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Global attractivity in concave or sublinear monotone infinite delay differential equations ☆

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A B S T R A C T

We study the dynamical behavior of the trajectories defined by a recurrent family of monotone functional differential equations with infinite delay and concave or sublinear nonlinearities. We analyze different sceneries which require the existence of a lower solution and of a bounded trajectory ordered in an appropriate way, for which we prove the existence of a globally asymptotically stable minimal set given by a 1-cover of the base flow. We apply these results to the description of the long term dynamics of a nonautonomous model representing a stage-structured population growth without irreducibility assumptions on the coefficient matrices.

1. Introduction

A large number of mathematical models describing different phenomena in engineering, biology, economics and other applied sciences present some monotonicity properties with respect to the state argument, which permits to apply the theory of monotone dynamical systems to their analysis. When some additional physical conditions occur, the increasing rate of the vector field which defines the differential equation decreases (or increases) as the state argument increases, so that the model exhibits concave (or convex) nonlinearities. There are also well-known phenomena in applied

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sciences for which only positive state arguments make sense, and for which the dynamics can be essentially described by a sublinear vector field. Sublinear, concave and convex monotone semiflows have been extensively studied in the literature. The works of Krasnoselskii [20,21], Hirsch [17,18], Selgrade [31], Smith [35], Takáç [36], Krause and Ranft [23], Krause and Nussbaum [22], Zhao and Jing [40], Freedman and Zhao [13], and references therein, provide a basic theory for autonomous and periodic monotone differential equations with concave or sublinear nonlinearities as well as for their discrete analogs. It is important to note that their proofs of the existence of a constant or periodic solution which is globally asymptotically stable require some conditions of strong monotonicity, strong concavity or strong sublinearity.

More recently, Zhao [39], Jiang and Zhao [19], Novo, Obaya and Sanz [26], and Novo, Núñez and Obaya [25] have obtained versions of this result valid for recurrent nonautonomous monotone differential equations. All these papers make use of a skew-product formulation which requires a compact minimal flow on the base and an ordered normal Banach space on the fiber. In [39] and [19], the authors study sublinear monotone differential equations and use methods of topological dynamics as well as the properties of the part metric in the interior of the positive cone. In [26] and [25] convex monotone functional differential equations with finite delay are considered, and methods of differentiable dynamics are applied in order to prove the exponential stability of the recurrent solutions by means of an ergodic representation theorem. We point out that in [25] the *strong* condition required for the global stability relies on the existence of a strong semiequilibrium instead of on the strong monotonicity or strong concavity of the semiflow. The bases for an alternative monotone theory for random dynamical systems are established by Arnold and Chueshov [4,5] and Chueshov [8].

In this paper we give a version of the result above mentioned, valid for recurrent monotone functional differential equations with infinite delay and concave or sublinear nonlinearities. In the line of the results of Novo, Obaya and Sanz [27] and Muñoz, Novo and Obaya [24], the fiber of our phase space is the set *BU* of the bounded and uniformly continuous m-dimensional functions on the negative half-line, endowed with the supremum norm. Under natural conditions on the vector field, every bounded trajectory is relatively compact for the compact-open topology, and its omega limit set admits a flow extension. When the vector field satisfies a quasimonotone condition and is concave or sublinear with respect to its state argument, the solutions of the functional differential equation define a monotone and concave or sublinear semiflow on *BU*. But there is an important difference with respect to those types of semiflows considered in the previous works before cited: since every trajectory always remembers its whole past, this semiflow satisfies neither a strong monotonicity nor a strong nonlinearity condition. For this reason we formulate the conditions of concavity or sublinearity on the vector field instead of on the semiflow. Similarly, the definitions of lower solution and strong lower solution, which are natural concepts in this monotone setting, can be also given in terms of the vector field. Roughly speaking, a lower solution is a solution of a differential inequality, and it determines a positively invariant region of the phase space which is relevant from a dynamical point of view.

We begin by analyzing the dynamics in the concave case. For it, we describe two different dynamical sceneries which allow us to prove the existence, on the positively invariant region determined by a lower solution, of a minimal set given by a globally asymptotically stable copy of the base flow. The first one requires the vector field to be concave, the lower solution to be strong, and the existence of a bounded trajectory which is above the graph of the lower solution. In the second scenery, the vector field is strongly concave, and the bounded trajectory whose existence we assume must be strongly above the graph of the lower solution. Then we prove that the second one of these sceneries has an analogue in the sublinear situation: the existence of a minimal set given by a globally asymptotically stable copy of the base flow is guaranteed by the assumptions of strong sublinearity of the vector field and the existence of a strongly positive bounded semiorbit. In particular, these hypotheses mean that the null function is a lower solution. Note that the results are optimal in the general settings we consider: when the delay is infinite, asymptotical stability does not imply exponential stability, even under some differentiability assumptions.

We apply the previous results to establish the existence of a unique positive recurrent attracting solution for a nonautonomous version of some population dynamics models, intensively analyzed in the literature. Different mathematical models representing stage-structured population growth are formulated and analyzed using methods of the theory of autonomous monotone differential equations by Aiello and Freedman [1], Freedman and Wu [12], Aiello, Freedman and Wu [2], Wu, Freedman and Miller [38], and Freedman and Peng [11], among others. Following [38] we consider a population growth model of a single species with dispersal in a multi-patch environment, assuming that the life of the individuals crosses an immature stage before reaching the matureness, and that this second stage is the only one in which reproduction is possible. We allow the presence of a stochastic component to determine the maturation period, so that an infinite delay element appears in the evolution equations. The fundamental difference in our approach concerns the birth and death rates as well as the net exchange rates among different patches: we assume them to be recurrent, bounded and uniformly continuous functions instead of constants. In addition, we suppress an irreducibility condition, used in the previous models in order to obtain a kind of strongly monotone semiflow. Obviously, a more realistic model is obtained in this way. And in this case we can go further than in the general one. The physical conditions on this problem allow us to define the vector field and to study the corresponding trajectories in a standard fading memory Banach space. The restriction of the norm topology of this space to the closure of a solution which is globally defined and bounded agrees with the compact-open topology, and we can apply the spectral theory for infinite-dimensional linear skew-product semiflows developed by Chow and Leiva [6,7] and Sacker and Sell [30] in order to deduce the exponential stability of the positive recurrent solution previously found.

Let us sketch the remaining pages of this paper. In Section 2, after explaining the type of infinite delay functional differential equations we work with, we state and prove the main results of the paper under concavity assumptions, concerning the existence of a unique equilibrium with strong properties of attraction. The same result is proved in Section 3 in the case of a strongly sublinear vector field. Sections 4 and 5 contain the application of this result to the nonautonomous stage-structured population growth model. In the first one we apply our results to show the existence of nonautonomous equilibria with some properties of asymptotic attraction for both the mature and immature populations, while the last section refines the attractivity result showing that in fact the convergence is of exponential type.

Finally, we close the introduction by recalling some standard concepts and basic results of topological dynamics.

Let Ω be a complete metric space. A (*real and continuous*) *global flow* on Ω is a continuous map $\sigma : \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \sigma(t, \omega)$ satisfying $\sigma_0 = \mathrm{Id}$ and $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for each $s, t \in \mathbb{R}$, where $\sigma_t(\omega) = \sigma(t, \omega)$. By replacing $\mathbb{R}$ by $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geqslant 0\}$, we obtain the definition of a (*real and continuous*) *global semiflow* on Ω. When the map σ is defined, continuous, and satisfies the previous properties on an open subset of $\mathbb{R} \times \Omega$ (resp. $\mathbb{R}^+ \times \Omega$) containing $\{0\} \times \Omega$, we talk about a *local flow* (resp. *local semiflow*).

Let $(\Omega, \sigma, \mathbb{R})$ be a global flow. The *orbit* of the point ω is $\{\sigma_t(\omega) \mid t \in \mathbb{R}\}$. A subset $\Omega_1 \subset \Omega$ is *σ-invariant* if $\sigma_t(\Omega_1) = \Omega_1$ for every $t \in \mathbb{R}$. A σ-invariant subset $\Omega_1 \subset \Omega$ is *minimal* if it is compact and does not contain properly any other compact σ-invariant set, which is equivalent to saying that the orbit of any one of its elements is dense in it. The continuous flow $(\Omega, \sigma, \mathbb{R})$ is *recurrent* or *minimal* if Ω itself is minimal.

In the case of a semiflow $(\Omega, \sigma, \mathbb{R}^+)$, we call (*positive*) *semiorbit* of $\omega \in \Omega$ to the set $\{\sigma_t(\omega) \mid t \geqslant 0\}$; a subset Ω_1 of Ω is *positively σ-invariant* if $\sigma_t(\Omega_1) \subset \Omega_1$ for all $t \geqslant 0$; a positively σ-invariant subset $K \subset \Omega$ is *minimal* if it is compact and it does not contain properly any closed, positively σ-invariant subset; and $(\Omega, \sigma, \mathbb{R}^+)$ is a *minimal* semiflow if Ω itself is minimal.

A *flow extension* of the semiflow $(\Omega, \sigma, \mathbb{R}^+)$ is a continuous flow $(\Omega, \widetilde{\sigma}, \mathbb{R})$ such that $\widetilde{\sigma}(t, \omega) = \sigma(t, \omega)$ for each $\omega \in \Omega$ and $t \geqslant 0$. A compact positively σ-invariant subset *admits a flow extension* if the restricted semiflow does. Actually, as proved by Shen and Yi [32], a positively σ-invariant compact set K admits a flow extension if every point in K admits a unique backward orbit which remains inside the set K. A *backward orbit* of a point $\omega \in \Omega$ is a continuous map $\psi : \mathbb{R}^- \to \Omega$ such that $\psi(0) = \omega$ and for each $s \leqslant 0$ it is $\sigma(t, \psi(s)) = \psi(s + t)$ whenever $0 \leqslant t \leqslant -s$.

Finally, if the semiorbit of $\omega_0 \in \Omega$ for the semiflow σ is relatively compact, we can consider the *omega limit set* of ω_0, given by those points $\omega \in \Omega$ such that $\omega = \lim_{n\to\infty} \sigma(t_n, \omega_0)$ for some sequence $(t_n) \uparrow \infty$. The omega limit set is nonempty, compact, connected and positively σ-invariant, and each one of its points admits a backward orbit inside this set.

The reader can find the basic properties on topological dynamics here summarized in Ellis [9], Sacker and Sell [28], Shen and Yi [32] and references therein.

2. Concave monotone differential equations with infinite delay

Let $\sigma : \mathbb{R} \times \Omega \to \Omega$, $(t,\omega) \mapsto \sigma(t,\omega) \equiv \omega \cdot t$ be a real continuous global flow on a compact metric space Ω. Throughout the paper we assume this flow to be minimal. We will work with a family of infinite delay differential equations defined along the σ-orbits under some fundamental monotonicity and concavity or sublinearity assumptions. The order in the phase space, that we are describing in what follows, relies on the usual partial strong order relation in $\mathbb{R}^m$,

$$\begin{aligned}
v \leqslant w \quad &\Longleftrightarrow \quad v_j \leqslant w_j \quad \text{for } j = 1, \ldots, m, \\
v < w \quad &\Longleftrightarrow \quad v \leqslant w \quad \text{and} \quad v_j < w_j \quad \text{for some } j \in \{1, \ldots, m\}, \\
v \ll w \quad &\Longleftrightarrow \quad v_j < w_j \quad \text{for } j = 1, \ldots, m,
\end{aligned}$$

where v_j represents the jth component of any point $v \in \mathbb{R}^m$. We work with the maximum norm in $\mathbb{R}^m$, $\|v\| = \max_{j=1,\ldots,m} |v_j|$, which is *monotone* for this ordering: $0 \leqslant v \leqslant w \Rightarrow \|v\| \leqslant \|w\|$. The relations $\geqslant$, $>$, $\gg$ are defined in the obvious way.

We endow the set $X = C((-\infty, 0], \mathbb{R}^m)$ with the compact-open topology, i.e., the topology of uniform convergence over compact subsets. Then X is a Fréchet space and the topology is equivalent to the metric topology given by the distance

$$\mathsf{d}(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x - y|_n}{1 + |x - y|_n}, \quad x, y \in X,$$

for the nondecreasing family of seminorms $|x|_n = \sup_{s \in [-n,0]} \|x(s)\|$, with $n \in \mathbb{N}$. Let $BU \subset X$ be the Banach space

$$BU = \{x \in X \mid x \text{ is bounded and uniformly continuous}\}$$

endowed with the supremum norm $\|x\|_\infty = \sup_{s \in (-\infty,0]} \|x(s)\|$. The positive cone

$$BU_+ = \bigl\{x \in BU \bigm| x(s) \geqslant 0 \text{ for each } s \in (-\infty, 0]\bigr\}$$

(with nonempty interior) defines a partial strong order relation on BU, given by

$$\begin{aligned}
x \leqslant y \quad &\Longleftrightarrow \quad x(s) \leqslant y(s) \quad \text{for each } s \in (-\infty, 0], \\
x < y \quad &\Longleftrightarrow \quad x \leqslant y \quad \text{and} \quad x \neq y, \\
x \ll y \quad &\Longleftrightarrow \quad \exists \delta > 0 \quad \text{with } x \leqslant y - \delta J,
\end{aligned} \tag{2.1}$$

for which the norm in BU is also monotone. The symbol J represents either the vector $(1, 1, \ldots, 1)$ of $\mathbb{R}^m$ or the constant map $(-\infty, 0] \to \mathbb{R}^m$, $s \mapsto (1, 1, \ldots, 1)$ of BU. Again we define relations $\geqslant$, $>$, $\gg$ in the obvious way. To complete the notation, we denote $B_r = \{x \in BU \mid \|x\|_\infty \leqslant r\}$ for $r > 0$.

In what follows we will work with BU endowed with the norm $\|\cdot\|_\infty$ as well as with the metric topology as a subset of X. We will write BU^{d} when this second topology is considered. Similarly, the symbol $\lim^{\mathsf{d}}_{n\to\infty}$ will represent either convergence in BU^{d} or in $\Omega \times BU^{\mathsf{d}}$.

As said in the introduction, this section is devoted to the concave monotone case. Let us describe the family of nonautonomous infinite delay functional differential equations we work with. As usual, given a negative half-line $I \subset \mathbb{R}$, a point $t \in I$, and a continuous function $z : I \to \mathbb{R}^m$, z_t will denote the element of X defined by $z_t(s) = z(t+s)$ for $s \in (-\infty, 0]$. Our equations are

$$z'(t) = F(\omega \cdot t, z_t), \quad t \geqslant 0, \ \omega \in \Omega, \tag{2.2}$$

with $F : \Omega \times BU \to \mathbb{R}^m$, $(\omega, x) \mapsto F(\omega, x)$. Several conditions of the following list will be assumed on F:

(C1) F is continuous on $\Omega \times BU$ (considering the norm topology on BU),
(C2) there exists the linear differential operator $F_x : \Omega \times BU \to \mathcal{L}(BU, \mathbb{R}^m)$ and it is continuous (considering the norm $\|\cdot\|_\infty$ in BU and the associated one in $\mathcal{L}(BU, \mathbb{R}^m)$, also denoted by $\|\cdot\|_\infty$),
(C3) for each $r > 0$, $F(\Omega \times B_r)$ is a bounded subset of $\mathbb{R}^m$ and $F_x(\Omega \times B_r)$ is a bounded subset of $\mathcal{L}(BU, \mathbb{R}^m)$,
(C4) for each $r > 0$, the function $\Omega \times B_r^{\mathrm{d}} \to \mathbb{R}^m$, $(\omega, x) \mapsto F(\omega, x)$ is continuous (i.e., if $\lim_{n\to\infty} \omega_n = \omega$ and $\lim^{\mathrm{d}}_{n\to\infty} x_n = x$ with $x_n, x \in B_r$, then $\lim_{n\to\infty} F(\omega_n, x_n) = F(\omega, x)$),
(C5) for each $r_1 > 0$ and $r_2 > 0$, the function $\Omega \times B_{r_1}^{\mathrm{d}} \times B_{r_2}^{\mathrm{d}} \to R^m$, $(\omega, x, v) \mapsto F_x(\omega, x)v$ is continuous (i.e., $\lim_{n\to\infty} \omega_n = \omega$, $\lim^{\mathrm{d}}_{n\to\infty} x_n = x$ with $x_n, x \in B_{r_1}$ and $\lim^{\mathrm{d}}_{n\to\infty} v_n = v$ with $v_n, v \in B_{r_2}$, imply $\lim_{n\to\infty} F_x(\omega_n, x_n)v_n = F_x(\omega, x)v$),
(C6) quasimonotone condition: if $x_1, x_2 \in BU$ with $x_1 \leqslant x_2$ and $(x_1)_j(0) = (x_2)_j(0)$ holds for some $j \in \{1, \dots, m\}$, then $F_j(\omega, x_1) \leqslant F_j(\omega, x_2)$ for each $\omega \in \Omega$,
(C7) concavity condition: if $x_1, x_2 \in BU$ with $x_1 \leqslant x_2$, then $F_x(\omega, x_2)(x_2 - x_1) \leqslant F(\omega, x_2) - F(\omega, x_1) \leqslant F_x(\omega, x_1)(x_2 - x_1)$ for each $\omega \in \Omega$ (which, since F is differentiable, is equivalent to $F(\omega, \lambda x_1 + (1-\lambda)x_2) \geqslant \lambda F(\omega, x_1) + (1-\lambda)F(\omega, x_2)$ for each $(\omega, x) \in \Omega \times BU$ and $\lambda \in [0, 1]$; see Amann [3]),
(C8) strong concavity condition: if $x_1, x_2 \in BU$ with $x_1 \ll x_2$, $F_x(\omega, x_2)(x_2 - x_1) \ll F(\omega, x_2) - F(\omega, x_1)$ for each $\omega \in \Omega$.

Note that $F(\omega_n, x_n) - F(\omega, x) = \int_0^1 F_x(\omega_n, \lambda x_n + (1-\lambda)x)(x_n - x)\, d\lambda + F(\omega_n, x) - F(\omega, x)$, and hence condition (C4) follows from (C1), (C3) and (C5).

Condition (C1) and the local Lipschitz character of F with respect to x guaranteed by (C2) and (C3) ensure that for each $\omega \in \Omega$ and each $x \in BU$ there exists a unique function $z(\cdot, \omega, x) : (-\infty, \alpha) \to \mathbb{R}^m$ which solves Eq. (2.2) for $t \in [0, \alpha)$, which is maximal in the sense that it cannot be extended to α, and which satisfies $z(s, \omega, x) = x(s)$ for each $s \in (-\infty, 0]$. Note that $\alpha = \alpha(\omega, x)$. If in addition the solution is bounded (i.e., if $\sup_{t\in(-\infty,\alpha)} \|z(t, \omega, x)\| < \infty$), then $\alpha = \infty$. (See Hale and Kato [14] and Hino, Murakami and Naito [16].) We define $u(\cdot, \omega, x) : [0, \alpha) \to BU$ by $u(t, \omega, x)(s) = z(t+s, \omega, x)$ for $s \in (-\infty, 0]$ and note that the family (2.2) induces a local skew-product semiflow

$$\tau : \mathbb{R}^+ \times \Omega \times BU \to \Omega \times BU, \quad (t, \omega, x) \mapsto \big(\omega \cdot t, u(t, \omega, x)\big).$$

It is proved in Novo, Obaya and Sanz [27] that, under conditions (C1)–(C3), a bounded τ-semiorbit $\{(\omega_0 \cdot t, u(t, \omega_0, x_0)) \mid t \geqslant 0\}$ has a well-defined omega limit set for the product metric, namely

$$K = \left\{(\omega, x) \in \Omega \times BU \;\middle|\; \exists (t_n) \uparrow \infty \text{ with } (\omega, x) = \lim^{\mathrm{d}}_{n\to\infty} \big(\omega_0 \cdot t_n, u(t_n, \omega_0, x_0)\big)\right\},$$

and in addition K is compact in $\Omega \times BU^{\mathrm{d}}$. When condition (C4) is also assumed, the restriction of the semiflow τ to K is continuous for the product metric, K is a positively τ-invariant set, and it admits a flow extension, which is also continuous. In particular, $u(t, \omega, x)$ is defined for every $t \in \mathbb{R}$ and every $(\omega, x) \in K$.

From now on we assume conditions (C1)–(C5) on F. Let $y(\cdot, \omega, x, v) : (-\infty, \alpha) \to \mathbb{R}^m$ (with $\alpha = \alpha(\omega, x)$) be the unique solution of the variational equation along the semiorbit of (ω, x)

$$y'(t) = F_x\big(\omega \cdot t, u(t, \omega, x)\big) y_t \tag{2.3}$$

satisfying $y(s,\omega,x,v) = v(s)$ for every $s \in (-\infty, 0]$. If whenever it makes sense we denote by $u_x(t,\omega,x) \in \mathcal{L}(BU, BU)$ the linear differential operator with respect to x, it turns out that $(u_x(t,\omega,x)v)(s) = y(t+s,\omega,x,v)$, $s \in (-\infty,0]$, $t \in (0,\alpha)$. The proof of this result can be found in Hale and Verduyn Lunel [15] for equations with finite delay, and it also works in the infinite delay case. Note that hypothesis (C2) on F ensures the continuity of the map $\Omega \times BU \times BU \to \mathbb{R}^m$, $(\omega,x,v) \mapsto F_x(\omega,x)v$, which is linear in v. In other words, the coefficient function of the family of Eqs. (2.3) satisfies condition (C1), while the linearity of the map with respect to its state argument v ensures that it also satisfies conditions (C2)–(C5) and (C7) (replacing Ω by $\Omega \times BU$). In particular, the cocycle property also holds for u_x, now over the flow τ on $\Omega \times BU$; that is, for every $(\omega,x) \in \Omega \times BU$,

$$u_x(t_1+t_2,\omega,x) = u_x\big(t_1, \tau(t_2,\omega,x)\big) \circ u_x(t_2,\omega,x) \tag{2.4}$$

for those values of t_1 and t_2 for which all the terms are defined. Note finally that the quasimonotone hypothesis (C6) of $F(\omega,x)$ with respect to x ensures the analogous property for $F_x(\omega,x)v$ with respect to v, and that the strong concavity condition (C8) never holds for (2.3).

As said before, the conditions we will impose ensure the monotonicity and concavity of the semiflow τ, as shown in the next lemma. Although the proof is standard, a sketch is included. The interested reader can find in Wu [37], Smith [34], Arnold and Chueshov [4,5], Jiang and Zhao [19], Novo, Obaya and Sanz [26] and references therein the basic properties of monotone and concave (or convex) semiflows.

Lemma 2.1. *Assume that conditions* (C1)–(C5) *on F hold. Then,*

(i) *under condition* (C6) *the semiflow τ is monotone*; *that is, for each $\omega \in \Omega$ and $x_1, x_2 \in BU$ with $x_1 \leqslant x_2$ it holds that $u(t,\omega,x_1) \leqslant u(t,\omega,x_2)$ for those values of $t \geqslant 0$ for which both terms are defined. Consequently, $u_x(t,\omega,x_1)v \geqslant 0$ for every $v \geqslant 0$ whenever it is defined.*
(ii) *Under conditions* (C6) *and* (C7)*, the semiflow τ is concave*; *that is, for each $\omega \in \Omega$ and $x_1, x_2 \in BU$ with $x_1 \leqslant x_2$,*

$$u_x(t,\omega,x_2)(x_2-x_1) \leqslant u(t,\omega,x_2) - u(t,\omega,x_1) \leqslant u_x(t,\omega,x_1)(x_2-x_1) \tag{2.5}$$

for those values of $t \geqslant 0$ for which all the terms are defined.

Proof. (i) It is well known (see e.g. [37,34]) that the quasimonotone condition (C6) implies the monotonicity of the semiflow. The positiveness of the differential operators $u_x(t,\omega,x)$ is an immediate consequence of this property under the presence of differentiability conditions (not required for the monotonicity).

(ii) Arguing as in Novo, Obaya and Sanz [26], we prove that the semiflow inherits the concavity of the map F: for those $t \geqslant 0$ for which all the terms are defined,

$$u\big(t,\omega,\lambda x_1 + (1-\lambda)x_2\big) \geqslant \lambda u(t,\omega,x_1) + (1-\lambda)u(t,\omega,x_2)$$

for any $\lambda \in [0,1]$, $\omega \in \Omega$ and $x, y \in BU$ with $x \leqslant y$. The differentiability of the map $u(t,\omega,x)$ with respect to x makes this inequality equivalent to (2.5) (see [3]). □

As explained in the introduction, we are interested in establishing conditions ensuring the existence of a nonautonomous equilibrium (a metric copy of the base) with strong attracting properties. These conditions are based on the existence of a lower solution or a strong lower solution.

Definition 2.2. A *metric copy of the base* for τ is a τ-positively invariant compact set $K \subset \Omega \times BU^{\mathrm{d}}$ which agrees with the graph of a continuous function $e : \Omega \to BU^{\mathrm{d}}$: $K = \{(\omega, e(\omega)) \mid \omega \in \Omega\}$. In particular, the semiflow admits a flow extension on K and the map e is τ-invariant: $e(\omega\cdot t) = u(t,\omega,e(\omega))$ for every $t \in \mathbb{R}$ and $\omega \in \Omega$.

Remarks 2.3. (1) As a consequence of this invariance condition, $e(\omega)(t+s) = e(\omega \cdot t)(s)$ for every $s \in (-\infty, 0]$, $t \in (-\infty, -s]$ and $\omega \in \Omega$. It is also clear that the minimality of the base flow guarantees that a metric copy of the base is minimal for the restriction of the semiflow to it.

(2) The function e is a *continuous equilibrium* for τ in the language of Chueshov [8], Novo, Núñez and Obaya [25] and Novo, Obaya and Sanz [27]. So that when giving conditions which ensure the existence of a metric copy of the base we are in fact describing situations in which a continuous nonautonomous equilibrium exists.

Definition 2.4. Let $\widetilde{a} : \Omega \to \mathbb{R}^m$ be a continuous function. We say that $\widetilde{a}$ *is* C^1 *along the* σ*-orbits* if for every $\omega \in \Omega$ the function $\mathbb{R} \to \mathbb{R}^m$, $s \mapsto \widetilde{a}'(\omega \cdot s) = (d/dt)\widetilde{a}(\omega \cdot (s+t))|_{t=0}$ exists and is continuous. We say that $\widetilde{a}$ is a *lower solution* for the family of Eqs. (2.2) if it is C^1 along the σ-orbits and the function $a : \Omega \to BU$ given by $a(\omega)(s) = \widetilde{a}(\omega \cdot s)$ for $s \in (-\infty, 0]$ satisfies that $u(t, \omega, a(\omega))$ is defined for any $t \geqslant 0$ and that $\widetilde{a}'(\omega) \leqslant F(\omega, a(\omega))$ for every $\omega \in \Omega$. We say that a lower solution $\widetilde{a} : \Omega \to \mathbb{R}^m$ is *strong* if $\widetilde{a}'(\omega) \ll F(\omega, a(\omega))$ for every $\omega \in \Omega$.

Remarks 2.5. (1) The continuity of the lower solution $\widetilde{a} : \Omega \to \mathbb{R}^m$ ensures that the map $a : \Omega \to BU^{\mathrm{d}}$ is well defined, continuous and norm-bounded.

(2) The idea of lower solution is closely related to the idea of *subequilibrium* appearing in [8,25,27]. In fact, the function $a : \Omega \to BU$ satisfies

$$a(\omega \cdot t) \leqslant u\big(t, \omega, a(\omega)\big) \quad \text{for every } \omega \in \Omega \text{ and } t \geqslant 0.$$

This assertion follows easily from a standard comparison argument for equations satisfying the quasi-monotone condition (C6). See for instance the proof of Proposition 4.4(i) of [25]. However, the concept of semiequilibrium is more general: there exist subequilibria not associated to lower solutions. In the case of infinite delay, the subequilibrium defined from a strong lower solution is not *strong* in the sense of [25]. However it inherits from the strong character of the lower solution the properties we need to prove the first result of this section.

Theorem 2.6. *Assume that conditions* (C1)–(C7) *hold and a strong lower solution* $\widetilde{a} : \Omega \to \mathbb{R}^m$ *exists. Assume also the existence of a subset* $K \subset \Omega \times BU$ *satisfying*

(k1) *K is compact in $\Omega \times BU^{\mathrm{d}}$,*
(k2) *K is positively τ-invariant and the restriction of the semiflow τ to K admits a flow extension,*
(k3) *K is "above a": $a(\omega) \leqslant x$ for any $(\omega, x) \in K$.*

Then K is a metric copy of the base and the unique set satisfying these properties.

In addition, all the semiorbits corresponding to initial data (ω, x) with $a(\omega) \leqslant x$ are globally defined and approach asymptotically K in $\Omega \times BU^{\mathrm{d}}$; i.e., if $K = \{(\omega, e(\omega)) \mid \omega \in \Omega\}$, then $\lim_{t\to\infty} \mathrm{d}(e(\omega \cdot t), u(t, \omega, x)) = 0$.

Proof. Note that the compactness of K in $\Omega \times BU^{\mathrm{d}}$ and the fact that it admits a flow extension imply the existence of $r > 0$ such that $K \subset \Omega \times B_r$: the compactness of $\{x(0) \mid (\omega, x) \in K\}$ in $\mathbb{R}^m$ provides $r > 0$ with $\|x(0)\| \leqslant r$ for every $(\omega, x) \in K$. Now given $(\omega, x) \in K$ and $s \in (-\infty, 0]$ we have $(\omega \cdot s, u(s, \omega, x)) \in K$ and $x(s) = u(s, \omega, x)(0)$. Corollary 4.3 of [27] then shows the continuity of the restriction of τ to K in the product metric.

Note also that, in fact, K is "strongly above a": there exists $\delta > 0$ such that $a(\omega) + \delta J \leqslant x$ for any $(\omega, x) \in K$. This follows from the equality $x(s) - a(\omega)(s) = u(s, \omega, x)(0) - a(\omega \cdot s)(0)$ for any $(\omega, x) \in K$ and $s \in (-\infty, 0]$ (due to the flow extension in K), from the continuity on $\Omega \times BU^{\mathrm{d}}$ of the map $K \to \mathbb{R}^m$, $(\omega, x) \mapsto x(0) - a(\omega)(0)$ (see Remark 2.5(1)), and from the fact that the image of every point is strongly positive (and hence larger than δJ for a $\delta > 0$), which we check by contradiction using that a is a strong lower solution: if there is $(\omega^*, x^*) \in K$ and $j \in \{1, \dots, m\}$ with $0 = x_j^*(0) - a_j(\omega^*)(0) = z_j(0, \omega^*, x^*) - \widetilde{a}_j(\omega^*)$, since $\widetilde{a}_j'(\omega^*) < F_j(\omega^*, a(\omega^*)) \leqslant F_j(\omega^*, x^*) = z_j'(0, \omega^*, x^*)$, we find that $\widetilde{a}_j(\omega^* \cdot l) > z_j(l, \omega^*, x^*)$ for some $l < 0$, contradicting (k3).

We begin by proving that the family

$$D = \{u_x(t, \omega, x)J \mid t \geqslant 0 \text{ and } (\omega, x) \in K\} \tag{2.6}$$

is relatively compact in BU^{d}. On the one hand, it is uniformly bounded: according to Lemma 2.1 and Remark 2.5(2), given any $t \geqslant 0$ and $(\omega, x) \in K$ (with $a(\omega) + \delta J \leqslant x$, as just checked),

$$\begin{aligned} 0 \leqslant \delta u_x(t, \omega, x)J &\leqslant u_x(t, \omega, x)\big(x - a(\omega)\big) \\ &\leqslant u(t, \omega, x) - u\big(t, \omega, a(\omega)\big) \leqslant u(t, \omega, x) - a(\omega \cdot t); \end{aligned}$$

hence, from the boundedness of K and a (see Remark 2.5(1)), we conclude that there exists a common $k > 0$ such that $0 \leqslant u_x(t, \omega, x)J \leqslant kJ$. The monotonicity of the norm in BU proves the uniform boundedness. On the other hand, D is equicontinuous: if $y(t, \omega, x, J) = (u_x(t, \omega, x)J)(0)$ represents the solution of the corresponding equation (2.3), then $(u_x(t, \omega, x)J)(s) = y(t + s, \omega, x, J)$, with $y(t + s, \omega, x, J) = J$ if $t + s \leqslant 0$ (so that its derivative is zero for $s \in (-\infty, -t^-])$ and

$$\big\|(d/ds)y(t + s, \omega, x, J)\big\| = \big\|F_x\big(\tau(t + s, \omega, x)\big)\big(u_x(t + s, \omega, x)J\big)\big\| \leqslant lk$$

for $s \in [-t^+, \infty)$, where $l = \sup_{(\omega,x) \in \Omega \times B_r} \|F_x(\omega, x)\|_\infty$, finite by condition (C3). Arzelà–Ascoli theorem and the fact that the closure of D in metric remains in BU, easily deduced, prove the assertion.

The main step of this proof is to check that

$$\lim_{t \to \infty} y(t, \omega, x, J) = 0 \quad \text{uniformly in } (\omega, x) \in K. \tag{2.7}$$

This property will follow easily once we have proved that $O \subseteq K \times \{0\}$, where

$$\begin{aligned} O = \Big\{(\omega, x, v) \in K \times BU \;\Big|\; &\exists (t_n) \uparrow \infty \text{ and } \big((\omega_n, x_n)\big) \subset K \\ &\text{with } (\omega, x, v) = \lim_{n \to \infty}{}^{\mathrm{d}} \big(\tau(t_n, \omega_n, x_n), u_x(t_n, \omega_n, x_n)J\big)\Big\}. \end{aligned} \tag{2.8}$$

Here $\lim^{\mathrm{d}}$ means that the sequences $(u(t_n, \omega_n, x_n))$ and $(u_x(t_n, \omega_n, x_n)J)$ converge in BU^{d}. Note that, since D is relatively compact, O is a nonempty subset of $K \times BU$. Clearly, O is compact in $K \times BU^{\mathrm{d}}$. The boundedness of D and condition (C5) ensure that the restriction of the semiflow

$$\phi : \mathbb{R}^+ \times K \times BU \to K \times BU, \quad (t, \omega, x, v) \mapsto \big(\tau(t, \omega, x), u_x(t, \omega, x)v\big)$$

to O is continuous for the product metric (see Corollary 4.3 in [27]). In particular, O is positively ϕ-invariant. Besides, it admits a flow extension, since any one of its points admits a unique backward orbit. The uniqueness is due to the infinite delay, while the existence is checked as follows: a point $(\omega, x, v) \in O$ is the limit in the product metric of a sequence $(\phi(t_n, \omega_n, x_n, J))$ with $((\omega_n, x_n)) \subset K$ and $(t_n) \uparrow \infty$. Given $s > 0$ we consider the sequence $(\phi(t_n - s, \omega_n, x_n, J))$, assuming without restriction that $t_n - s > 0$ for every n. The compactness of K and the relatively compactness of D ensure the existence of a subsequence, say $(\phi(t_j - s, \omega_j, x_j, J))$, which converges in $\Omega \times BU^{\mathrm{d}} \times BU^{\mathrm{d}}$ to the point (ω_*, x_*, v_*). Then $\phi(s, \omega_*, x_*, v_*) = (\omega, x, v)$.

We reason by contradiction assuming that $O \nsubseteq K \times \{0\}$. The map

$$h : O \to \mathbb{R}, \quad (\omega, x, v) \mapsto \sup_{1 \leqslant j \leqslant m} \frac{v_j(0)}{x_j(0) - \widetilde{a}_j(\omega)}$$

is well defined (recall that $x(0)-\widetilde{a}(\omega)=x(0)-a(\omega)(0)\geqslant\delta J\gg 0$), nonnegative and continuous. Hence it reaches its maximum value $\widetilde{\alpha}$ at a point $(\widetilde{\omega},\widetilde{x},\widetilde{v})\in O$. Our contradiction hypothesis means that $\widetilde{\alpha}>0$. We assume without restriction that $\widetilde{\alpha}=\widetilde{v}_1(0)/(\widetilde{x}_1(0)-\widetilde{a}_1(\widetilde{\omega}))$. Now, for $j=1,\dots,m$, we take $\alpha_j(t)$ as the real number satisfying $y_j(t,\widetilde{\omega},\widetilde{x},\widetilde{v})=\alpha_j(t)(z_j(t,\widetilde{\omega},\widetilde{x})-\widetilde{a}_j(\widetilde{\omega}\cdot t))$. As seen before, α_j is defined for every $t\in\mathbb{R}$, and it is clearly a C^1 function. Note also that $\widetilde{\alpha}=\alpha_1(0)=\max\{\alpha_j(t)\mid 1\leqslant j\leqslant m,\ t\in\mathbb{R}\}$. However, as we are going to prove, $\alpha_1'(0)<0$, which gives the contradiction we search.

The differential equations (2.3) and (2.2) respectively satisfied by $y(t,\widetilde{\omega},\widetilde{x},\widetilde{v})$ and $z(t,\widetilde{\omega},\widetilde{x})$ show that $y_1'(0,\widetilde{\omega},\widetilde{x},\widetilde{v})=(F_x(\widetilde{\omega},\widetilde{x}))_1\widetilde{v}$ and $z_1'(0,\widetilde{\omega},\widetilde{x})=F_1(\widetilde{\omega},\widetilde{x})$. Therefore

$$\alpha_1'(0)\big(\widetilde{x}_1(0)-a_1(\widetilde{\omega})(0)\big)=\big(F_x(\widetilde{\omega},\widetilde{x})\big)_1\widetilde{v}-\widetilde{\alpha}\big(F_1(\widetilde{\omega},\widetilde{x})-\widetilde{a}_1'(\widetilde{\omega})\big). \tag{2.9}$$

The fact that $\widetilde{a}$ is a strong lower solution and the concavity condition (C7) provide

$$\begin{aligned}\widetilde{\alpha}\big(F_1(\widetilde{\omega},\widetilde{x})-\widetilde{a}_1'(\widetilde{\omega})\big)&>\widetilde{\alpha}\big(F_1(\widetilde{\omega},\widetilde{x})-F_1\big(\widetilde{\omega},a(\widetilde{\omega})\big)\big)\\&\geqslant\big(F_x(\widetilde{\omega},\widetilde{x})\big)_1\big(\widetilde{\alpha}\big(\widetilde{x}-a(\widetilde{\omega})\big)\big)\geqslant\big(F_x(\widetilde{\omega},\widetilde{x})\big)_1\widetilde{v}.\end{aligned} \tag{2.10}$$

To check the last inequality, note first that (C6) ensures that $(F_x(\widetilde{\omega},\widetilde{x}))_j w\geqslant 0$ whenever $w\geqslant 0$ and $w_j(0)=0$; and second that $\widetilde{w}=\widetilde{\alpha}(\widetilde{x}-a(\widetilde{\omega}))-\widetilde{v}$ satisfies

$$\begin{aligned}\widetilde{w}_1(0)&=\widetilde{\alpha}\big(\widetilde{x}_1(0)-\widetilde{a}_1(\widetilde{\omega})\big)-\widetilde{v}_1(0)=0,\\ \widetilde{w}_j(s)&=\widetilde{\alpha}\big(\widetilde{x}_j(s)-\widetilde{a}_j(\widetilde{\omega}\cdot s)\big)-\widetilde{v}_j(s)\geqslant\alpha_j(s)\big(z_j(s,\widetilde{\omega},\widetilde{x})-\widetilde{a}_j(\widetilde{\omega}\cdot s)\big)-\widetilde{v}_j(s)\\&=y_j(s,\widetilde{\omega},\widetilde{x},\widetilde{v})-\widetilde{v}_j(s)=0\quad\text{for every } s\in(-\infty,0]\text{ and } 1\leqslant j\leqslant m.\end{aligned}$$

Combining (2.9) and (2.10) we conclude that $\alpha_1'(0)<0$. Assertion (2.7) is proved.

Now we can complete the proof of the first two assertions. Let $k_1,k_2\in\mathbb{R}$ satisfy $k_1J\leqslant x\leqslant k_2J$ for every $(\omega,x)\in K$. Then, by Lemma 2.1, for $t>0$,

$$\begin{aligned}0&\leqslant z(t,\omega,k_2J)-z(t,\omega,x)\leqslant\big(u_x(t,\omega,x)(k_2J-x)\big)(0)\\&\leqslant\big(u_x(t,\omega,x)\big((k_2-k_1)J\big)\big)(0)=(k_2-k_1)y(t,\omega,x,J).\end{aligned}$$

The last term is bounded for every t as a consequence of (2.7). Consequently the monotonicity of the norm in $\mathbb{R}^m$ and the boundedness of $z(t,\omega,x)$ for $(\omega,x)\in K$ ensure that $z(t,\omega,k_2J)$ is bounded and hence defined for every $t>0$. Then, again by (2.7),

$$\lim_{t\to\infty}\big(z(t,\omega,k_2J)-z(t,\omega,x)\big)=0\quad\text{uniformly in } (\omega,x)\in K.$$

Given any $\varrho>0$ we take $t_*>0$ such that $\|z(t,\omega,k_2J)-z(t,\omega,x)\|\leqslant\varrho$ for every $t\geqslant t_*$ and every $(\omega,x)\in K$. We take now $(\omega,x_1),(\omega,x_2)\in K$ and fix $s\in(-\infty,0]$. Then $\|x_1(s)-x_2(s)\|=\|z(t_*,\omega\cdot(-t_*+s),u(-t_*+s,\omega\cdot(-t_*+s),x_1))-z(t_*,\omega\cdot(-t_*+s),u(-t_*+s,\omega\cdot(-t_*+s),x_2))\|\leqslant 2\varrho$. Hence $x_1=x_2$, from where we deduce that K is a metric copy of the base. The same argument precludes the existence of a set with properties (k1), (k2) and (k3) and different from K.

Let $e:\Omega\to BU$ be the map satisfying $K=\{(\omega,e(\omega))\mid\omega\in\Omega\}$. Take now $(\omega_0,x_0)\in\Omega\times BU$ with $x_0\geqslant a(\omega_0)$. We first prove that $z(t,\omega_0,x_0)$ is defined for every $t\in\mathbb{R}$: choose $k_3\in\mathbb{R}$ such that $x_0\leqslant k_3J$ and $e(\omega)\leqslant k_3J$ for every $\omega\in\Omega$, and recall that, as seen before, $z(t,\omega_0,k_3J)$ is defined for every $t\in\mathbb{R}$; then, if $t\geqslant 0$,

$$\widetilde{a}(\omega_0\cdot t)\leqslant z\big(t,\omega_0,a(\omega_0)\big)\leqslant z(t,\omega_0,x_0)\leqslant z(t,\omega_0,k_3J),$$

from where the assertion follows easily. Let K_1 be the omega limit set of (ω_0, x_0) in $\Omega \times BU^{\mathrm{d}}$, and take $(\omega, x) \in K_1$. Then $(\omega, x) = \lim^{\mathrm{d}}_{n\to\infty}(\omega_0 \cdot t_n, u(t_n, \omega_0, x_0))$ for a sequence $(t_n) \uparrow \infty$. By Remark 2.5(2) and Lemma 2.1,

$$a(\omega_0 \cdot t_n) \leqslant u\big(t_n, \omega_0, a(\omega_0)\big) \leqslant u(t_n, \omega_0, x_0),$$

and hence the continuity of a ensures that $a(\omega) \leqslant x$. Consequently, the set K_1 satisfies conditions (k1), (k2) and (k3). By the uniqueness before checked, $K_1 = K$. From here it follows easily the asymptotical convergence stated in the theorem, whose proof is hence complete. □

The strong character of the lower solution required in the previous theorem can be replaced by the strong concavity condition of the vector field F, as the next result shows.

Theorem 2.7. *Assume that conditions* (C1)–(C8) *hold and a lower solution* $\widetilde{a} : \Omega \to \mathbb{R}^m$ *exists. Assume also the existence of a subset* $K \subset \Omega \times BU$ *satisfying*

(k1) *K is compact in $\Omega \times BU^{\mathrm{d}}$,*
(k2) *K is positively τ-invariant and the restriction of the semiflow τ to K admits a flow extension,*
($\widetilde{\mathrm{k3}}$) *K is "strongly above a": $a(\omega) \ll x$ for any $(\omega, x) \in K$.*

Then K is a metric copy of the base and the unique set satisfying these properties.

In addition, all the semiorbits corresponding to initial data (ω, x) with $a(\omega) \ll x$ are globally defined and approach asymptotically K in $\Omega \times BU^{\mathrm{d}}$.

Proof. The proof of the first assertion is almost identical to the one of Theorem 2.6: checking the existence of $\delta > 0$ with $a(\omega) + \delta J \leqslant x$ for every $(\omega, x) \in K$ is easier, and the strict inequality in the chain of inequalities (2.10) is now the second instead of the first. The proof of the second assertion starts by taking $(\omega_0, x_0) \in \Omega \times BU$ with $x_0 \gg a(\omega_0)$ and is identical to the corresponding proof in Theorem 2.6 except for the way of checking that the omega limit set K_1 satisfies condition ($\widetilde{\mathrm{k3}}$). We look for $\lambda \in [0, 1)$ such that $x_0 \geqslant \lambda a(\omega_0) + (1 - \lambda)e(\omega_0)$. Then, the monotonicity and the concavity of the semiflow (see Lemma 2.1 and its proof) and Remark 2.5(2) allow us to ensure that for any $t > 0$,

$$u(t, \omega_0, x_0) \geqslant u\big(t, \omega_0, \lambda a(\omega_0) + (1 - \lambda)e(\omega_0)\big) \geqslant \lambda a(\omega_0 \cdot t) + (1 - \lambda)e(\omega_0 \cdot t)$$

and

$$u(t, \omega_0, x_0) - a(\omega_0 \cdot t) \geqslant (1 - \lambda)\big(e(\omega_0 \cdot t) - a(\omega_0 \cdot t)\big) \geqslant (1 - \lambda)\delta J.$$

Hence, the definition of K_1 and the continuity of a ensure that $a(\omega) + (1 - \lambda)\delta J \leqslant x$ for every $(\omega, x) \in K_1$, and ($\widetilde{\mathrm{k3}}$) is satisfied. □

Remark 2.8. One defines *upper solution* and *strong upper solution* in an analogous way. In fact, Theorems 2.6 and 2.7 can be symmetrically formulated and proved in the case of existence of an upper solution if the concavity conditions on F are replaced by their convex analogs.

3. Sublinear monotone differential equations with infinite delay

The purpose of this section is to analyze the conditions ensuring the existence of a unique (and asymptotically stable) copy of the base when the concavity hypotheses are replaced by some sublinearity properties. So that from now on we keep the hypotheses on the base flow and work with the family of equations

$$z'(t) = F(\omega \cdot t, z_t), \quad t \geqslant 0, \;\; \omega \in \Omega, \tag{3.1}$$

with the function $F : \Omega \times BU_+ \to \mathbb{R}^m$, $(\omega, x) \mapsto F(\omega, x)$ satisfying the following list of conditions. We denote $B_r^+ = B_r \cap BU_+$.

(S1) F is continuous on $\Omega \times BU_+$ (considering the norm topology on BU), and $F(\omega, 0) \geqslant 0$ for every $\omega \in \Omega$,
(S2) there exists the linear differential operator $F_x : \Omega \times \operatorname{Int} BU_+ \to \mathcal{L}(BU, \mathbb{R}^m)$ and it is continuous (considering the norm $\|\cdot\|_\infty$ in BU and the associated one in $\mathcal{L}(BU, \mathbb{R}^m)$),
(S3) for each $r > 0$, $F(\Omega \times B_r^+)$ is a bounded subset of $\mathbb{R}^m$ and $F_x(\Omega \times \operatorname{Int} B_r^+)$ is a bounded subset of $\mathcal{L}(BU, \mathbb{R}^m)$,
(S4) for each $r > 0$, the function $\Omega \times B_r^{+\mathrm{d}} \to \mathbb{R}^m$, $(\omega, x) \mapsto F(\omega, x)$ is continuous (i.e., if $\lim_{n\to\infty} \omega_n = \omega$ and $\lim^{\mathrm{d}}_{n\to\infty} x_n = x$ with $x_n, x \in B_r^+$, then $\lim_{n\to\infty} F(\omega_n, x_n) = F(\omega, x)$),
(S5) for each $r_1 > 0$ and $r_2 > 0$, the map $\Omega \times \operatorname{Int} B_{r_1}^{+\mathrm{d}} \times B_{r_2}^{\mathrm{d}} \to R^m$, $(\omega, x, v) \mapsto F_x(\omega, x)v$ is continuous (i.e., $\lim_{n\to\infty} F_x(\omega_n, x_n)v_n = F_x(\omega, x)v$ in the case that $\lim_{n\to\infty} \omega_n = \omega$, $\lim^{\mathrm{d}}_{n\to\infty} x_n = x$ and $\lim^{\mathrm{d}}_{n\to\infty} v_n = v$ with $x_n, x \in \operatorname{Int} B_{r_1}^+$ and $v_n, v \in B_{r_2}$),
(S6) quasimonotone condition: if $x_1, x_2 \in BU_+$ with $x_1 \leqslant x_2$ and $(x_1)_j(0) = (x_2)_j(0)$ holds for some $j \in \{1, \dots, m\}$, then $F_j(\omega, x_1) \leqslant F_j(\omega, x_2)$ for each $\omega \in \Omega$,
(S7) sublinearity condition: if $x \in BU_+$ with $x \gg 0$, then $F_x(\omega, x)x \leqslant F(\omega, x)$ for each $\omega \in \Omega$ (which, since F is differentiable, is equivalent to $F(\omega, \lambda x) \geqslant \lambda F(\omega, x)$ for each $(\omega, x) \in \Omega \times BU_+$ and $\lambda \in [0, 1]$; see [8]),
(S8) strong sublinearity condition: if $x \in BU_+$ with $x \gg 0$, then $F_x(\omega, x)x \ll F(\omega, x)$ for each $\omega \in \Omega$.

We also keep the notation established in the previous section. As there, conditions (S1)–(S6) ensure the local existence and monotonicity of $u(t, \omega, x)$ for $(\omega, x) \in \Omega \times BU_+$ and $t \in [0, \alpha)$, with $\alpha = \alpha(\omega, x)$. That is, the family of Eqs. (3.1) induces a local skew-product semiflow

$$\tau : \mathbb{R}^+ \times \Omega \times BU_+ \to \Omega \times BU_+, \quad (t, \omega, x) \mapsto \big(\omega \cdot t, u(t, \omega, x)\big). \tag{3.2}$$

The conclusions deduced from (C1)–(C6) concerning the existence and characteristics of omega limit sets for bounded trajectories and the properties of the solutions of the corresponding variational equations (2.3), also hold in this case. In addition,

Lemma 3.1. *Assume that conditions* (S1)–(S7) *on F hold. Then the semiflow τ is sublinear; that is, if $(\omega, x) \in \Omega \times BU_+$, it holds that $u_x(t, \omega, x)x \leqslant u(t, \omega, x)$ for those values of $t \geqslant 0$ for which both terms are defined.*

Proof. Having in mind that $F(\omega, \lambda x) \geqslant \lambda F(\omega, x)$ if $\lambda \in [0, 1]$, a standard argument of comparison of solutions provides

$$u(t, \omega, \lambda x) \geqslant \lambda u(t, \omega, x) \tag{3.3}$$

for $\omega \in \Omega$, $x \gg 0$ and $\lambda \in [0, 1]$ for those $t \geqslant 0$ for which both functions are defined. Since u is C^1 in x, (3.3) holds for $x \geqslant 0$ and is equivalent to the assertion. □

The fact that $F(\omega, 0) \geqslant 0$ ensures that the constant function $a \equiv 0$ defines a subequilibrium for τ (defined as in the previous section) if $u(t, \omega, 0)$ is globally defined for every $\omega \in \Omega$, which in particular happens if there exists a globally defined positive semiorbit. In this sense, the result proved in the following theorem is the version of Theorem 2.7 for the strongly sublinear setting.

Theorem 3.2. *Assume that conditions* (S1)–(S8) *hold. Assume also the existence of a subset $K \subset \Omega \times BU_+$ satisfying*

(k1) *K is compact in $\Omega \times BU_+^{\mathrm{d}}$,*
(k2) *K is positively τ-invariant and the restriction of the semiflow τ to K admits a flow extension,*
($\widehat{\text{k3}}$) *$x \gg 0$ for every $(\omega, x) \in K$.*

Then K is a metric copy of the base and the unique set satisfying these properties.

In addition, all the semiorbits corresponding to initial data (ω, x) with $x \gg 0$ are globally defined and approach asymptotically K in $\Omega \times BU_+^{\mathrm{d}}$.

Proof. The beginning of the proof follows the scheme of the one of Theorem 2.6. Reasoning as there, we show the existence of $r > 0$ such that $K \subset \Omega \times B_r^+$, while the compactness of K, the continuity of the (strongly positive) map $K \to \mathbb{R}^m$, $(\omega, x) \mapsto x(0)$ for the product metric on K, and the flow extension in K ensure the existence of $\delta > 0$ with $x \geqslant \delta J$ for every $(\omega, x) \in K$. To check the next step, that is, the relative compactness in BU_+^{d} of the family D defined by (2.6), the only modification refers to its uniform boundedness: since, by Lemmas 2.1 and 3.1,

$$0 \leqslant u_x(t, \omega, x)(\delta J) \leqslant u_x(t, \omega, x)x \leqslant u(t, \omega, x)$$

for every $(\omega, x) \in K$ and $t \geqslant 0$, we obtain $\|u_x(t, \omega, x)J\|_\infty \leqslant r/\delta$. Finally, to prove assertion (2.7), that is, $\lim_{t\to\infty} y(t, \omega, x, J) = 0$ uniformly in $(\omega, x) \in K$, we repeat everything for $a \equiv 0$ excepting the analogue of (2.10), which now becomes

$$\widetilde{\alpha} F_1(\widetilde{\omega}, \widetilde{x}) > \widetilde{\alpha}\big(F_x(\widetilde{\omega}, \widetilde{x})\big)_1 \widetilde{x} \geqslant \big(F_x(\widetilde{\omega}, \widetilde{x})\big)_1 \widetilde{v}.$$

Here we use (S8) for the first inequality and (S6) for the second one.

Once obtained these fundamental preliminary results, the rest of the proof requires some additional work. Given any point $(\widetilde{\omega}, \widetilde{x}) \in \Omega \times BU_+$ with $\widetilde{x} \gg 0$, we choose $(\widetilde{\omega}, x) \in K$ and take $0 < \lambda < 1$ with $\lambda x \leqslant \widetilde{x} \leqslant \lambda^{-1} x$. The sublinearity and monotonicity properties of τ (see also the proof of Lemma 3.1) and the lower and upper bounds for K ensure that

$$\begin{aligned} \delta\lambda J &\leqslant \lambda u(t, \widetilde{\omega}, x) \leqslant u(t, \widetilde{\omega}, \lambda x) \leqslant u(t, \widetilde{\omega}, \widetilde{x}) \\ &\leqslant u\big(t, \widetilde{\omega}, \lambda^{-1} x\big) \leqslant \lambda^{-1} u(t, \widetilde{\omega}, x) \leqslant r\lambda^{-1} J. \end{aligned}$$

Consequently, the semiorbit of $(\widetilde{\omega}, \widetilde{x})$ is globally defined, and its omega limit set satisfies conditions (k1), (k2) and $(\widehat{\mathrm{k3}})$.

Let us now define

$$\widetilde{K} = \big\{\big(\omega, \lambda x_1 + (1-\lambda)x_2\big) \in \Omega \times BU_+ \;\big|\; (\omega, x_1), (\omega, x_2) \in K \text{ and } \lambda \in [0, 1]\big\},$$

which is clearly a new compact subset of $\Omega \times BU_+^{\mathrm{d}}$ satisfying $\delta J \leqslant x \leqslant r J$ for every $(\omega, x) \in \widetilde{K}$, and

$$\widetilde{D} = \big\{u(t, \omega, x) \;\big|\; t \geqslant 0 \text{ and } (\omega, x) \in \widetilde{K}\big\} \subset BU.$$

Since, as checked before, there exist $\widetilde{\delta} > 0$ and $\widetilde{r} > 0$ such that $\widetilde{\delta} J \leqslant u(t, \omega, \delta J) \leqslant u(t, \omega, r J) \leqslant \widetilde{r} J$ for every $t \geqslant 0$, we deduce from the monotonicity of τ that $\widetilde{\delta} J \leqslant x \leqslant \widetilde{r} J$ for every $x \in \widetilde{D}$. The monotonicity of the norm ensures that the family $\widetilde{D}$ is uniformly bounded. In addition, it is equicontinuous at every compact subinterval $[l, 0] \subset (-\infty, 0]$. This follows from the equicontinuity of $\widetilde{K}$ in such intervals, in turn deduced from its compactness in $\Omega \times BU^{\mathrm{d}}$, and from condition (S3) on the vector field F. Since the metric closure of $\widetilde{D}$ remains in BU, Arzelà–Ascoli theorem shows that $\widetilde{D}$ is relatively compact in BU^{d}. (A more detailed proof of a similar property is done in Proposition 4.1 of [27].)

It follows easily from the relative compactness of $\widetilde{D}$ that the set

$$\widetilde{O} = \Big\{(\omega, x) \in \Omega \times BU_+ \;\Big|\; \exists (t_n) \uparrow \infty \text{ and } \big((\omega_n, x_n)\big) \subset \widetilde{K} \text{ with } (\omega, x) = \lim_{n\to\infty}{}^{\mathrm{d}} \big(\omega_n \cdot t_n, u(t_n, \omega_n, x_n)\big)\Big\}$$

is a compact subset of $\Omega \times BU_+^{\mathrm{d}}$, with $\widetilde{\delta} J \leqslant u(t, \omega, x) \leqslant \widetilde{r} J$ for every $(\omega, x) \in \widetilde{O}$ and $t \geqslant 0$. Corollary 4.3 of [27] ensures that the restriction of τ to $\widetilde{O}$ is continuous for the product metric. Hence the set is positively τ-invariant. In addition, $\widetilde{O}$ admits a flow extension, as checked as the analogous property for the set O given by (2.8). Consequently, $\widetilde{O}$ satisfies (k1), (k2) and ($\widehat{\text{k3}}$), which as seen before implies that

$$\lim_{t\to\infty} y(t,\omega,x,J)=0 \quad \text{uniformly in } (\omega,x)\in\widetilde{O}. \tag{3.4}$$

This property will be fundamental to prove the following one, from which the statements of the theorem will be easily deduced: given any $\varepsilon>0$ there exists $t_\varepsilon>0$ such that

$$\mathsf{d}\big(u(t_\varepsilon,\omega,x_1),u(t_\varepsilon,\omega,x_2)\big)\leqslant\varepsilon \quad \text{for every } (\omega,x_1),(\omega,x_2)\in K. \tag{3.5}$$

In turn, (3.5) requires some previous work. We fix a constant $c>0$ such that

$$-cJ\leqslant u_x\big(t,\omega,\lambda x_1+(1-\lambda)x_2\big)(x_1-x_2)\leqslant cJ \tag{3.6}$$

for every $(\omega,x_1),(\omega,x_2)\in K$, $\lambda\in[0,1]$ and $t\geqslant 0$. The existence of this constant follows from $(\delta-r)J\leqslant x_1-x_2\leqslant(r-\delta)J$ and from

$$0\leqslant u_x(t,\omega,x)\delta J\leqslant u_x(t,\omega,x)x\leqslant u(t,\omega,x)\leqslant u(t,\omega,rJ)\leqslant\widetilde{r}J \tag{3.7}$$

for $t\geqslant 0$ and $\omega\in\Omega$ if $\delta J\leqslant x\leqslant rJ$, which implies that $0\leqslant u_x(t,\omega,x)J\leqslant(\widetilde{r}/\delta)J$.

Now we fix $\varepsilon>0$ and choose $n_\varepsilon>0$ with

$$\mathsf{d}(x_1,x_2)\leqslant\frac{\varepsilon}{2}+\sup_{s\in[-n_\varepsilon,0]}\big\|x_1(s)-x_2(s)\big\| \tag{3.8}$$

for every pair of points $x_1,x_2\in BU$. Property (3.4) ensures the existence of $\widetilde{t}_\varepsilon>0$ such that $\|y(t,\omega,x,J)\|<\varepsilon/(2c)$ for every $(\omega,x)\in\widetilde{O}$ and $t\geqslant\widetilde{t}_\varepsilon$. Therefore, there exists a constant $\rho_\varepsilon>0$ such that

$$\big\|\big(u_x(t,\omega,x)J\big)(0)\big\|=\big\|y(t,\omega,x,J)\big\|<\frac{\varepsilon}{2c} \quad \text{for } t\in[\widetilde{t}_\varepsilon,\widetilde{t}_\varepsilon+n_\varepsilon] \tag{3.9}$$

when $\widetilde{\delta}J\leqslant x\leqslant\widetilde{r}J$ and $\bar{\mathsf{d}}((\omega,x),\widetilde{O})\leqslant\rho_\varepsilon$. The symbol $\bar{\mathsf{d}}$ represents the product distance in $\Omega\times BU$. The existence of ρ_ε follows from the compactness of $\widetilde{O}$ and from the following property of continuity in the product metric of the restriction of the cocycle u_x, which is proved in Proposition 4.2 of [27]: if for $(\omega,x)\in BU$ it is $(\omega,x)=\lim^{\mathsf{d}}_{n\to\infty}(\omega_n,x_n)$, with $(\omega_n,x_n)\subset\Omega\times B_{r_*}$, and there is $t_0>0$ with $\|y(t,\omega_n,x_n,J)\|\leqslant r_*$ for every $t\in[0,t_0]$ and $n\in\mathbb{N}$, then $u_x(t,\omega,x)J=\lim^{\mathsf{d}}_{n\to\infty}u_x(t,\omega_n,x_n)J$. (A common bound $r_*\geqslant\widetilde{r}$ for $\|u_x(t,\omega,x)J\|_\infty$ when $t\geqslant 0$, $\omega\in\Omega$ and $\widetilde{\delta}J\leqslant x\leqslant\widetilde{r}J$ is obtained by repeating the argument used in (3.7).) And finally, there exists $\bar{t}_\varepsilon>0$ such that $\bar{\mathsf{d}}(\tau(t,\omega,x),\widetilde{O})<\rho_\varepsilon$ for every $(\omega,x)\in\widetilde{K}$ whenever $t\geqslant\bar{t}_\varepsilon$, as immediately deduced by contradiction from the relative compactness of $\widetilde{D}$ and the definition of $\widetilde{O}$.

Let us now take $(\omega,x_1),(\omega,x_2)\in K$ and $t\in[\widetilde{t}_\varepsilon,\widetilde{t}_\varepsilon+n_\varepsilon]$. Then

$$u(t+\bar{t}_\varepsilon,\omega,x_1)(0)-u(t+\bar{t}_\varepsilon,\omega,x_2)(0)=\int_0^1\big(u_x\big(t+\bar{t}_\varepsilon,\omega,\lambda x_1+(1-\lambda)x_2\big)(x_1-x_2)\big)(0)\,d\lambda.$$

Since, according to the cocycle property (2.4) for u_x,

$$\begin{aligned}&u_x\big(t+\bar{t}_\varepsilon,\omega,\lambda x_1+(1-\lambda)x_2\big)(x_1-x_2)\\&\quad=u_x\big(t,\tau\big(\bar{t}_\varepsilon,\omega,\lambda x_1+(1-\lambda)x_2\big)\big)\big(u_x\big(\bar{t}_\varepsilon,\omega,\lambda x_1+(1-\lambda)x_2\big)(x_1-x_2)\big)\end{aligned}$$

and (3.6) holds, we deduce from the monotonicity of u_x ensured by Lemma 2.1, the choices of $\bar{t}_\varepsilon$ and ρ_ε and (3.9) that

$$\left\|u(t+\bar{t}_\varepsilon,\omega,x_1)(0)-u(t+\bar{t}_\varepsilon,\omega,x_2)(0)\right\| \leqslant c\int_0^1 \left\|\left(u_x\left(t,\tau\left(\bar{t}_\varepsilon,\omega,\lambda x_1+(1-\lambda)x_2\right)\right)J\right)(0)\right\|d\lambda \leqslant \frac{\varepsilon}{2}.$$

This and (3.8) show that (3.5) holds for $t_\varepsilon = n_\varepsilon + \widetilde{t}_\varepsilon + \bar{t}_\varepsilon$.

We can complete the proof of the theorem. To check that K is a copy of the base, i.e., that each one of its sections reduces to a point, we take $\varepsilon > 0$ and write (ω, x_1) and (ω, x_2) in K as $\tau(t_\varepsilon, \omega\cdot(-t_\varepsilon), u(-t_\varepsilon,\omega,x_1))$ and $\tau(t_\varepsilon,\omega\cdot(-t_\varepsilon),u(-t_\varepsilon,\omega,x_2))$, with t_ε provided by (3.5), which hence shows that $\mathrm{d}(x_1,x_2) < \varepsilon$ and therefore that $x_1 = x_2$. To check that K is the unique set satisfying (k1), (k2) and $(\widehat{\mathrm{k3}})$ note that the union of two of those sets also satisfies the three properties, and hence it is a copy of the base. Finally, as seen before, the omega limit set of the semiorbit starting at any (ω, x) with $x \gg 0$ satisfies (k1), (k2) and $(\widehat{\mathrm{k3}})$, and hence it agrees with K. □

Remarks 3.3. (1) Assuming that the initial vector field F satisfies hypotheses (C1)–(C6) and (C8), it is possible to determine regularity conditions on a lower solution $\widetilde{a}$ ensuring that the new vector field $\widetilde{F}(\omega,x) = F(\omega, x + a(\omega)) - \widetilde{a}'(\omega)$ satisfies properties (S1)–(S6) and (S8). In this sense Theorem 3.2 weakens the conditions of Theorem 2.7 in those situations for which such a lower solution is *a priori* known.

(2) There are well-known examples of sublinear vector fields admitting an infinite number of minimal sets for which $\widetilde{a} \equiv 0$ is a strong lower solution. This means that Theorem 2.6 does not have an analogue in the sublinear setting.

4. A nonautonomous stage-structured population growth model

The results previously obtained allow us to establish the existence of a unique positive attracting recurrent state for a nonautonomous model describing a stage-structured population growth.

As explained in the introduction, our model is a nonautonomous version of the one described by Wu, Freedman and Miller in [38], which in turn generalizes the previous models of Aiello and Freedman [1], Freedman and Wu [12], and Aiello, Freedman and Wu [2]. The equations we will work with are hence time-dependent versions of those appearing in [38]. However, the way in which they are obtained presents some additional points of difficulty in our nonautonomous framework. For this reason we explain with some detail the ideas taking the equations initially obtained for the model to a form in which our results can be applied. We slightly modify the arguments of the mentioned authors.

Let m be the number of patches, and represent by $I_j(t)$ and $M_j(t)$ the number of immature and mature individuals in the j patch for $j = 1,\dots,m$. We make the following assumptions:

- the birth rate of the immature population in each patch is proportional to the number of mature individuals, $\widetilde{\alpha}_j(t)$ being the proportionality value in time t;
- the death rate of the immature population in each patch is proportional to the number of immature individuals, $\widetilde{\beta}_j(t)$ being the proportionality value in time t;
- the death rate of the mature population in each patch is of logistic nature: proportional to the square of the number of mature individuals, $\widetilde{\gamma}_j(t)$ being the proportionality value in time t;
- the net exchange rates of mature and immature populations from the k patch to the j patch are proportional to the differences $M_k - M_j$ and $I_k - I_j$, $\widetilde{\epsilon}_{jk}(t)$ and $\widetilde{\eta}_{jk}(t)$ being the proportionality values in time t;
- the probability distribution of the maturation period in all the patches is given by a positive and normalized Borel measure μ on $[0,\infty)$. This means that an individual has matured after a period t of its life with probability $\mu[0,t]$.

The evolution equations then take the form

$$M_j'(t) = -\widetilde{\gamma}_j(t)M_j^2(t) + \sum_{k\neq j} \widetilde{\epsilon}_{jk}(t)\big(M_k(t) - M_j(t)\big) + p_j(t),$$

$$I_j'(t) = -\widetilde{\beta}_j(t)I_j(t) + \sum_{k\neq j} \widetilde{\eta}_{jk}(t)\big(I_k(t) - I_j(t)\big) + \widetilde{\alpha}_j(t)M_j(t) - p_j(t),$$

$p_j(t)$ representing the maturation rate in the j patch. This model includes the fixed maturation period case, in which μ is the Dirac measure concentrated in the maturating time t^*.

We also assume all the functions $\widetilde{\alpha}_j, \widetilde{\beta}_j, \widetilde{\gamma}_j, \widetilde{\eta}_{jk}, \widetilde{\epsilon}_{jk} : \mathbb{R} \to \mathbb{R}$ to be bounded and uniformly continuous, that there exists $\delta > 0$ with $\widetilde{\alpha}_j > \delta$, $\widetilde{\beta}_j > \delta$, $\widetilde{\gamma}_j > \delta$, and that $\widetilde{\eta}_{jk} \geqslant 0$ and $\widetilde{\epsilon}_{jk} \geqslant 0$. Moreover, we assume that they are recurrent: if Ω is the common hull for all these functions, then the translation flow σ on Ω is minimal. This is the case if, for instance, these coefficient functions are almost periodic or almost automorphic. We represent by $\alpha_j, \beta_j, \gamma_j, \eta_{jk}, \epsilon_{jk} : \Omega \to \mathbb{R}$ the corresponding (continuous) operators of evaluation in time 0. In this way we obtain a $2m$-dimensional system of evolution equations for each element $\omega \in \Omega$, namely

$$M_j'(t) = -\gamma_j(\omega\cdot t)M_j^2(t) + \sum_{k\neq j} \epsilon_{jk}(\omega\cdot t)\big(M_k(t) - M_j(t)\big) + p_j(t), \tag{4.1}$$

$$I_j'(t) = -\beta_j(\omega\cdot t)I_j(t) + \sum_{k\neq j} \eta_{jk}(\omega\cdot t)\big(I_k(t) - I_j(t)\big) + \alpha_j(\omega\cdot t)M_j(t) - p_j(t). \tag{4.2}$$

Note that the initial system is one of the previous ones: it corresponds to the initial vector function $\omega_* \in \Omega$ with components $\widetilde{\alpha}_1, \dots, \widetilde{\epsilon}_{d,d-1}$.

In what follows we fix an element $\omega \in \Omega$. Our next purpose is to obtain a representation for $p_j(t)$ suitable to apply our results to the rewritten equations. Note first that

$$p_j(t) = \frac{d}{dh} \int_{-\infty}^{t} y_j(t,s,h)\, d\mu(t-s)\bigg|_{h=0^+}, \tag{4.3}$$

$y_j(t,s,h)$ being the number of immature individuals living in time $t > s$ in the j patch who were born at any of the patches in the interval of time $[s-h, s]$ for $h > 0$. This is a consequence of the fact that the number of maturating individuals in the j patch in the period $[t-h, t]$ is precisely $\int_{-\infty}^{t} y_j(t,s,h)\, d\mu(t-s)$: the integral, for $s \in (-\infty, t]$, of those immature individuals who were born in the period $[s-h, s]$ with the maturation probability corresponding to the time $t-s$.

The definition of $y_j(t,s,h)$ shows that if $h > 0$ is small enough (so that we can ignore the migrations and the deaths), then

$$y_j(s,s,h) = \int_{s-h}^{s} \alpha_j(\omega\cdot r)M_j(r)\, dr. \tag{4.4}$$

In addition, since $y_j(t,s,h)$ only makes sense if the maturation time of those individuals is longer than $t-s$,

$$\frac{d}{dt} y_j(t,s,h) = -\beta_j(\omega\cdot t)y_j(t,s,h) + \sum_{k\neq j} \eta_{jk}(\omega\cdot t)\big(y_k(t,s,h) - y_j(t,s,h)\big).$$

We write the previous m linear ODEs in system form,

$$\frac{d}{dt}\begin{bmatrix} y_1(t,s,h) \\ \vdots \\ y_m(t,s,h) \end{bmatrix} = A(\omega\cdot t)\begin{bmatrix} y_1(t,s,h) \\ \vdots \\ y_m(t,s,h) \end{bmatrix}, \tag{4.5}$$

the entries of the matrix $A(\omega)=[a_{jk}(\omega)]$ being $a_{jk}(\omega)=\eta_{jk}(\omega)$ for $j\neq k$ and $a_{jj}(\omega)=-\beta_j(\omega)-\sum_{k\neq j}\eta_{jk}(\omega)$. Note that the matrix $A(\omega\cdot t)$ is negatively diagonally dominant by rows for every $t\in\mathbb{R}$, and hence a hyperbolic matrix for which the stable bundle at $+\infty$ is $\Omega\times\mathbb{R}^m$ (see Fink [10] and Sacker and Sell [29]). In addition, since the nondiagonal entries of the matrix A are nonnegative, the linear system (4.5) is cooperative and the induced flow on $\Omega\times\mathbb{R}^m$ is monotone (see Smith [34]). Let $U_\omega(t)$ be the fundamental matrix solution of the linear system $y'=A(\omega\cdot t)y$ with $U_\omega(0)=\mathrm{Id}_m$, which is defined for every $t\in\mathbb{R}$ and satisfies the linear cocycle property $U_\omega(t+s)=U_{\omega\cdot t}(s)U_\omega(t)$. Then, if $Y_\omega(t,s)=U_\omega(t)U_\omega^{-1}(s)$ for $t\geqslant s$, we have $(d/dt)Y_\omega(t,s)=A(\omega\cdot t)Y_\omega(t,s)$ and $Y_\omega(s,s)=\mathrm{Id}_m$ and, by (4.5) and (4.4),

$$\begin{bmatrix} y_1(t,s,h) \\ \vdots \\ y_m(t,s,h) \end{bmatrix} = Y_\omega(t,s)\begin{bmatrix} y_1(s,s,h) \\ \vdots \\ y_m(s,s,h) \end{bmatrix} = Y_\omega(t,s)\begin{bmatrix} \int_{s-h}^{s}\alpha_1(\omega\cdot r)M_1(r)\,dr \\ \vdots \\ \int_{s-h}^{s}\alpha_m(\omega\cdot r)M_m(r)\,dr \end{bmatrix}.$$

In addition, since $Y_\omega(t,t+s)=U_{\omega\cdot t}^{-1}(s)$ for every $s\leqslant 0$,

$$\begin{bmatrix} y_1(t,t+s,h) \\ \vdots \\ y_m(t,t+s,h) \end{bmatrix} = U_{\omega\cdot t}^{-1}(s)\begin{bmatrix} \int_{t+s-h}^{t+s}\alpha_1(\omega\cdot r)M_1(r)\,dr \\ \vdots \\ \int_{t+s-h}^{t+s}\alpha_m(\omega\cdot r)M_m(r)\,dr \end{bmatrix}.$$

We write $U_\omega^{-1}(s)=[u_{jk}(\omega,s)]$. The following remarks are fundamental in what follows. Note first that the entries of this matrix $U_\omega^{-1}(s)$ satisfy

$$u_{jk}(\omega,s)\geqslant 0 \quad\text{and}\quad u_{jj}(\omega,s)>0 \tag{4.6}$$

for $s\leqslant 0$. This follows from the conditions $U_\omega^{-1}(s)=Y_\omega(0,s)$ and $Y_\omega(s,s)=\mathrm{Id}_m$ and from the monotonicity and the componentwise separating property of cooperative systems of linear ODEs like $y'=A(\omega\cdot s)y$ (see Smith [34] and Shen and Zhao [33]). In addition, due to the hyperbolic character of the matrix A before mentioned, it turns out (see again [29]) that there exist constants $k\geqslant 1$ and $\varrho>0$ with

$$\left\|U_\omega(t)U_\omega^{-1}(s)\right\|\leqslant ke^{-\varrho(t-s)} \tag{4.7}$$

for every $\omega\in\Omega$ and $t\geqslant s$ (where we consider the matrix norm associated to the maximum norm in $\mathbb{R}^m$), which in particular means that $\lim_{s\to-\infty}U_\omega^{-1}(s)=0$ exponentially uniformly in Ω. Coming back to our equations, note that

$$\begin{aligned}\int_{-\infty}^{t} y_j(t,s,h)\,d\mu(t-s) &= \int_{-\infty}^{0} y_j(t,t+s,h)\,d\mu(-s)\\ &= \sum_{k=1}^{m}\int_{-\infty}^{0} u_{jk}(\omega\cdot t,s)\left(\int_{t+s-h}^{t+s}\alpha_k(\omega\cdot r)M_k(r)\,dr\right)d\mu(-s),\end{aligned}$$

so that the integral is defined as long as the functions $M_k : (-\infty, t] \to \mathbb{R}$ are bounded, as deduced from inequality (4.7) for $t = 0$. Consequently, relation (4.3) shows that the last term in Eqs. (4.1) and (4.2) corresponding to the j patch can be written as

$$p_j(t) = \sum_{k=1}^{m} \int_{-\infty}^{0} u_{jk}(\omega \cdot t, s)\alpha_k\big(\omega \cdot (t+s)\big)M_k(t+s)\, d\mu(-s). \tag{4.8}$$

Once obtained the expression of $p_j(t)$, we can explicitly rewrite the evolution equations. Let us denote $M = [M_1, \dots, M_m]^T$ and $I = [I_1, \dots, I_m]^T$, and consider them as elements of BU and $\mathbb{R}^m$ respectively. We define $H_j : \Omega \times BU \to \mathbb{R}$, $F_j : \Omega \times BU \to \mathbb{R}$ and $G_j : \Omega \times \mathbb{R}^m \times BU \to \mathbb{R}$ by

$$H_j(\omega, M) = \sum_{k=1}^{m} \int_{-\infty}^{0} u_{jk}(\omega, s)\alpha_k(\omega \cdot s)M_k(s)\, d\mu(-s),$$

$$F_j(\omega, M) = -\gamma_j(\omega)M_j^2(0) + \sum_{k \neq j} \epsilon_{jk}(\omega)\big(M_k(0) - M_j(0)\big) + H_j(\omega, M),$$

$$G_j(\omega, I, M) = -\beta_j(\omega)I_j + \sum_{k \neq j} \eta_{jk}(\omega)(I_k - I_j) + \alpha_j(\omega)M_j(0) - H_j(\omega, M)$$

and, finally, we represent $F = [F_1, \dots, F_m]^T$ and $G = [G_1, \dots, G_m]^T$. Then Eqs. (4.1) and (4.2) for the fixed element ω can be reformulated as

$$M'(t) = F(\omega \cdot t, M_t), \tag{4.9}$$

$$I'(t) = G\big(\omega \cdot t, I(t), M_t\big), \tag{4.10}$$

these expressions describing simultaneously the evolution of the populations in all the patches. In the fixed maturation period case the equation we obtain is of fixed finite delay type. We point out that the symmetry conditions $\varepsilon_{jk} = \varepsilon_{kj}$ and $\eta_{jk} = \eta_{kj}$ are not necessary in what follows, although they are logical properties for the model.

Now we let ω vary in Ω. Note that the family of Eqs. (4.9) does not depend on the immature population. So that in order to establish the existence of a global nonautonomous equilibrium for the mature and immature populations we begin by analyzing the mature one.

We consider (4.9) as a family of equations of type (2.2). The following result shows that it defines a global semiflow on $\Omega \times BU_+$. Note that only the elements of the positive cone BU_+ represent possible populations.

Proposition 4.1. *The function F satisfies $F(\omega, 0) \geqslant 0$ and all the hypotheses* (C1)–(C8)*, and the family of Eqs.* (4.9) *defines a monotone and concave local semiflow on $\Omega \times BU$ and a monotone and concave global semiflow on $\Omega \times BU_+$.*

Proof. We omit the proof of the first assertion (which in particular means that F also satisfies (S1)–(S8)). The monotonicity and concavity of the semiflow are guaranteed by Lemma 2.1. Finally, the global character of the restriction to $\Omega \times BU_+$ follows from the boundedness of any semiorbit, which in turn is deduced again from the monotonicity, having in mind that kJ is an upper solution when k is large enough to ensure that $F(\omega, kJ) \leqslant 0$. □

Our next result, Theorem 4.2, proves the existence of a unique nonautonomous equilibrium (see Remark 2.3(2)) for the corresponding semiflow which is strongly positive and which attracts asymptotically any semiorbit starting at a strongly positive initial mature population. We point out that, although we apply Theorem 2.6 to prove this result, it could also be obtained as a consequence of Theorems 2.7 or 3.2. In fact, the results obtained in Section 3 provide a description of the dynamics of population models similar to the one we are considering but for which the death rates in the different patches of the mature population are given by suitable strongly sublinear functions.

Theorem 4.2. *There exists a unique metric copy of the base for the semiflow defined by* (4.9) *on* $\Omega \times BU_+$, $K_M = \{(\omega, M_*(\omega)) \mid \omega \in \Omega\}$ *with* $M_*(\omega) \gg 0$, *such that the semiorbit starting at any point* (ω, x) *with* $x \gg 0$ *approaches asymptotically* K_M *in* $\Omega \times BU^{\mathrm{d}}$ *as* $t \to \infty$.

Proof. Note to begin that, for $j = 1, \dots, m$ and $\omega \in \Omega$,

$$\sum_{k=1}^{m} \int_{-\infty}^{0} u_{jk}(\omega, s)\alpha_k(\omega \cdot s)\, d\mu(-s) > \eta > 0,$$

since this expression defines a function which is continuous in ω, inequalities (4.6) hold, and the functions $\alpha_1, \dots, \alpha_m$ are strictly positive (recall that $\int_{-\infty}^{0} d\mu(-s) = 1$). Having in mind that $\gamma_j > 0$ for $j = 1, \dots, m$, we deduce the existence of $\varepsilon > 0$ small enough and $k > 0$ large enough such that $F(\omega, \varepsilon J) \gg 0$ and $F(\omega, kJ) \ll 0$. It follows easily (see Remark 2.5(2)) that $\varepsilon J \leqslant u(t, \omega, \varepsilon J) \leqslant u(t, \omega, kJ) \leqslant kJ$ for every $t \geqslant 0$. Let $K_M \subset \Omega \times BU$ be the omega limit set of the semiorbit starting at (ω, kJ). Then $\varepsilon J \leqslant x \leqslant kJ$ for every $(\omega, x) \in K_M$, and hence K_M satisfies (k1), (k2) and (k3) of Theorem 2.6 for the strong lower solution $\widetilde{a} : \Omega \to \mathbb{R}^m$, $\omega \mapsto \varepsilon J$. Proposition 4.1 and these facts prove our statement. □

Let us now analyze the situation for the immature population. As explained in Remark 2.3(1), $(M_*(\omega))_t(s) = M_*(\omega)(t+s) = M_*(\omega \cdot t)(s)$ for every $t \in \mathbb{R}$ and $s \in (-\infty, 0]$. Substituting now the variable M by the function $M_*(\omega)$ in Eq. (4.10) we obtain the family of m-dimensional linear systems of ODEs

$$I'(t) = G\big(\omega \cdot t, I(t), M_*(\omega \cdot t)\big) = A(\omega \cdot t)I(t) + L(\omega \cdot t), \tag{4.11}$$

where $L : \Omega \to \mathbb{R}^m$ is the continuous function with components

$$L_j(\omega) = \alpha_j(\omega)M_*(\omega)_j(0) - H_j\big(\omega, M_*(\omega)\big)$$

for $j = 1, \dots, m$; that is, denoting $R(\omega) = \begin{bmatrix} \alpha_1(\omega)(M_*)_1(\omega)(0) \\ \vdots \\ \alpha_m(\omega)(M_*)_m(\omega)(0) \end{bmatrix}$, we have

$$L(\omega) = R(\omega) - \int_{-\infty}^{0} U_\omega^{-1}(s)R(\omega \cdot s)\, d\mu(-s),$$

and we obtain a linear equation that the immature population must satisfy when the mature one is in the equilibrium situation described by K_M. The continuity of α_j and M_* and condition (4.7) for $t = 0$ ensure that $L(\omega)$ is bounded and continuous in Ω.

Note that the family (4.11) of linear *ordinary* differential equations induces a global flow on $\Omega \times \mathbb{R}^m$. Clearly, in order to obtain a nonautonomous equilibrium for the whole (mature and immature) population we need to obtain a nonautonomous equilibrium in $\mathbb{R}^m$ (or a copy of the base) for Eq. (4.11). That is, the graph of a continuous function $\widetilde{I} : \Omega \to \mathbb{R}^m$ such that $\mathbb{R} \to \mathbb{R}^m$, $t \mapsto \widetilde{I}(\omega \cdot t)$ solves (4.11) for any $\omega \in \Omega$. This is the goal of the next result, which completes this section. Note also that a possible stable situation for the immature population, as in the mature case, only makes sense for the model if it corresponds to a nonnegative solution.

Theorem 4.3. *There exists a unique copy of the base for the flow defined on $\Omega \times \mathbb{R}^m$ by (4.11), $K_I = \{(\omega, \widetilde{I}(\omega)) \mid \omega \in \Omega\} \subset \Omega \times \mathbb{R}^m$, given by the strongly positive function*

$$\widetilde{I}(\omega) = \int_{-\infty}^{0} \left(\int_{s}^{0} U_\omega^{-1}(r) R(\omega \cdot r)\, dr \right) d\mu(-s).$$

In addition, any orbit approaches exponentially K_I in $\Omega \times \mathbb{R}^m$ as $t \to \infty$.

Proof. Condition (4.7) for $t = 0$ and the boundedness of $L(\omega)$ ensure that the function $\widetilde{I} : \Omega \to \mathbb{R}^m$ given by

$$\widetilde{I}(\omega) = \int_{-\infty}^{0} U_\omega^{-1}(l) L(\omega \cdot l)\, dl \tag{4.12}$$

is well defined and bounded. It is also easy to deduce that it is continuous on Ω. It is also well known that it provides a solution of (4.11) when evaluated along the corresponding base orbit; in other words, the map $\mathbb{R} \to \mathbb{R}^m$, $t \mapsto \widetilde{I}(\omega \cdot t) = \int_{-\infty}^{0} U_{\omega\cdot t}^{-1}(l) L((\omega \cdot t) \cdot l)\, dl = \int_{-\infty}^{t} U_\omega(t) U_\omega^{-1}(l) L(\omega \cdot l)\, dl$ satisfies the equation. This means that the compact set $K_I = \{(\omega, \widetilde{I}(\omega)) \mid \omega \in \Omega\} \subset \Omega \times \mathbb{R}^m$ is a copy of the base for the flow defined on $\Omega \times \mathbb{R}^m$ by (4.11). In addition, due to the hyperbolic character of the matrix A, any other solution of the equation approaches exponentially $\widetilde{I}$ in $\mathbb{R}^m$ (see Fink [10]); that is, there exist real constants $k > 0$, $\varrho > 0$ (the ones appearing in (4.7)) such that $\|\widetilde{I}(\omega \cdot t) - z_I(t, \omega, c)\| \leqslant k e^{-\varrho t} \|\widetilde{I}(\omega) - c\|$ for every $t \geqslant 0$, where $c \in \mathbb{R}^m$ and $z_I(t, \omega, c)$ represents the solution of (4.11) with $z_I(0, \omega, c) = c$. This shows that K_I attracts exponentially all the possible initial states in $\Omega \times \mathbb{R}^m$ as time increases, and hence it is the unique copy of the base.

Now we follow the idea of Theorem 3.3 of Freedman and Wu [12] in order to check that $\widetilde{I}$ has the expression stated, and hence it corresponds to a strongly positive immature population. Note to begin that, from the definitions of U and L,

$$\begin{aligned}
U_\omega^{-1}(l) L(\omega \cdot l) &= U_\omega^{-1}(l) \left(R(\omega \cdot l) - \int_{-\infty}^{0} U_{\omega\cdot l}^{-1}(s) R\big(\omega \cdot (l+s)\big)\, d\mu(-s) \right) \\
&= \int_{-\infty}^{0} \big(U_\omega^{-1}(l) R(\omega \cdot l) - U_\omega^{-1}(l+s) R\big(\omega \cdot (l+s)\big)\big)\, d\mu(-s) \\
&= \int_{-\infty}^{0} \left(\frac{d}{dl} \int_{l+s}^{l} U_\omega^{-1}(r) R(\omega \cdot r)\, dr \right) d\mu(-s).
\end{aligned}$$

Substituting in (4.12) and applying Fubini's theorem, we obtain

$$\widetilde{I}(\omega) = \int_{-\infty}^{0} \left(\int_{-\infty}^{0} \left(\frac{d}{dl} \int_{l+s}^{l} U_\omega^{-1}(r) R(\omega \cdot r)\, dr \right) dl \right) d\mu(-s).$$

Consequently,

$$\widetilde{I}(\omega) = \int_{-\infty}^{0} \left(\int_{s}^{0} U_\omega^{-1}(r) R(\omega \cdot r)\, dr - \lim_{l \to -\infty} \int_{l+s}^{l} U_\omega^{-1}(r) R(\omega \cdot r)\, dr \right) d\mu(-s)$$

$$= \int_{-\infty}^{0} \left(\int_{s}^{0} U_\omega^{-1}(r) R(\omega \cdot r)\, dr \right) d\mu(-s),$$

as asserted. The last equality follows easily from (4.7) for $t = 0$. The positiveness of $\widetilde{I}(\omega)$ follows then from the one of $R(\omega)$ and from (4.6). The proof is complete. □

5. A theorem on global exponential stability

The results of Section 4 show that the compact set $\{(\omega, M_*(\omega), \widetilde{I}(\omega)) \mid \omega \in \Omega\} \subset \Omega \times BU^{\mathrm{d}} \times \mathbb{R}^m$ is the unique nonautonomous equilibrium for the semiflow induced in $\Omega \times BU_+ \times \mathbb{R}^m$ by the family of $2m$-dimensional systems of equations composed by those of (4.9) and (4.10). The last section of the paper is devoted to obtain the optimal result concerning the attractivity properties of this copy of the base: not only does it attract asymptotically in $\Omega \times BU^{\mathrm{d}} \times \mathbb{R}^m$ any semiorbit starting at a strongly positive initial mature population, but in fact, the values in time t of the mature and immature populations approach exponentially their corresponding values in the nonautonomous equilibrium.

Throughout this section, for the semiflow given on $\Omega \times BU$ by the family of Eqs. (4.9) we use a notation similar to the one established in Section 2 for Eqs. (2.2): the semiflow is $\tau_M(t, \omega, x) = (\omega \cdot t, u(t, \omega, x))$, the solution in $\mathbb{R}^m$ of the equation is $z_M(t, \omega, x) = u(t, \omega, x)(0)$ (and hence $z_M(s, \omega, x) = x(s)$ for every $s \in (-\infty, 0]$), the linear differential operator with respect to x is $u_x(t, \omega, x) \in \mathcal{L}(BU, BU)$ (with $u_x(0, \omega, x)v = v$), and $y(t, \omega, x, v) = (u_x(t, \omega, x)v)(0)$ is the solution of the variational equation (2.3) satisfying $y(s, \omega, x, v) = v(s)$ for every $s \in (-\infty, 0]$. Recall that Theorem 4.2 proves that all these functions are defined for any $t > 0$ in the case that $x \gg 0$.

Let us define $M^\delta(\omega) = M_*(\omega) - \delta J$ for $\delta \geqslant 0$, where M_* is the continuous equilibrium obtained in Theorem 4.2. Our main tool to prove the exponential stability will be the analysis of the solutions of the linear systems

$$y'(t) = F_x\big(\omega \cdot t, M^\delta(\omega \cdot t)\big) y_t \tag{5.1}$$

obtained from (4.9), with j component given by

$$y_j'(t) = -2\gamma_j(\omega \cdot t) M_j^\delta(\omega)(t) y_j(t) + \sum_{k \neq j} \epsilon_{jk}(\omega \cdot t)\big(y_k(t) - y_j(t)\big) + H_j(\omega \cdot t, y_t). \tag{5.2}$$

Given $v \in BU$, we denote by $y^\delta(t, \omega, v)$ the value in t of the solution of (5.1) satisfying $y^\delta(s, \omega, v) = v(s)$ for $s \in (-\infty, 0]$, and by $w^\delta(t, \omega)v$ the element of BU given by $(w^\delta(t, \omega)v)(s) = y^\delta(t + s, \omega, v)$. Note that y^δ and w^δ are linear in v. Therefore

$$\phi^\delta : \mathbb{R}^+ \times \Omega \times BU \to \Omega \times BU, \quad (t, \omega, v) \mapsto \big(\omega \cdot t, w^\delta(t, \omega)v\big)$$

defines a linear skew-product semiflow, which is monotone since the coefficient function of (5.1) satisfies the quasimonotone condition (C6). Note also that $w^0(t, \omega)v = u_x(t, \omega, M_*(\omega))v$.

To complete the notation related to Eqs. (4.9) and (5.1), we define $\widetilde{M} : \Omega \to \mathbb{R}^m$, $\omega \mapsto M_*(\omega)(0)$ and $\widetilde{M}^\delta : \Omega \to \mathbb{R}^m$, $\omega \mapsto M^\delta(\omega)(0)$. Recall that $\widetilde{M}'(\omega \cdot t) = F(\omega \cdot t, M_*(\omega \cdot t))$ and note that $t \mapsto \widetilde{M}^\delta(\omega \cdot t) = \widetilde{M}(\omega \cdot t) - \delta J$ does not define a solution of (4.9) if $\delta \neq 0$. Finally we fix constants $\varepsilon_* > 0$ and $k_* > 0$ with

$$\varepsilon_* J \leqslant M_*(\omega) \leqslant k_* J \quad \text{for every } \omega \in \Omega. \tag{5.3}$$

Let us now consider the immature population. For $(\omega, x, c) \in \Omega \times BU \times \mathbb{R}^m$ given, we represent by $z_I(t, \omega, x, c)$ the solution of the ODE

$$I'(t) = G\big(\omega \cdot t, I(t), u(t, \omega, x)\big)$$

with $z_I(0, \omega, x, c) = c$. Note that, for $\omega \in \Omega$ fixed, this solution represents the immature population in time t when the initial values of the mature and immature populations are x and c respectively. As in Section 4, the former equation can be rewritten as

$$I'(t) = A(\omega \cdot t)I(t) + L\big(\tau_M(t, \omega, x)\big) \tag{5.4}$$

with

$$L_j(\omega, x) = \alpha_j(\omega)x_j(0) - \sum_{k=1}^{m} \int_{-\infty}^{0} u_{jk}(\omega, s)\alpha_k(\omega \cdot s)x_k(s)\, d\mu(-s).$$

In particular, $z_I(t, \omega, x, c)$ is defined whenever $z_M(t, \omega, x)$ is, which is for any $t \in \mathbb{R}$ in the case that $x \gg 0$. Recall that the continuous equilibrium (in $\mathbb{R}^m$) obtained in Theorem 4.3 for Eq. (5.4) corresponding to $(\omega, M_*(\omega))$ is represented by $\widetilde{I}(\omega)$, with $\widetilde{I}(\omega \cdot t) = z_I(t, \omega, M_*(\omega), \widetilde{I}(\omega))$ for $t \in \mathbb{R}$ and $\omega \in \Omega$.

The purpose of this section is to prove the following result.

Theorem 5.1. *For any $\varepsilon > 0$ there exist constants $\eta_\varepsilon > 1$ and $\rho > 0$ such that, if $x \geqslant \varepsilon J$, then*

(i) $\|\widetilde{M}(\omega \cdot t) - z_M(t, \omega, x)\| \leqslant \eta_\varepsilon e^{-\rho t}\|M_*(\omega) - x\|_\infty$,
(ii) $\|\widetilde{I}(\omega \cdot t) - z_I(t, \omega, x, c)\| \leqslant \eta_\varepsilon e^{-\rho t}(\|M_*(\omega) - x\|_\infty + \|\widetilde{I}(\omega) - c\|)$

for any $t \in \mathbb{R}$, $\omega \in \Omega$ and $c \in \mathbb{R}^m$.

This theorem will follow as a corollary of several results. The first one describes a basic and fundamental property of uniformity in the asymptotical approach to the set K_M.

Proposition 5.2. *Given $\delta > 0$ and $\varepsilon > 0$ with $\varepsilon J \leqslant M_*(\omega)$ for every $\omega \in \Omega$, there exists $t_0 = t_0(\delta, \varepsilon)$ such that $z_M(t, \omega, x) \geqslant \widetilde{M}^\delta(\omega \cdot t)$ for every $t \geqslant t_0$ and $(\omega, x) \in \Omega \times BU$ with $x \geqslant \varepsilon J$.*

Proof. The proof is basically a consequence of Theorem 2.6. As a first step, we prove the following uniformity property: given $\varrho > 0$ and $\varepsilon > 0$ there exists $t_1 = t_1(\varrho, \varepsilon)$ such that $\mathrm{d}(u(t, \omega, \varepsilon J), M_*(\omega \cdot t)) < \varrho$ for every $\omega \in \Omega$ and $t \geqslant t_1$. We can assume that ε is small enough to guarantee that $F(\omega, \varepsilon J) \gg 0$ (see the proof of Theorem 4.2). Remark 2.5(2) and the monotonicity of the semiflow then ensure that

$$\varepsilon J \leqslant u(t, \omega, \varepsilon J) \leqslant u\big(t, \omega, M_*(\omega)\big) = M_*(\omega \cdot t) \leqslant k_* J \tag{5.5}$$

for every $t \geqslant 0$ and $\omega \in \Omega$, with k_* satisfying (5.3). Consequently, any sequence $(u(t_n, \omega_n, \varepsilon J))$ is uniformly bounded. From here, equality $u(t_n, \omega_n, \varepsilon J)(s) = z_M(t_n + s, \omega_n, \varepsilon J)$, and condition (C3) on the coefficient function F of Eq. (4.9), we deduce that the sequence is also equicontinuous (see the proof of Theorem 2.6 for a similar argument). Arzelà–Ascoli theorem shows that any sequence has a subsequence which converges in metric, and it is easily checked that the limit remains in BU. Now we define

$$K = \Big\{(\omega, x) \in \Omega \times BU \;\Big|\; \exists (t_n) \uparrow \infty \text{ and } (\omega_n) \subset \Omega \text{ with } (\omega, x) = \lim_{n\to\infty}{}^{\mathrm{d}}\, \tau_M(t_n, \omega_n, \varepsilon J)\Big\}.$$

By using the existence of convergent subsequences one proves that K is compact in $\Omega \times BU^{\mathrm{d}}$, that it is positively τ_M-invariant, and that the semiflow restricted to it admits a flow extension. In addition, (5.5) shows that $x \geqslant \varepsilon J$ for every $(\omega, x) \in K$. This means that K satisfies all the conditions of Theorem 2.6 with respect to the strong lower solution $\widetilde{a} : \Omega \to \mathbb{R}^m$, $\omega \mapsto \varepsilon J$. Hence, by uniqueness, $K = K_M$.

We complete the proof of the mentioned property by contradiction. Assume the existence of sequences $(t_n) \uparrow \infty$ and $(\omega_n) \subset \Omega$ with $\mathrm{d}(u(t_n, \omega_n, \varepsilon J), M_*(\omega_n \cdot t_n)) \geqslant \varrho$. We can assume (by taking a new subsequence if needed) that $(\omega_n \cdot t_n)$ converges to a point $\omega^* \in \Omega$. And, as asserted before, there exists a subsequence of $(u(t_n, \omega_n, \varepsilon J))$ which converges in metric to a point x^*. So that we find a point (ω^*, x^*) which is in K but at a positive distance of K_M: $\mathrm{d}(x^*, M_*(\omega^*)) \geqslant \varrho$. And this is impossible since both sets agree.

Now, given $\delta > 0$ we define $\varrho = \delta/(2 + 2k_*)$ and $t_0(\delta, \varepsilon) = t_1(\varrho, \varepsilon)$. Then, for $t \geqslant t_0$ and $\omega \in \Omega$,

$$\begin{aligned} \varrho > \mathrm{d}\big(M_*(\omega \cdot t), u(t, \omega, \varepsilon J)\big) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|M_*(\omega \cdot t) - u(t, \omega, \varepsilon J)|_n}{1 + |M_*(\omega \cdot t) - u(t, \omega, \varepsilon J)|_n} \\ &\geqslant \frac{1}{2(1 + k_*)} \big|M_*(\omega \cdot t) - u(t, \omega, \varepsilon J)\big|_1 \geqslant \frac{1}{2 + 2k_*} \big\|\widetilde{M}(\omega \cdot t) - z_M(t, \omega, \varepsilon J)\big\|, \end{aligned}$$

and hence necessarily $z_M(t, \omega, \varepsilon J) \geqslant \widetilde{M}(\omega \cdot t) - \delta J$. The monotonicity of the semiflow guarantees the same property for $z_M(t, \omega, x)$ for $t \geqslant t_0$ if $x \geqslant \varepsilon J$. This completes the proof of the proposition. □

The next result, Proposition 5.3, shows that it makes sense to consider the semiflow induced by the family of Eqs. (5.1) on spaces which are larger than $\Omega \times BU$. Given $\varsigma > 0$, we define

$$C_\varsigma = \Big\{ v \in C\big((-\infty, 0], \mathbb{R}^m\big) \;\Big|\; \text{there exists } \lim_{s \to -\infty} \big\|v(s)\big\| e^{\varsigma s} \Big\},$$

a Banach space for the norm $\|v\|_\varsigma = \sup_{s \in (-\infty, 0]} \|v(s)\| e^{\varsigma s}$. On the *fading memory phase space* C_ς we consider the same pointwise partial order relation as in BU, defined by (2.1).

In order to find the values of ς for which (5.1) defines a semiflow on $\Omega \times C_\varsigma$, we recall relation (4.7), which provides $k > 1$ and $\varrho > 0$ such that, for $s \leqslant 0$,

$$\big\|U_\omega^{-1}(s)\big\| \leqslant k e^{\varrho s} \quad \text{and hence} \quad u_{jk}(\omega, s) \leqslant k e^{\varrho s}. \tag{5.6}$$

Proposition 5.3. *For $\delta \geqslant 0$ and $\varsigma \leqslant \varrho$, the family of Eqs.* (5.1) *defines a linear continuous semiflow in $\Omega \times C_\varsigma$, namely*

$$\phi_\varsigma^\delta : \mathbb{R}^+ \times \Omega \times C_\varsigma \to \Omega \times C_\varsigma, \quad (t, \omega, v) \mapsto \big(\omega \cdot t, w^\delta(t, \omega) v\big),$$

which is monotone: *if $v_1 \leqslant v_2$ in C_ς then $w^\delta(t, \omega) v_1 \leqslant w^\delta(t, \omega) v_2$ for every $t \geqslant 0$.*

Proof. First of all, let us check that the coefficient function of (5.1) is well defined on $\Omega \times C_\varsigma$. Having a look at Eq. (5.2), we see that it is enough to apply (5.6) in order to check that, for $v \in C_\varsigma$,

$$\begin{aligned} \big|H_j(\omega, v)\big| &= \left| \sum_{k=1}^{m} \int_{-\infty}^{0} u_{jk}(\omega, s) \alpha_k(\omega \cdot s) v_k(s) \, d\mu(-s) \right| \\ &\leqslant k\alpha_* \|v\|_\varsigma \int_{-\infty}^{0} e^{(\varrho - \varsigma)s} \, d\mu(-s) \leqslant k\alpha_* \|v\|_\varsigma, \end{aligned} \tag{5.7}$$

where $\alpha_* \geqslant \alpha_j(\omega)$ for $j = 1, \dots, m$ and $\omega \in \Omega$. That is, $H_j : \Omega \times C_\varsigma \to \mathbb{R}$ is well defined and continuous for $j = 1, \dots, m$. The linearity of the family (5.1) ensures that the semiflow ϕ_ς^δ is well defined and continuous (see [14] and [16]).

The quasimonotone condition (C6) satisfied by F ensures the same property for the coefficient function of Eq. (5.1) also on the space $\Omega \times C_\varsigma$. This is enough to guarantee the monotonicity of the semiflow ϕ_ς^δ. The proof is complete. □

The next technical result shows the equivalence of different topologies in the omega limit set of a ϕ_ς^δ-semiorbit satisfying a boundedness condition.

Lemma 5.4.

(i) *If a sequence $(v_n) \subset C_\varsigma$ converges to $v \in C_\varsigma$ in $\|\cdot\|_\varsigma$, then it converges uniformly on the compact subsets of $(-\infty, 0]$.*

Let us fix $\varsigma \leqslant \varrho$ and assume the existence of a point $(\widetilde{\omega}, \widetilde{v}) \in \Omega \times C_\varsigma$ and a constant $l > 0$ such that $\|y^\delta(t, \widetilde{\omega}, \widetilde{v})\| \leqslant l$ for every $t \geqslant 0$. Then,

(ii) *the sequence $(w^\delta(t_n, \widetilde{\omega})\widetilde{v})$ with $(t_n) \uparrow \infty$ converges to $v_* \in C_\varsigma$ in $\|\cdot\|_\varsigma$ if and only if it converges uniformly on the compact subsets of $(-\infty, 0]$. In addition, in this case, $v_* \in BU$.*

(iii) *The omega limit set D of $(\widetilde{\omega}, \widetilde{v})$ for the semiflow ϕ_ς^δ is a well-defined compact subset of $\Omega \times C_\varsigma$ contained in $\Omega \times BU$. In addition, the restriction to D of the topologies of $\Omega \times C_\varsigma$ and $\Omega \times BU^{\mathrm{d}}$ agree.*

Proof. The proof of (i) is very easy. In order to prove the reciprocal property in (ii), assume that $(w^\delta(t_n, \widetilde{\omega})\widetilde{v})$ converges to v_* uniformly on the compact subsets of $(-\infty, 0]$. Note first that $\|v_*(s)\| = \lim_{n\to\infty} \|y^\delta(t_n + s, \widetilde{\omega}, \widetilde{v})\| \leqslant l$. Now, given $\varepsilon > 0$, we look for $s_0 \in (-\infty, 0]$ such that $2le^{\varsigma s_0} \leqslant \varepsilon$ and n_0 such that $\|(w^\delta(t_n, \widetilde{\omega})\widetilde{v})(s) - v_*(s)\| \leqslant \varepsilon$ for every $s \in [s_0, 0]$ and $n \geqslant n_0$. Then, for these values of n, $\|(w^\delta(t_n, \widetilde{\omega})\widetilde{v})(s) - v_*(s)\|e^{\varsigma s} \leqslant \varepsilon$ for $s \in (-\infty, 0]$, which proves the convergence in C_ς. In order to check that $v_* \in BU$, note that $\sup_{t\geqslant 0} \|(y^\delta)'(t, \widetilde{\omega}, \widetilde{v})\| < \infty$, which in turn follows from the assumption $\|y^\delta(t, \widetilde{\omega}, \widetilde{v})\| \leqslant l$ for $t \geqslant 0$ (and hence $\|(y^\delta)_t(\cdot, \widetilde{\omega}, \widetilde{v})\|_\varsigma = \|w^\delta(t, \widetilde{\omega})\widetilde{v}\|_\varsigma \leqslant l + \|\widetilde{v}\|_\varsigma$ for every $t \geqslant 0$), the form of Eq. (5.2), and relation (5.7).

Let us now concentrate on (iii). The fact that $\sup_{t\geqslant 0} \|(y^\delta)'(t, \widetilde{\omega}, \widetilde{v})\| < \infty$, statement (ii), and a standard application of Arzelà–Ascoli theorem ensure the relative compactness in $\Omega \times C_\varsigma$ of the set $\{w^\delta(t, \widetilde{\omega})\widetilde{v} \mid t \geqslant 0\}$. This guarantees the existence and compactness in $\Omega \times C_\varsigma$ of the omega limit set D, which, according to (ii), is contained in $\Omega \times BU$. Now consider the map $(D, \|\cdot\|_\varsigma) \to (D, \mathrm{d})$, $(\widetilde{\omega}, \widetilde{v}) \mapsto (\widetilde{\omega}, \widetilde{v})$. Statement (i) ensures its continuity, so that the image is also a compact set; and hence the (bijective) map is bicontinuous. This means that both topologies are equivalent over D, as asserted. □

The previous results are fundamental tools in the proof of the following theorem, which describes several properties of the semiflows ϕ^δ and ϕ_ς^δ. In turn, these properties will allow us to prove Theorem 5.1. We point out that, although a specific monotone theory for semiflows on fading memory phase spaces exists (see [37]), the proof of Theorem 5.5 is based on the results for the semiflow on $\Omega \times BU$ obtained in Section 2, without any requirement on strong monotonicity.

Theorem 5.5. *Let $\delta \geqslant 0$ satisfy $M_*(\omega) - 2\delta J \gg 0$ for every $\omega \in \Omega$. Then,*

(i) *$K_0 = \Omega \times \{0\}$ is the only positively ϕ^δ-invariant compact subset of $\Omega \times BU^{\mathrm{d}}$ which admits a flow extension, and all the semiorbits approach asymptotically K_0 in $\Omega \times BU^{\mathrm{d}}$; i.e., $\lim^{\mathrm{d}}_{n\to\infty} w^\delta(t, \omega)v = 0$ for every $(\omega, v) \in \Omega \times BU$.*

(ii) *There exists $\theta < \rho$ such that for every ς with $\theta < \varsigma \leqslant \varrho$ the continuous function $N : \Omega \to C_\varsigma$ given by $N(\omega)(s) = \widetilde{M}(\omega \cdot s)e^{-\theta s}$ for $s \in (-\infty, 0]$ satisfies $y^\delta(t, \omega, N(\omega)) \leqslant \widetilde{M}(\omega \cdot t)$ for $t \geqslant 0$ and $\omega \in \Omega$. In particular,*

$$\left\| y^\delta\big(t, \omega, N(\omega)\big)\right\| \leqslant k_* \quad \text{and} \quad \left\| w^\delta(t, \omega)N(\omega)\right\|_\varsigma \leqslant k_* \quad \text{for } t \geqslant 0,$$

where k_* satisfies (5.3)*.*

(iii) *Let us take ς with $\theta < \varsigma \leqslant \varrho$. For any $\omega \in \Omega$, there exists the omega limit set in $\Omega \times C_\varsigma$ of $(\omega, N(\omega))$ for the semiflow ϕ^δ_ς, and it agrees with K_0.*

(iv) *For any $\omega \in \Omega$, the norm in C_θ of the linear operator $w^\delta(t, \omega)$, namely*

$$\left\| w^\delta(t, \omega)\right\|_\theta := \sup_{\|v\|_\theta \leqslant 1} \left\| w^\delta(t, \omega)v\right\|_\theta,$$

converges to 0 as $t \to \infty$.

(v) *The solutions in C_θ of the linear equations* (5.1) *converge exponentially to* 0 *as time increases. That is, there exist constants $\kappa > 1$ and $\rho > 0$ such that, for every $t \geqslant 0$, $\omega \in \Omega$ and $v \in C_\theta$, $\|w^\delta(t, \omega)v\|_\theta \leqslant \kappa e^{-\rho t}\|v\|_\theta$ and $\|y^\delta(t, \omega, v)\| \leqslant \kappa e^{-\rho t}\|v\|_\theta$.*

Proof. (i) In order to apply Theorem 2.6, let us check that the function $-\widetilde{M} : \Omega \to \mathbb{R}^m$, $\omega \mapsto -M_*(\omega)(0)$ defines a strong lower solution for the linear equations (5.1). We know that $-\widetilde{M}'(\omega) = -F(\omega, M_*(\omega))$, so that we only have to check that

$$-F\big(\omega, M_*(\omega)\big) - F_x\big(\omega, M^\delta(\omega)\big)\big(-M_*(\omega)\big) \ll 0.$$

The j-component of this difference is given by

$$\gamma_j(\omega)\big(\widetilde{M}_j(\omega)\big)^2 - 2\gamma_j(\omega)\widetilde{M}^\delta_j(\omega)\widetilde{M}_j(\omega) = \gamma_j(\omega)\widetilde{M}_j(\omega)\big(2\delta - \widetilde{M}_j(\omega)\big),$$

which is strictly negative by the choice of δ. In addition, Eq. (5.1) satisfies the concavity hypothesis (C7), since it is linear. Applying Theorem 2.6 to the positively ϕ^δ-invariant set $K_0 = \Omega \times \{0\}$ (a compact in $\Omega \times BU^{\mathrm{d}}$) and the strong lower solution $-\widetilde{M}$ we conclude that all the semiorbits starting above $-K_M$ approach K_0 asymptotically in metric: $\lim_{t\to\infty} \mathrm{d}(w^\delta(t, \omega)v, 0) = 0$ for any $(\omega, v) \in \Omega \times BU$ with $v \geqslant -M_*(\omega)$. By linearity the same property holds for every $(\omega, v) \in \Omega \times BU$, since there exists $l \in \mathbb{R}$ with $lv \geqslant -M_*(\omega)$. The uniqueness of K_0 follows immediately, and completes the proof of (i).

(ii) We assume that $\theta < \varrho$ to define N. Note first that $N(\omega) \in C_\varsigma$ if $\varsigma > \theta$, since $\|N(\omega)(s)\|e^{\varsigma s} = \|\widetilde{M}(\omega \cdot s)\|e^{(\varsigma-\theta)s}$ tends to zero as $s \to -\infty$. In addition, $\|N(\omega)\|_\varsigma \leqslant k_*$.

Let us fix $\widetilde{\omega} \in \Omega$ and $j \in \{1, \dots, m\}$, define $n(t) = \widetilde{M}_j(\widetilde{\omega}\cdot t) - y^\delta_j(t, \widetilde{\omega}, N(\widetilde{\omega}))$, and note that $n(0) = 0$. Our next purpose is to find θ small enough to get $n'(0) > 0$. Eqs. (4.9) and (5.1) respectively satisfied by $\widetilde{M}(\widetilde{\omega} \cdot t)$ and $y^\delta(t, \widetilde{\omega}, N(\widetilde{\omega}))$ show that

$$\begin{aligned} n'(0) = {} & -\gamma_j(\widetilde{\omega})\widetilde{M}_j(\widetilde{\omega})\big(\widetilde{M}_j(\widetilde{\omega}) - 2\widetilde{M}^\delta_j(\widetilde{\omega})\big) \\ & + \sum_{k=1}^m \int_{-\infty}^0 u_{jk}(\widetilde{\omega}, s)\alpha_k(\widetilde{\omega} \cdot s)\widetilde{M}_k(\widetilde{\omega} \cdot s)\big(1 - e^{-\theta s}\big)\, d\mu(-s). \end{aligned}$$

The choice of δ ensures the existence of $l > 0$ such that

$$-\gamma_j(\widetilde{\omega})\widetilde{M}_j(\widetilde{\omega})\big(\widetilde{M}_j(\widetilde{\omega}) - 2\widetilde{M}^\delta_j(\widetilde{\omega})\big) = -\gamma_j(\widetilde{\omega})\widetilde{M}_j(\widetilde{\omega})\big(2\delta - \widetilde{M}_j(\widetilde{\omega})\big) > l.$$

On the other hand, by the boundedness of $\widetilde{\alpha}_k(\widetilde{\omega})$ and $\widetilde{M}_k(\widetilde{\omega})$ and relation (5.6), the function

$$h(s,\theta) := \sum_{k=1}^{m} u_{jk}(\widetilde{\omega}, s)\alpha_k(\widetilde{\omega}\cdot s)\widetilde{M}_k(\widetilde{\omega}\cdot s)\big(1-e^{-\theta s}\big)$$

satisfies $\lim_{\theta\to 0^+} h(s,\theta) = 0$ for every $s \in [-\infty, 0]$ and

$$\big|h(s,\theta)\big| \leqslant k_1 e^{\varrho s}\big(e^{-\theta s}-1\big) \leqslant k_1 e^{(\varrho-\theta)s} \leqslant k_1$$

for $s \in (-\infty, 0]$. Applying dominated convergence theorem we conclude that

$$\lim_{\theta\to 0^+} \int_{-\infty}^{0} h(s,\theta)\, d\mu(-s) = 0.$$

So that there exists θ such that

$$\sum_{k=1}^{m} \int_{-\infty}^{0} u_{jk}(\widetilde{\omega}, s)\alpha_k(\widetilde{\omega}\cdot s)\widetilde{M}_k(\widetilde{\omega}\cdot s)\big(1-e^{-\theta s}\big)\, d\mu(-s) > -l/2,$$

from where our assertion follows. Consequently, $n(t) > 0$ for $t > 0$ small enough. Note also that θ can be chosen independent of $\widetilde{\omega}$ and j.

Let us now define

$$J = \big\{t_0 \geqslant 0 \;\big|\; \widetilde{M}(\widetilde{\omega}\cdot t) - y^\delta\big(t, \widetilde{\omega}, N(\widetilde{\omega})\big) \geqslant 0 \text{ for every } t \in [0, t_0]\big\}$$

and $t_* = \sup J$. The property previously proved shows that $t_* > 0$. The first assertion in (ii) is equivalent to show that, in fact, $t_* = \infty$, what we do in what follows. We assume by contradiction that $t_* < \infty$. We first check that

$$N(\widetilde{\omega}\cdot t_*) \geqslant w^\delta(t_*, \widetilde{\omega})N(\widetilde{\omega}): \tag{5.8}$$

for $s \in [-t_*, 0]$, by definition of t_*,

$$\begin{aligned} N(\widetilde{\omega}\cdot t_*)(s) &= \widetilde{M}\big(\widetilde{\omega}\cdot(t_*+s)\big)e^{-\theta s} \geqslant \widetilde{M}\big(\widetilde{\omega}\cdot(t_*+s)\big) \\ &\geqslant y^\delta\big(t_*+s, \widetilde{\omega}, N(\widetilde{\omega})\big) = \big(w^\delta(t_*, \widetilde{\omega})N(\widetilde{\omega})\big)(s), \end{aligned}$$

and, for $s \in (-\infty, -t_*)$,

$$\begin{aligned} N(\widetilde{\omega}\cdot t_*)(s) &= \widetilde{M}\big(\widetilde{\omega}\cdot(t_*+s)\big)e^{-\theta s} \geqslant \widetilde{M}\big(\widetilde{\omega}\cdot(t_*+s)\big)e^{-\theta(t_*+s)} \\ &= N(\widetilde{\omega})(t_*+s) = \big(w^\delta(t_*, \widetilde{\omega})N(\widetilde{\omega})\big)(s). \end{aligned}$$

Now, by reasoning as before for the point $\widetilde{\omega}\cdot t_*$ we find $t_*^1 > 0$ with $\widetilde{M}(\widetilde{\omega}\cdot(t_*+t)) \geqslant y^\delta(t, \widetilde{\omega}\cdot t_*, N(\widetilde{\omega}\cdot t_*))$ for $t \in (0, t_*^1]$. Hence relation (5.8) and the monotonicity of the semiflow ϕ_ς^δ guaranteed by Proposition 5.3 show that

$$\widetilde{M}\big(\widetilde{\omega}\cdot(t_*+t)\big) \geqslant y^\delta\big(t, \widetilde{\omega}\cdot t_*, w^\delta(t_*, \widetilde{\omega})N(\widetilde{\omega})\big) = y^\delta\big(t_*+t, \widetilde{\omega}, N(\widetilde{\omega})\big)$$

for $t \in (0, t_*^1]$, impossible by definition of t_*.

The last assertions in (ii) follow immediately from the first one.

(iii) Let us fix $\widetilde{\omega} \in \Omega$. Statement (ii) allows us to apply Lemma 5.4(iii) in order to conclude that the omega limit set $D \subset \Omega \times C_\varsigma$ of $(\widetilde{\omega}, N(\widetilde{\omega}))$ for the semiflow ϕ^δ_ς exists and is a compact subset also in $\Omega \times BU^{\mathrm{d}}$. Since the semiflows ϕ^δ_ς and ϕ^δ agree when restricted to D, the restriction of ϕ^δ to D admits a flow extension, and by (i) we conclude that $D = K_0$, as asserted.

(iv) We work again for a fixed point $\widetilde{\omega} \in \Omega$. As a consequence of (iii) and Lemma 5.4(i), we know that $w^\delta(t, \widetilde{\omega})N(\widetilde{\omega})$ converges to 0 uniformly on the compact subsets of $(-\infty, 0]$ as $t \to \infty$. The definition of $N(\widetilde{\omega})$ then shows that $-\varepsilon_*^{-1}N(\widetilde{\omega}) \leqslant v \leqslant \varepsilon_*^{-1}N(\widetilde{\omega})$ whenever $\|v\|_\theta \leqslant 1$, where ε_* satisfies (5.3). The monotonicity ensured by Proposition 5.3 leads us to

$$-\varepsilon_*^{-1} w^\delta(t, \widetilde{\omega})N(\widetilde{\omega}) \leqslant w^\delta(t, \widetilde{\omega})v \leqslant \varepsilon_*^{-1} w^\delta(t, \widetilde{\omega})N(\widetilde{\omega}). \tag{5.9}$$

This shows that $w^\delta(t, \widetilde{\omega})v$ converges to 0 as $t \to \infty$ uniformly on the compact subsets of $(-\infty, 0]$, being this convergence uniform in the set $\|v\|_\theta \leqslant 1$. Using now Lemma 5.4(ii), we conclude that $\lim_{t\to\infty} w^\delta(t, \widetilde{\omega})v = 0$ in C_θ, and the argument there used shows that this convergence is uniform in $\|v\|_\theta \leqslant 1$, which proves (iv).

(v) Once proved (iv), the spectral theory for infinite-dimensional linear skew-product semiflows of Chow and Leiva [6,7] and Sacker and Sell [30] shows the existence of constants $\kappa \geqslant 1$ and $\rho > 0$ such that $\|w^\delta(t, \omega)\|_\theta \leqslant \kappa e^{-\rho t}$ for every $t \geqslant 0$ and $\omega \in \Omega$. Consequently, $\|w^\delta(t, \omega)v\|_\theta \leqslant \kappa e^{-\rho t}\|v\|_\theta$ and hence, evaluating at $s = 0$, $\|y^\delta(t, \omega, v)\| \leqslant \kappa e^{-\rho t}\|v\|_\theta$. This completes the proof of the theorem. □

We are finally in a position to prove the main result of the section.

Proof of Theorem 5.1. (i) We can assume without restriction that $\varepsilon \leqslant \varepsilon_*$ (the constant appearing in (5.3)). Let us fix $(\omega, x) \in \Omega \times BU$ with $x \geqslant \varepsilon J$. Assume first that $x \geqslant M_*(\omega)$. The monotonicity and concavity of the semiflow τ_M (see Lemma 2.1) show that

$$0 \leqslant z_M(t, \omega, x) - \widetilde{M}(\omega \cdot t) \leqslant u_x\big(t, \omega, M_*(\omega)\big)\big(x - M_*(\omega)\big)(0) = y^0\big(t, \omega, x - M_*(\omega)\big),$$

and Theorem 5.5(v) for $\delta = 0$ gives the searched inequality for $\eta_\varepsilon = \kappa$ (recall that $\|v\|_\theta \leqslant \|v\|_\infty$ for $v \in BU$).

Let us now consider the case $\varepsilon J \leqslant x < M_*(\omega)$. We fix $\delta > 0$ with $M_*(\omega) - 2\delta J \gg 0$ and take the minimum time $t_1 = t_1(x) \geqslant 0$ satisfying $z_M(t, \omega, x) \geqslant \widetilde{M}^\delta(\omega \cdot t)$ for every $t \geqslant t_1$. Proposition 5.2 guarantees the existence of $t_0(\varepsilon)$ (δ is fixed) independent of x such that $t_1 \leqslant t_0(\varepsilon)$. As before, the monotonicity and concavity of the semiflow ensure

$$0 \leqslant \widetilde{M}\big(\omega \cdot (t + t_1)\big) - z_M(t + t_1, \omega, x) \leqslant y\big(t + t_1, \omega, x, M_*(\omega) - x\big)$$

for any $t \geqslant 0$. According to the notation established in Section 2, the function

$$t \mapsto y\big(t + t_1, \omega, x, M_*(\omega) - x\big) = \big(u_x(t + t_1, \omega, x)\big(M_*(\omega) - x\big)\big)(0)$$

satisfies the variational equation obtained by linearizing (4.9) along the τ-semiorbit $(\omega \cdot t, u(t, \omega, x))$, whose jth component is

$$\begin{aligned} y_j'(t) = {} & -2\gamma_j\big(\omega \cdot (t + t_1)\big)(z_M)_j(t + t_1, \omega, x)y_j(t) \\ & + \sum_{k\neq j} \epsilon_{jk}\big(\omega \cdot (t + t_1)\big)\big(y_k(t) - y_j(t)\big) + H_j\big(\omega \cdot (t + t_1), y_t\big). \end{aligned} \tag{5.10}$$

On the other hand, system (5.1) for $\omega \cdot t_1$ has jth component

$$y_j'(t) = -2\gamma_j\big(\omega\cdot(t+t_1)\big)\widetilde{M}_j^{\delta}\big(\omega\cdot(t+t_1)\big)y_j(t) + \sum_{k\neq j}\epsilon_{jk}\big(\omega\cdot(t+t_1)\big)\big(y_k(t)-y_j(t)\big) + H_j\big(\omega\cdot(t+t_1), y_t\big).$$

Since $z_M(t+t_1,\omega,x) \geqslant \widetilde{M}^{\delta}(\omega\cdot(t+t_1))$ for $t \geqslant 0$, a standard argument of comparison of solutions shows that, if $v = u_x(t_1,\omega,x)(M_*(\omega)-x)$, then

$$y\big(t+t_1,\omega,x,M_*(\omega)-x\big) = y\big(t,\omega\cdot t_1,u(t_1,\omega,x),v\big) \leqslant y^{\delta}(t,\omega\cdot t_1,v)$$

for $t \geqslant 0$. Consequently, $0 \leqslant \widetilde{M}(\omega\cdot(t+t_1)) - z_M(t+t_1,\omega,x) \leqslant y^{\delta}(t,\omega\cdot t_1,v)$. Theorem 5.5(v) hence proves that

$$\big\|\widetilde{M}\big(\omega\cdot(t+t_1)\big) - z_M(t+t_1,\omega,x)\big\| \leqslant \kappa e^{-\rho t}\|v\|_\infty \quad \text{for } t \geqslant 0. \tag{5.11}$$

In the case that $t_1 = 0$, $v = M_*(\omega) - x$ and the statement of the theorem holds for $\eta_\varepsilon = \kappa$. Assume now that $t_1 > 0$. Then, on the one hand, there exists η_ε, independent of ω and x with $t_1(x) > 0$, such that $\|M_*(\omega)-x\|_\infty \geqslant \eta_\varepsilon$: just take η_ε such that if $\|M_*(\omega)-x\|_\infty \leqslant \eta_\varepsilon$ then $\|M_*(\omega\cdot t)-u(t,\omega,x)\|_\infty \leqslant \delta/2$ for $t \in [0,t_0(\varepsilon)]$ and $\omega \in \Omega$ (see Proposition 4.2 of [27]). Therefore,

$$\big\|\widetilde{M}(\omega\cdot t) - z_M(t,\omega,x)\big\| \leqslant k_*\eta_\varepsilon^{-1}e^{\rho(t_0(\varepsilon)-t)}\big\|M_*(\omega)-x\big\|_\infty \tag{5.12}$$

for $t \in [0,t_1]$. And, on the other hand, Eqs. (5.10), the monotonicity of τ_M and a new argument of comparison of solutions show that $0 \leqslant u_x(t_1,\omega,x)J \leqslant u_x(t_1,\omega,0)J$. Hence, since $0 \leqslant M_*(\omega) - x \leqslant \|M_*(\omega)-x\|_\infty J$, we have by monotonicity and linearity that $\|v\|_\infty \leqslant \|u_x(t_1,\omega,0)J\|_\infty \|M_*(\omega)-x\|_\infty$. This and relations (5.11) and (5.12) show that statement (i) holds for every x with $0 \leqslant x \leqslant M_*(\omega)$ for $\eta_\varepsilon = \kappa_\varepsilon$, with

$$\kappa_\varepsilon = \max\Big(k_*\eta_\varepsilon^{-1}e^{\rho t_0(\varepsilon)}, \kappa e^{\rho t_0(\varepsilon)} \max_{t\in[0,t_0(\varepsilon)],\omega\in\Omega}\big\|u_x(t,\omega,0)J\big\|_\infty\Big).$$

Finally, in the general case, given any $x \geqslant \varepsilon J$ we look for $x_1, x_2 \in BU$ with $\varepsilon J \leqslant x_1 \leqslant x \leqslant x_2$, $x_1 \leqslant M_*(\omega) \leqslant x_2$ and $\|M_*(\omega)-x_1\|_\infty \leqslant \|M_*(\omega)-x_2\|_\infty = \|M_*(\omega)-x\|_\infty$. This can be done, for instance, by taking $x_2 = M_*(\omega) + \|M_*(\omega)-x\|_\infty J$, and $x_1 = \max(\varepsilon J, M_*(\omega) - \|M_*(\omega)-x\|_\infty J)$ (defining the maximum component by component). An easy application of the monotonicity of the semiflow and the previously proved properties shows the stated inequality for $\eta_\varepsilon = \max(\kappa,\kappa_\varepsilon)$ and completes the proof of the first assertion of the theorem.

(ii) Let us now analyze the evolution of the immature population. Note that we can assume without restriction that $\rho < \varrho$, where ρ is the constant satisfying (i) and ϱ is the one of (4.7) and (5.6). Once $z_M(t,\omega,x)$ is known, the solution of the linear equation (4.2) is given by

$$z_I(t,\omega,x,c) = U_\omega(t)\left(c + \int_0^t U_\omega^{-1}(l)L\big(\tau(l,\omega,x)\big)\,dl\right),$$

and hence

$$\widetilde{I}(\omega\cdot t) = z_I\big(t,\omega,M_*(\omega),\widetilde{I}(\omega)\big) = U_\omega(t)\left(\widetilde{I}(\omega) + \int_0^t U_\omega^{-1}(l)L\big(\tau\big(l,\omega,M_*(\omega)\big)\big)\,dl\right).$$

According to (5.6),

$$\big\|U_\omega(t)\big(\widetilde{I}(\omega)-c\big)\big\| \leqslant ke^{-\rho t}\big\|\widetilde{I}(\omega)-c\big\|. \tag{5.13}$$

Let α_* satisfy $\alpha_* \geqslant \alpha_j(\omega)$ for every $\omega \in \Omega$ and $j \in \{1, \dots, m\}$. By using statement (i) of the theorem and relation (5.6), one checks that, for $l \geqslant 0$,

$$\begin{aligned}\big\|L\big(\tau\big(l,\omega,M_*(\omega)\big)\big)-L\big(\tau(l,\omega,x)\big)\big\| &\leqslant \alpha_*\left(\kappa_\varepsilon e^{-\rho l}+k\kappa_\varepsilon\int_{-\infty}^{0} e^{\varrho s}e^{-\rho(l+s)}\,ds\right)\big\|M_*(\omega)-x\big\|_\infty\\ &\leqslant \widetilde{\kappa}_\varepsilon e^{-\rho l}\big\|M_*(\omega)-x\big\|_\infty,\end{aligned}$$

for a large enough constant $\widetilde{\kappa}_\varepsilon$ independent of l. Consequently, using (4.7),

$$\begin{aligned}\left\|U_\omega(t)\int_0^t U_\omega^{-1}(l)\big(L\big(\tau\big(l,\omega,M_*(\omega)\big)\big)-L\big(\tau(l,\omega,x)\big)\big)\,dl\right\| &\leqslant k\widetilde{\kappa}_\varepsilon e^{-\varrho t}\left(\int_0^t e^{(\varrho-\rho)l}\,dl\right)\big\|M_*(\omega)-x\big\|_\infty\\ &< \frac{k\widetilde{\kappa}_\varepsilon}{\varrho-\rho}e^{-\rho t}\big\|M_*(\omega)-x\big\|_\infty.\end{aligned}$$

This relation, (5.13), and the expressions of $z_I(t,\omega,x,c)$ and $\widetilde{I}(\omega\cdot t)$ show that the statements of the theorem hold for ρ and a large enough η_ε. The proof is complete. □

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