded for $\hat{a} \neq \hat{0}$, as noted above, so that r_a can be written in terms of the Cartan integers $a^*(b) = r_a(b) = b - a^*(b)\, a$.

Henceforth, we assume that no pair of type T_2 occurs in R. We then proceed to prove the Decomposition Theorem announced in the Introduction.

6.2. THEOREM. *Let R be a Lie rootsystem excluding T_2 and let $\hat{a}_1, \ldots, \hat{a}_r$ be a base for the classical rootsystem $\hat{R}$. Then $S = \{n_1 a_1 + \cdots + n_r a_r \mid n_i \in \mathbb{Z}, n_1 \hat{a}_1 + \cdots + n_r \hat{a}_r \in \hat{R}\}$ is a classical rootsystem isomorphic to $\hat{R}$ and $R \subset R_0 + S$, where R_0 is a Witt rootsystem given by $R_0 = \{a \in R \mid \hat{a} = \hat{0}\}$.*

Proof. From Table I, $b^*(a) = 0, -1, -1, -1, -2, -3, 0, 0, -2$ in types $A \bigvee A$, A_2, B_2 long, G_2 long, B_2 short, B_2 short, $W \bigvee A$, $W \oplus A$ mixed, $W \oplus A$ classical, since R excludes T_2. These are the Cartan integers, up to sign, for all classical b and all classical or Witt a. Since $b^0(c+d) = b^0(c) + b^0(d)$ $(c, d \in kR)$, it follows that $b^*(c+d) = b^*(c) + b^*(d)$ $(c, d \in kR)$ for $b \in R^{\cdot}$. This is verified for $p > 7$ by considering the integers $b^*(c)$ modulo p: $b^*(c+d) \equiv b^*(c) + b^*(d)$ modulo p with $-3 \leqslant b^*(c), b^*(d), b^*(c+d) \leqslant 3$ implies $b^*(c+d) = b^*(c) + b^*(d)$. For $p = 5$ and 7, it follows from the characteristic 5 and 7 theory developed in Section 1 of this paper. Consequently, the groupoid dual $H = \mathrm{Hom}(R, \mathbb{R})$ of R over $\mathbb{R}$ contains b^* $(b \in R^{\cdot})$.

Consider the $\mathbb{Z}$-closure mapping $R \to \hat{R} = \{\hat{a} \mid a \in R\}$ described in Section 4, which is a groupoid homomorphism from R to the classical rootsystem $\hat{R}$. Note that $\hat{a} = \hat{b} \Leftrightarrow fa) = f(b)$ for all $f \in H = \mathrm{Hom}(R, \mathbb{R})$. We claim that $R^0 = \{a \in R \mid a \text{ is a Witt root}\}$ is the kernel $\{a \in R \mid \hat{a} = \hat{0}\rangle$ of $R \to \hat{R}$. Since $\hat{a}, 2\hat{a}, \ldots, (p-1)\,\hat{a} \in \hat{R}$ for $a \in R^0$, and since $\hat{R}$ is classical, we have $\hat{a} = \hat{0}$ for $a \in R^0$. Next, let $b \in R^{\cdot}$. Then $b^* \in H = \mathrm{Hom}(R, \mathbb{R})$, as observed above, so that $\hat{b}(b^*)=b^*(b)=2$. It follows that $\hat{b}\neq\hat{0}$. Thus, $R^0=R-R^{\cdot}$ is the kernel $\{a\in R|\hat{a}=\hat{0}\}$ as asserted.

Since R^0 consists of Witt roots, and since $\{a\in R|\hat{a}=\hat{0}\}$ is a Lie rootsystem, R^0 is a Witt Lie rootsystem.

We now construct a copy S of the classical rootsystem $\hat{R}$ in R such that $R\subset R^0+S$. A part of this construction was done in collaboration with M. Haileh. Let $\hat{f}\in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}\hat{R},\mathbb{R})$ be *regular* on the rootsystem $\hat{R}$, that is, $\hat{f}(\hat{a})\neq\hat{0}$ for $\hat{a}\in\hat{R}-\{\hat{0}\}$. Define $f\colon R\to\mathbb{R}$ by $f(a)=\hat{f}(\hat{a})$. Let $\langle\hat{a},\hat{b}\rangle$ be a positive definite symmetric bilinear form on $\mathbb{R}\hat{R}$ such that $\hat{a}^*(\hat{b})=2(\langle\hat{a},\hat{b}\rangle/\langle\hat{a},\hat{a}\rangle$ $(\hat{a},\hat{b}\in\hat{R}-\{\hat{0}\})$, define $\langle a,b\rangle=\langle\hat{a},\hat{b}\rangle$ $(a\in R^{\cdot},\, b\in R)$ and note that $a^*(b)=\hat{a}^*(\hat{b})=\langle\hat{a},\hat{b}\rangle=\langle a,b\rangle$ $(a\in R^{\cdot}, b\in R)$, by Theorem 1.1. Observe, accordingly that if $\langle a,b\rangle>0$, then $b-a\in R(a\in R^{\cdot},\ b\in R)$. Let $\hat{R}^+=\{a\in\hat{R}|\hat{f}(\hat{a})>0\}$ and $R^+=\{a\in R|f(a)>0\}$.

Let $a_1,\ldots,a_r$ be any elements of R such that $\hat{\pi}=\{\hat{a}_1,\ldots,\hat{a}_r\}$ is the set of simple roots $\hat{R}^+$ in the classical rootsystem $\hat{R}$. We claim first that $S^+=\{n_1a_1+\cdots+n_ra_r|n_i\in\mathbb{Z},\ n_1\hat{a}_1+\cdots+n_r\hat{a}_r\in\hat{R}^+\}$ is contained in R^+ and that $\widehat{R_b(a_j)}=R_{\hat{b}}(\hat{a}_j)$ for $\hat{b}=\sum n_i\hat{a}_i\in S^+$ and $\hat{b}=\sum n_i\hat{a}_i\in\hat{R}^+$. We proceed by induction on the height $h(\hat{b})=\sum n_i$ of $\hat{b}$. If $h(\hat{b})=1$, $\hat{b}=\hat{a}_i$ and $a_i\in S^+$ for some i. Moreover, $a_i-a_j\notin R$, since $\hat{a}_i-\hat{a}_j\notin\hat{R}$: $a_i-a_j\in R\Rightarrow\hat{a}_i-\hat{a}_j=\hat{a}_i+(\widehat{-a_j})=\widehat{a_i-a_j}\in R$. Since $a_j^*(a_i)=\hat{a}_j^*(\hat{a}_i)$, by Theorem 1.1, it follows that $R_{a_i}(a_j)=\{a_i,\ldots,\ a_i-a_j^*(a_i)\,a_j\}$ maps onto $\hat{R}_{\hat{a}_i}(\hat{a}_j)=\{\hat{a}_i,\ldots,\hat{a}_i-\hat{a}_j^*(\hat{a}_i)(\hat{a}_j)\}$, which establishes over assertion for $\hat{b}=\hat{a}_i$ and $h(\hat{b})=1$. Next, let $h(\hat{b})>1$ and suppose that our assertion has been established for $h-1$. Since $\widehat{a_1},\ldots,\widehat{a_r}$, $\hat{b}$ are linearly dependent elements of $\hat{R}^+$ and $\langle\widehat{a_i},\widehat{a_j}\rangle<0$ for all $i\neq j$, we have $\langle\hat{b},\widehat{a_i}\rangle>0$ for some i. But then $\hat{b}-\hat{a}_i=\sum_n j\hat{a}_j-\hat{a}_i\in\hat{R}^+$, with $h(\hat{b}-\hat{a}_i)=h(\hat{b})-1$. By induction, therefore, $c=\sum n_ja_j-a_i$ is in R and $\widehat{R_c(a_i)}=R_{\hat{c}}(\widehat{a_i})=R_{\hat{b}-\hat{a}_i}(\widehat{a_i})=R_{\hat{b}}(\widehat{a_i})\ni\hat{b}=\hat{c}+\widehat{a_i}$.

We do not yet know that $\sum n_ja_j\in R$. However, we know that some element of $R_c(a_i)$ maps to $\hat{c}+\hat{a}_i$, and this element, by virtue of its f-value, must be $c+a_i=\sum n_ja_j =_{\mathrm{def}} b$. This said, we may conclude that $b=\sum n_ja_j\in R^+$ and, moreover, that $\widehat{R_b(a_i)}=\widehat{R_{b-a_i}(a_i)}=\widehat{R_c(a_i)}=R_{\hat{b}}(\hat{a}_i)$, by what was shown above. It remains to show that $R_b(a_j)=R_{\hat{b}}(\hat{a}_j)$ for $j\neq i$. If $\hat{b}-\hat{a}_j\in\hat{R}^+$, we argue by induction, just as in the case $j=i$ above. If $\hat{b}-\hat{a}_j\notin\hat{R}^+$, then $b-a_j\notin R$, in which case $\widehat{R_b(a_j)}=R_{\hat{b}}(\widehat{a_j})$, since $a_j^*(\hat{b})=\hat{a}_j*(\hat{b})$, with details as in the similar case encountered above. We conclude, by induction that $S^+\subset R^+$ and $\widehat{R_b(a_j)}=R_{\hat{b}}(\widehat{a_j})=R_b(a_j)$ for $b\in S^+$, as asserted.

Implicitly derived in the above considerations is the decisive identity $\hat{b}=\sum n_i\widehat{a_i}$, valid for any $\sum n_ia_i\in\hat{R}^+$ and $b=\sum n_i\widehat{a_i}\in S^+$. This is based on the implicit iterative construction/reconstruction of elements $b\in S^+/\hat{b}\in\hat{R}^+$ as

$b = a_{i_1} + \cdots + a_{i_h}$ and $\hat{b} = \hat{a}_{i_1} + \cdots + \hat{a}_{i_h}$, where all partial sums are roots in S^+, respectively in $\hat{R}^+$.

We claim next that $R \subset R^0 + S$. Suppose not, and take $b \in R^+ - (R^0 + S)$ with $f(b)$ minimal, noting that $\hat{b} \in \hat{R}^+$. We claim, firstly, that $b - a_i \in R$ for some $1 \leqslant i \leqslant r$. For this, note that $\hat{a}_1, \ldots, \hat{a}_r$, $\hat{b} \in \hat{R}^+$ are linearly dependent and $\langle \widehat{a_i}, \widehat{a_j} \rangle < 0$ for all $1 \leqslant i \neq j \leqslant r$, so that $\langle \hat{b}, \widehat{a_i} \rangle > 0$ for some i. But then $\langle b, a_i \rangle > 0$ for some i, so that $b - a_i \in R$. Next, we observe that $b \in R^0 + S$, contrary to assumption. In the case $\hat{b} = \hat{a}_i$, we have $b - a_i \in R$ (see above) and $\hat{b} - \hat{a}_i = \hat{0}$, so that $b - a_i \in R^0$ and $b = (b - a_i) + a_i \in R^0 + S$. In the case $b \neq a_i$, $b - a_i \in R$, we have $b - a_i \in R$ (as above), $\hat{b} - \widehat{a_i} \in \hat{R}^+$. But then $b - a_i \in R^+$ with $f(b - a_i) < f(b)$. By minimality of $f(b)$, $b - a_i \in R^0 + S$. But then $b \in R^0 + S$. To see this, write $b - a_i = a + \sum n_j a_j$, where $a \in R^0$ and $\sum n_j a_j \in \hat{R}^+$. Then $\hat{b} - \hat{a}_i = \sum n_j \widehat{a_j} \in \hat{R}^+$ and $\hat{b} = n_1 \hat{a}_1 + \cdots + n_{i-1} \hat{a}_{i-1} + (n_i + 1)\ \hat{a}_i + n_{i+1} \hat{a}_i + \cdots + n_r \hat{a}_r$. Thus, $b = a + n_1 a_1 + \cdots + n_{i-1} a_{i-1} + (n_i + 1)\ a_i + n_{i+1} a_i + \cdots + n_r a_r$ and $b \in R^0 + S$, by the definition of S, contrary to assumption. Thus, $R \subset R^0 + S$, as asserted.

Finally, S has at most as many elements as $\hat{R}$, by its constructional definition, and $\hat{R} \subset \widehat{R^0 + S} = \hat{S}$, so that $S \to \hat{S}$ is surjective from S to $\hat{R}$. It follows that $S \to \hat{S}$ is bijective from S to $\hat{R}$, so that $S \to \hat{R}$ is an isomorphism, by Theorem 6.1, which completes the proof of Theorem 6.2, since the rootsystem $\hat{R}$ is classical. ∎

Finally, we briefly consider subsystems.

Consider a subset σ of $\pi = \{a_1, \ldots, a_r\}$, where $\hat{\pi} = \{\hat{a}_1, \ldots, \hat{a}_r\}$ is a simple system of $\hat{R}$, say $\sigma = \{a_1, \ldots, a_k\}$. Construct $R_\sigma = R_{a_1 \cdots a_k} = \{n_1 a_1 + \cdots + n_k a_k \mid n_1 \hat{a}_1 + \cdots + n_k \hat{a}_k \in \hat{R}\}$. Then $\hat{R}_\sigma = \hat{R} \cap (\mathbb{Z}\hat{a}_1 + \cdots + \mathbb{Z}\hat{a}_k) = \hat{R}\hat{a}_1 \cdots \hat{a}_k$ is a classical rootsystem and $\hat{R}_\sigma^{-1} =_{\text{def}} \{a \in R \mid \hat{a} \in \hat{R}_\sigma\}$ is a rootsystem R^σ. Relative to the global closure maps $R \to \hat{R}$, the same arguments as above show that $R^\sigma \subset R^{\sigma^0} + R_\sigma$, where R_σ, as defined above, is a classical subsystem of R^σ. Both the global closure map and the closure map defined relative to R^σ map R_σ isomorphically to the image (closure in either sense) of R^σ.

Taking $\hat{\pi} = \hat{\pi}_1 \cup \cdots \cup \hat{\pi}_n$ to be the decomposition of $\hat{\pi}$ into connected components, and letting $R^i = R^{\pi_i}$ and $R_i = R_{\pi_i}$, one now easily sees that:

(1) $R = R^1 \cup \cdots \cup R^n$;

(2) $a, b \in R^i$, $a + b \in R \Rightarrow a + b \in R^i$;

(3) $R^i \subset R^{i0} + R_i$ with R_i irreducible and classical and R^{i0} Witt.

Next, take any $b \in R - R^0$, so that $\hat{b} \neq \hat{0}$, and take a simple system $\hat{\pi} = \{\hat{a}_1, \ldots, \hat{a}_r\}$ for $\hat{R}$ such that $\hat{b} = \hat{a}_1$. This is possible, since $\hat{R}$ is reduced, and a classical rootsystem. In this case, R_b as defined above is $R_b = \{-b, 0, b\}$. We write $R^b = R^{\{b\}} = \hat{R}_b^{-1} = \{a \in R \mid \hat{a} \in \hat{R}_b\}$. Then $R^b \subset R^{b0} + R_b$. Let

$a \in R^{b0}$. If $a+b \in R$, then $Rab = R \cap (\mathbb{Z}a + \mathbb{Z}b)$ is type $W \oplus A$, by the exclusion of T_2 and, consequently, $a-b$ is in R^b as well. It follows that $a+R_b \subset R^{b0}$ if $a+b \in R$, so that $R^b = \bigcup_{a \in R^{b0}, a+b \in R} a+R_b$. In particular, we have $R = \bigcup_{a \in R^0, b \in R; a+b \in R} a+R_b$, where $R^1 = R_\pi$ and $a+R_b = a+\{-b, 0, b\}$, by the above arguments and the inclusion $R \subset R^0 + R^1$ established in Theorem 6.2.

II. LIE ALGEBRAS

7. PRELIMINARIES

Throughout Part II of this paper, k denotes a field of characteristic $p > 3$. The following results of Part I on Lie rootsystems (V, R) and Cartan functions $a^0 \in \mathrm{Hom}_k(V, k)$ play key roles in Part II.

7.1. PROPOSITION. *A reduced nonzero symmetryset in $\mathbb{Z}_p$ whose symmetries are the reversals $r_a(b) = -b$ must be $\{-a, 0, a\}$ for some $a \in \mathbb{Z}_p$.*

7.2. THEOREM. *Let R be a classical Lie rootsystem or reduced symmetryset. Then R is isomorphic as groupoid to a rootsystem in the sense of Bourbaki.*

7.3. THEOREM. *Let R have rank 1. Then $R = \{-a, 0, a\}$ or R is a subgroup of k^+.*

7.4. THEOREM. *Let c, $d \in R$ be k-linearly independent. Then $R \subset D$ is classical or one of W (irreducible rank 1), $W \cup A$, $W \cup W$ (reducible rank 2), $W \oplus A$, $W \oplus W$, S_2, T_2 (irreducible rank 2), where $W = \{0, a, ..., (p-1)a\} = \mathbb{Z}a$, $A = \{-b, 0, b\}$, $W \oplus A = \{ia + jb \mid j = \pm 1 \text{ or } 0\}$, $W \oplus W = \mathbb{Z}_p^2$, $S_2 = \{(i, j) \mid i + j \neq 0\}$, $T_2 = S_2 \cup \{(m, -m), (0, 0), (-m, m)\}$.*

7.5. THEOREM. *Let $R \to R|_{H_\infty} = R_\infty$ be the core map and let $a_1, ..., a_n$ $(n \geqslant 2)$ be k-independent elements of R such that $a_{1\infty} = \cdots = a_{n\infty}$. Then $Ra_1 \cdots a_n = R \cap (\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n)$ is of type S_n.*

We need the following theorems on representations of 3 dimensional Lie algebras. Theorem 1.6 is Lemma II.2.2 of Seligman [8].

7.6. THEOREM (Seligman). *Let V be a module for the three dimensional Lie algebra $L = ke + kh + kf$ with $[e, f] = h$, $[h, e] = e$, $[h, f] = -f$. Suppose that the set of characteristic roots is $\{b, b+1, ..., b+j\}$. Then $j = p-1$ or $2b = -j$.*

In any L-module V, L a Lie algebra over k, we define $(x-a)\,v = xv - av$ and $(x-a)^{i+1}v = (x-a)(x-a)^i v$ recursively for $x \in L$, $a \in k$, $v \in V$; and we let $V_a^i(x) = \{x \in V \mid (x-a)^i\, v = 0\}$.

7.7. THEOREM. *Let $L = ke + kh + kf$, where $[e, f] = h$ and $[h, L] = 0$. Let V be an L-module such that $e^{p-1}V = 0$. Then $h^n V = 0$ for some n.*

Proof. Let $b \in k$ be an eigenvalue for h on V. Choose $v \in V - \{0\}$ satisfying $hv = bv$, subject to the constraint that the corresponding integer n such that $e^n v \neq 0$ and $e^{n+1}v = 0$ is maximal. Note that $n+1 \leqslant p-1$. Define $[e^n, f]\; v = e^n(fv) - f(e^n v)$, and note that $[e^{n+1}, f]\; v = 0$, since $h(fv) = bv$ and, consequently, $e^{n+1}(fv) = 0$ by the constraint on v. One can show, by induction, that $0 = [e^{n+1}, f]\; v = (n+1)\, be^n v$. Thus $0 = (n+1)\, b$ and $b = 0$. Consequently, h is nilpotent on V. ▌

8. REDUCTIVE LIE ALGEBRAS

Let L be a Lie algebra. Let $L = L_1 \not\geqslant L_2 \not\geqslant \cdots \not\geqslant L_{n+1} = 0$ be a maximally refined chain of ideals of L and $\bar{L}_1 \oplus \cdots \oplus \bar{L}_n$ where $\bar{L}_i = L_i/L_{i+1}$ $(1 \leqslant i \leqslant n)$. Then the ideal $\text{Nil}\, L = \{x \in L \mid [x, L_i] \subset L_{i+1}\ (1 \leqslant i \leqslant n)\}$ consists of nilpotent elements and is called the *nil radical* of L. Note that $L/\text{Nil}\, L$ has the faithful completely reducible module $\bar{L} = \bar{L}_1 \oplus \cdots \oplus \bar{L}_n$. It follows, as in the proof of Theorem 8.2 below, that $\text{Nil}\, L$ contains every other ideal I of L such that $ad_L I$ consists of nilpotent elements: $I\bar{L} = \{\bar{0}\}$ and, therefore, $I \subset \text{Nil}\, L$. That is, $\text{Nil}\, L$ is the unique maximal ideal such that $\text{ad Nil}\, L$ consists of nilpotent elements.

8.1. DEFINITION. L is *reductive* if $\text{Nil}\, L$ is central in L.

8.2. THEOREM. *L is reductive if and only if* ad L *has a faithful completely reducible representation which preserves nilpotency of elements of* ad L.

Proof. If L is reductive, the representation afforded to $\text{ad}\, L$ by the L-module $\bar{L} = \bar{L}_1 \oplus \cdots \oplus \bar{L}_n$ is such a faithful completely reducible representation. Conversely, let $V = N_1 \oplus \cdots \oplus V_n$ be a representation for $\text{ad}\, L$ with nonzero irreducible submodules $V_1, \ldots, V_n$. Let $\text{ad}\, N$ be an ideal of $\text{ad}\, L$ consisting of nilpotent elements, and assume that $\text{ad}\, N$ acts by nilpotent transformations on V. Since $\text{ad}\, N$ is an ideal of $\text{ad}\, L$, $V_{io} = \{v \in V_i \mid (\text{ad}\, N)\; v = 0\}$ is a nonzero $\text{ad}\, L$-submodule of V_i, so that $V_i = V_{io}$ for $1 \leqslant i \leqslant n$. Thus $(\text{ad}\, N)\; V = 0$. It follows that $\text{ad}\, N = \{0\}$ and N is central in L if V is faithful. Thus, L is reductive. ▌

8.3. THEOREM. *L is reductive if and only if every solvable ideal is central in L.*

Proof. One direction is trivial, since Nil L is nilpotent and therefore solvable. For the other, suppose that L is reductive and let I be a solvable ideal of L. We show by induction on the dimension of I that I is central. Suppose first that I is nilpotent. Then ad I is an ideal consisting of nilpotent elements since $[I,..., [I, L]...] \subset I^n = \{0\}$. Thus, I is central, by Definition 8.1. Next, suppose that the assertion is true for solvable ideals of lower dimension than that of I and let J be the ideal $J = [I, I]$. By induction, J is central in L. Thus, I is nilpotent. But then I is central, as shown above. ▮

8.4. DEFINITION. L is *semisimple* if every solvable ideal of L is 0.

8.5. COROLLARY. *L is semisimple if and only if L is reductive with center* 0 *if and only if* Nil $L = 0$.

Proof. One direction of the first implication is clear. For the other, suppose that L is reductive with center 0, and let I be a solvable ideal of L. Then I is central, by Theorem 8.3. Thus $I = \{0\}$. The remaining implication follows easily. ▮

8.6. DEFINITION. Core $L = L^\infty/\text{Nil } L^\infty$, where $L^\infty = \bigcap_{i=1}^\infty L^i$.

Since Center L is the kernel of ad: $L \to \text{Der } L$, and since $[d, \text{ad } x] = \text{ad } d(x)$ for $d \in \text{Der } L$, $x \in L$, $C = \text{Center } L$ is stabilized by Der L. It follows that $\text{Der}(L, C) = \{d \in \text{Der } L | d(L) \subset C\} = \text{Hom}(L/L^{(1)}, C)$ is an ideal in Der L: $d \in \text{Der}(L, C)$, $e \in \text{Der } L \Rightarrow [d, e] = de - ed$ maps L to C and $L^{(1)}$ to 0. Note that $\text{Der}(L, C) \cap \text{ad } L = \text{Center ad } L$: $\text{ad } x(L) \subset C \Leftrightarrow [x, L] \subset C \Leftrightarrow [\text{ad } x, \text{ad } L] = 0$. We can now easily prove the following theorem.

8.7. THEOREM. *Let L be reductive. Then the solvable radical of* Der L *is contained in* $\text{Der}(L, C)$.

Proof. Let I be a solvable ideal of Der L, so that $[I, \text{ad } L] = \text{ad } I(L) \subset I \cap \text{ad } L$ is solvable ideal of ad L. Then $I(L) + C$ is central in L, since L is reductive, so that $I(L) \subset C$ and $I \subset \text{Der}(L, C)$. ▮

8.8. COROLLARY. *Suppose that L is semisimple or that L is reductive and* idempotent *in the sense that* $L = L^2$. *Then* Der L *is semisimple.*

Proof. In either case, $\text{Der}(L, C) = \text{Hom}(L/L^{(1)}, C = 0$. ▮

We now consider Cartan decomposition $L = \sum_{a \in R} L_a$ of L with Cartan subalgebra $H = L_0$. Note that $H_\infty = H \cap L^\infty = \sum_{a \in R - \{0\}} [L_{-a}, L_a]$ is a

Cartan sub-algebra of L^∞ if and only if $[H_\infty, L_b] = L_b$ for all $b \in R - \{0\}$, in which case $\text{ad}_L H_\infty$ contains Center ad $_L L^\infty$.

8.9. THEOREM. *Let* $\text{Der}(L^\infty, C)$ *denote the set of derivations of L mapping L^∞ into C. Then* $\text{Der}(L^\infty, C)$ *is ideal of* $\text{Der}\, L$ *and* $\text{Der}(L^\infty, C) \cap \text{ad}\, L^\infty = \text{Center ad}\, L^\infty$.

Proof. We have $\text{Der}\, L = D \trianglerighteq \text{ad}\, L > \text{ad}\, H$ with $[D, \text{ad}\, H] \subset \text{ad}\, L$, so that $D = D_0 + \text{ad}\, L^\infty$, where $D_0 = D_0(\text{ad ad}\, H)$. We then have $D = \text{Der}\, L = \sum_{b \in R} D_b$ with $D_b = \text{ad}\, L_b$ $(b \in R - \{0\})$. It suffices to show that $[D_0, \text{Der}(L^\infty, C)] \subset \text{Der}(L^\infty, C)$, since $[\text{ad}\, L^\infty, \text{Der}(L^\infty, C)] = 0$. Thus, take $d_0 \in D_0$, $d \in \text{Der}(L^\infty, C)$ and observe that $[d_0, d] = d_0 d - d d_0$ maps L_a $(a \in R - \{0\})$ to C: $d_0 d(L_a) \subset d_0(C) \subset C$ and $d d_0(L_a) \subset d(L_a) \subset C$. Thus, $[d_0, d]$ maps L^∞ to C and $[d_0, d] \in \text{Der}(L^\infty, C)$. ∎

A Lie algebra L is *complete* if L has center 0 and $\text{Der}\, L = \text{ad}\, L$.

8.10. THEOREM (Schenkman [7]). *Let L be a Lie algebra over any field such that $L = L^2$ and Center $L = 0$. Then* Der L *is complete.*

8.11. COROLLARY. *Let L be semisimple* with $L^2 = L$. *Then* Der L *is complete semisimple.*

8.12. COROLLARY. *Let L be simple. Then* Der L *is complete semisimple.*

Block [3] defines Socle L as the sum of the minimal ideals of L and shows, for L semisimple, that $L \to \text{ad}\, L|_{\text{Socle}\, L}$ is injective. Up to "adjoint identifications," this shows that Der Socle $L \geqslant L \geqslant$ Socle L. For L semisimple, Socle L is idempotent and semisimple. Thus, Der Socle L is complete, by Theorem 8.10. Block [3] determines Der Socle L in terms of the simple Lie algebras of Socle L.

8.13. THEOREM. *Let L be semisimple. Then* Der I *is complete semisimple and* Der $I \geqslant L \geqslant I$ *(up to identifications) for $I = L^{(\infty)}$, in fact, for any idempotent ideal I of L containing* Socle L. *If I is* Der L*-stable, then* Der L *is the normalizer of L in* Der I.

9. REFLECTIVE AND TORAL LIE ALGEBRAS

The contents of this section generalize the theory of Seligman [8, Chap. II, Sects. 1–4], and set the stage for the remainder of the paper.

Let L be a finite dimensional Lie algebra over k with split Cartan subalgebra $H = L_0$ and rootspace decomposition $L = \sum_{a \in R} L_a$. Thus, $L_a =$

$\{x \in L \mid (\mathrm{ad}\, h - a(h))^{\dim L} x = 0$ for all $h \in H\}$ for a in the additive group H^* of functions from H to K, and $R = R(L, H) = \{a \in H^* \mid L_a \neq \{0\}\}$ is the set of roots of H on L.

Let V be an L-module, $V_b = \{v \in V \mid (h - b(h))^{\dim V}\ v = 0$ for all $h \in H\}$ and $S(V, H) = \{b \in H^* \mid V_b = \{0\}\}$, the set of weights of H in V.

The following representation theorem is equivalent to the two representation theorems 7.6 and 7.7 for three dimensional algebras, as shown in the proof.

9.1. THEOREM. *Let $a \in R$ and let $S_b(a) = \{b - ra, \ldots, b + qa\}$ be a bounded a-orbit in $S = S(V, H)$. Suppose that $h = [e, f]$, $[h, e] = a(h)\ e$, $[h, f] = -a(h) f$ with $e \in L_a$, $f \in L_{-a}$. Then*

(1) $2b(h) = (r - q)\ a(h)$;

(2) *if $a(h) \neq 0$, then the reflection r_a reversing $S_b(a)$ is given by $r_a(c) = c - 2(c(h)/a(h))\ a (c \in S_b(a))$.*

Proof. For (1), suppose first that $a(h) \neq 0$ and let $h' = a(h)^{-1} h$. Then $2(b(h') - r = -(q + r)$, by Theorem 7.6, so that $2b(h') = r - q$ and $2b(h) = (r - q)\ a(h)$. Suppose next that $a(h) = 0$. Then $[e, f] = h$, $[h, e] = 0$, $[hf] = 0$ and $W = \sum_{c \in S_b(a)} V_c$ is a module for $N = ke + kh + kf$ such that $e^{p-1} W = 0$. It follows from Theorem 7.7 that $h^n W = 0$ for some n, so that $b(h) = 0$. This proves (1) for $a(h) = 0$.

For (2), let $c = b + ia$ and observe that $c - 2(c(h)/a(h))\ a = (b + ia) - 2((b(h)/a(h)) + i)\, a$. Since $2(b(h)/a(h)) = r - q$ by (1), it follows that $c - 2(c(h)/a(h))\ a = b + (q - r - i)\ a = r_a(b + ia)$. ∎

We let R_* be the set of those $a \in R$ such that $R_b(a)$ is bounded for all $b \in R$. We also define $L_a^1 = \{x \in L_a \mid [h, x] = a(h)\, x$ for all $h \in H\}$.

9.2. THEOREM. *Let $a \in R_*$ and suppose that $h = [e, f]$ with $e \in L_a'$, $f \in L_{-a}^1$. Then*

(1) *if $a(h) = 0$, ad h is nilpotent;*

(2) *if $a(h) \neq 0$, then $r_a(c) = c - 2(c(h)/a(h))\ a$ is a symmetry of R at a;*

(3) *if $a(h) \neq 0$, then $2a \notin R$.*

Proof. For (1), suppose that $a(h) = 0$. Since the $R_b(a)$ are bounded $(b \in R)$, k has characteristic $P \neq 2$. Since $2b(h) = (r - q)\ a(h) = 0$, by Theorem 9.1, $b(h) = 0$ for all $b \in R$. Thus, ad h is nilpotent. Note that (2) follows directly from Theorem 9.1. For (3), consider $V = kf + H + L_a + \cdots L_{qa}$, where $a, \ldots, qa \in R$ and $(q + 1)\ a \notin R$. Let $S = S(V, H)$, so that $S_0(a) = \{-a, 0, a, \ldots, qa\}$. Then $qa = r_a(-a) = a$, by (2). Thus, $S_0(a) = \{-a, 0, a\}$ and $2a \notin R$. ∎

We recall the definition of classical Lie algebra.

9.3. DEFINITION (Seligman [8]). A Lie algebra L with split Cartan subalgebra H is *classical* if L has center 0, $L^{(1)}=L$, ad H is diagonalizable on L, $[L_a L_{-a}]$ is one dimensional and $R_b(a)$ $(b \in R)$ is bounded for all $a \in R-\{0\}$.

We now introduce the reflective Lie algebras as generalizations of classical Lie algebras. In our definition, $[L_a^1, L_a^1]$ denotes the span of $\{[e,f] | e \in L_a^1, f \in L_{-a}^1\}$. This theory generalizes part of Winter [10].

9.4. DEFINITION. A Lie algebra L with split Cartan subalgebra H is *reflective* if ad$[L_a^1, L_{-a}^1]$ has some nonnilpotent element and $R_b(a)$ $(b \in R)$ is bounded for all $a \in R-\{0\}$.

Note that there are no reflective Lie algebras in characteristics 2 and 3, by the boundedness condition.

We let $L^\infty = \bigcap_{i=1}^\infty L^i$ and Core $L = L^\infty/\text{Nil } L^\infty$. If $L=L^{(1)}=L^\infty$, Recall that L is *idempotent*. For any Cartan subalgebra H of L, we let $H_\infty = H \cap L^\infty$. Then $L = H + L^\infty$ and $H_\infty = \sum_{a \in R-\{0\}} [L_a, L_{-a}]$, where $L = \sum_{a \in R-\{0\}} L_a$ is the Cartan decomposition of L with $L_0 = H$.

The following theorem shows that reflective Lie algebras (L, H) are roughly classified by corresponding classical rootsystems $R(L, H)$, defined and described in Section 1. For classical Lie algebras $(L, H$, this classification $(L, H) \to R(L, H)$ is "up to isomorphism," e.g., by a version of Theorem 3.7.4.9 of Winter [9]: $(L_1, H_1) \cong (L_2, H_2)$ if and only if $R(L_1, H_1) \cong R(L_2, H_2)$ for (L_1, H_1), (L_2, H_2) classical.

9.5. THEOREM. *Let L be a reflective Lie algebra with split Cartan subalgebra $L_0 = H$ and Cartan decomposition $L = \sum_{a \in R} L_a$. Then*

(1) *R is a classical rootsystem*;

(2) dim $L_a =$ dim $[L_a, L_{-a}] = 1$ *for all* $a \in R-\{0\}$ *and* $[L_a, L_b] = L_{a+b}$ *for all* a, b, $a+b \in R-\{0\}$;

Proof. Consider the set $\{[e,f] | e \in L_a^1, f \in L_{-a}^1\}$ and note that W is commutative: $[[x,y], [e,f]] = [[[x,y], e], f] + [e, [[x,y], f]] = a([x,y])[e,f] - a([x,y])[e,f] = 0$. Since the span ad$[L_a^1, L_{-a}^1]$ of ad W has dome nonnilpotent element, ad W must therefore contain a nonnilpotent element ad h_a. Let $h_a = [e_a, f_a]$ with $e_a \in L_a^1, f_a \in L_{-0}^1$. When the context is clear, we abbreviate $h = h_a$, $e = e_a$, $f = f_a$. Note that $a(h) \neq 0$, by Theorem 9.2, so that $r_a(c) = c - 2(c(h)/a(h))\, a$ is a symmetry of R at a by the same Theorem 9.2. Note that $[H, f] = kf$, since $f \in L_{-a}^1$ and $[h, f] = a(h) f \neq 0$. It follows that $2a \notin R$, by Theorem 9.2. Thus, R is a reduced symmetryset in the sense of Theorem 1.2. Since R is reduced with bounded orbits, the characteristic of k is not 2 or 3. Thus, R is a reduced rootsystem by Theorem 7.2.

Consider $L^\infty = H_\infty + \sum_{a\in R-\{0\}} L_a$ and $H_\infty = H\cap L^\infty = \sum_{a\in R-\{0\}} [L_a, L_{-a}]$. Note that $[L_a, L_a] = \{0\}$, since $2a\notin R$. Take $h_a = h = [e, f] \in [L_a^1, L_{-a}^1]$ as above, with $a(h)\neq 0$. For $u\in L_a^1$, note that $-a(h)\, u = [u, h] = [u, [e, f]] = [e, [u, f]] + 0 = -a([u, f])\, e$ and $u = (a([u, f])/a(h))\, e \in ke$. Thus, $L_a^1 = ke$. We claim that $L_a = ke$. Suppose that $L_a \supsetneq ke$. Since $(\operatorname{ad} h - a(h))^{\dim L} L_a = 0$, there exists $u\in L_a - ke$ such that $(\operatorname{ad} h - a(h))\, u = ce$ and $[h, u] - a(h)\, u = ce$ for some $c\in k$. But then $-a(h)\, u - ce = [u, h] = [u, [e, f]] = [e, [u, f]] + 0 = -a([u, f])\, e$ and $u = (a([u, f]) - c/a(h))\, e \in ke$, a contradiction. Thus, $L_a = ke$. This establishes that L_a and $[L_a, L_{-a}]$ are one dimensional for all $a\in R-\{0\}$.

Now let a, b, $a+b\in R-\{0\}$, $S_b(a) = \{b-ra, ..., b+qa\}$, $T = \{b-ra, ..., b\}$, $V = \sum_{c\in T} L_c$. If $[L_a, L_b] = 0$, then V is a module for $ke + kh + kf = L_a + [L_a, L_{-a}] + L_{-a}$, so that $r_a(b-ra) = b$. But this is impossible, since $r_a(b-ra) = b+qa$ with $q\geqslant 1$. Thus $[L_a, L_b]\neq 0$, so that $[L_a, L_b] = L_{a+b}$. ∎

Next, we introduce toral Lie algebras as generalizations of reflective Lie algebras.

9.6. Definition. A Lie algebra L with split Cartan subalgebra H is *toral* if $\dim L_a = 1$ and $a([L_a, L_{-a}])\neq 0$ for all $a\in R-\{0\}$.

The *algebras of Block* are those toral Lie algebras (L, H) which are idempotent, have center 0 and have ad H diagonalizable. The algebras of Block are classified in Block [4] for $p>5$.

9.7. Proposition. *Let L be toral. Then $L^\infty\cap$ Center L = Center $L^\infty \cap$ Center H, and H_∞ is a Cartan subalgebra of L^∞. Moreover,* $\operatorname{ad}_L H_\infty$ *is diagonalizable.*

Proof. Each ad $h\in$ ad H is diagonalizable on the $L_a (a\in R-\{0\})$, therefore on the algebra L^∞ generated by them. Thus, $[H, H_\infty] = 0$. But then ad H_∞ is diagonalizable on L^∞ and 0 on H. ∎

We now determine Nil L. For this, observe that Kern $R = \{h\in H \mid a(h) = 0$ for all $a\in R\}$ is contained in the centralizer $C_L(L^\infty) = \{x\in L \mid [x, L^\infty] = 0\}$. For Kern R centralizes the generators L_a $(a\in R-\{0\})$ for L^∞, so that Kern $R\subset C_L(L^\infty)$. Conversely, any $x\in C_L(L^\infty)$ centralizes the L_a $(a\in R-\{0\})$. Writing $x = \sum_{b\in R} x_b$ with $x_b\in L_b$, $0 = [x, e_a] = \sum_{b\in R} [x_b, e_a]$, which implies that $0 = [x_b, L_a]$ for $L_a = ke_a (a\in R-\{0\}, b\in R)$. Thus, $[x_b, L_{-b}] = 0$, which implies that $x_b = 0 (b\in R-\{0\})$ and, therefore, that $x = x_0\in H$. Thus, $C_L(L^\infty)\subset H$ and, therefore, $C_L(L^\infty) \subset \text{Kern } R$. Thus, Kern $R = C_L(L^\infty)$ is an ideal of L contained in H centralizing L^∞. As such, Kern $R \subset \text{Nil } L$. Conversely, Nil L is ad H-stable and is, therefore, a sum of $(\text{Nil } L) \cap H$ and certain of the one dimensional spaces L_a. But $[L_a, L_{-a}] \not\subset \text{Nil } L$, since $a([L_a, L_{-a}]) \neq 0$, whereas $[(\text{Nil } L), L_{-a}] \subseteq \text{Nil } L$ for $a \in R - \{0\}$. It follows that Nil $L \subset H$ and, therefore, that Nil $L \subset \text{Kern } R$. This establishes the following theorem.

9.8. THEOREM. *Let L be toral. Then* Nil $L = \text{Kern } R = C_L(L^\infty)$, *where* Kern $R = \{H \in H | a(h) = 0$ *for all* $a \in R\}$ *and* $C_L(L^\infty)$ *is the centralizer in L of* L^∞.

9.9. COROLLARY. *Let L be toral and idempotent. Then L is reductive.*

9.10. COROLLARY. *Let L be toral and H abelian. Then L is reductive.*

9.11. PROPOSITION. *Let L be toral with center* 0. *Then H is abelian, L is reductive and* Core $L = L^{(1)}$, *that is,* $L^{(1)} = L^\infty$ *and* $L^{(1)}$ *has center* 0.

Proof. Suppose that H is not abelian, and choose a nonzero element $h \in H^{(1)} \cap \text{Center } H$. Then $[h, L_a] = 0$ for all $a \in R$ and $h \in \text{Center } L$, so Center $L \neq \{0\}$ in contradiction to the hypothesis. Thus, H is abelian. It follows that L is reductive, by Corollary 9.10. Finally L^∞ has center 0, since $L = H + L^\infty$ has center 0 and H is abelian. Thus, $L^{(1)} = H^{(1)} + L^\infty = L^\infty$. ▮

9.12. PROPOSITION. *Let L be toral and reductive. Then* Core ad $L = (\text{ad } L)^{(1)}$.

Proof. Since $H^{(1)} \subset \text{Kern } R = \text{Nil } L$ and Nil L is central, $0 = \text{ad } H^{(1)} = [\text{ad } H, \text{ad } H]$. Thus, ad H is abelian and $(\text{ad } L)^{(1)} = \text{ad } H^{(1)} + \text{ad } L^\infty = \text{ad } L^\infty$. Since ad L^∞ is idempotent, it suffices to show that it had center 0. Let ad h be central in ad L^∞, so that $h \in \text{Kern } R \subset \text{Nil } L \subset \text{Center } L$. Then ad $h = 0$. ▮

The following theorem shows that the rough classification of reflective Lie algebras (L, H) by their rootsystems, discussed in the paragraph preceding Theorem 9.5, is equivalent to a rough classification of reflective Lie algebras L by their (classical) cores. The latter classification is independent of a split Cartan subalgebra H of L.

9.13. THEOREM. *Let L be reflective. Then* Core $L = L^\infty/\text{Nil } L^\infty$ *is classical and isomorphic to* $(L/\text{Nil } L)^{(1)}$, and the root systems of L and Core L are canonically isomorphic.

Proof. Let $\tilde{L} = L/\text{Nil } L$, $\tilde{H} = (H + \text{Nil } L)/\text{Nil } L$. Then $\tilde{H}$ is a split Cartan subalgebra of $\tilde{L}$, dim $\tilde{L}_a = 1$ and $a([\tilde{L}_a, \tilde{L}_{-a}]) \neq 0$ for all nonzero roots a of $\tilde{L}$ and $\tilde{L}$ is toral, by Theorem 9.5 and Definition 9.6. By Theorem 9.8, $\tilde{L}$ has center 0, since any central element x would lie in $\{\tilde{x} \in \tilde{H} | a(x) = 0$ for all roots $a\} \subset$ Nil L/Nil $L = \{\tilde{0}\}$. Thus, Core $\tilde{L} = \tilde{L}^{(1)}$, by Proposition 9.11. Since L^∞ is idempotent, the homomorphism $L^\infty \to \tilde{L}$ has image $\tilde{L}^{(1)} =$ Core $\tilde{L}$ and, therefore, Kernel Nil L^∞. Thus, Core L is isomorphic to $(L/\text{Nil } L)^{(1)}$. Consider the mappings $R_L \to R_{L^\infty}$ (restriction to H_∞) and $R_{L^\infty} \to R_{\text{Core } L}$ (reduction mod Nil L), where R_L, R_{L^∞}, $R_{\text{Core } L}$ are the sets of roots for (L, H), (L^∞, H_∞), (Core L, H_∞/Nil L^∞), respectively. We know, by Theorem 1.7, that $\{a^\oplus | a \in R - \{0\}\}$ separates R. Since $a^\oplus(b) = 2(b(h_a)/a(h_a))$ with $h_a \in L^\infty$, it follows that $\{a^\oplus | a \in R_\infty - \{0\}\}$ separates R_∞ and, moreover, L^∞ is reflective and Core L classical, since $a(h_a) \neq 0$ $(a \in R - \{0\})$. Thus, the mappings $R_L \to R_{L^\infty} \to R_{\text{Core } L}$ are bijections. It therefore follows from Winter [16], Theorem 2.4, that they are isomorphisms of groupoids. ∎

We say that $L = \sum_{a \in R} L_a$ is *weakly reflective* if $R_b(a)$ is bounded and $[L'_{-a}, L'_a]$ has a nonnilpotent element for all $a, b \in R$ such that $L_a \not\subset$ Nil L^∞ and $L_b \not\subset$ Nil L^∞. The above results lead easily to the following version of part of them which, by the conjugacy of Cartan subalgebras of classical Lie algebras is an "invariant characterization." The proof is based on passage from L to L^∞.

9.14. THEOREM. *A Lie algebra L is weakly reflective with respect to some split Cartan subalgebra H if and only if* Core *L is classical.*

10. THE NILPOTENT ROOTS OF L

For a Lie algebra $L = \sum_{a \in R} L_a$ with orbits $R_b(a)$ $(a \in R - \{0\}, b \in R)$ bounded, L is reflective if and only if the set $\text{Nil}^1 R =_{\text{def}} \{c \in R - \{0\} | [L^1_{-c}, L^1_c]$ consists of ad-nilpotent elements$\} \cup \{0\}$ of *nilpotent roots* of (L, H) is $\{0\}$. Note, in this connection, that $0 \in \text{Nil}^1 R$ is not an anomaly, since $[L^1_0, L^1_0] = 0$.

For ad L_0 diagonalizable, $\text{Nil}^1 R = \text{Nil } R$, where $\text{Nil } R =_{\text{def}} \{c \in R - \{0\} | \{L_{-c}, L_c]$ consists of ad-nilpotent elements$\} \cup \{0\}$.

Without assuming a condition that ad L_0 be diagonalizable, we now show that the subalgebra $L_{\text{Nil } R}$ generated by $\{L_c | c \in \text{Nil } R - \{0\}$ is ad-nilpotent on L, provided that the orbits $R_b(a)$ $(a \in R - \{0\}, b \in R)$ are bounded. Note, in this connection, that $0 \in$ Nil R is an anomaly for certain Lie algebras L, even when $L = L_0 \oplus L^\infty = L_0 \oplus L_a$, where L^∞ is abelian and ad L_0 is irreducible on $L^\infty = L_a$: $[L_0, L_0]$ need not be ad-nilpotent.

10.1. THEOREM. *Suppose that the orbits* $R_b(a)$ $(a \in R - \{0\},\ b \in R)$ *are bounded. Then* $L_{\mathrm{Nil}\,R}$ *is* ad-*nilpotent on* L.

Proof. Let $S = \mathrm{Nil}\ R - \{0\}$. Observe that the subalgebra $H_{\mathrm{Nil}\,R}$ generated by the "weakly closed" st $\bigcup_{c \in S} [L_{-c}, L_c]$ is ad-nilpotent on L, by the Jacobson–Engel theorem (Jacobson [5]). We claim that the "weakly closed" set $W = \bigcup_{n=1}^{\infty} W_n$ of commutators $[x_1, ..., x_n] \in W_n$ (with any legal arrangements of brackets) $(n = 1, 2, ..., c_i \in S, x_i \in L_{c_i})$ consists of ad-nilpotent elements. Consider $x = [x_1, ..., x_n] \in W_n$ of weight $\sum_1^n c_i = 0$, $x_i \in L_{c_i}$, $c_i \in S$. After successive factorizations $x = [[x_1 \cdots x_m], [x_{m+1}, ..., x_n]]$ of x and generated terms thereof, and successive use of the Jacobi identity in conjunction therewith, x can be written as a linear combination of terms of W_n of the form $x' = [x'_1, [x'_2, ..., x'_n]]$ of weight $0 = \sum_1^n c'_i$, $x_i = L_{c'_i}$, $c'_i \in S$. But then $x' \in [L_{c'_i}, L_{-c'_i}] \subset H^S$, so that $x \in H^S$, as a linear combination of the generated terms x', and ad x is nilpotent on L, as an element of ad H^S. Finally, consider an element $x = [x_1, ..., x_m] \in W_n$ of weight $c = \sum_1^n c_i \neq 0$. Then ad x is nilpotent on L, by the boundedness of orbits $R_b(c)$ $(b \in R)$. Since ad W is a weakly closed set of nilpotent linear transformations of L, it follows that ad L^S is nilpotent on L, by the Jacobson–Engel Theorem. ▌

We now identify Nil R precisely in the case of Lie algebras L of characteristic 0.

10.2. THEOREM. *Let* $L = \sum_{a \in R} L_a$ *be a Lie algebra of characteristic* 0. *Then* Nil $R = \{c \in R - \{0\} \mid L_c \subset \mathrm{Nil}\ L\} \cup \{0\}$.

Proof. Since $L/\mathrm{Nil}\ L$ is reductive, the theory of reductive Lie algebras of characteristic 0 implies that $L_c \subset \mathrm{Nil}\ L$ for $c \in \mathrm{Nil}\ R$. For the other direction, let $h \in [L_{-c}, L_c]$, where ad h is not nilpotent on L. By Theorem 3.5.1 of Winter [9], we have $c(h) \neq 0$, $\mathrm{Tr}(\mathrm{ad}\ h)^2 \neq 0$ and $h \notin L^{\perp}$ when $L^{\perp}$ is the radical of the killing form on L. It follows that $\bar{h} = h + \mathrm{Rad}\ L$ is nonzero in $\bar{L} = L/\mathrm{Rad}\ L$, where Rad L is the solvable radical of L. Since $\bar{h} \in [\bar{L}_{-c}, \bar{L}_c]$ and $c(h) \neq 0$, we have shown that $c \notin \mathrm{Nil}\ R$ implies that $L_c \not\subset \mathrm{Nil}\ L$. ▌

The following corollary to Theorem 10.2 is straightforward. In it, $\langle L_{\mathrm{Nil}\,R} \rangle$ denotes the ideal of L generated by $L_{\mathrm{Nil}\,R}$ and Core $R = R - \mathrm{Nil}\ R \cup \{0\}$.

10.3. COROLLARY. *Let* $L = \sum_{a \in R} L_a$ *be a split Lie algebra of characteristic* 0. *Then* $L/\langle L_{\mathrm{Nil}\,R} \rangle$ *is reductive with rootsystem canonically isomorphic to* Core R.

11. Symmetric Lie Algebras

11.1. Definition. Given a Lie algebra $L = \sum_{a \in R} L_a$, we let $L_a^1 = \{x \in L_a \mid [h, x] = a(h)\, x \text{ for all } h \in L_0\}$ $(a \neq 0)$ and $L_0^1 = L_0$, and we let L^1 be the subalgebra $L^1 = \sum_{a \in R} L_a^1$. We say that L is *symmetric* if $a([L_{-a}^1, L_a^1]) \neq 0$ for all $a \in R - \{0\}$.

11.2. Proposition. *Let $L = \sum_{a \in R} L_a$ be symmetric and let $a \in R - \{0\}$, $b \in R$ with $R_b(a)$ bounded. Then $a + b \in R \Rightarrow [L_a^1, L_b^1] \neq 0$.*

Proof. Let $R_b(a) = \{b - ra, ..., b + qa\}$ and suppose that $[L_a^1, L_b^1] = 0$. Let $T = \{b - ra, ..., b\}$ and consider $V = \sum_{c \in T} L_c^1$. Then V is a module for $L^{(a)} = kf_a + kh_a + ke_a$ with $0 \neq h_a = [e_a, f_a] \in [L_a^1, L_{-a}^1]$, so that $r_a(c) = c - (c(h_a)/a(h_a))\, a$ maps $b - ra$ to $b + qa$ and to b. Thus, $q = 0$ and $a + b \notin R$.

11.3. Theorem. *Let $L = \sum_{a \in R} L_a$ be symmetric. Then $\mathbb{Z}_p a \cap R$ is either $\{-a, 0, a\}$ or $\{-a, 0, a, ..., (p-2)\, a\}$ for any $a \in R - \{0\}$, that is, $R = R^\circ \cup R^\cdot$ in the sense of Section 2.*

Proof. Suppose that $S = \mathbb{Z}_p a \cap R$ is not $\{-a, 0, a, ..., (p-2)\, a\}$. Then $S_b(c) = R_b(c)$ is bounded for all $b \in S - \{0\}$, $c \in S$. Let $b \in S - \{0\}$. Then the orbits $R_b(c)$ are stable under the reversal $r_c(b) = b - 2(b(h_c)/c(h_c))\, c = ic - 2(ic(h_c)/c(h_c))\, c = -ic = -b$. Thus, S is a symmetryset in $\mathbb{Z}_p$, all of whose orbits $S_b(a)$ are bounded. Let $a \in S - \{0\}$ and consider the module $L_{-ra} + \cdots + L_0 + ke_a$ for $ke_a + kh_a + kf_a$, where $S_0(a) = \{-ra, ..., ra\}$. Then $T = \{-ra, ..., 0, a\}$ is stable under r_a, so that $a = r_a(-ra) = ra$ and $r = 1$. It follows that S is reduced. But then $S = \{-a, 0, a\}$, by Proposition 7.1. ∎

11.4. Theorem. *Let $L = \sum_{a \in R} L_a$ be symmetric and let $H = L_0$. Then:*

(1) $R \subset \mathrm{Hom}_k(H, k)$;

(2) ad H *is triangulable on* L;

(3) L^∞ *is symmetric with Cartan subalgebra* H_∞;

(4) Core L *is symmetric*;

(5) $L/(\mathrm{Nil})\, L$ *has center* 0;

(6) $L^1 = \sum_{a \in R} L_a^1$ *is symmetric and the Cartan subalgebra* $\bar{H}^1$ *of* $\bar{L}^1 = L^1/\mathrm{Nil}\, L^1$ *is* ad-*diagonalizable.*

Proof. For (1) and (2), observe that $\sum_{a \in R - \{0\}} L_a^1$ is a module for ad H which is annihilated by $(\mathrm{ad}\, H)^{(1)}$. Thus, $(\mathrm{ad}\, H)^{(1)}$ is upper triangulable with only zeros on the diagonal, by Engel's Theorem. It follows that ad H is triangulable on L. For (3), note that $[H_\infty, L_a] = L_a$, so that H_∞ is a Cartan subalgebra of L^∞. For (4), note that if $h \in [L_{-a}^1, L_a^1]$ with $a(h) \neq 0$, then $h \in L^\infty - \text{Nil } L$; for otherwise Nil L contains $L^1_{-a} + kh + L^1_a$, since Nil L is an ideal which would contain h, contradicting the nilpotence of Nil L. Similarly, $h \in L^\infty - \text{Nil } L^\infty$. Thus Core $L = L^\infty/\text{Nil } L^\infty$ is symmetric. If L is toral, $L^\infty \to L/\text{Nil } L$ has image $(L/\text{Nil } L)^{(1)}$ and kernel Nil L^∞, by a straightforward verification. For (5), let $h + \text{Nil } L \in \text{Center } L/\text{Nil } L$. Then $[h, L_a] \subset \text{Nil } L$ $(a \in R - \{0\})$. Since L is symmetric, $L_a \not\subset$ Nil L $(a \in R - \{0\})$. Thus, $a(h) = 0$, for otherwise $L_a = [h, L_a] \subset \text{Nil } L$ $(a \in R - \{0\})$. It follows that the ideal $kh + \text{Nil } L$ is ad-nilpotent on L, so that $h \in \text{Nil } L$, by the maximality of Nil L. Thus, $L/\text{Nil } L$ has center 0. For (6), note that if L has center 0 and ad H is diagonalizable on the L_a $(a \in R - \{0\})$, then ad H is diagonalizable since H is then abelian: $h \in H^{(1)} \cap \text{Center } H \Rightarrow h$ central in $L \Rightarrow h = 0$. ▌

11.5. THEOREM. *Let $L = \sum_{a \in R} L_a$ be symmetric with $0 \neq \mathbb{Z}_p a \subset R$. Then $L_1(a)/\text{Solv } L^{1(a)}$ is the Witt algebra W_1 for $L^{1(a)} = H + \sum_{i=1}^{p-1} L^1_{ai}$.*

Proof. Since $L^{1(a)}$ is symmetric $\bar{L} = L^{1(a)}/\text{Nil } L^{1(a)}$ is symmetric with center 0, by Theorem 11.4. Let $\bar{H}$ be the image of H in $\bar{L}$. It follows that $\bar{H}$ is abelian, for otherwise any $h \in \bar{H}^{(1)} \cap \text{Center } \bar{H}$ is central in $\bar{L}$. But then $\bar{H}$ is ad-diagonalizable on $\bar{L}$. Since Center $\bar{L} = 0$, it follows that $\bar{H}$ has dimension 1. Let S be a maximal proper ideal of $L^{1(a)}$ containing Nil $L^{1(a)}$. If $\bar{H} \subset \bar{S}$, then $\bar{L} = \bar{L}_0(\text{ad } \bar{H}) + \bar{S} = \bar{H} + \bar{S} = \bar{S}$ and $L = S$. Thus, $\bar{H} \not\subset \bar{S}$. Since $\dim \bar{H} = 1$, it follows that $\bar{H} \cap \bar{S} = \bar{0}$ and $\bar{S} = \sum_{i=1}^{p-1} \bar{S}_{ia}$, where $\bar{S}_{ia} = \bar{S}_{ia}(\text{ad } \bar{H})$. It then follows that $\text{ad}_{\bar{S}} \bar{S}_{ia}$ is nil $(1 \leqslant i \leqslant p-1)$ and $\bar{S}$ is nilpotent. Thus, $S = \text{Solv}_L 1(a)$. It follows that $L^{1(a)}/\text{Solv } L^{1(a)}$ is simple of rank 1 and toral rank 1, so that it is W_1, by Kaplansky [3]. ▌

11.6. THEOREM. *Let $L = \sum_{a \in R} L_a$ be a symmetric Lie algebra. Then (L_0^*, R) is a Lie rootsystem in the sense of Section* 2

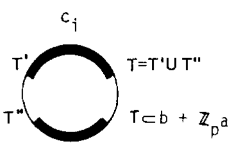
Proof. Let $a \in R - \{0\}$ and choose $h_a \in [L^1_{-a}, L^1_a]$ with $a(h_a) \neq 0$. Define $a^0 \in \text{Hom}_k(L_0^*, k)$ by $a^0(v) = 2(v(h_a)/a(h_a))$, let $r_a(v) = v - a^0(v)\, a$ $(v \in L_0^*)$, and note that $a^0(a) = 2$ and $r_a R_b(a) = R_b(a)$ for every bounded a-orbit $R_b(a)$ $(b \in R)$, by Theorem 3.1. Thus, (L_0^*, R) is a rootsystem in the sense of Section 2, and it remains to verify the supplemental "Lie" conditions that $R = R^0 \cup R^{\cdot}$ and each "Witt orbit" $R_b(a)$ $(a \in R^0 - \{0\}, b \in R)$ has 1 or $p-1$ or p elements. The condition $R = R^0 \cup R^{\cdot}$ was proved in Theorem 5.3. Next, consider a "Witt orbit" $R_b(a)$ $(a \in R^0 - \{0\}, b \in R)$. We must show that $R_b(a)$ has $1, p-1$ or p elements. Accordingly, we may, without loss of generality, assume that $1 < |(R_b(a)| \leqslant p-1$. To show that $|R_b()a)| = p-1$, we may replace $L = \sum_{c \in R} L_c$ by another symmetric Lie algebra having corresponding $a \in R^w$, $b \in R$ and $R_b(a)$ of the same length. It follows that we can, successively, replace L by $\sum_{i,j=0}^{p-1} L^1_{ia+jb}$ with $L^1_0 =_{\text{def}} \sum_{i,j=0}^{p-1}$

$[L^1_{ia+jb}, L^1_{-ia-jb}]$, L/Center L. Consequently, we may assume, with no loss of generality, that $L = \sum_{i,j=0}^{p-1} L^1_{ia+jb}$ and Center $L = \{0\}$.

For each $1 \leqslant i \leqslant p-1$, choose $e_i \in L^1_{ia}$, $f_i \in L^1_{-ia}$, $h_i = [e_i, f_i]$ such that $a(h_i) = 1$ and define $r_{ia}(v) = v - 2(v(h_i)/ia(h_i))\ ia = v - 2v(h_i)\ a$ $(v \in L_0^*)$. Note that $r_i(a) = -a$ $(1 \leqslant i \leqslant p-1)$. By Theorem 9.1 and the assumption $|R_b(a)| \leqslant p-1$, the ia-orbits $T_{b'}(ia)$ $(b' \in T)$ of $T = R \cap (b + \mathbb{Z}_p a)$ are r_i-stable for any $1 \leqslant i \leqslant p-1$. It follows, in particular, that T contains $r_i(b) = b - 2b(h_i)\, a$, so that $\mathbb{Z}_p$ contains $b(h_i)$. Define $c_i = b - b(h_i)\ a \in b + \mathbb{Z}_p a$ and note that $r_{ia}(c_i) = c_i$ since $c_i(h_i) = 0$, for $1 \leqslant i \leqslant p-1$. Since, for $1 \leqslant i \leqslant p-1$, we have $r_i(c_i) = c_i$ and $r_i T_{b'}(ia) = T_{b'}(ia)(b' \in T)$, one can easily verify that:

(1) T has either 1 or 2 ia-orbits,

(2) if T has 2 ia-orbits T', T'', then T has an odd number of elements and $T'' = T - T'$; and then one of T', T'' has an odd number of elements and contains c_i, and the other has an even number of elements.



We claim, for each $1 \leqslant i \leqslant p-1$ and each $b' \in T$, that the ia-orbit $T_{b'}(ia)$ is stable under each r_j $(1 \leqslant j \leqslant p-1)$. By (1) and (2) above, $T = T_{b'}(ia)$ (case of one ia-orbit) or $T' = T_{b'}(ia)$ and the ia-orbits of T are T' and $T'' = T - T'$, where one of T', T'' has odd number of elements and the other has an even number of elements. Since $T = r_j(T) = r_j(T') \cup r_j(T'')$ and since $r_j(ia) = -ia$, one can easily verify that $r_j(T')$, $r_j(T'')$ are ia-orbits of T, thus that they are T', T'' in one of the orders T', T'' or T'', T'. But r_j preserves "odd" and "even" numbers of elements. It follows that $r_j(T') = T'$ and $r_j(T'') = T''$ for $1 \leqslant j \leqslant p-1$.

Take one fixed a-orbit $T_{b'}(a)$ of T. Since it is stable under $r_1, ..., r_{p-1}$ and $r_j(a) = -a$ $(1 \leqslant j \leqslant p-1)$, each of the $r_1, ..., r_{p-1}$ reverse the a-orbit $T_{b'}(a)$. It follows that $r_1(b') = \cdots = r_{p-1}(b')$ and $b'(h_1) = \cdots = b'(h_{p-1})$ for all $b' \in T$. Consequently, we have $b(h_i - h_j) = a(h_i - h_j) = 0$ for all $1 \leqslant i \leqslant p-1$. Since $L = \sum_{i,j=0}^{p-1} L^1_{ia+jb}$ with $L^1_0 =_{\text{def}} \sum_{i,j=0}^{p-1} [L^1_{ia+jb}, L^1_{-ia-jb}]$, it follows that $h_i - h_j \in$ Center $L = \{0\}$ and $h_i = h_j$ for all $1 \leqslant i, j \leqslant p-1$.

Finally, we let h denote h_1, so that $h = h_i$ for $1 \leqslant i \leqslant p-1$ and $a(h) = 1$. By the flexibility in the choice of h_i above, it follows that $e \in L_{ia}$, $f \in L_{-ia}$ with $a[e, f] = 1$ implies that $h = [e, f]$, for $1 \leqslant i \leqslant p-1$. We claim that $e' \in L_{ia}$, $f' \in L_{-ia}$, $a([e', f']) = 0$ implies that $[e', f'] = 0$ for $1 \leqslant i \leqslant p-1$. To see this, let $1 \leqslant i \leqslant p-1$, choose $e \in L_{ia}$, $f \in L_{-ia}$ such that $h = [e, f]$ and suppose that $e' \in L_{ia}$, $f' \in L_{-ia}$, $h' = [e', f]$, $a(h') = 0$. We claim that

$h'=0$. To see this, let $h''=[e',f]$, $h'''=[e,f']$. Consider first the case where $a(h'')=a(h''')=0$. Then $1=a(h+h'')=a([e+e',f])$, so that $h=[e+e',f]$ as observed above. But then $h=h+h''$ and $h''=0$. Similarly, $h'''=0$. It follows that $[e+e',f+f']=h+h'+h''+h'''=h+h'$. Since $1=a(h)=(a(h+h')=a([e+e',f+f'])$, it follows from the discussion above that $h=h+h'$ and $h'=0$. Thus, $[e',f']=0$ in the present case. Next, consider the case where one of $a(h'')$, $a(h''')$ is not zero. We may then assume with no loss of generality that $a(h'')\neq 0$, for otherwise we can interchange h'', h'''. By replacing e' by $(1/a(h''))\,e'$, we may also assume that $1=a(h'')=a[e',f])$. But then $h''=h$, by our earlier discussion. But then $h+h'=[e',f]+[e',f']=[e',f+f']$ and $1=a([e',f+f'])$ implies that $[e',f+f']=h$, by our earlier discussion, so that $h+h'=h$ and $h'=0$.

By the preceding paragraph, we have $L_0^1 =_{\text{def}} \sum_{i,j=0}^{p-1} L_{ia+jb}^1=kh$, that is, the Cartan subalgebra $H=L_0^1$ of L is one dimensional. Let $L^a=\sum_{i=0}^{p-1} L_{ia}=\sum_{i=0}^{p-1} L_{ia}^1$ and let $S=\text{Solv}\, L^a$. We observed in Theorem 11.5 that L/S is the Witt algebra W_1. Since S is a proper ideal of L, we have $H\not\subset S$. Since $\dim H=1$, it follows that $H\cap S=\{0\}$. Consequently, $S=\sum_{i=1}^{p-1} S_{ia}$. Regard $V=\sum_{c\in T} L_c$ as L-module via adjoints, where $T=R\cap(b+\mathbb{Z}_p a)\subsetneqq b+\mathbb{Z}_p a$. Let f: $L\to \text{Hom}\, V$ be the associated representation. Since $T\subsetneqq b+\mathbb{Z}_b a$, $f(S_{ia})$ consists of nilpotent transformations of V for $1\leqslant i\leqslant p-1$. By the theorem of Jacobson [5] on weakly closed sets of linear transformations, it follows that $f(S)$ consists of nilpotent linear transformations of V. Letting $\bar{V}$ be any irreducible subquotient $\bar{V}=V_i/V_{i+1}$, where $V_1,\dots,V_r$ is a composition series for V, and letting $\bar{f}$: $L\to\text{Hom}\,\bar{V}$ be the associated representation of L, we claim that $\bar{f}(S)\,\bar{V}=\{\bar{0}\}$. We use the notation $\bar{v}=v+V_{i+1}\in\bar{V}$ for $v\in V_i$. Note that since $\bar{f}(S)$ is a Lie algebra of nilpotent linear transformations of $\bar{V}$, $\bar{V}=\{v\in\bar{V}\mid \bar{f}(S)\,\bar{v}=0\}$ is nonzero. One sees easily that $\bar{V}_0$ is an L-submodule of $\bar{V}$, since S is an ideal of L:

$$\bar{f}(S)\,\bar{v}=0\Rightarrow \bar{f}(S)[\bar{f}(L)\,\bar{v}]=0.$$

Since $\bar{V}$ is irreducible, $\bar{V}=\bar{V}_0$ and $\bar{f}(S)\,\bar{V}=\{0\}$.

Since $\bar{f}(S)\,\bar{V}=\{\bar{0}\}$, we may regard $\bar{V}$ as a module for $W_1=L/S$ where the module action is given by

$$(x+s)\,\bar{v}=\overline{[x,v]}$$

for $x+S\in L/S$, $\bar{v}=v+V_{i+1}\in\bar{V}$, $\overline{[x,v]}=[x,v]+V_{i+1}\in\bar{V}$. We let $\bar{V}=\sum_{c\in T'}\bar{V}_c$ be the root decomposition of $\bar{V}$ with respect to $H\subset L$ and $H+S/S=\bar{H}$ in $L/S=W_1$, and regard T' as a subset of R. The restricted irreducible W_1-modules are shown in Block [4] to have dimensions 1, $p-1$ or p and one-dimensional weight spaces. It follows that $|T'|=1$, $p-1$ or p. Since $1<|R_b(a)|$, one of $b-a$, $b+a$ is in R. Consequently, we can choose $\bar{V}$ such that $\bar{V}_b\neq\{\bar{0}\}$ and either $\bar{V}_{b-a}\neq\{\bar{0}\}$ or $V_{b+a}\neq\{\bar{0}\}$, by

Proposition 11.2. For this $\bar{V}$, $|T'| \neq 1$, so that $|T'| = p-1$ or $|T'| = p$. But then $p-1 \leqslant |T'| \leqslant |R_b(a)| \leqslant p-1$, so that $|R_b(a)| = p-1$ as was to be proved. ∎

11.7. COROLLARY. *Let L be a symmetric Lie algebra. Then L is reflective if and only if* $\mathbb{Z}_p a \not\subset R$ *for all* $a \in R - \{0\}$ *if and only if R is classical*: $R = R^{\cdot} \cup \{0\}$.

Proof. The latter condition is equivalent, by Theorem 11.6, to the condition that the Lie rootsystem R is classical, that is, $R = R^{\cdot} \cup \{0\}$, which in turn is equivalent, by the results stated in Section 7, to the condition that the orbits $R_b(a)$ ($a \in R - \{0\}$, $b \in R$) of the Lie rootsystem R are all bounded. ∎

We close by noting that part of Theorem 9.5 generalizes as follows, by the same arguments.

11.8. COROLLARY. *Let* $L = \sum_{a \in R} L_a$ *be symmetric. Then* $\dim L_a = 1$ *for all* $a \in R^{\cdot}$.

12. EXCLUSION OF SUBTYPES OF R AND L

Let R be a Lie rootsystem and/or $L = \sum_{a \in R} L_a$ a symmetric Lie algebra. We have observed that L is reflective and Core L is classical if and only if R is classical: R has no $Ra = R \cap \mathbb{Z}a$ of type W_1. The latter condition can be restated "R excludes W_1" in the following language.

12.1. DEFINITION. Let S be a rootsystem of rank r. Then R *excludes* S if $Ra_1 \cdots a_r = R \cap (\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_r)$ is not isomorphic to S for any $a_1, \ldots, a_r \in R$. $L = \sum_{a \in R} L_a$ *excludes* S if R excludes S.

12.2. THEOREM. *Let R be an irreducible Lie rootsystem. Then*:

(1) *R is classical or* $R = R^0$ *if and only if R excludes* $W \oplus A$ *and* T_2;

(2) *R is classical or rank* 1 *if and only if R excludes* $W \oplus A$, $W \oplus W$ *and* S_2.

Proof. One direction for both (1) and (2) is clear. For the other, suppose that R excludes $W \oplus A$ and T_2. Suppose that $a \in R^0$, $b \in R^{\cdot}$ and consider Rab. If $a \neq 0$, then $Rab = W \cup A$ and $a + b \notin R$, since the possibilities $W \oplus A$, T_2 are excluded. It follows that $a, a' \in R^0$ and $a + a' \in R$ implies $a + a' \in R^0$: $a + a' =_{\text{def}} -b \in R^{\cdot} \Rightarrow a + b \in R \Rightarrow a = 0 \Rightarrow a + a' = a' \in R^0$. Similarly, $b, b' \in R^{\cdot}$ and $b + b' \in R$ implies $b + b' \in R \dot{\cup} \{0\}$: $b + b' =$

$-a \in R^0 \Rightarrow a+b \in R \Rightarrow a=0 \Rightarrow b+b'=0 \in R^{\cdot} \cup \{0\}$. Since R is irreducible, it follows that $R=R^0$ or $R=R^{\cdot} \cup \{0\}$. This proves (1). For (2), suppose that R excludes $W \oplus A$, $W \oplus W$ and S_2. Then R also excludes T_2, so that R is classical or $R=R^0$. Let $a \in R-\{0\}$ and define $S=R \cap ka$, $T=(R-S) \cup \{0\}$. Take $b \in T$ and note that $b \neq 0$ implies $Rab = W \cup W$ and $a+b \notin R$, by exclusion of $W \oplus W$. It follows that $a, a' \in S$, $a+a' \in R$ implies $a+a' \in S$ and $b, b' \in T$, $b+b' \in R$ implies $b+b' \in T$, as in earlier arguments. By irreducibility of R, therefore, $R=S$ and R has rank 1. ∎

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