On a computation of plurigenera of a canonical threefold

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Abstract

For a canonical threefold \( X \), we know \( h^0(X, \mathcal{O}_X(nK_X)) \geq 1 \) for a sufficiently large \( n \). When \( \chi(\mathcal{O}_X) > 0 \), there are few known results about the integer \( n \). This paper introduces an algorithm for computing plurigenera. Furthermore, when \( \chi(\mathcal{O}_X) \) is small, especially 1 and 2, plurigenera are computed. This produces \( h^0(X, \mathcal{O}_X(nK_X)) \geq 1 \) for \( n \geq 7 \) and \( h^0(X, \mathcal{O}_X(nK_X)) \geq 2 \) for \( n \geq 10 \) when \( \chi(\mathcal{O}_X) = 1 \). Also, \( h^0(X, \mathcal{O}_X(nK_X)) \geq 1 \) for \( n \geq 14 \) and \( h^0(X, \mathcal{O}_X(nK_X)) \geq 2 \) for \( n \geq 20 \) with 8 possible exceptional cases when \( \chi(\mathcal{O}_X) = 2 \).

Keywords: Canonical threefold; Plurigenera

Throughout this paper \( X \) is assumed to be a projective threefold with only canonical singularities and an ample canonical divisor \( K_X \) over the complex number field \( \mathbb{C} \), i.e., a canonical threefold.

It is well known that \( H^0(X, \mathcal{O}_X(mK_X)) \) does not vanish and generates a birational map for a sufficiently large \( m \). In a case of surface \( X \) of general type, \( H^0(X, \mathcal{O}_X(mK_X)) \) does not vanish for \( m \geq 2 \) and \( H^0(X, \mathcal{O}_X(mK_X)) \) generates a birational map for \( m \geq 5 \). In a case of threefold, when \( \chi(\mathcal{O}_X) \leq 0 \), it is easy to have such integer \( m \) (see Fletcher [1]); however, when \( \chi(\mathcal{O}_X) > 0 \), it is not easy to produce such integer \( m \). If there exists an integer \( n \) such that \( h^0(X, \mathcal{O}_X(nK_X)) \geq 2 \), the integer \( m \) can be induced using Kollár’s technique (see Kollár [3]). However, when \( \chi(\mathcal{O}_X) > 0 \), it is not easy to obtain an integer \( n \) such that \( h^0(X, \mathcal{O}_X(nK_X)) \geq 1 \).
M. Reid and A.R. Fletcher described the formula for $\chi(\mathcal{O}_X(nK_X))$. Combining the formula for $\chi(\mathcal{O}_X(nK_X))$ with a vanishing theorem, it is possible to compute $h^0(X, \mathcal{O}_X(nK_X))$. In [1], A.R. Fletcher showed that $h^0(X, \mathcal{O}_X(12K_X)) \geq 1$ and $h^0(X, \mathcal{O}_X(24K_X)) \geq 2$ when $\chi(\mathcal{O}_X) = 1$.

The formula for $\chi(\mathcal{O}_X(nK_X))$ is as follows:

$$\chi(\mathcal{O}_X(nK_X)) = \frac{n(n-1)(2n-1)}{12}K_X^3 + (1 - 2n)\chi(\mathcal{O}_X) + \sum_Q l(Q, n),$$

where the summation is over a basket of singularities. Although singularities in a basket are not necessarily singularities in $X$, the singularities in $X$ make the contribution as if they were in a basket. For detailed explanations about a basket of singularities, see Reid [5], Fletcher [1] or Kawamata [2]. The exact formula for $l(Q, n)$ is as follows:

$$l(Q, n) = \sum_{i=1}^{n-1} \frac{i\bar{b}(r - i\bar{b})}{2r},$$

where $Q$ is a singularity of type $\frac{1}{r}(1, -1, b)$, $r$ and $b$ are relatively prime, and $\bar{b}$ is the least residue of $ib$ modulo $r$.

For the sake of simplicity, denote $\frac{i\bar{b}(r - i\bar{b})}{2r}$ by $g(Q, i)$ and $\sum_Q l(Q, n)$ by $l(n)$. Then $l(n)$ can be expressed as follows:

$$l(n) = \sum_Q l(Q, n) = \sum_Q \sum_{i=1}^{n-1} g(Q, i).$$

The singularity type $\frac{1}{r}(1, -1, b)$ can be denoted by $b/r$ unless there is some confusion. Moreover, identify the singularity type $b/r$ with the number $b/r$ in the interval $(0, 1]$. By identifying the type $b/r$ with the number $b/r$ in $(0, 1]$, our situation is defined more effectively for the computation of $l(n)$.

The following proposition is a standard application of the Kawamata–Viehweg Vanishing Theorem.

**Proposition 1.** For all $n \geq 2$,

$$p_n := h^0(X, \mathcal{O}_X(nK_X)) = \frac{n(n-1)(2n-1)}{12}K_X^3 + (1 - 2n)\chi(\mathcal{O}_X) + \sum_Q l(Q, n).$$

Even though there is a formula for $h^0(X, \mathcal{O}_X(nK_X))$, it is not easy to compute $h^0(X, \mathcal{O}_X(nK_X))$ because there is no information about the basket of singularities. The following lemmas are needed to compute the plurigenera of $X$.

**Lemma 1.** Let $Q$ be a point of type $b/r$. Let $k = \min\{b, r - b\}$. Then $\bar{b}(r - \bar{b}) = \bar{k}(r - \bar{k})$ for a positive integer $i$.

**Proof.** If $k = r - b$, then $\bar{k} \equiv i\bar{r} - i\bar{b} \equiv -\bar{b} \mod r$. The graph of $x(r - x)$ yields $\bar{b}(r - \bar{b}) = i\bar{k}(r - i\bar{k})$. □
Note that \( k \leq \frac{r}{2} \). To compute \( p_n \), by Lemma 1, it may be assumed that the basket consists of points related only to types \( \frac{k}{r} (k \leq \frac{r}{2}) \) because \( \frac{b}{r} \) and \( \frac{k}{r} \) produce the same value for \( g(Q, i) \).

**Lemma 2.** Let \( \{kj/r_j\} \) be a basket of singularities of \( X \). Then

\[
\sum kj = 10\chi(O_X) + (5p_2 - p_3),
\]

where the summation is over the basket of singularities.

**Proof.** By Proposition 1, it is possible to compute \( p_3 - 5p_2 \). Recall that \( k \leq \frac{r}{2} \).

\[
p_3 - 5p_2 = 10\chi(O_X) + l(3) - 5l(2)
= 10\chi(O_X) + \sum (g(Q, 2) - 4g(Q, 1))
= 10\chi(O_X) + \sum \frac{2kj(r_j - 2kj) - 4kj(r_j - kj)}{2r_j}
= 10\chi(O_X) - \sum kj.
\]

Lemma 2 is one of the ways by which an upper bound is given for a number of points in a basket. For example, when \( \chi(O_X) = 1 \) and \( p_2 = 0 \), the basket cannot contain more than 10 points.

**Lemma 3.** Let \( \{kj/r_j\} \) be a basket of singularities of \( X \). Then

\[
4\chi(O_X) + (3p_2 - p_3) < \sum \frac{k_j^2}{r_j} \leq 3 \sum \frac{r_j^2 - 1}{r_j} - 68\chi(O_X) + (3p_2 - p_3),
\]

where the summation is over the basket of singularities.

**Proof.** To prove the left inequality, computation of the following equation is done below.

\[
5p_2 - 3p_3 = -5K_X^3 + \sum (2g(Q, 1) - 3g(Q, 2))
= -5K_X^3 - 2\sum kj + 5\sum \frac{k_j^2}{r_j}
= -5K_X^3 - 20\chi(O_X) - 10p_2 + 2p_3 + 5\sum \frac{k_j^2}{r_j},
\]

since \( \sum kj = 10\chi(O_X) + 5p_2 - p_3 \) by Lemma 2. Thus,

\[
5K_X^3 = -20\chi(O_X) + 5\sum \frac{k_j^2}{r_j} - 15p_2 + 5p_3.
\]

Since \( K_X^3 > 0 \), the left inequality is induced.
To prove the right inequality, by the result of R. Barlow (see also Kawamata [2] or Reid [5])

$$\rho^* K_X \cdot c_2(Y) = \sum r^2 - \frac{1}{r} - 24 \chi(\mathcal{O}_X),$$

where $\rho : Y \rightarrow X$ is a resolution of singularities of $X$.

$$\chi(\mathcal{O}_X) = \frac{1}{24} \sum r_j^2 - \frac{1}{r_j} - \frac{1}{72} K_X^3$$

$$\leq \frac{1}{24} \sum r_j^2 - \frac{1}{r_j} - \frac{1}{72} K_X^3$$

$$= \frac{1}{24} \sum r_j^2 - \frac{1}{r_j} - \frac{1}{72} \left( -4 \chi(\mathcal{O}_X) + \sum \frac{k_j^2}{r_j} - 3p_2 + p_3 \right),$$

where the second inequality is Miyaoka–Yau inequality (see Miyaoka [4]) and the last equality is proved just above. Hence,

$$\sum \frac{k_j^2}{r_j} \leq 3 \sum \frac{r_j^2 - 1}{r_j} - 68 \chi(\mathcal{O}_X) + (3p_2 - p_3). \quad \square$$

For the next lemma, some new notation is introduced.

Let $2 \leq m \leq n \leq N$ and the basket of singularities of $X$ be the union of $S_1$ and $S_2$, where $S_2$ is the set of points $< \frac{1}{N-1}$.

In the formula of $l(m)$, the sum $\sum_{Q \in S_1} l(Q, m)$ over $S_1$ is denoted by $l(m)^+$ and the remaining part $\sum_{Q \in S_2} l(Q, m)$ by $l(m)^-$. In addition, the following term is expressed by $K^3_{X,m,n}$:

$$\frac{12}{(m-1)m(2m-1)} \left( p_m + (2m-1)\chi(\mathcal{O}_X) - l(m)^+ \right) + \left( \frac{3}{2n-1} - \frac{3}{2m-1} \right) \sum_{S_2} k.$$

In Lemma 4, $K^3_{X,m,n}$ acts like the real $K^3_{X}$ in the formula of a plurigenus. Once $p_m$ ($m \leq n$) is known, by Lemma 4, $p_n$ can be computed even though complete information about some points, like $S_2$, in the basket is unavailable.

**Lemma 4.** With above assumptions, $p_n$ is given as follows:

$$p_n = \frac{n(n-1)(2n-1)}{12} K^3_{X,m,n} - (2n-1)\chi(\mathcal{O}_X) + l(n)^+.$$

**Proof.** $K^3_{X}$ can be induced from $p_m$, which yields

$$K^3_{X} = \frac{12}{(m-1)m(2m-1)} \left( p_m + (2m-1)\chi(\mathcal{O}_X) - l(m) \right)$$

$$= \frac{12}{(m-1)m(2m-1)} \left( p_m + (2m-1)\chi(\mathcal{O}_X) - l(m)^+ - l(m)^- \right).$$
Since a point in $S_2$ is less than $\frac{1}{N-1}$,

\[
l(m)^- = \sum_{S_2} \sum_{i=1}^{m-1} \frac{i k(r - i k)}{2r}
\]

\[
= \sum_{S_2} \sum_{i=1}^{m-1} \frac{i k(r - i k)}{2r}
\]

\[
= \sum_{S_2} \sum_{i=1}^{m-1} \left( \frac{i k^2}{2} - i k^2 \frac{k}{2r} \right)
\]

\[
= \sum_{S_2} \left( \frac{(m-1)m}{4} k - \frac{m(m-1)(2m-1) k^2}{12} \frac{1}{r} \right).
\]

Thus,

\[
K^3_X = \frac{12}{(m-1)m(2m-1)} (p_m + (2m-1) \chi(\mathcal{O}_X) - l(m)^+)
\]

\[
- \frac{3}{2m-1} \sum_{S_2} k + \sum_{S_2} k^2 \frac{1}{r}.
\]

Similarly, from $p_n$,

\[
K^3_X = \frac{12}{(n-1)n(2n-1)} (p_n + (2n-1) \chi(\mathcal{O}_X) - l(n)^+)
\]

\[
- \frac{3}{2n-1} \sum_{S_2} k + \sum_{S_2} k^2 \frac{1}{r}.
\]

By comparing the two and rearranging the terms, it is seen that

\[
p_n = \frac{n(n-1)(2n-1)}{12} K^3_{X,m,n} - (2n-1) \chi(\mathcal{O}_X) + l(n)^+.
\]

The main theorem is given as follows:

**Main Theorem.** Let $X$ be a canonical threefold.

(1) When $\chi(\mathcal{O}_X) = 1$, the following is obtained:
   (i) $p_n \geq 1$ for $n \geq 7$,
   (ii) $p_n \geq 2$ for $n \geq 10$.

(2) When $\chi(\mathcal{O}_X) = 2$, the following is obtained:
   (i) $p_{12} \geq 1$ and $p_n \geq 1$ for $n \geq 14$ with possible exceptional cases 1, \ldots, 6 shown in Table 1,
(ii) \( p_{18} \geq 2 \) and \( p_n \geq 2 \) for \( n \geq 20 \) with possible exceptional cases 4, \ldots, 8 shown in Table 1.

In Table 1, the notation \( k/r \times n \) means \( n \) points related to type \( k/r \). Table 1 describes possible exceptional baskets to the main theorem. Note that it does not imply the existence of canonical threefold which has a given basket.

Recall that it is assumed that the basket consists of points related only to types \( k/r \) (\( k \leq \frac{r}{2} \)) by Lemma 1. Thus, in fact, type \( k/r \) in Table 1 stands for either a point of type \( k/r \) or a point of type \( (r-k)/r \). For example, \( 3/7 \times 2 \) in Table 1 stands for one of the following three cases:

1. \{two points of type 3/7\},
2. \{one point of type 3/7 and one point of type 4/7\}, or
3. \{two points of type 4/7\}.

**Remark 1.** For an arbitrary canonical threefold of \( \chi(\mathcal{O}_X) = 2 \), Table 1 shows that \( p_{12} \geq 1 \), \( p_{16} \geq 1 \), \( p_{18} \geq 1 \) and \( p_n \geq 1 \) for \( n \geq 20 \). Table 1 shows also that \( p_{20} \geq 2 \), \( p_{23} \geq 2 \), \( p_{24} \geq 2 \) and \( p_n \geq 2 \) for \( n \geq 26 \). Thus, the 8 possible baskets described above are very exceptional.

**Remark 2.** When \( \chi(\mathcal{O}_X) = 1 \), the number of possible baskets for \( p_6 = 0 \) is less than or equal to 13. When \( \chi(\mathcal{O}_X) = 2 \), the number of possible baskets for \( p_{13} = 0 \) is less than or equal to 26.

The main idea for a proof consists of four steps and is very combinatorial. Hence, it is easily done through computer programming.

Each step will be described under the assumption \( p_4 = 0 \), just for illustrative purposes.

**Step 1.** Find an appropriate linear combination of \( p_n \)’s to eliminate the term \( K_X^3 \).

\( p_4 = 0 \) implies \( p_2 = 0 \). Consider the following equation:

\[
0 = -p_4 + 14p_2 = -35\chi(\mathcal{O}_X) - I(4) + 14l(2).
\]
Since $-l(4) + 14l(2) = \sum (-g(Q, 3) - g(Q, 2) + 13g(Q, 1))$,
\[
\sum_Q (-g(Q, 3) - g(Q, 2) + 13g(Q, 1)) = 35\chi(O_X),
\]
where the summation is over the basket of singularities. Recall that $Q$ is a point in the interval $(0, 1/2]$.

Denote $-g(Q, 3) - g(Q, 2) + 13g(Q, 1)$ by $Eq(Q)$. Now, the problem of ‘finding a basket of singularities’ has changed to ‘finding a partition $\sum_Q Eq(Q)$ of $35\chi(O_X)$ using points in the interval $(0, 1/2]$’.

**Step 2.** Find all possible candidates for a basket of singularities of $X$ which satisfy $\sum_Q Eq(Q) = 35\chi(O_X)$.

When $Q$ is a point of type $k/r$, the formula for $Eq(Q)$ is as follows:
\[
Eq(Q) = -g(Q, 3) - g(Q, 2) + 13g(Q, 1) = \begin{cases} 4k & \text{if } k/r < 1/3, \\ r + k & \text{if } 1/3 \leq k/r \leq 1/2. \end{cases}
\]

Notice that $Eq(Q)$ is always positive. To find all possible candidates for a basket of singularities, it is enough to consider only points $Q$ at which the values of $Eq$ are less than or equal to $35\chi(O_X)$.

Thus, by following procedures (1) and (2), Step 2 is complete:

1. Find all the points in the interval $(0, 1/2]$ at which the values of $Eq$ are less than or equal to $35\chi(O_X)$.
2. Find all possible candidates for a basket of singularities of $X$ which consist of points in $BL$ and satisfy $\sum_Q Eq(Q) = 35\chi(O_X)$.

Since the summation $\sum_Q Eq(Q)$ is over points $Q$ in a basket, to reduce computation time, a good upper bound for number of points in a basket is needed. Three ways to find an upper bound will be presented next.

Lemma 2 is one of ways to give an upper bound. For the case $p_4 = 0$, $\sum k_i = 10\chi(O_X) - p_3$ since $p_2 = 0$. A basket cannot contain more than $10\chi(O_X)$ points since $k_i \geq 1$. Hence, one of the upper bounds is $10\chi(O_X)$. Lemma 2 is very useful when $p_2$ is known.

Another way to attain an upper bound is to compute $\frac{35\chi(O_X)}{\min\{Eq(Q)\}}$ since $\sum_Q Eq(Q) = 35\chi(O_X)$. Since the formula for $Eq(Q)$ is explicitly given, it is easy to find the minimum of $Eq(Q)$. The minimum is 3 which occurs at the point $1/2$, so one of the upper bound is $35\chi(O_X)/3$. This upper bound is not as good as an upper bound given by Lemma 2, but is useful when $p_2$ is unknown.

A third way comes from the following:
\[
p_n - p_{n+1} = -\frac{n^2}{2}K_X^3 + 2\chi(O_X) - \sum g(Q, n).
\]

If $p_n \geq p_{n+1}$, another upper bound $\frac{2\chi(O_X) - p_n + p_{n+1}}{\min\{g(Q, n)\}}$ is found since $K_X^3 > 0$. This results in a fairly good upper bound, but caution is needed because the $\min\{g(Q, n)\}$ can be a zero.
Notice that there are an infinite number of points in the subinterval \((0,1/3)\) at which the value of \(\text{Eq}(Q)\) is \(4k\), since \(\text{Eq}(Q)\) is independent of \(r\) for a point in the interval \((0,1/3)\). This kind of point in the basket will be denoted by \(k/R\). It means that a point of type \(k/R\) stands for infinitely many points in the subinterval \((0,1/3)\) at which the value of \(\text{Eq}(Q)\) is \(4k\). For example, when \(\chi(O_X) = 2\), the basket \(\{1/2 \times 5, 1/3, 3/7 \times 2, 4/11, 2/R_1, 2/R_2\}\) satisfies \(\sum_O \text{Eq}(Q) = 35\chi(O_X)\). \(R_1\) and \(R_2\) should be determined.

**Step 3.** Classify all candidates by determining whether or not \(p_n \geq 1\) (or 2) for necessary \(n\).

Since it is claimed that \(p_n \geq 1\) for \(n \geq 7\) when \(\chi(O_X) = 1\), it is enough to check all candidates for \(7 \leq n \leq 13\). Once \(p_n \geq 1\) for \(7 \leq n \leq 13\), it can easily be shown that \(p_n \geq 1\) by induction for \(n \geq 14\). For example, \(p_{15} \geq p_7 + p_8 - 1\).

The difficulty in this step is that the candidate may contain a point of type \(k/R\). Without any information about \(R\), it is not possible to compute \(p_n\). For example, to compute \(p_7\), even though \(k\) is known, it is not possible to compute \(6k\) since \(R\) is unknown.

For \(7 \leq n \leq 13\), the maximal multiple of \(k\) is \(12k\) in \(p_{13}\). Thus, to compute \(p_n\) for \(7 \leq n \leq 13\), divide the interval \((0,1/3)\) into two subintervals \((0,1/12)\) and \((1/12,1/3)\).

For example, let us take \(2/R\). First, to be a point in the interval \([1/12,1/3]\), \(R\) must be between 6 and 25. Thus, there are 9 possible values for \(R\) since 2 and \(R\) are relatively prime. Hence, \(i/k\) (\(i = 1,\ldots, 12\)) can be computed for each of 9 values of \(R\). Second, if \(2/R\) is a point in \((0,1/13)\), then it is possible to compute \(p_n\) for \(7 \leq n \leq 13\) without any information about \(R\). By assuming \(m = 4\) and \(N = 13\) in Lemma 4, \(p_n\) for \(7 \leq n \leq 13\) can be computed. In conclusion, \(p_n\) can be computed eventually in every case.

For the case \(\chi(O_X) = 2\), the same procedures are followed.

**Step 4.** Filter the candidates which fail to pass the test \(p_n \geq 1\) or 2.

Some baskets satisfy all the conditions, yet still cannot exist. To filter such candidates, there are some tools including Lemma 3.

For example, when \(\chi(O_X) = 1\), \(p_2 = 0\) and \(p_3 = 0\), a basket \(\{4/11, 2/5 \times 2, 1/2 \times 2\}\) passes Steps 1, 2 and 3. It is easily seen that \(\sum k_j^2/r_j > 4\) but \(3 \sum r_j^2 - 1 - 68 < 3\). This basket does not satisfy the right inequality in Lemma 3, thus it cannot exist.

Another example is \(\{\chi(O_X) = 2, K^3_X = \frac{1}{2784}\},\) and a basket \(\{1/2 \times 4, 1/3 \times 2, 2/5, 3/7, 3/8, 5/13, 1/5\}\). This example passes Steps 1, 2 and 3. However, \(p_{17} = 0\) although \(p_5 = 1\) and \(p_{12} = 1\). Hence, it cannot exist.

For all these steps, a computer software which can do symbolic computations was employed.

**Proof of Main Theorem.** To prove the theorem, the problem is divided into three cases. To deal with each case, the four steps described above are going to be utilized.

**Case 1.** \(p_4 = 0\).

**Case 2.** \(p_4 \neq 0, p_7 = 0\).

**Case 3.** \(p_4 \neq 0, p_7 \neq 0\).

**Case 1.** \(p_4 = 0\).

In Step 1, the linear combination \(-p_4 + 14p_2 = 0\) was used.
In Step 2, when $\chi(\mathcal{O}_X) = 1$, BL of 41 points was obtained to find all the possible candidates for a basket of singularities. When $\chi(\mathcal{O}_X) = 2$, BL of 143 points was obtained.

In Steps 3 and 4, it was determined that $p_n \geq 1$ for $n \geq 7$ and $p_n \geq 2$ for $n \geq 10$ when $\chi(\mathcal{O}_X) = 1$.

When $\chi(\mathcal{O}_X) = 2$, it was determined that $p_{12} \geq 1$, $p_n \geq 1$ for $n \geq 14$ and $p_{18} \geq 2$ and $p_n \geq 2$ for $n \geq 20$ with the following possible exceptional baskets:

1. $\chi(\mathcal{O}_X) = 2$, $K_X^3 = 1/462$, {1/2 × 5, 1/3, 3/7 × 2, 4/11, 2/7 × 2}.
   
   With these data, $p_{15} = 0$.

2. $\chi(\mathcal{O}_X) = 2$, $K_X^3 = 1/840$, {1/2 × 5, 1/3, 3/7 × 2, 3/8, 3/10, 2/7}.
   
   With these data, $p_{15} = 0$.

3. $\chi(\mathcal{O}_X) = 2$, $K_X^3 = 1/1680$, {1/2 × 6, 1/3 × 2, 3/7 × 2, 5/16, 1/5}.
   
   With these data, $p_{17} = 0$.

4. $\chi(\mathcal{O}_X) = 2$, $K_X^3 = 1/1170$, {1/2 × 4, 1/3 × 3, 2/5, 4/9, 1/4 × 2, 5/13}.
   
   With these data, $p_{14} = p_{17} = p_{19} = 0$ and $p_{21} = p_{22} = 1$.

5. $\chi(\mathcal{O}_X) = 2$, $K_X^3 = 1/1680$, {1/2 × 5, 1/3 × 7/16, 2/7, 1/5}.
   
   With these data, $p_{17} = p_{19} = 0$ and $p_{22} = 1$.

6. $\chi(\mathcal{O}_X) = 2$, $K_X^3 = 1/2856$, {1/2 × 5, 1/3, 3/7 × 2, 3/8, 5/17}.
   
   With these data, $p_{15} = 0$ and $p_{18} = p_{22} = p_{25} = 1$.

7. $\chi(\mathcal{O}_X) = 2$, $K_X^3 = 1/1020$, {1/2 × 5, 2/5, 5/12, 6/17, 1/4 × 2}.
   
   With these data, $p_{18} = p_{21} = 1$.

8. $\chi(\mathcal{O}_X) = 2$, $K_X^3 = 1/714$, {1/2 × 5, 1/3, 5/14, 7/17, 1/4 × 2}.
   
   With these data, $p_{18} = 1$.

**Case 2.** $p_4 \neq 0$, $p_7 = 0$.

$p_3 = 0$ since $p_4 \neq 0$ and $p_7 = 0$. For Case 2, the linear combination $-5p_7 + 91p_3 = 0$ was used. Four steps described above were followed and all the possible baskets were obtained. All the possible baskets for this case showed:

- When $\chi(\mathcal{O}_X) = 1$, $p_n \geq 1$ for $n \geq 7$ and $p_n \geq 2$ for $n \geq 10$.
- When $\chi(\mathcal{O}_X) = 2$, $p_{12} \geq 1$ and $p_n \geq 1$ for $n \geq 14$. Moreover, $p_{18} \geq 2$ and $p_n \geq 2$ for $n \geq 20$.

**Case 3.** $p_4 \neq 0$, $p_7 \neq 0$.

First, let us investigate the case $\chi(\mathcal{O}_X) = 1$.

It is clear that $p_8 \geq 1$, $p_{11} \geq 1$ and $p_{12} \geq 1$ since $p_4 \geq 1$ and $p_7 \geq 1$. If $p_9 \geq 1$ and $p_{10} \geq 1$, then $p_n \geq 1$ ($n \geq 13$) can be shown by inducting from $p_i$ ($i = 7, \ldots, 10$). For example, $p_{13} \geq p_4 + p_9 - 1$. Thus, the first claim for the case $\chi(\mathcal{O}_X) = 1$ can be proved.

To get a contradiction, it is assumed that $p_9 = 0$ or $p_{10} = 0$, then $p_2 = p_3 = p_5 = 0$ since $p_9 = 0$ or $p_{10} = 0$. The linear combination $-p_5 + 6p_3 = 0$ was used and all the steps described above were followed; however no candidate was produced which gave $p_9 = 0$ or $p_{10} = 0$.

Therefore, $p_n \geq 1$ for $n \geq 7$ when $\chi(\mathcal{O}_X) = 1$.

Next, check the second claim for the case $\chi(\mathcal{O}_X) = 1$.

When $p_4 = 1$, there are two subcases (1) $p_4 = p_2 = 1$, (2) $p_4 = 1$, $p_2 = 0$. For both subcases, the linear combination $-p_4 + 14p_2$ was used and all steps were followed. In both cases, no candidate gave $p_n = 1$ for some $n \geq 10$. In fact, to show this, it is enough to check $p_n$ for $10 \leq n \leq 13$ since $p_4 = 1$. 


When $p_4 \geq 2$, then $p_n \geq 2$ for $n \geq 11$ since $p_n \geq 1$ for $n \geq 7$. Only one information about $p_{10}$ is so far known, i.e., $p_{10} \geq 1$. If $p_{10} = 1$, then $p_6 = p_3 = p_2 = 0$ since $p_4 \geq 2$. The linear combination $-p_6 + 11p_3 = 0$ was used for the case $p_{10} = 1$, $p_6 = 0$ and $p_3 = 0$; however, no candidate gave $p_{10} = 1$. Hence, if $p_4 \geq 2$, then $p_n \geq 2$ for $n \geq 11$.

Therefore, $p_n \geq 2$ for $n \geq 10$ since $p_6 \geq 1$ and $p_3 \geq 1$ for $n \geq 7$. Only one information about $p_1$ is so far known, i.e., $p_1 \geq 1$. If $p_1 = 1$, then $p_2 = p_3 = p_4 = 0$ since $p_4 \geq 2$. The linear combination $-p_6 + 11p_3 = 0$ was used for the case $p_1 = 1$, $p_6 = 0$ and $p_3 = 0$; however, no candidate gave $p_1 = 1$. Hence, if $p_4 \geq 2$, then $p_n \geq 2$ for $n \geq 10$.

Therefore, $p_n \geq 2$ for $n \geq 10$ when $\chi(\mathcal{O}_X) = 1$.

Now assume $\chi(\mathcal{O}_X) = 2$.

Since $p_4 \geq 1$ and $p_7 \geq 1$, it is clear that $p_{12} \geq 1$, $p_{14} \geq 1$, $p_{15} \geq 1$ and $p_{16} \geq 1$. Hence, if $p_{17} \geq 1$, then the first claim for the case $\chi(\mathcal{O}_X) = 2$ is proved since $p_n \geq 1$ ($n \geq 18$) can be shown by inducing from $p_i$ ($i = 14, 15, 16, 17$). To get a contradiction, it is assumed that $p_{17} = 0$. Then, $p_2 = p_3 = p_5 = 0$ since $p_{12} \geq 1$, $p_{14} \geq 1$ and $p_{15} \geq 1$. Using the linear combination $-p_5 + 6p_3 = 0$ and following all the steps resulted in no candidate giving $p_{17} = 0$.

Therefore, $p_{12} \geq 1$ and $p_n \geq 1$ for $n \geq 14$ when $\chi(\mathcal{O}_X) = 2$.

Next, check the second claim for the case $\chi(\mathcal{O}_X) = 2$.

If $p_4 \geq 2$, then $p_n \geq 2$ for $n \geq 18$ since $p_n \geq 1$ for $n \geq 14$. Thus, $p_4 = 1$ may be assumed. When $p_4 = 1$, two subcases (1) $p_4 = p_2 = 1$ and (2) $p_4 = 1$, $p_2 = 0$ are produced. For both cases, the linear combination $-p_4 + 14p_2$ was used and all steps carried out. In both cases, there is no candidate which gives $p_{18} = 1$ or $p_n = 1$ for some $n \geq 20$.

Therefore, the main theorem is proved. □

**Remark 3.** Although the proof is quite awkward, the technique for the proof is simple and combinatorial. Hence, it can be applied to the case $\chi(\mathcal{O}_X) \geq 3$, for which more computations are needed.

**References**


