Product structure of the fuzzy *n*-ary factor polygroup

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ABSTRACT

Fuzzy *n*-ary hypergroups were introduced as suitable generalizations of fuzzy polygroups and a special case of fuzzy *n*-ary hyper groups. The aim of this paper is to introduce the notion of fuzzy *n*-ary factor polygroups of a polygroup. Based on the fuzzy *n*-ary factor group, we also study the product structures of the generating fuzzy factor *n*-ary groups. At the end of the paper, we prove the fundamental theorem of isomorphism of fuzzy *n*-ary groups.

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1. Introduction

Since the concept of hypergroups was first introduced by Marty in his pioneering paper [1] in 1934, the properties of hypergroups have been studied by many scholars around the world. Moreover, over the past decades hypergroups have been widely used in the fields of algebra, geometry, convexity and computer science [2–4]. Polygroups, as an important subclass of hypergroups, were introduced by Ghadiri and Waphare [5] and Comer [6,7]. Researches in polygroups have also produced a large number of papers. The concepts of matrix representations of polygroups over hyperrings and the polygroup hyperring were introduced by Davvaz and Pousalavati [8], and the concept of fuzzy rough polygroups was introduced in [9].

Up to now, *n*-ary operations have been investigated by many authors, and the applications of *n*-ary systems in the theory of automata [10] and quantum groups [11]. The concept of *n*-ary polygroups, which was a suitable generalization of polygroups, was first introduced by Ghadiri and Waphare [12]. Based on *n*-ary polygroups, the concept of a fuzzy *n*-ary subgroup was introduced by Davvaz [13]. In this paper, we propose the notion of fuzzy *n*-ary factor polygroups. By using the concept of fuzzy *n*-ary factor polygroups, we then study the product structures of fuzzy *n*-ary polygroups. Additionally, we prove the fundamental theorem of isomorphism of fuzzy *n*-ary groups.

The remainder of the paper is structured as following. In Section 2, we review some results about fuzzy *n*-ary polygroups. In Section 3 we introduce the concept of the fuzzy *n*-ary factor group. In Section 4, we study the product structures of fuzzy *n*-ary polygroups, their properties, and the homomorphism image of the fuzzy *n*-ary polygroups. In the last section, a conclusion is presented.

2. Preliminaries

First, we recall some basic definitions and propositions, which will be used in our paper. In this paper, $A[\alpha]$ will denote the set $\{x : A(x) \geq \alpha\}$.

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Definition 2.1 ([5]). An n-ary polygroup is a multivalued system $\langle P, f, e, ^{-1} \rangle$, where $e \in P$, $^{-1}$ is a unitary operation on P, f is an n-ary hyperoperation on P, and the following axioms hold for all $i, j \in 1, \ldots, n, x_1, \ldots, x_{2n-1}, x \in P$:

(1)
$$f\left(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f(x_i^{n+j-1}), x_{n+j}^{2n-1}\right).$$

(2) e is a unique element, such that $f(\underbrace{e, \ldots, e}_{i}, x, \underbrace{e, \ldots, e}_{n-i}) = x$.

(3) $x \in f(x_1^n)$ implies $x_i \in f(x_{i-1}^{-1}, \ldots, x_1^{-1}, x_n^{-1}, \ldots, x_{i+1}^{-1})$.

Below we present an example of 2-ary polygroups.

Example 2.1. Let $P = \{e, x, y, z\}$ be a set endowed with a 2-ary hyper-operation f as follows:

$$
\begin{array}{llll}
f(e, e) = e & f(e, x) = x & f(e, y) = y & f(e, z) = z \\
f(x, e) = x & f(x, x) = \{x, y, z\} & f(x, y) = P & f(x, z) = P \\
f(y, e) = y & f(y, x) = P & f(y, y) = \{x, y, z\} & f(y, z) = P \\
f(z, e) = z & f(z, x) = P & f(z, y) = P & f(z, z) = \{x, y, z\}.
\end{array}
$$

For any $x_i \in P$ ($i = 1, \ldots, 3$), we have

$$f(f(x_1, x_2), x_3) = f(x_1, f(x_2, x_3)),$$

i.e., f is associative. We suppose that $^{-1} : P \to P$ is the identity function on P. We have

$$e^{-1} = e, \qquad x^{-1} = x, \qquad y^{-1} = y, \qquad z^{-1} = z.$$

It is easy to see that $a \in f(x_1, x_2)$ implies that

$$x_1 \in f(a, x_2^{-1}), \qquad x_2 \in f(x_1^{-1}, a),$$

for every $x_i \in P$ ($i = 1, \ldots, 3$). Therefore $H = \langle P, f, e, ^{-1} \rangle$ is a 2-ary polygroup.

Definition 2.2 ([13]). Let P be an n-ary polygroup. A fuzzy subset μ of P is called a fuzzy n-ary subpolygroup of P if the following axioms hold:

(1)
$$\min\{\mu(x_1), \ldots, \mu(x_n)\} \leq \inf_{z \in f(x_1^n)} \{\mu(z)\}.$$

(2) $\mu(x) \leq \mu(x^{-1})$ for all $x \in P$.

Example 2.2. Let H be a fuzzy subpolygroup of a polygroup P. If for all $x_1, x_2, \ldots, x_n \in P$, we define $f(x_1^n) = x_1, \ldots, x_n$ then (P, f) is an n-ary polygroup with the same scalar identity, and H is a fuzzy n-ary polygroup.

Definition 2.3 ([13]). Let μ be a fuzzy n-ary subpolygroup of P. Then μ is said to be normal if, for all $x, y \in P$,

$$\mu(z) = \mu(z'), \quad \forall z \in f(x, y, e^*), \ \forall z' \in f(y, x, e^*).$$

It is obvious that if μ is normal then, for all $x, y \in P$,

$$\mu(z) = \mu(z'), \quad \forall z \in f(x, y, e^*), \ \forall z' \in f(y, x, e^*).$$

Definition 2.4 ([13]). Let μ be a fuzzy n-ary subpolygroup of P. Then the following conditions are equivalent:

(1) μ is normal.
(2) For all $x, y \in P$ and for all $z \in f(x, y, x^{-1}, e^*)$, $\mu(z) = \mu(y)$.
(3) For all $x, y \in P$ and for all $z \in f(x, y, x^{-1}, e^*)$, $\mu(z) \geq \mu(y)$.
(4) For all $x, y \in P$ and for all $z \in f(x^{-1}, y^{-1}, x, y, e^*)$, $\mu(z) \geq \mu(y)$.

Proposition 2.1 ([13]). *Let $\varphi : P_1 \to P_2$ be a strong homomorphism.*
(1) *If μ is a fuzzy n-ary subpolygroup of P_1, then $\varphi(\mu)$ is a fuzzy n-ary subpolygroup of P_2.*
(2) *If λ is a fuzzy n-ary subpolygroup of P_2, then $\varphi^{-1}(\lambda)$ is a fuzzy n-ary subpolygroup of P_1.*

Definition 2.5 ([13]). Let $\langle P_1, f, e_1, ^{-1} \rangle$ and $\langle P_2, f, e_2, ^{-1} \rangle$ be two n-ary polygroups. Let μ, λ be fuzzy n-ary subpolygroups of P_1, P_2, respectively. Then the direct product $\mu \times \lambda$ is the fuzzy subset of $P_1 \times P_2$ defined by

$$(\mu \times \lambda)(x, y) = \min\{\mu(x), \lambda(y)\}, \quad \text{for all } (x, y) \in P_1 \times P_2.$$

Proposition 2.2 ([13]). *Let $\langle P_1, f, e_1, ^{-1} \rangle$ and $\langle P_2, f, e_2, ^{-1} \rangle$ be two n-ary polygroups, and μ, λ be fuzzy n-ary subpolygroups of P_1, P_2, respectively. Then $\mu \times \lambda$ is the fuzzy n-ary subpolygroup of $P_1 \times P_2$.*

3. Fuzzy *n*-ary factor polygroup

Definition 3.1. Let $\langle P, f, e, ^{-1} \rangle$ be an n-ary polygroup, and B be a fuzzy n-ary subpolygroup of P, and $a \in P$. Then $f(a, B, e^*)$ ($f(B, a, e^*)$) is called a left (right) fuzzy n-ary coset of B in P defined as follows:

$$f(a, B, e^*)(x) = B(f(a^{-1}, x, e^*)), \quad \text{for any } x \in P \qquad (f(B, a, e^*)(x) = B(f(x, a^{-1}, e^*)), \quad \text{for any } x \in P)$$
$$B(f(a^{-1}, x, e^*)) = \sup_{z \in f(a^{-1}, x, e^*)} \{B(z)\}.$$

It is clear that B is normal iff $f(a, B, e^*) = f(B, a, e^*)$ for all $a \in P$.

Proposition 3.1. *Let $\langle P, f, e, ^{-1} \rangle$ be an n-ary polygroup; if B is a fuzzy n-ary subpolygroup of P, then, for any a, b in P, $f(a, B, e^*) = f(b, B, e^*)$ iff $B(f(a^{-1}, b, e^*)) = B(e)$.*

Proof. For any $x \in P$, let $f(a, B, e^*)(x) = f(b, B, e^*)(x)$; then $B(f(a^{-1}, x, e^*)) = B(f(b^{-1}, x, e^*))$. Let $x = b$; then

$$B(f(a^{-1}, b, e^*)) = B(f(b^{-1}, b, e^*)).$$

Since $e \in f(b^{-1}, b, e^*)$, for any $z \in f(b^{-1}, b, e^*)$, we have $B(e) \geq B(z)$, so $B(f(b^{-1}, b, e^*)) = B(e)$. Hence

$$B(f(a^{-1}, b, e^*)) = B(e).$$

Conversely, assume that $B(f(a^{-1}, b, e^*)) = B(e)$.

(1) For any $z \in f(a^{-1}, x, e^*) \subseteq f(a^{-1}, b, b^{-1}, x, e^*) = f(f(a^{-1}, b, e^*), f(b^{-1}, x, e^*), e^*)$, then

$$B(z) \geq \min\{B(f(a^{-1}, b, e^*)), B(f(b^{-1}, x, e^*)), B(e)\}.$$

Since $B(f(a^{-1}, b, e^*)) = B(e)$, we have $B(z) \geq B(f(b^{-1}, x, e^*))$; hence

$$B(f(a^{-1}, x, e^*)) = \sup_{z \in f(a^{-1}, x, e^*)} \{B(z)\} \geq B(f(b^{-1}, x, e^*)).$$

(2) For any $z \in f(b^{-1}, x, e^*) \subseteq f(b^{-1}, a, a^{-1}, x, e^*) = f(f(b^{-1}, a, e^*), f(a^{-1}, x, e^*), e^*)$, then

$$B(z) \geq \min\{B(f(b^{-1}, a, e^*)), B(f(a^{-1}, x, e^*)), B(e)\}.$$

Since $B(f(b^{-1}, a, e^*)) \geq B(f(a^{-1}, b, e^*)) = B(e)$, we have $B(z) \geq B(f(a^{-1}, x, e^*))$; hence

$$B(f(b^{-1}, x, e^*)) = \sup_{z \in f(b^{-1}, x, e^*)} \{B(z)\} \geq B(f(a^{-1}, x, e^*)).$$

So we have $B(f(a^{-1}, x, e^*)) = B(f(b^{-1}, x, e^*))$. Hence $f(a, B, e^*) = f(b, B, e^*)$. $\square$

Proposition 3.2. *Let $\langle P, f, e, ^{-1} \rangle$ be an n-ary polygroup, and B be a fuzzy n-ary subpolygroup of P. For any a, b in P, if $f(a, B, e^*) = f(b, B, e^*)$ then $f(a^{-1}, B, e^*) = f(b^{-1}, B, e^*)$.*

Proposition 3.3. *Let $\langle P, f, e, ^{-1} \rangle$ be an n-ary polygroup, and B be a fuzzy n-ary subpolygroup of P. For any a, b in P, let $X = \{x : f(x, B, e^*) = f(a, B, e^*)\}$, $Y = \{x : f(x, B, e^*) = f(b, B, e^*)\}$, $Z = \{x : f(x, B, e^*) = f(a, b, B, e^*)\}$. Then $f(X, Y, e^*) = Z$.*

Proof. Clearly $f(X, Y, e^*)$ is included in Z.
    Let $x \in Z$; so $f(x, B, e^*) = f(a, b, B, e^*)$. Then $B(f(x^{-1}, a, b, e^*)) = B(e)$ or $B(f(f(a^{-1}, x, e^*), b, e^*)) = B(e)$. Hence

$$f(a^{-1}, x, e^*) \in Y.$$

Thus $x \in f(a^{-1}, a, x, e^*) \subseteq f(a, f(a^{-1}, x, e^*), e^*) \in f(X, Y, e^*)$. Hence

$$f(X, Y, e^*) = Z.$$

We consider the fuzzy n-ary factor polygroup as a fuzzy set. Let $\langle P, f, e, ^{-1} \rangle$ be an n-ary polygroup, and B be a fuzzy n-ary normal subpolygroup of P. For $f(a_1, B, e^*), \ldots, f(a_n, B, e^*)$ in P/B, we define

$$P/B = \{f(a, B, e^*) \mid a \in P\},$$
$$F : \underbrace{P/B \times P/B \times \cdots \times P/B}_{n} \to \vartheta^*(P/B).$$
$$F(f(a_1, B, e^*), \ldots, f(a_n, B, e^*)) = \{f(a, B, e^*) \mid a \in f(a_1^n)\},$$

and

$$^{-l} : P/B \to P/B$$
$$f(a, B, e^*)^{-l} = f(a^{-1}, B, e^*). \quad \square$$

Proposition 3.4. *Let* $\langle P, f, e, ^{-1} \rangle$ *be an n-ary polygroup, and B be a fuzzy n-ary subpolygroup of P.* $\langle P/B, F, B, ^{-l} \rangle$ *is an n-ary polygroup.*

Proof. For all $i, j \in 1, \ldots, n, a_1, \ldots, a_{2n-1} \in P, f(a_1, B, e^*), \ldots, f(a_{2n-1}, B, e^*) \in P/B$,

(1) $F(f(a_1, B, e^*), \ldots, f(a_{i-1}, B, e^*), F(f(a_i, B, e^*), \ldots, f(a_{n+i-1}, B, e^*)), f(a_{n+i}, B, e^*), \ldots, f(a_{2n-1}, B, e^*))$

$\quad = \{F(f(a_1, B, e^*), \ldots, f(a_{i-1}, B, e^*), f(a, B, e^*), f(a_{n+i}, B, e^*), \ldots, f(a_{2n-1}, B, e^*)) \mid a \in f(a_i^{n+i-1})\}$

$\quad = \{f(a', B, e^*) \mid a \in f(a_i^{n+i-1}), a' \in f(a_1^{i-1}, a, a_{n+i}^{2n-1})\}$

$\quad = \{f(a', B, e^*) \mid a' \in f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1})\}.$

We can also get

$\quad F(f(a_1, B, e^*), \ldots, f(a_{j-1}, B, e^*), F(f(a_j, B, e^*), \ldots, f(a_{n+j-1}, B, e^*)), f(a_{n+j}, B, e^*), \ldots, f(a_{2n-1}, B, e^*))$

$\quad = \{f(a'', B, e^*) \mid a'' \in f(a_1^{j-1}, f(a_j^{n+j-1}), a_{n+j}^{2n-1})\}.$

Since P is an n-ary polygroup, $f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1}) = f(a_1^{j-1}, f(a_j^{n+j-1}), a_{n+j}^{2n-1})$. Hence

$\quad F(f(a_1, B, e^*), \ldots, f(a_{i-1}, B, e^*), F(f(a_i, B, e^*), \ldots, f(a_{n+i-1}, B, e^*)), f(a_{n+i}, B, e^*), \ldots, f(a_{2n-1}, B, e^*))$

$\quad = F(f(a_1, B, e^*), \ldots, f(a_{j-1}, B, e^*), F(f(a_j, B, e^*), \ldots, f(a_{n+j-1}, B, e^*)), f(a_{n+j}, B, e^*), \ldots, f(a_{2n-1}, B, e^*)).$

(2) $F(\underbrace{B, \ldots, B}_{i-1}, f(a, B, e^*), \underbrace{B, \ldots, B}_{n-i}) = F(\underbrace{f(e, B, e^*), \ldots, f(e, B, e^*)}_{i-1}, f(a, B, e^*), \underbrace{f(e, B, e^*), \ldots, f(e, B, e^*)}_{n-i})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \left\{ f(a', B, e^*) \mid a' \in f(\underbrace{e, \ldots, e}_{i-1}, a, \underbrace{e, \ldots, e}_{n-i}) \right\}.$

Since $f(\underbrace{e, \ldots, e}_{i-1}, a, \underbrace{e, \ldots, e}_{n-i}) = a$, we have

$\quad F(\underbrace{B, \ldots, B}_{i-1}, f(a, B, e^*), \underbrace{B, \ldots, B}_{n-i}) = f(a, B, e^*).$

(3) $F(f(a_1, B, e^*), \ldots, f(a_n, B, e^*)) = \{f(a, B, e^*) \mid a \in f(a_1^n)\}$. Since P is an n-ary polygroup, then

$\quad a_i \in f(a_{i-1}^{-1}, \ldots, a_1^{-1}, a, a_n^{-1}, \ldots, a_{i+1}^{-1}).$

Hence

$\quad f(a_i, B, e^*) \in F(f(a_{i-1}^{-1}, B, e^*), \ldots, f(a_1^{-1}, B, e^*), f(a, B, e^*), f(a_n^{-1}, B, e^*), \ldots, f(a_{i+1}^{-1}, B, e^*))$

$\qquad\qquad\quad = F(f(a_{i-1}, B, e^*)^{-l}, \ldots, f(a_1, B, e^*)^{-l}, f(a, B, e^*), f(a_n, B, e^*)^{-l}, \ldots, f(a_{i+1}, B, e^*)^{-l}).$

Thus $\langle P/B, F, B, ^{-l} \rangle$ is an n-ary polygroup.

Let A be a fuzzy n-ary subpolygroup of P, $\langle P, f, e, ^{-1} \rangle$ be an n-ary polygroup, and B be a fuzzy n-ary subpolygroup of P; then A/B is a fuzzy n-ary set on P/B defined as follows:

$\quad A/B : P/B \to [0, 1]$ satisfying

$\quad A/B(f(a, B, e^*)) = \sup\limits_{f(x, B, e^*) = f(a, B, e^*)} A(x), \quad$ for any $f(a, B, e^*)$ in P/B. $\quad\square$

Theorem 3.5. *The above A/B is a fuzzy n-ary subpolygroup of P/B.*

Proof. (1) For any $f(a, B, e^*) \in F(f(a_1, B, e^*), \ldots, f(a_n, B, e^*)), a \in f(a_1^n)$, we have

$$\inf\{A/B(f(a, B, e^*))\} = \inf\left\{ \sup\limits_{f(x, B, e^*) = f(a, B, e^*)} A(x) \right\} = \sup\limits_{f(x, B, e^*) = f(a, B, e^*)} \left\{ \inf\limits_{x \in f(x_1^n)} A(x) \right\}.$$

By Definition 2.2,

$$\inf\{A/B(f(a, B, e^*))\} \geq \sup\limits_{f(x_1, B, e^*) = f(a_1, B, e^*), \ldots, f(x_n, B, e^*) = f(a_n, B, e^*)} \min\{A(x_1), A(x_2), \ldots, A(x_n)\}$$

$$= \min\left\{ \sup\limits_{f(x_1, B, e^*) = f(a_1, B, e^*)} A(x_1), \ldots, \sup\limits_{f(x_n, B, e^*) = f(a_n, B, e^*)} A(x_n) \right\}$$

$$= \min\{A/B(f(a_1, B, e^*)), \ldots, A/B(f(a_n, B, e^*))\}.$$

(2) For any $f(a, B, e^*) \in P/B$, by Definition 2.2, for any $x \in P$, we have $A(x^{-1}) \geq A(x)$. Hence

$$
\begin{aligned}
A/B(f(a, B, e^*)^{-I}) &= A/B(f(a^{-1}, B, e^*)) \\
&= \sup_{f(x^{-1}, B, e^*) = f(a^{-1}, B, e^*)} A(x^{-1}) \\
&\geq \sup_{f(x, B, e^*) = f(a, B, e^*)} A(x) \\
&= A/B(f(a, B, e^*)).
\end{aligned}
$$

Hence A/B is a fuzzy n-ary subpolygroup of P/B. □

Definition 3.2. The above fuzzy n-ary subpolygroup A/B is called the fuzzy n-ary factor polygroup of A with respect to B.

Example 3.1. Consider Example 2.1 and let G be a fuzzy n-ary polygroup; then H/G is a fuzzy n-ary factor polygroup of H with respect to G.

4. The structure of product fuzzy n-ary factor polygroups

Definition 4.1. Let $\langle P, f, e, ^{-1} \rangle$ be an n-ary polygroup, and A be a fuzzy n-ary set of P. A is said to have the sup property if, for any nonempty subset $X \subset P$, there is a $x' \in X$ such that $A(x') = \sup_{x \in X} A(x)$.

Proposition 4.1. *If A has the sup property, then $(A/B)[\alpha] = A[\alpha]/B$ for any $\alpha \in [0, 1]$.*

Proof. (1) For any $f(a, B, e^*) \in A/B[\alpha]$, we have

$$
A/B(f(a, B, e^*)) = \sup_{f(x, B, e^*) = f(a, B, e^*)} A(x).
$$

Since A has the sup property, there exists $x' \in P$ such that $f(x', B, e^*) = f(a, B, e^*)$ and $A(x') = \sup_{x \in X} A(x)$. Hence

$$
A/B(f(a, B, e^*)) = A(x') \geq \alpha,
$$

which implies that $x' \in A[\alpha]$. Therefore $f(x', B, e^*) \in A[\alpha]/B, f(a, B, e^*) \in A[\alpha]/B$. Hence

$$
(A/B)[\alpha] \subseteq A[\alpha]/B.
$$

(2) For any $f(a, B, e^*) \in A[\alpha]/B$, which implies $A[a] \geq \alpha$,

$$
A/B(f(a, B, e^*)) = \sup_{f(x, B, e^*) = f(a, B, e^*)} A(x).
$$

Since $f(x, B, e^*) = f(a, B, e^*) \in A[\alpha]/B, A[x] \geq \alpha$. Hence

$$
A/B(f(a, B, e^*)) = \sup_{f(x, B, e^*) = f(a, B, e^*)} A(x) \geq \alpha.
$$

So $f(a, B, e^*) \in A/B[\alpha]$. By the above result, we obtain $A[\alpha]/B \subseteq (A/B)[\alpha]$.

Hence $(A/B)[\alpha] = A[\alpha]/B$ for any $\alpha \in [0, 1]$. □

Proposition 4.2. *Let $\langle P, f, e, ^{-1} \rangle$, $\langle P', f', e', ^{-1} \rangle$ be an n-ary polygroup, and A, B be fuzzy n-ary subpolygroups of P and P', respectively. If A has the sup property and there is a mapping g such that $g(A[\alpha]) = B[\alpha]$ for any $\alpha \in [0, 1]$, then $g(A) = B$.*

Proof. Let $y \in P'$ and $\alpha = B(y)$. Then $y \in B[\alpha] = g(A[\alpha])$. Thus there exists $x' \in A[\alpha]$ such that $g(x') = y$. Hence

$$
g(A)(y) = \sup_{g(x) = y} A(x) \geq A(x') \geq \alpha = B(y).
$$

Suppose that $B(y) < \sup_{g(x) = y} A(x)$. Since A has the sup property, there exists $x'' \in P'$ such that

$$
\sup_{g(x) = y} A(x) = A(x''), \qquad g(x'') = y.
$$

Let $\beta = A(x'')$, then $x'' \in A[\beta]$. Hence $y = g(x'') \in g(A[\beta])$ and y is not in $B[\beta]$, which is a contradiction. Hence $B(y) = g(A)(y)$. Thus $g(A) = B$. □

Proposition 4.3. *Let A, B be separately fuzzy n-ary subpolygroups of P, P'. For any $\alpha \in [0, 1], (A \times B)[\alpha] = A[\alpha] \times B[\alpha]$ holds.*

Proof. For any $(x, y) \in (A \times B)[\alpha]$, we have $(A \times B)(x, y) = \min\{A(x), B(y)\} \geq \alpha$. Hence $A(x) \geq \alpha, B(y) \geq \alpha$, which implies that $x \in A[\alpha]$ and $y \in A[\alpha]$. Therefore $(x, y) \in A[\alpha] \times B[\alpha]$.

By the above result, we obtain

$$(A \times B)[\alpha] \subseteq A[\alpha] \times B[\alpha].$$

Similarly we can also get $A[\alpha] \times B[\alpha] \subseteq (A \times B)[\alpha]$. Hence

$$(A \times B)[\alpha] = A[\alpha] \times B[\alpha]. \quad \square$$

Proposition 4.4. *Let $\langle P, f, e, ^{-1} \rangle$ be an n-ary polygroup, A be a fuzzy n-ary subpolygroup of P, and B be a fuzzy n-ary normal subpolygroup of P. If A has the sup property, then A/B has the sup property.*

Proof. For any X included in A/B,

$$\sup_{f(x,B,e^*) \subseteq X} A/B(f(x, B, e^*)) = \sup_{f(x,B,e^*) \subseteq X} \sup_{f(y,B,e^*)=f(x,B,e^*)} A(y).$$

Since A has the sup property, then, for every $f(x, B, e^*)$ in X, there is y_x such that

$$A/B(f(x, B, e^*)) = \sup_{f(y,B,e^*)=f(x,B,e^*)} A(y) = A(y_x)$$

holds.

Hence

$$\sup_{f(x,B,e^*) \subseteq X} A/B(f(x, B, e^*)) = \sup_{f(x,B,e^*) \subseteq X} A(y_x) = \sup_{y_x \in Y_x} A(y_x).$$

Hence Y_x is the collection of all y_x as above. Still by A having the sup property, there exists a y_* in Y_x such that

$$\sup_{y_x \in Y_x} A(y_x) = A(y_*) = \sup_{x_* \in X} A/B(f(x_*, B, e^*)).$$

Hence

$$\sup_{f(x,B,e^*) \subseteq X} A/B(f(x, B, e^*)) = A(y_*) = A/B(f(x_*, B, e^*)).$$

Thus A/B has the sup property. $\quad \square$

Theorem 4.5. *Let $\langle P, f, e, ^{-1} \rangle, \langle P', f', e', ^{-1} \rangle$ be an n-ary polygroup. Suppose that A, B are fuzzy n-ary subpolygroups of P and P', respectively, with the sup property. Suppose that A', B' are separately fuzzy n-ary normal subpolygroups of P and P', A'(e) = B'(e'). Then $A \times B/A' \times B'$ is isomorphic to $A/A' \times B/B'$.*

Proof. In [13], Davvaz et al. have proved that $A \times B$ is a fuzzy n-ary subpolygroup of $P \times P'$. We can also prove that $A' \times B'$ is a fuzzy n-ary normal subpolygroup of $P \times P'$.

Consider any nonempty set $T = X \times Y \subseteq P \times P'$. Since A, B have the sup property, there exists $x' \in X, y' \in Y$ such that

$$A(x') = \sup_{x \in X} A(x), \qquad B(y') = \sup_{y \in Y} B(y).$$

For any (x, y) in T,

$$(A \times B)(x, y) = \min\{A(x), B(y)\} \leq \min\{A(x'), B(y')\},$$

such that

$$(A \times B)(x', y') = \sup_{(x,y) \in T} (A \times B)(x, y).$$

Hence $A \times B$ has the sup property.

Now we come to prove that $A[\alpha] \times B[\alpha]/A' \times B'$ is isomorphic to $A[\alpha]/A' \times B[\alpha]/B'$ for any $\alpha \in [0, 1]$. It is clear that $A[\alpha] \times B[\alpha]/A' \times B'$ is empty iff $A[\alpha]/A' \times B[\alpha]/B'$ is empty. Suppose that both of them are nonempty. Let

$$f\alpha : A[\alpha] \times B[\alpha]/A' \times B' \rightarrow A[\alpha]/A' \times B[\alpha]/B' \quad \text{satisfy}$$
$$f\alpha((f \times f')((a, b), A' \times B', (e, e')^*)) = (f(a, A', e^*), f'(b, B', e^*)), \quad a \in A[\alpha], \; b \in B[\alpha].$$

If

$$(f \times f')((a, b), A' \times B', (e, e')^*) = (f \times f')((c, d), A' \times B', (e, e')^*), \tag{1}$$

then

$$A' \times B'((f \times f')((a, b), (c, d), (e, e')^*)) = A' \times B'(e, e')$$
$$A' \times B'(f(a^{-1}, c, e^*), f'(b^{-1}, d, e^*)) = A' \times B'(e, e')$$
$$\min\{A'(f(a^{-1}, c, e^*)), B'(f'(b^{-1}, d, e^*))\} = \min\{A'(e) \times B'(e')\}.$$

Since $A'(e) = B'(e')$, we have

$$A'(f(a^{-1}, c, e^*)) = A'(e), \qquad B'\{(f'(b^{-1}, d, e^*))\} = B'(e').$$

Therefore

$$f(a, A', e^*) = f(c, A', e^*), \qquad f'(b, B', e^*) = f'(d, B', e^*).$$

Hence

$$(f(a, A', e^*), f'(b, B', e^*)) = (f(c, A', e^*), f'(d, B', e^*)). \tag{2}$$

Thus $f\alpha$ is an one-valued mapping.

It is clear that, if the formula (2) holds, then the formula (1) holds, so $f\alpha$ is a monomorphism. For any $(f \times f')((x, y), A' \times B', (e, e')^*)$, $(f \times f')((x', y'), A' \times B', (e, e')^*) \in A[\alpha] \times B[\alpha]/A' \times B'$, we have

$$f\alpha((f \times f')((x, y), A' \times B', (e, e')^*)(f \times f')((x', y'), A' \times B', (e, e')^*))$$
$$= f\alpha((f \times f')(f(x, x', e^*), f'(y, y', e^*), A' \times B', (e, e')^*))$$
$$= (f(x, x', A', e^*), f'(y, y', B', e^*))$$
$$= ((f \times f')((f(x, A', e^*), f'(y, B', e^*)), (f(x', A', e^*), f'(y', B', e^*)), (e, e')^*))$$
$$= f\alpha((f \times f')((x, y), A' \times B', (e, e')^*)f\alpha(f \times f')((x', y'), A' \times B', (e, e')^*)).$$

Hence $f\alpha$ is an isomorphism.

It is clear that the restriction of f_0 on $(A \times B/A' \times B')[\alpha]$ is $f\alpha$. For any $\alpha \in [0, 1]$, we have

$$\begin{aligned}(A \times B/A' \times B')[\alpha] &= (A \times B/A')[\alpha] \times B' \\ &= A[\alpha] \times B[\alpha]/A' \times B' \\ &= A[\alpha]/A' \times B[\alpha]/B' \\ &= (A/A')[\alpha] \times (B/B')[\alpha] \\ &= (A/A' \times B/B')[\alpha].\end{aligned}$$

Hence $A \times B/A' \times B'$ is isomorphic to $A/A' \times B/B'$ by Proposition 4.2. $\square$

Definition 4.2. Let $\langle P, f, e, ^{-1} \rangle$ be an n-ary polygroup, and A, B be separately fuzzy n-ary subsets of nonempty set P. The inner product AB is the fuzzy n-ary subset of P defined by

$$AB(x) = \sup_{x \in f(a, b, e^*)} \{\min\{A(a), B(b)\}\}, \quad x \in P.$$

Proposition 4.6. *Let A, B be fuzzy n-ary subsets of P. If A, B have the sup property, then $AB[\alpha] = f(A[\alpha], B[\alpha], e^*)$.*

Proof. It is clear that $f(A[\alpha], B[\alpha], e^*) \subseteq (AB)[\alpha]$. Let $x \in (AB)[\alpha]$; then

$$(AB)(x) = \sup_{x \in f(x, y^{-1}, y, e^*)} \min\{A(f(x, y^{-1}, e^*)), B(y)\} \geq \alpha.$$

Since A, B have the sup property, we can get a y' such that

$$\min\{A(f(x, y'^{-1}, e^*)), B(y')\} \geq \alpha.$$

Hence

$$A(f(x, y'^{-1}, e^*)) \geq \alpha, \qquad B(y') \geq \alpha,$$

such that

$$f(x, y'^{-1}, e^*) \in A[\alpha], \quad y' \in B[\alpha].$$

Therefore

$$x \in f(x, y'^{-1}, y, e^*) = f(f(x, y'^{-1}, e^*), y, e^*) \subseteq f(A[\alpha], B[\alpha], e^*),$$

such that

$$AB[\alpha] \subseteq f(A[\alpha], B[\alpha], e^*).$$

Thus

$$AB[\alpha] = f(A[\alpha], B[\alpha], e^*). \quad \square$$

Proposition 4.7. *Let A, B be fuzzy n-ary subpolygroups of P such that AB is the fuzzy n-ary subpolygroup of P. Let C be a fuzzy n-ary normal subpolygroup of P. If A, B have the sup property, then $AB/C = (A/C)(B/C)$.*

Proof. It is easy to prove that AB has the sup property. For any $\alpha \in [0, 1]$,

$$
\begin{aligned}
(AB/C)[\alpha] &= (AB)[\alpha]/C \\
&= f(A[\alpha], B[\alpha], C, e^*) \\
&= f(f(A[\alpha], C, e^*), f(B[\alpha], C, e^*), e^*) \\
&= f((A/C)[\alpha], (B/C)[\alpha], e^*) \\
&= ((A/C)(B/C))[\alpha].
\end{aligned}
$$

Hence

$$AB/C = (A/C)(B/C). \quad \square$$

Theorem 4.8. *Let A, B be fuzzy n-ary subpolygroups of P with the sup property such that AB is the fuzzy n-ary subpolygroup of P. Let C be a fuzzy n-ary normal subpolygroup of P. Then AB/C is homomorphic to $A \times B/C \times C$.*

Proof. For any $\alpha \in [0, 1]$,

$$
\begin{aligned}
(A \times B/C \times C)[\alpha] &= (A \times B)[\alpha]/C \times C \\
&= A[\alpha] \times B[\alpha]/C \times C \\
&= A[\alpha]/C \times B[\alpha]/C
\end{aligned}
$$

$$
\begin{aligned}
(AB/C)[\alpha] &= (AB)[\alpha]/C \\
&= f(A[\alpha], B[\alpha], e^*)/C \\
&= f(A[\alpha], B[\alpha], C, e^*).
\end{aligned}
$$

Let $g\alpha : A[\alpha]/C \times B[\alpha]/C \to f(A[\alpha], B[\alpha], e^*)/C$ satisfy $g\alpha((f(a, C, e^*), f(b, C, e^*))) = f(a, b, C, e^*)$.

If

$$(f(a, C, e^*), f(b, C, e^*)) = (f(a', C, e^*), f(b', C, e^*)),$$

then

$$f(a, C, e^*) = f(a', C, e^*), \qquad f(b, C, e^*) = f(b', C, e^*),$$

such that

$$C(f(a^{-1}, a', e^*)) = C(e), \qquad C(f(b^{-1}, b', e^*)) = C(e).$$

For any $x \in f(f(a, b, e^*)^{-1}, a', b', e^*) = f(b^{-1}, a^{-1}, a', b', e^*)$, so

$$
\begin{aligned}
C(f(b^{-1}, a^{-1}, a', b', e^*)) &= C(f(b', b^{-1}, a^{-1}, a', e^*)) \\
&= C(f(f(b', b^{-1}, e^*), f(a^{-1}, a', e^*), e^*)).
\end{aligned}
$$

For any $y \in f(f(b', b^{-1}, e^*), f(a^{-1}, a', e^*), e^*)$, we have

$$
\begin{aligned}
C(y) &\geq \min\{C(f(b', b^{-1}, e^*)), C(f(a^{-1}, a', e^*))\} \\
&= \min\{C(f(b^{-1}, b', e^*)), C(f(a^{-1}, a', e^*))\} \\
&= C(e).
\end{aligned}
$$

Hence

$$C(f(b^{-1}, a^{-1}, a', b', e^*)) = C(f(f(a, b, a)^{-1}, a', b', e^*)) = C(e).$$

Therefore $f(a, b, C, e^*) = f(a', b', C, e^*)$; thus $g\alpha$ is an one-valued mapping.

It is clear that $g\alpha$ is a homomorphism. Hence $f\alpha g\alpha$ is a homomorphism from $(A \times B/C \times C)[\alpha]$ onto $(AB/C)[\alpha]$, $f\alpha$ is as in Theorem 4.5. $\quad \square$

It is easy to prove that the restriction of $f_0 g_0$ on $(A \times B/C \times C)[\alpha]$ is $f\alpha g\alpha$. Hence AB/C is homomorphic to $A \times B/C \times C$.

5. Conclusions

In this paper we concentrate our study on the algebraic properties of fuzzy n-ary factor polygroups. We study the product structures of fuzzy n-ary polygroups and present their properties. Our future work on this topic will be focused on other algebraic structures such as the lattice, ring and field.

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