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SOME PROPERTIES OF κ-COMPLETE IDEALS DEFINED IN TERMS OF INFINITE GAMES

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We consider several infinite games involving a given κ-complete ideal over a regular uncountable cardinal κ. We give a new characterization of precipitous ideals and introduce the class of weakly precipitous and pseudo-precipitous ideals. We also define the notion of degree of functions and functionals and compare it with the Galvin–Hajnal norm.

0. Introduction

In [7], K. Prikry and the present author introduced a class of ideals, called *precipitous*, which the subsequent work of W. Mitchell and others proved to be the correct generalization of κ-complete ultrafilters for successor cardinals. In [5], this author observed that precipitous ideals can be characterized in terms of an infinite game, due to F. Galvin (which was a generalization of Banach's game [0]). Galvin's game is investigated in the paper [3].

The present paper has three parts. In Section 1, we modify Galvin's game and obtain a related game which, unlike Galvin's game, is always determined, and show that precipitous ideals can be characterized in terms of the modified game. We also address ourselves to the question what relation there is between the Galvin–Hajnal norm of an ordinal function f, and the ordinal represented by f in the generic ultrapower. We obtain new equivalences for precipitous ideals, and raise some new questions.

Section 2 is devoted to a related game, invented by S. Shelah for his work on arithmetic of singular cardinals. We call the ideals characterized in terms of Shelah's game *weakly precipitous*.

In Section 3 we introduce a new related game, and the corresponding class of ideals, which we call *pseudo-precipitous*. We show that like the precipitous ideals, pseudo-precipitous ideals can be defined in terms of generic ultrapowers.

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The relations between the three classes of ideals known to us are as follows:

precipitous ↘
weakly precipitous.
pseudo-precipitous ↗

It is known that 'precipitous' is equiconsistent with 'measurable cardinals'. Although we show that 'pseudo-precipitous' is at least as strong as 'measurable', we have only been able to prove its consistency from something stronger, namely 'κ^+-saturated'. Weakly precipitous ideals are weaker; by Shelah's result they follow from 'Ramsey'. Although we believe that 'weakly precipitous' is a large cardinal property, we were unable to prove it.

1. Precipitous ideals

Throughout the paper, let κ be a regular uncountable cardinal, and let I be a nontrivial κ-complete ideal over κ; i.e. $I \subset \mathcal{P}(\kappa)$ and

(1.1) (i) if $X \subseteq Y$ and $Y \in I$, then $X \in I$,
(ii) if $\gamma < \kappa$ and $\{X_\xi : \xi < \gamma\} \subset I$, then $\bigcup_{\xi<\gamma} X_\xi \in I$,
(iii) $\{\alpha\} \in I$ for all $\alpha < \kappa$.

Let I^+ denote the set $\{S \subseteq \kappa : S \notin I\}$; we call the elements of I^+ *sets of positive measure.* (We often use the measure theoretic terminology; e.g. "for almost all $\alpha \in S$" means that the set of all contrary $\alpha \in S$ belongs to I). I is called *normal* if for every $S \in I^+$ and every function f on S, if $f(\alpha) < \alpha$ for all $\alpha \in S$, then f is constant on some set $T \subseteq S$ of positive measure.

Let $S \in I^+$. A collection W of subsets of S is an *I-partition* of S if

(1.2) (i) $X \in I^+$ for every $X \in W$,
(ii) if $X, Y \in W$ and $X \neq Y$, then $X \cap Y \in I$,
(iii) W is maximal: if $X \subseteq S$ has positive measure, then $X \cap Y \in I^+$ for some $Y \in W$.

If λ is a cardinal, then I is *λ-saturated* if there is no I-partition of κ of size λ. We denote by $\operatorname{sat}(I)$ the least λ such that I is λ-saturated.

If W and Z are two partitions of S, then $W \leqslant Z$ means that

(1.3) for every $X \in Z$ there is $Y \in W$ such that $Y \subseteq X$.

An *ordinal function* is a function whose values are ordinal numbers. An *I-function* is a function whose domain is a set $S \in I^+$. If S is a set of positive measure, a *functional* on S is a collection F of ordinal I-functions such that

(1.4) (i) $W_F = \{\operatorname{dom}(f) : f \in F\}$ is an I-partition of S,
(ii) if $f, g \in F$ and $f \neq g$, then $\operatorname{dom}(f) \neq \operatorname{dom}(g)$.

If F and G are two functionals on S, then $F < G$ means that

(1.5) (i) $W_F \leqslant W_G$,
(ii) if $f \in F$ and $g \in G$ are such that $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$, then $f(\alpha) < g(\alpha)$ for all $\alpha \in \mathrm{dom}(f)$

Let P_I denote the set I^+ of all sets of positive measure partially ordered by inclusion, and consider P_I as a notion of forcing. (Equivalently, we may consider the Boolean valued model via the completion of the Boolean algebra $\mathcal{P}(\kappa)/I$). A generic set $G \subset P_I$ is an ultrafilter over the ground model and so we consider the ultrapower $\mathrm{Ult}_G(V)$ of V by G. We call this ultrapower a *generic ultrapower*.

1.1. The game $\mathcal{G}(I)$

We consider an infinite game between two players, One and Two. Player One moves first and chooses a set A_0 of positive measure. Then Two chooses a set $B_0 \subseteq A_0$ of positive measure. Then One chooses $A_1 \subseteq B_0$ of positive measure, and so on. Thus they produce a sequence

$$A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \supseteq \cdots$$

of sets of positive measure. Player One wins if and only if

$$\bigcap_{n=0}^{\infty} A_n = \emptyset.$$

Theorem 1.1. *Let I be a nontrivial κ-complete ideal over a regular uncountable cardinal κ. The following properties are equivalent:*

(1) *If S is a set of positive measure and if $W_0 \geqslant W_1 \geqslant W_2 \geqslant \cdots$ is a sequence of I-partitions of S, then there exists a sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ of sets $X_0 \in W_0$, $X_1 \in W_1$, $X_2 \in W_2, \ldots$, such that $\bigcap_{n=0}^{\infty} X_n \neq \emptyset$.*

(2) *There is no infinite descending sequence $F_0 > F_1 > F_2 > \cdots$ of functionals on S, for any $S \in I^+$.*

(3) *Every condition $S \in P_I$ forces that the generic ultrapower* $\mathrm{Ult}_G(V)$ *is well-founded.*

(4) *Player One does not have a winning strategy in the game $\mathcal{G}(I)$.*

(For the proof of Theorem 1.1 see [8] and [3].) If I has these properties it is called *precipitous*.

1.2. The games $\mathcal{G}_1(I)$ and $\mathcal{G}_2(I)$

Again, we consider two players, One and Two. This time, One starts the game by choosing an ordinal I-function f_0. Then Two chooses a set $B_0 \subseteq \mathrm{dom}(f_0)$ of positive measure. Then One picks an ordinal I-function f_1 such that $\mathrm{dom}(f_1) \subseteq B_0$, and $f_1(\alpha) < f_0(\alpha)$ for all $\alpha \in \mathrm{dom}(f_1)$. They continue in this fashion and produce a sequence $f_0, f_1, \ldots$ of I-functions such that for every n, $\mathrm{dom}(f_{n+1}) \subseteq \mathrm{dom}(f_n)$ and

$f_{n+1}(\alpha) < f_n(\alpha)$ for all $\alpha \in \mathrm{dom}(f_{n+1})$. If Player One can continue the play indefinitely, he wins; otherwise, Two wins.

Clearly, this game is determined: If Two does not have a winning strategy, then One does; he simply goes on and keeps making moves that witness that Two does not have a winning strategy. (The game is open is a suitable topology.)

Theorem 1.2. *Player One has a winning strategy in the game $\mathscr{G}_1(I)$ if and only if Player One has a winning strategy in the game $\mathscr{G}(I)$.*

Corollary. *I is precipitous if and only if*

(5) *Player Two has a winning strategy in the game $\mathscr{G}_1(I)$.*

Proof. It is easy to see that if One has a winning strategy in $\mathscr{G}_1$, then One has a winning strategy in $\mathscr{G}$: The winning moves of One in $\mathscr{G}$ are simply the sets $A_0, A_1, \ldots$ which are the domains of the functions $f_0, f_1, \ldots$, the winning moves of One in $\mathscr{G}_1$. Clearly, the intersection $\bigcap_{n=0}^{\infty} A_n$ has to be empty as otherwise we would have a descending sequence of ordinals $f_0(\alpha) > f_1(\alpha) > \cdots$ for any α in the intersection. Thus let us assume that One has a winning strategy in the game $\mathscr{G}(I)$, and let us show that there is a winning strategy for One in the (more difficult) game $\mathscr{G}_1(I)$.

Let σ be a winning strategy for One in $\mathscr{G}$. Let us consider the set T of all finite sequences

$$\langle A_0, B_0, A_1, B_1, \ldots, A_n \rangle$$

of odd length $(2n+1)$ of sets of positive measure such that $A_0 = \sigma(\langle\ \rangle)$, $B_0 \subseteq A_0$, $A_1 = \sigma(\langle A_0, B_0 \rangle)$, $B_1 \subseteq A_1$, $A_2 = \sigma(\langle A_0, B_0, A_1, B_1 \rangle)$, etc.; i.e. of finite plays in the game $\mathscr{G}$ in which One uses the strategy σ. The set T ordered by extension of sequences is a tree.

Every path in the tree T consitutes a play $\langle A_0, B_0, A_1, B_1, \ldots \rangle$ in $\mathscr{G}$, and since the A_n's are obtained by One's winning strategy σ, the intersection $\bigcap_{n=0}^{\infty} A_n$ is empty. Let $\alpha \in \kappa$, and let T_α consist of all those $\langle A_0, B_0, \ldots, A_n \rangle \in T$ such that $\alpha \in A_n$. It follows that T_α is a well-founded tree. Consequently, there is a rank function ρ_α associated with the tree T_α. Clearly,

$$\rho_\alpha(\langle A_0, B_0, \ldots, A_{n-1} \rangle) > \rho_\alpha(\langle A_0, B_0, \ldots, A_{n-1}, B_{n-1}, A_n \rangle) \quad \text{for any } \langle A_0, B_0, \ldots, A_n \rangle \in T_\alpha .$$

Now we construct a winning strategy τ for One in $\mathscr{G}_1$: As for the first move $f_0 = \tau(\langle\ \rangle)$, let $A_0 = \sigma(\langle\ \rangle)$, and let f_0 be the following function on A_0: for $\alpha \in A_0$ let $f_0(\alpha) = \rho_\alpha(\langle A_0 \rangle)$.

By induction on n, let us assume that we have constructed τ for all plays $\langle f_0, B_0, \ldots, f_{k-1}, B_{k-1} \rangle$ in $\mathscr{G}_1$ of all lengths $2k < 2n$, and let $\langle f_0, B_0, \ldots, f_{n-1}, B_{n-1} \rangle$ be a play of length $2n$. Let $A_n = \sigma(\langle A_0, B_0, \ldots, B_{n-1} \rangle)$, where $A_i = \mathrm{dom}(f_i)$ for all

$i<n$, and let f_n be the following function on A_n: for $\alpha \in A_n$ let $f_n(\alpha)=\rho_\alpha(\langle A_0, B_0, \ldots, B_{n-1}, A_n\rangle)$. It follows that $f_n(\alpha)<f_{n-1}(\alpha)$ for all $\alpha \in \mathrm{dom}(f_n)$, and hence f_n is a legal move for Player One. Consequently, τ is a winning strategy for One in the game $\mathscr{G}_1$. □

Next we consider the following game $\mathscr{G}_2(I)$; later on in this section we shall look at its refinement. The moves of Player One are the same as in the game $\mathscr{G}_1$, namely his nth move is an I-function f_n such that $\mathrm{dom}(f_n) \subseteq B_{n-1}$ and $f_n(\alpha)< f_{n-1}(\alpha)$ for all $\alpha \in \mathrm{dom}(f_n)$. Player Two's nth move is again a set of positive measure $B_n \subseteq \mathrm{dom}(f_{n-1})$, but he also has to choose an ordinal number α_n, such that $\alpha_n<\alpha_{n-1}$. Clearly, not both players can make legal moves indefinitely; whoever makes the last move wins.

Theorem 1.3. *Player Two has a winning strategy in the game $\mathscr{G}_2(I)$ if and only if Player Two has a winning strategy in the game $\mathscr{G}_1(I)$.*

Corollary. *I is precipitous if and only if*

(6) *Player Two has a winning strategy in $\mathscr{G}_2(I)$.*

Proof. Since the game $\mathscr{G}_2$ is more difficult for Player Two than $\mathscr{G}_1$, it suffices to prove that if Two has a winning strategy in $\mathscr{G}_1$, then Two has a winning strategy in $\mathscr{G}_2$.

Thus assume that Two has a winning strategy in $\mathscr{G}_1$. By Theorem 1.2, I is a precipitous ideal and so the generic ultrapower is well-founded. We shall now describe a winning strategy for Two in $\mathscr{G}_2$: When Player One plays an I-function f_n, every condition $S \subseteq \mathrm{dom}(f_n)$ in P_I forces that f_n represents an ordinal number. Let α_n be the least ordinal forced by some condition to be represented by f_n, and let $B_n \subseteq \mathrm{dom}(f_n)$ be such a condition. It is clear that $B_n \Vdash [f_n]_G<[f_{n-1}]_G$ and so $\alpha_n<\alpha_{n-1}$. Hence Two is assured to have the last move. □

(We remark in passing that one can directly produce a winning strategy for Two in $\mathscr{G}_2$ from a winning strategy for Two in $\mathscr{G}_1$ by considering the well founded tree of finite plays in $\mathscr{G}_1$ in which Two employs his winning strategy.)

1.3. Degrees of functionals

In [2], Galvin and Hajnal defined the *norm* of an ordinal function on κ. If f and g are ordinal functions, let $f<_I g$ mean that $\{\alpha: f(\alpha) \geqslant g(\alpha)\} \in I$. The relation $<_I$ is well-founded, and

$$\|f\|=\|f\|_I=\sup\{\|g\|_I+1: g<_I f\}$$

is the rank of f in this well-founded relation.

For $S \in I^+$, let

$$\|f\|_S=\|f\|_{I \restriction S}$$

where $I \restriction S$ is the ideal defined by $(I \restriction S)^+ = I^+ \cap P(S)$. Clearly, $\|f\|_S \geq \|f\|$ for all $S \in I^+$.

If α is an ordinal number, then the *αth function*, if it exists, is the unique (mod I) function f_α such that $\|f_\alpha\|_S = \alpha$ for all $S \in I^+$. It is clear that for every $\alpha < \kappa$ the αth function exists, namely the constant function with value α. If I is normal, then the αth function exists for each $\alpha < \kappa^+$ (by induction: use diagonalization at stages of cofinality κ). A result of A. Hajnal [4] states that if $V = L$, then the κ^+th function does not exist for the dual of the filter of closed unbounded sets.

It is easy to see that if the αth function exists for all α, then I is precipitous and that for every α and every $S \in P_I$, S forces that f_α represents α in the generic ultrapower. It is not known whether the functions f_α exist for every precipitous ideal (probably not). (It has been observed by Levinski [10] that the precipitous ideal on ω_1 constructed by Mitchell in [6] does have this property.)

Let f be a function of norm α, and assume that $S \Vdash f$ respresents an ordinal number. Then, as one can easily verify, $S \Vdash f$ represents an ordinal greater than or equal to α. There does not seem to be however any further relation between the norm and the ordinal represented by the function (except in case of *normed ideals* to be discussed in 1.5). Thus we define the *degree* of an ordinal function, and more generally, the *degree of a functional.*

Let F be a functional on κ. We say that F has a degree if the relation $<$ (defined in (1.5)) is well-founded below F, i.e. on the set $\{G : G < F\}$. The *degree* of F

$$\deg(F) = \deg_I(F) = \sup\{\deg_I(G) + 1 : G < F\}$$

is the length of this well-founded relation. If f is an ordinal function on κ, then $\deg(f) = \deg(\{f\})$ (as $\{f\}$ is a functional on κ), if the right hand side is defined. We also define

$$\deg_S(F) = \deg_{I \restriction S}(F), \qquad \deg_S(f) = \deg_{I \restriction S}(f)$$

for all $S \in I^+$.

It follows from Theorem 1.1(2) that I is precipitous if and only if every functional on every $S \in I^+$ has a degree. Similarly we have:

Proposition. *Every functional has a degree if and only if $I \restriction S$ is precipitous for some $S \in I^+$.*

Proof. Let $S \in I^+$ be such that $I \restriction S$ is precipitous. There can be no descending sequence of functionals on κ since the same sequence (restricted to S) would be a descending sequence with respect to the ideal $I \restriction S$, contradicting the precipitousness of $I \restriction S$. Hence every functional has a degree.

Conversely, if no $I \restriction S$ is precipitous, then the set $\{S \in I^+$: there is a descending sequence of functionals on $S\}$, is dense. Thus there is an I-partition W such that for each $S \in W$ we have a descending sequence $F_0^S > F_1^S > \cdots$ on S. Now it is easy to build up a descending sequence of functionals on κ. Hence not every functional has a degree. $\square$

Theorem 1.4. *The degree of a functional F is the least ordinal α such that for some $f \in F$ and some set $S \subseteq \mathrm{dom}(f)$ of positive measure, $S \Vdash f$ represents α in the generic ultrapower.*

Proof. We shall prove the theorem by induction on α. Let α be an ordinal and let F be a functional on κ. First note that the set of all $S \in I^+$ with the following property is dense: either $S \subseteq \mathrm{dom}(f)$ for some $f \in F$ and S forces that f represents an ordinal in Ult_G, or there is a descending sequence of functionals on S below F. Thus there exists an I-partition $W \leqslant W_F$ such that for each $S \in W$ either there is an α_S such that $S \Vdash f$ represents α_S (where $f \in F$ is such that $S \subseteq \mathrm{dom}(f)$), or there is a descending sequence $F_0^S > F_1^S > \cdots$ of functionals on S below F.

If there is no $S \in W$ of the first kind, then F does not have a degree as we can use the F_n^S to build up a descending sequence of functionals below F. If there is such S, let α be the least value of all such α_S. It suffices to prove that $\deg(F) = \alpha$.

If $G < F$, let $S \in W$ be such that $\alpha_S = \alpha$ and let $g \in G$ be such that $T = \mathrm{dom}(g) \cap S$ has positive measure. Then T forces that g represents an ordinal less than α, and by the induction hypothesis we have $\deg(G) < \alpha$. It follows that $\deg(F) \leqslant \alpha$.

To show that $\deg(F) \geqslant \alpha$, let $\beta < \alpha$. For each $S \in W$ of the first kind, let G_S be a functional on S such that for each $g \in G_S$, $\mathrm{dom}(g) \Vdash g$ represents β. For each $S \in W$ of the second kind, let $G_S = F_0^S$. Let G be the functional built up from the G_S, $S \in W$. Then β is the least β for which some S forces an element of G to represent β; by the induction hypothesis, $\deg(G) = \beta$. Since $G < F$, we have $\deg(F) > \beta$. Consequently, $\deg(F) \geqslant \alpha$. $\square$

Let α be an ordinal. The *αth functional*, it if exists, is a functional F_α such that $\deg_S(F_\alpha) = \alpha$ for all $S \in I^+$. It is easy to see that F_α is unique (mod I) in the sense that if F_α and F'_α are both such functionals then, whenever $f \in F_\alpha$ and $f' \in F'_\alpha$,

$$\{\xi \in \mathrm{dom}(f) \cap \mathrm{dom}(f') : f(\xi) \neq f'(\xi)\} \in I.$$

The following proposition is an immediate corollary of Theorem 1.

Proposition. *F is the αth functional if and only if for every $f \in F$, $\mathrm{dom}(f) \Vdash f$ represents α.* $\square$

One consequence of this is that if $\alpha < \beta$ and if F_β exists, then F_α exists. Another consequence is:

Corollary. *I is precipitous if and only if*

(7) *For every α, the αth functional exists.*

1.4. The game $\mathcal{G}_3(I, f, \alpha)$

We shall now consider a refinement of the game $\mathcal{G}_2(I)$. Let f be an ordinal function on κ, and let α be an ordinal number. The game $\mathcal{G}_3$ is played as follows:

One	Two
f_0	
	B_0, α_0
f_1	
	B_1, α_1
$\vdots$	$\vdots$

Each f_n is an I-function, and each B_n is a set of positive measure and $\mathrm{dom}(f_0) \supseteq B_0 \supseteq \mathrm{dom}(f_1) \supseteq B_1 \supseteq \cdots$. Moreover, $\alpha_0 > \alpha_1 > \cdots$, and $f_{n+1}(\xi) < f_n(\xi)$ for all n and all $\xi \in \mathrm{dom}(f_{n+1})$. In addition to these rules (as in $\mathcal{G}_2$), the first moves have to be such that $\alpha_0 < \alpha$ and $f_0(\xi) < f(\xi)$ for all $\xi \in \mathrm{dom}(f_0)$. Whichever player makes the last legal move wins.

By Theorem 1.3, I is precipitous if and only if for every f there is α such that Two has a winning strategy in $\mathcal{G}_3(I, f, \alpha)$. The following theorem gives a more detalied correlation:

Theorem 1.5. *Player Two has a winning strategy in the game $\mathcal{G}_3(I, f, \alpha)$ if and only if $\mathrm{deg}_S(f) \leq \alpha$ for all $S \in I^+$.*

Proof. First let us assume that $\mathrm{deg}_S(f) \leq \alpha$ for all $S \in I^+$. We shall describe a winning strategy for Two in $\mathcal{G}_3$. When One plays an I-function f_0 with domain A_0, then because $\mathrm{deg}_{A_0}(f) \leq \alpha$ we have $\mathrm{deg}_{A_0}(f_0) < \alpha$. Therefore there is, by Theorem 1.4, a set $B_0 \subseteq A_0$ of positive measure, and an ordinal α_0 such that B_0 forces that f_0 represents α_0. In other words, $\mathrm{deg}_S(f_0) = \alpha_0$ for all $S \subseteq B_0$ in I^+. Let Two play B_0, α_0. Then when One plays f_1, we can similarly find $B_1 \subseteq \mathrm{dom}(f_1)$ and $\alpha_1 < \alpha_0$ such that $\mathrm{deg}_S(f_1) = \alpha_1$ for all $S \subseteq B_1$, and so on. This way, Two can keep making legal moves and therefore Two wins.

Conversely, let us assume that for some $S \in I^+$, it is not the case that $\mathrm{deg}_S(f) \leq \alpha$. We shall describe a winning strategy for One in $\mathcal{G}_3$. Let $S \in I^+$ be such that $\mathrm{deg}_S(f) \nleq \alpha$. If $\mathrm{deg}_S(f)$ does not exist, there is a descending sequence $F_0 > F_1 > \cdots$ of functionals on S below f. So let One choose $f_0 \in F_0$, and then let him play as follows: whenever Two plays B_n, α_n, One picks some $g \in F_{n+1}$ such that $\mathrm{dom}(g) \cap B_n \in I^+$, and plays $f_n = g \restriction B_n$. This strategy wins. If $\mathrm{deg}_S(f) > \alpha$, there is a functional F on S such that $\mathrm{deg}_S F = \alpha$, and by Theorem 1.4 there is $g_0 \in F$ and $A_0 \subseteq \mathrm{dom}(g_0)$ such that A_0 forces that g_0 represents α. Let One play $f_0 = g_0 \restriction A_0$. Now $\mathrm{deg}_S(f_0) = \alpha$ for all $S \subseteq A_0$ and when Two plays $B_0 \subseteq A_0$ and $\alpha_0 < \alpha$, One can similarly find f_1 with domain $A_1 \subseteq B_0$ such that $\mathrm{deg}_S(f_1) = \alpha_0$ for all $S \subseteq A_1$. And so on: when Two plays B_n, α_n, One responds by playing f_{n+1} such that $B_n \supseteq \mathrm{dom}(f_{n+1}) \Vdash f_{n+1}$ represents α_n. Thus One has a winning strategy. $\square$

1.5. Normed ideals

Let f be an ordinal function. If $\deg(f)$ exists, then clearly $\|f\| \leqslant \deg(f)$. The ideal I is *normed* if for every ordinal function f, $\deg(f) = \|f\|$.

Lemma. *Let I be a normal ideal. For every I-function f, if* $\mathrm{dom}(f) = S$, *then* $\deg_S(f) = \|f\|_S$.

Proof. Let $S = \mathrm{dom}(f)$ and assume that $-S \in I^+$. Let $\alpha = \|f\|_S$. Let g be the extension of f to κ defined by $g(\xi) = \alpha^+$ for all $\xi \in -S$. Clearly, $\|g\| = \alpha$, and hence $\deg(g) = \alpha$. By Theorem 1.4, there is $T \in I^+$ which forces that g represents α. Now it is clearly impossible that $T \subseteq -S$ (too many constant functions); hence we may assume that $T \subseteq S$ and so T forces that f represents α. By Theorem 1.4 again we have that $\deg_S(f) \leqslant \alpha$. And so $\deg_S(f) = \|f\|_S$. □

If F is a functional on a set $S \in I^+$, then

$$\deg_S(F) = \min\{\deg_{\mathrm{dom}(f)}(f) : f \in F\}.$$

Hence if I is normed, every functional on every set in I^+ has a degree and so we have

Corollary. *If I is normed, then I is precipitous.*

We don't know whether every precipitous ideal is normed (probably not). Also, it is easy to prove that if the αth function exists for every α, then I is normed, but we don't know if the converse is true (again probably not).

In [10], Levinski calls I 'normé' if

(1.6) for every ordinal function f on κ there is $S \in I^+$ such that $\|f\|_T = \|f\|$ for all $T \subseteq S$.

The reason why I call the ideal normed is that Levinski's condition is equivalent to my definition:

Proposition. *I is normed if and only if it satisfies the condition* (1.6).

Proof. Let I be normed and let f be an ordinal function on κ. Let $\|f\| = \alpha$. Since $\deg(f) = \alpha$, there is $S \in I^+$ such that $S \Vdash f$ represents α. Hence $\deg_T(f) = \alpha$ for all $T \subseteq S$ and so $\|f\|_T = \|f\|$ for all $T \subseteq S$.

Let I satisfy the condition (1.6). First note that (1.6) implies a somewhat stronger condition:

(1.7) for every I-function f there is $S \subseteq \mathrm{dom}(f)$ such that $\|f\|_T = \|f\|_{\mathrm{dom}(f)}$ for all $T \subseteq S$.

(To see this, extend f to κ by letting $f(\xi)=\|f\|^+$ for all $\xi \notin \mathrm{dom}(f)$ and apply (1.6).)

I claim that (1.7) implies that I is precipitous. If not, there is a descending sequence of functionals $F_0 > F_1 > \cdots$ on some $S \in I^+$. Pick $f_0 \in F_0$ and let $S_0 \subseteq \mathrm{dom}(f_0)$ be such that $\|f_0\|_T = \|f_0\|_{S_0}$ for all $T \subseteq S_0$. Then pick $f_1 \in F_1$ and $S_1 \subseteq \mathrm{dom}(f_1)$ such that $S_1 \subseteq S_0$ and that $\|f_1\|_T = \|f_1\|_{S_1}$ for all $T \subseteq S_1$, and so on. Then we have $\|f_0\|_{S_0} = \|f_0\|_{S_1} > \|f_1\|_{S_1} = \|f_1\|_{S_2} > \|f_2\|_{S_2} = \cdots$, a descending sequence of ordinals, a contradiction.

Now let's prove that (1.7) implies that I is normed. We shall prove, by induction on α, that if f is an I-function if $S = \mathrm{dom}(I)$ and if $\mathrm{deg}_S(f) = \alpha$, then $\|f\|_S = \alpha$. Let f be an I-function such that $\mathrm{dom}(f) = S$ and $\mathrm{deg}_S(f) = \alpha$. Assume that $\|f\|_S = \beta < \alpha$. Let $T \subseteq S$ be such that $\|f\|_U = \beta$ for all $U \subseteq T$. Since $\mathrm{deg}_S(f) = \alpha$, there is an I-function g with domain $U \subseteq T$ such that $\mathrm{deg}_U(g) = \beta$ and $g(\xi) < f(\xi)$ for all $\xi \in U$. By the induction hypothesis, $\|g\|_U = \beta$, a contradiction since $\|g\|_U < \|f\| = \beta$. □

2. Weakly precipitous ideals

2.1. *Shelah's game $\mathcal{G}_4(I)$*

Let I be a normal κ-complete ideal over κ. In this game, the moves of Player One and Player Two are as follows:

One	Two
f_0	
	I_0
f_1	
	I_1
f_2	
	I_2
$\vdots$	$\vdots$

f_0 is an ordinal I-function. I_0 is a normal (κ-complete) ideal over κ such that $I_0 \supseteq I \restriction \mathrm{dom}(f_0)$. Then f_1 is an ordinal I_0-function such that $f_1(\alpha) < f_0(\alpha)$ for all $\alpha \in \mathrm{dom}(f_1)$. $I_1 \supseteq I_0 \restriction \mathrm{dom}(f_1)$ is a normal ideal, $f_2 < f_1$ is an I_1-function and so on. Player One wins if and only if he can continue making legal moves indefinitely.

The game $\mathcal{G}_4$ is determined. Note that this game is a generalization of the game $\mathcal{G}_1$, as the moves of player Two in $\mathcal{G}_1$ are in fact moves of Two in $\mathcal{G}_4$ with the additional specification that each I_n is equal to $I \restriction B_n$ for some $B_n \in I^+$. Thus if Two has a winning strategy in $\mathcal{G}_1$, Two has a winning strategy in $\mathcal{G}_4$. Thus

Definition. I is *weakly precipitous* if Player Two has a winning strategy in the game $\mathcal{G}_4(I)$.

And we have

Proposition. *If I is precipitous, then I is weakly precipitous.*

In Section 1 I have shown how the game $\mathcal{G}_1$ is related to Galvin's game (Theorem 1.2). One can prove a similar theorem for Shelah's game:

2.2. The game $\mathcal{G}_5(I)$

The moves of Players One and Two are as follows:

One	Two
A_0	
	I_0
A_1	
	I_1
A_2	
	I_2
⋮	
	⋮

Each A_n is such that $A_n \subseteq A_{n-1}$ and $A_n \in I_{n-1}^+$ (and $A_0 \in I^+$). Each I_n is a normal ideal such that $I_n \supseteq I_{n-1} \restriction A_n$ (and $I_0 \supseteq I \restriction A_0$). Player One wins if and only if $\bigcap_{i=0}^{\infty} A_n = \emptyset$.

Theorem 2.1. *Player One has a winning strategy in the game $\mathcal{G}_5(I)$ if and only if One has a winning strategy in the game $\mathcal{G}_4(I)$.*

Proof. The argument is exactly as in the proof of Theorem 1.2. □

The following is an unpublished theorem of S. Shelah [12].

Theorem 2.2. *If there is a Ramsey cardinal $\lambda > \kappa$, then there exists a normal κ-complete ideal I over κ such that Player Two has a winning strategy in the game $\mathcal{G}_4(I)$.*

Proof (sketch). First we define a certain filter $\mathcal{F}$ (due to Magidor [11]). Let E be the set of all $P \subset \lambda$ such that $|P| = \lambda$ and $\alpha_p = P \cap \kappa$ is an initial segment of κ. For every $F:[\lambda]^{<\omega} \to \lambda$ be the set of all $P \in E$ closed under F (i.e. $F([P]^{<\omega}) \subseteq P$). Since λ is Ramsey, each A_F is nonempty. Let $\mathcal{F}$ be the filter over E generated by the sets A_F for all $F:[\lambda]^{<\omega} \to \lambda$. By Magidor, $\mathcal{F}$ is κ-complete and is *normal* in the sense that if $f(P) \in P$ holds on a set of positive measure then f is constant on a set of positive measure. Let $\mathcal{J}$ be the dual of Magidor's filter.

Let $\pi: E \to \kappa$ be defined as follows: $\pi(P) = \alpha_p$. If $\mathcal{J}$ is a normal ideal over E, then $\pi[\mathcal{J}]$ is a normal ideal over κ. For any function $f:\kappa \to \lambda$, let $\hat{f}: E \to E$ denote the function $\hat{f}(P) = f(\pi(P))$th element of P.

Let $I = \pi[\mathcal{J}]$. We claim that Player Two has a winning strategy in the game $\mathcal{G}_4(I)$. An elementary submodel argument shows that it is enough to prove this only in the case when the first move of One is a function f_0 with values less than λ. Now the strategy is as follows. When One plays f_n, let $\mathcal{F}_n = \pi_{-1}(I_{n-1} \restriction \mathrm{dom}(f_n))$. $\mathcal{F}_n$ is a normal filter over E and so the function $\hat{f}_n$ is constant on a set $B_n \in \mathcal{J}_n$ (where $\mathcal{J}_n$ is the dual of $\mathcal{F}_n$). Moreover, its constant value γ_n is less than γ_{n-1}. So let Two play the ideal $\pi[\mathcal{J}_n \restriction B_n]$. This strategy wins, as $\gamma_0 > \gamma_1 > \gamma_2 > \cdots$. □

Remark. *The filter $\pi[\mathscr{F}]$ is the closed unbounded filter over κ.*

Proof. Let $F:[\lambda]^{<\omega}\to\lambda$. We shall show that the set $\pi[A_F]$ contains a closed unbounded set C. We construct C as follows: $C=\{\alpha_{P_\xi}:\xi<\kappa\}$ where α_{P_ξ} is a normal sequence: Let P_0 be any element of E closed under F. Given P_ξ, let $P_{\xi+1}$ be some element of E closed under F such that $\alpha_{P_{\xi+1}}>\alpha_{P_\xi}$. If ξ is a limit ordinal, let $P_\xi=\bigcup_{\eta<\xi}P_\eta$; clearly, P_ξ is closed under F. □

Corollary. *If there is a Ramsey cardinal $\lambda>\kappa$, then the ideal of thin subsets of κ is weakly precipitous.* □

To carry a weakly precipitous ideal is probably a large cardinal property of κ, but it is unknown even whether it contradicts $V=L$.

3. Pseudo-precipitous ideals

We shall now generalize the games $\mathscr{G}_4$ and $\mathscr{G}_5$. Let I be a normal κ-complete ideal over κ.

3.1. The game $\mathscr{G}_6(I)$

One	Two
I_0, f_0	J_0
I_1, f_1	J_1
I_2, f_2	J_2
⋮	⋮

The I_n and J_n are normal κ-complete ideals over κ such that $I\subseteq I_0\subseteq J_0\subseteq I_1\subseteq J_1\subseteq\cdots$. Each f_n is an ordinal I_n-function, and each J_n satisfies $J_n\supseteq I_n\restriction\mathrm{dom}(f_n)$. Moreover, $f_{n+1}(\alpha)<f_n(\alpha)$ for each n and each $\alpha\in\mathrm{dom}(f_{n+1})$. Player One wins if and only if he can continue making legal moves indefinitely. The game $\mathscr{G}_6$ is determined.

Definition. I is *pseudo-precipitous* if Player Two has a winning strategy in the game $\mathscr{G}_6(I)$.

3.2. The game $\mathscr{G}_7(I)$

One	Two
I_0, A_0	J_0
I_1, A_1	J_1
I_2, A_2	J_2
⋮	⋮

The I_n and J_n are normal κ-complete ideals over κ such that $I \subseteq I_0 \subseteq J_0 \subseteq I_1 \subseteq J_1 \cdots$. Each A_n a subset of κ such that $A_{n+1} \in J_n^+$ (and $A_0 \in I^+$), and each I_n satisfies $I_{n+1} \supseteq J_n \upharpoonright A_{n+1}$ (and $I_0 \supseteq I \upharpoonright A_0$). Player One wins if and only if $\bigcap_{n=0}^{\infty} A_n = \emptyset$.

First we notice that the proof of Theorem 1.2 easily generalizes to the present situation and we have:

Theorem 3.1. *Player One has a winning strategy in the game $\mathscr{G}_6(I)$ if and only if Player One has a winning strategy in the game $\mathscr{G}_7(I)$.* □

Secondly, the game $\mathscr{G}_6$ can be considered a modification of the game $\mathscr{G}_4$ by making the rules easier for Player One. This if Two has a winning strategy in $\mathscr{G}_6(I)$, Two also has a winning strategy in $\mathscr{G}_4(I)$ and we have:

Proposition. *Every pseudo-precipitous ideal is weakly precipitous.*

We want to determine how strong is the property "κ carries a pseudo-precipitous ideal". The vehicle for our investigations is the use of generic ultrapowers.

Let Q_I be the following notion of forcing. Foring conditions are normal κ-complete ideals J extending I, and J_1 is stronger than J_2 just in case $J_1 \supseteq J_2$.

Let G be a generic subset of Q_I. The union of all $J \in G$ is a normal ideal and is prime with respect to the ground model (an easy argument using genericity). Thus let $\mathscr{U}$ be the dual of this prime ideal; $\mathscr{U}$ is an ultrafilter for the ground model and we can form the ultrapower $\mathrm{Ult}_{\mathscr{U}}(V)$. We call $\mathscr{U}$ a *Q_I-generic ultrafilter*, and $\mathrm{Ult}_{\mathscr{U}}(V)$ a *Q_I-generic ultrapower.*

Theorem 3.2. *The ideal I is pseudo-precipitous if and only if every condition $J \in Q_I$ forces that the Q_I-generic ultrapower is well founded.*

Proof. First suppose that some condition J forces that the Q_I-generic ultrapower is not well-founded. We shall produce a winning strategy for Player One in the game $\mathscr{G}_6(I)$ thus showing that I is not pseudo-precipitous. J forces that there is a sequence $\vec{f}$ (in $V[G]$) of ordinal functions (in V) on κ, descending mod $\mathscr{U}$. Let $I_0 \in Q$ and f_0 (the opening move of Player One) be such that $I_0 \supseteq J \upharpoonright \mathrm{dom}(f_0)$ and that I_0 forces that f_0 is the 0th term of the sequence $\vec{f}$. For each n, when Two plays J_n, let I_{n+1} and f_{n+1} be such that $I_{n+1} \supseteq J_n \upharpoonright \mathrm{dom}(f_{n+1})$, that $f_{n+1}(\alpha) < f_n(\alpha)$ for all $\alpha \in \mathrm{dom}(f_{n+1})$ and that I_{n+1} forces that f_{n+1} is the $(n+1)$st term of the sequence $\vec{f}$. Clearly, this strategy wins for Player One.

Now suppose that every condition $J \in Q_I$ forces that $\mathrm{Ult}_{\mathscr{U}}(V)$ is well-founded. We shall show that Player One does not have a winning strategy in $\mathscr{G}_6(I)$, and thus I is pseudo-precipitous. Let σ be a strategy for One in $\mathscr{G}_6$. Let I_0, f_0 be the opening move of Player One (using σ). Let G be Q_I-generic over V such that

$I_0 \in G$; let us work in $V[G]$. Let $\mathcal{U}$ be the corresponding Q_I-generic ultrafilter and let Ult be the Q_I-generic ultrapower; let $j: V \to \text{Ult}$ be the corresponding elementary embedding. Since Ult is well-founded, we identify it with a transitive class in $V[G]$.

Let us assume that σ is a winning strategy; we shall reach a contradiction. Let $J_0 \in G$ be such that $J_0 \supseteq I_0$; such J_0 exists by the genericity of G. Let I_1, f_1 be the move by One using σ against J_0. Let $J_1 \in G$ be such that $J_1 \supseteq I_1$ and let I_2, f_2 be One's move using σ against J_0, J_1. And so on. This produces a sequence $I_0 \subseteq J_0 \subseteq I_1 \subseteq \cdots$ of conditions in G. It follows that for every n, $\text{dom}(f_n) \in \mathcal{U}$. Since $\mathcal{U}$ is normal, we have $\kappa \in \text{dom}(j(f_n))$ for all n. But this is a contradiction because we would have

$$j(f_0)(\kappa) > j(f)(\kappa) > j(f_2)(\kappa) > \cdots.$$

Hence σ is not a winning strategy and therefore I is pseudo-precipitous. □

The equivalence proved in Theorem 3.2 can be used to prove the following result:

Theorem 3.3. *If κ carries a pseudo-precipitous ideal, then κ is measurable in an inner model.*

Proof. Kunen's technique from [9] is applicable in this situation, as it is in the analogous case of precipitous ideals. In fact, the proof of Theorem 2 in [6] can be repeated here almost word by word, with appropriate changes. I suggest that the reader looks at the proof of Lemma 2.2 in [6]. Using the same notation as there, the proof boils down to showing that for every $X \in L[A]$, $X \in I^+$ implies $\kappa - X \in I$. This is established by the following chain of arguments:

$$I \restriction X \Vdash X \in \mathcal{U},$$

$$I \restriction X \Vdash \kappa \in j(X) \qquad (\mathcal{U} \text{ is normal}),$$

$$I \restriction X \Vdash (L[A] \Vdash \varphi(\kappa, E, A)),$$

$$L[A] \models \varphi(\kappa, E, A),$$

$$I \Vdash (L[A] \models \varphi(\kappa, E, A)),$$

$$I \Vdash X \in \mathcal{U},$$

$$\kappa - X \in I.$$

The rest of the proof is more or less like in [6]. □

We don't know how strong property it is for κ to carry a pseudo-precipitous ideal. The only result in the opposite direction is the following result:

Theorem 3.4. *If I is a normal κ^+-saturated κ-complete ideal over κ, then I is pseudo-precipitous.*

Proof. By Theorem 3.1 of [1], every normal $J \supseteq I$ is equal to $I \mid S$ for some $S \in I^+$. The game $\mathcal{G}_7(I)$ is thus reduced to the game $\mathcal{G}(I)$, and as I is precipitous, One does not have a winning strategy. Hence I is pseudo-precipitous. □

4. Open problems

[1. Is every precipitous ideal normed?]

[2. If I is normed, does the αth function exist for every α?]

3. If there is a weakly precipitous ideal
   (a) Is $V \neq L$?
   (b) Does $0^\#$ exist?
   [(c) Is there a Ramsey cardinal in K?]

4. How strong is the consistency of "there is a pseudo-precipitous ideal on $\aleph_1$"?

Problems 1 and 2 are solved (Fall 1980). The answer is "not necessarily". See the paper by Jech and Mitchell in this Journal [13].

With regard to Problem 3 Levinski (Fall 1980) showed that the following partition property, whose strength is between Ramsey and $0^\#$ suffices for a weakly precipitous ideal:

$$\kappa \to ((2^{2^{\aleph_1}})^+)^{<\omega}$$

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