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On list critical graphs

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ABSTRACT

In this paper we discuss some basic properties of k-list critical graphs. A graph G is *k-list critical* if there exists a list assignment L for G with $|L(v)| = k - 1$ for all vertices v of G such that every proper subgraph of G is L-colorable, but G itself is not L-colorable. This generalizes the usual definition of a k-chromatic critical graph, where $L(v) = \{1, \ldots, k-1\}$ for all vertices v of G. While the investigation of k-critical graphs is a well established part of coloring theory, not much is known about k-list critical graphs. Several unexpected phenomena occur, for instance a k-list critical graph may contain another one as a proper induced subgraph, with the same value of k. We also show that, for all $2 \leq p \leq k$, there is a minimal k-list critical graph with chromatic number p. Furthermore, we discuss the question, for which values of k and n is the complete graph K_n k-list critical. While this is the case for all $5 \leq k \leq n$, K_n is not 4-list critical if n is large.

1. Colorings and critical graphs

In the traditional model of graph coloring, we assign a color from a set C to each vertex of a graph in such a way that no two adjacent vertices have the same color. If the size of the color set C is large enough, such an assignment (coloring) exists. In practical applications, however, we often have a restriction on the colors that may be assigned to a vertex. One very natural and frequently occurring restriction is that the color of a vertex x must be chosen from a given allowed set (list) $L(x)$ of colors. A benchmark problem of high importance that leads to such type of questions is the well-known Frequency Assignment Problem.

It turns out that graphs critical with respect to lists behave in a very unusual way, as compared to the classical concept of 'color critical graphs'. Our main observations in this direction show that

- a list critical graph may contain other list critical graphs as (induced) subgraphs, with the same sizes of 'critical lists',
- a graph may be critical with respect to many different values of list sizes,
- for every $k \geq 5$, every complete graph on at least k vertices is k-list critical, whereas there are only finitely many 4-list critical complete graphs.

Results of a general nature are proved in Section 2, while the complete graphs are considered in Section 3. The necessary definitions and some basic properties are described in the rest of this introduction.

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1.1. List colorings

A *graph* $G = (V, E)$ consists of a finite set $V = V(G)$ of *vertices* and a set $E = E(G)$ of 2-element subsets of V, called *edges*. Thus, our graphs have no loops or multiple edges. An edge $\{u, v\}$ is usually written as uv or vu.

Let $G = (V, E)$ be a graph and let $k \geq 0$ be an integer. A *list assignment* L of G is a function that assigns to every vertex v of G a set (list) $L(v)$ of colors (usually each color is a positive integer). We say that L is a *k-assignment* if $|L(v)| = k$ for all $v \in V$. A *coloring* of G is a function φ that assigns a color to each vertex of G so that $\varphi(v) \neq \varphi(w)$ whenever $vw \in E$. An *L-coloring* of G is a coloring φ of G such that $\varphi(v) \in L(v)$ for all $v \in V$. If G admits an L-coloring, then G is called *L-colorable*. When $L(v) = [1, k]$ for all $v \in V$ (where $[1, k]$ denotes the set $\{1, \ldots, k\}$), the corresponding terms become *k-coloring* and *k-colorable*, respectively. The graph G is said to be *k-list colorable* if G is L-colorable for every k-assignment L of G.

The *chromatic number* of G, denoted by $\chi(G)$, is the least number k such that G is k-colorable. The *list chromatic number* or *choice number* of G, denoted by $\chi_\ell(G)$, is the least number k such that G is k-list colorable. Clearly, every graph G satisfies $\chi(G) \leq \chi_\ell(G)$.

The study of list coloring problems for graphs was initiated in the 1970s by Vizing [28] and, independently, by Erdős, Rubin and Taylor [8]. Both Vizing and Erdős, Rubin and Taylor observed that bipartite graphs, i.e., graphs with chromatic number at most 2, can have arbitrarily large list chromatic number. The gap between the two parameters $\chi_\ell(G)$ and $\chi(G)$ can thus be arbitrarily large. On the other hand, there are several classes of graphs for which it is conjectured that $\chi_\ell(G) = \chi(G)$. These include line graphs, total graphs, claw-free graphs, and squares of graphs. Classes of graphs for which it is known that $\chi_\ell(G) = \chi(G)$ are the chordal graphs (satisfying even a more general equation, cf. [27]), line graphs of bipartite multigraphs (Galvin [10]), and powers of cycles (Prowse and Woodall [20]).

One of the basic results in graph coloring is Brooks's theorem [2] from 1941, which asserts that complete graphs and odd cycles are the only connected graphs whose chromatic number is larger than their maximum degree. The choosability version of this result has been proved by Vizing [28] and, independently, by Erdős, Rubin and Taylor [8].

Theorem 1. *A graph G with maximum degree $\Delta \geq 1$ satisfies $\chi_\ell(G) \leq \Delta$ unless some component of G is a complete graph on $\Delta + 1$ vertices or $\Delta = 2$ and some component of G is an odd cycle.*

For detailed information on list colorings, we refer to the surveys [26,17].

1.2. Critical graphs

Criticality is a general concept in graph theory and can be defined with respect to various graph parameters or graph properties. The importance of the notion of criticality lies in the fact that problems for graphs may often be reduced to problems for critical graphs, and the structure of the latter is more restricted. Critical graphs with respect to the *chromatic number* were first defined and used by Dirac [5] in 1951.

A graph G is *L-critical*, where L is a list assignment for G, if every proper subgraph of G is L-colorable, but G itself is not L-colorable. When $L(v) = [1, k - 1]$ for all $v \in V$, where $k \geq 2$ is an integer, we also use the term *k-critical*. The graph G is *k-list critical* if there is a $(k - 1)$-assignment L for G such that G is L-critical.

Let us first discuss some elementary facts about list critical graphs. Obviously, every L-critical graph is non-empty as well as connected, and a connected graph G is L-critical if and only if G is not L-colorable, but $G - e$ is L-colorable for every edge e of G. Since every graph $G = (V, E)$ is $|V|$-list colorable, every k-critical graph has at least k vertices.

Let G be a graph and let L be a list assignment for G. The *L-core* of G is the unique maximal subgraph H of G such that $d_H(x) \geq |L(x)|$ for all $x \in V(H)$ (where $d_H(x)$ denotes the degree of vertex x in graph H). Clearly, H is an induced subgraph of G (possibly empty). If L is a k-assignment, the corresponding term becomes *k-core*. Note that the k-core of a graph G is empty if and only if G is $(k - 1)$-*degenerate*; that is, every (non-empty) subgraph of G has minimum degree at most $k - 1$.

Proposition 2. *Let G be a graph and let L be a list assignment for G. Then G is L-colorable if and only if the L-core of G is L-colorable.*

Proof. The "only if" part is evident. For the proof of the "if" part suppose that G is not L-colorable. Then there is a smallest subgraph H of G which is not L-colorable. Then H is non-empty and $H - y$ is L-colorable for every vertex y of H. Since H is not L-colorable, we conclude that $d_H(y) \geq |L(y)|$ for all $y \in V(H)$. Otherwise we can extend an L-coloring of $H - y$ to an L-coloring of H, a contradiction. Hence H is a subgraph of the L-core of G and, therefore, the L-core is not L-colorable, a contradiction. $\square$

2. Structure of list critical graphs

In the first part of this section we study list critical structures from an aspect that has no nontrivial analogy in 'classical' graph coloring: k-list critical graphs that are minimal under inclusion. After that, the subclass of those k-list critical graphs is investigated in which every uncolorable k-assignment has identical lists on all vertices.

2.1. Minimal list critical graphs

By definition, a graph G is k-critical if and only if every proper subgraph of G is $(k-1)$-colorable but G itself is not $(k-1)$-colorable. Since $\chi(G-x) \geq \chi(G) - 1$ for every vertex x of G, this implies that G is k-critical if and only if $\chi(H) < \chi(G) = k$ holds for every proper subgraph H of G. Hence, our definition of k-critical graphs (without lists) is equivalent to the definition introduced by G.A. Dirac.

Obviously, for a graph G and an integer $k \geq 1$, we have $\chi(G) \geq k$ if and only if G contains a k-critical subgraph. The complete graph K_k is an example of a k-critical graph and, for $k = 1, 2$, it is the only one. König's theorem [12] that a graph is bipartite if and only if it does not contain an odd cycle is equivalent to the statement that the only 3-critical graphs are the odd cycles. No reasonable characterization of 4-critical graphs, or equivalently of 3-colorability, seems possible.

Every k-critical graph is k-list critical. The following known example (see [14,15]) shows that the converse statement is not true. This example also shows that a k-list critical graph can contain another k-list critical graph as a proper subgraph. For k-critical graphs this is impossible.

Example 3. Let G be the graph obtained from two disjoint copies K, K' of the complete graph K_k of order $k \geq 3$ by adding an edge joining a vertex $x \in V(K)$ with a vertex $x' \in V(K')$. Furthermore, let L be the $(k-1)$-assignment for G defined by

$$L(v) = \begin{cases} \{1, \dots, k-1\} & \text{if } v \in V(G) - \{x, x'\}, \\ \{2, \dots, k\} & \text{if } v \in \{x, x'\}. \end{cases}$$

Then G is L-critical and hence k-list critical, but not k-critical. Furthermore, $\chi_\ell(G) = \chi_\ell(K_k) = k$ and K_k is k-list critical.

Proof. Since K and K' are complete graphs of order k, in every L-coloring of $K \cup K'$, both vertices x and x' receive the color k. Hence G is not L-colorable. If e is an edge of G, then the L-core of $G - e$ is either K or K' or $K \cup K'$. Hence, by Proposition 2, $G - e$ is L-colorable and, therefore, G is L-critical. Since $\Delta(G) \leq k$, Theorem 1 implies that $\chi_\ell(G) \leq k$ and hence $\chi_\ell(G) = k$. Obviously, $\chi_\ell(K_k) = k$ and K_k is k-list critical. $\square$

Let G be a graph and let L be a list assignment for G. Then G is not L-colorable if and only if G contains an L-critical subgraph. Hence, for $k \geq 1$, $\chi_\ell(G) \geq k$ if and only if G contains a k-list critical subgraph.

Obviously, for $k = 1, 2$, the complete graph K_k is the only k-list critical graph. But, as we shall see later, there are examples of k-list critical graphs with arbitrarily large list chromatic number. By a *minimal k-list critical* graph we mean a k-list critical graph that does not contain any k-list critical graph as a proper subgraph.

Proposition 4. *For $k \geq 1$, a graph G is minimal k-list critical if and only if $\chi_\ell(H) < \chi_\ell(G) = k$ for every proper subgraph H of G.*

Proof. For $k = 1$, this is evident and hence we assume that $k \geq 2$. If G is minimal k-list critical, then $\chi_\ell(G) \geq k > \chi_\ell(H)$ for every proper subgraph H of G and we only need to show that $\chi_\ell(G) = k$. To see this, let L be an arbitrary k-assignment for G. Choose a vertex $y \in V(G)$ and a color $c \in L(y)$. For the graph $H = G - y$, let L' be the list-assignment such that, for $x \in V(H)$, we have $L'(x) = L(x) \setminus \{c\}$ if $xy \in E(G)$ and $L'(x) = L(x)$ otherwise. Then $|L'(x)| \geq k - 1$ for every $x \in V(H)$. Since $\chi_\ell(H) \leq k - 1$, H is L'-colorable and hence G is L-colorable. This proves that $\chi_\ell(G) = k$.

Conversely, if $\chi_\ell(H) < \chi_\ell(G) = k$ for every proper subgraph H of G, then there is a (k-1)-assignment L such that G is not L-colorable, but every proper subgraph of G is L-colorable. Hence, since every proper subgraph is (k-1)-list colorable, G is minimal k-list critical. $\square$

Taking into account that $\chi_\ell(G) \leq \chi_\ell(G - e) + 1$ holds for every edge e of G, we obtain

Corollary 5. *For $k \geq 1$, a graph G is minimal k-list critical if and only if G is connected and $\chi_\ell(G - e) < k \leq \chi_\ell(G)$ for every edge e of G.*

2.2. Strong critical graphs

Let G be a graph and let $k \geq 1$ be an integer. By a *bad k-assignment* of G we mean a k-assignment L of G such that G is not L-colorable. The graph G is called *strong k-critical* if G is k-critical and if every bad $(k-1)$-assignment L for G is *constant*, i.e., there is a set C of $k-1$ colors such that $L(v) = C$ for every vertex v of G.

Now consider a strong k-critical graph G with $k \geq 1$. Since G is k-critical, we have $\chi_\ell(G) \geq \chi(G) = k$. Since every bad $(k-1)$-assignment for G is a constant $(k-1)$-assignment, we conclude that $G - e$ is $(k-1)$-list colorable for every edge e of G. Consequently, $\chi_\ell(G) = k$ and, by Proposition 4, we obtain that G is minimal k-list critical. Hence, we have the following result:

Proposition 6. *For $k \geq 1$, every strong k-critical graph is minimal k-list critical and has list chromatic number k.*

It is not hard to check that the complete graph K_k is strong k-critical, and also all odd cycles are strong 3-critical. Indeed, if L is a non-constant $(k-1)$-assignment on $G = K_k$, or on $G = C_{2t+1}$ for $k = 3$, then G has two *adjacent* vertices x, x' with unequal lists. Hence, there is a color $c \in L(x) \setminus L(x')$. We may assign c to x, and modify the lists of all neighbors v of x to $L'(v) = L(v) \setminus \{c\}$. The L'-core of $G-x$ is then empty, because x' can be removed first, and then the other vertices get deleted, too, e.g. in increasing order of their distance from x'. Consequently, G is L-colorable.

Another interesting class of strong k-critical graphs are the Dirac graphs or Gallai graphs, see Theorem 13.

Example 7. For $k \geq 3$, let G be a graph whose vertex set consists of three non-empty pairwise disjoint sets X, Y_1, Y_2 with $|Y_1| + |Y_2| = |X| + 1 = k-1$ and two additional vertices x_1, x_2 such that $X, Y_1 \cup Y_2$ are cliques in G not joined by any edge and $N_G(x_i) = X \cup Y_i$ for $i = 1, 2$, where $N_G(x)$ denotes the neighborhood of x in G. Then G is strong k-critical. Furthermore, $|V(G)| = 2k-1$ and $2|E(G)| = (k-1)|V(G)| + (k-3)$. We call such a graph a *Dirac graph* or *D_k-graph*, and write $G = D_k(X, Y_1, Y_2, x_1, x_2)$.

Proof. Let $G = D_k(X, Y_1, Y_2, x_1, x_2)$ be a Dirac graph for $k \geq 3$. Obviously, G has $2k-1$ vertices, $d_G(x) = k-1$ for all vertices $x \neq x_i, x_2$, and $2|E(G)| - (k-1)|V(G)| = d_G(x_1) + d_G(x_2) - 2(k-1) = k-3$. Furthermore it is easy to check that $G - e$ is $(k-2)$-degenerate for all edges e of G. Consequently, $\chi(H) \leq k-1$ holds for every proper subgraph H of G. That $\chi(G) \geq k$ holds follows from the fact that G has $2k-1$ vertices and independence number 2. Hence G is a k-critical.

To complete the proof, consider a list assignment L of G such that $|L(v)| \geq k-1$ for all $v \in V(G)$ and G is not L-colorable. We show by induction on $k \geq 3$ that L is a constant (k-1)-assignment, i.e., there is a set of $k-1$ colors C such that $L(v) = C$ for all $v \in V(G)$. If $k = 3$, then $G = C_5$ is a cycle of order 5 and it is easy to see that L is a constant 2-assignment.

Now assume that $k \geq 4$. Then Y_1 or Y_2 has at least two vertices. By symmetry, we may assume that $|Y_1| \geq 2$. We distinguish two cases.

Case 1: There is a vertex $y \in Y_1$ and a vertex $x \in X$ such that $L(x) \cap L(y)$ contains a color c. Let $G' = G - x - y$ and let L' be the list assignment for G' with $L'(v) = L(v) \setminus \{c\}$ for all $v \in V(G')$. Since G is not L-colorable, G' is not L' colorable. Furthermore, $G' = D_{k-1}(X \setminus \{x\}, Y_1 \setminus \{y\}, Y_2, x_1, x_2)$ is a Dirac graph and $|L'(v)| \geq k-2$ for all $v \in V(G')$. Then the induction hypothesis implies that there is a set C' of $k-2$ colors such that $L'(v) = C'$ for all $v \in V(G')$. Consequently, $C = C' \cup \{c\}$ is a set of $k-1$ colors and $L(v) = C$ for all $v \in V(G')$. Hence we need only to show that $L(x) = C$ and $L(y) = C$. Suppose this is false. Then there are colors $c_x \in L(x)$ and $c_y \in L(y)$ such that $c_x \notin C$ or $c_y \notin C$. Let L^* be the list assignment for G' such that

$$L^*(v) = C \setminus \{c_z \mid z \in \{x, y\},\ vz \in E(G)\}$$

for $v \in V(G')$. Then $|L^*(v)| \geq k-2$ for all $v \in V(G')$ and $L^*(x') \neq L^*(y')$ for $x' \in X \setminus \{x\}$ and $y' \in Y_1 \setminus \{y\}$. By the induction hypothesis it then follows that G' is L^*-colorable. Consequently, G is L-colorable, a contradiction. This proves that $L(x) = L(y) = C$ and, therefore, $L(v) = C$ for all $v \in V(G)$.

Case 2: $L(x) \cap L(y) = \emptyset$ for all $x \in X$ and $y \in Y_1$. Let $x \in X$ and $y \in Y_1$ be two vertices. Then $G' = G - x - y$ is a D_{k-1}-graph. By assumption, we have $|L(x) \cup L(y)| = |L(x)| + |L(y)| \geq 2(k-1)$. Hence there are two colors $c_x \in L(x)$ and $c_y \in L(y)$ such that $|L(x_1) \setminus \{c_x, c_y\}| \geq k-2$. Let L' be the list assignment for G' with

$$L'(v) = C \setminus \{c_z \mid z \in \{x, y\},\ vz \in E(G)\}$$

for $v \in V(G')$. Since x_1 is the only vertex of G that is adjacent to both x and y, we have $|L(v)| \geq k-2$ for all $v \in V(G')$. Since G is not L-colorable, G' is not L'-colorable. Then the induction hypothesis implies that there is a set C' of $k-2$ colors such that $L'(v) = C'$ for all $v \in V(G')$. Then there is a vertex $x' \in X \setminus \{x\}$ as well as a vertex $y' \in Y_1 \setminus \{y\}$ and we have $L(x') = C' \cup \{c_x\}$ and $L(y') = C' \cup \{c_y\}$. Consequently, $L(x') \cap L(y') \neq \emptyset$, contradicting the assumption of case 2. □

Example 8. For $k \geq 3$, let G be a graph whose vertex set consists of four non-empty pairwise disjoint sets X_1, X_2, Y_1, Y_2, where $|Y_1| + |Y_2| = |X_1| + |X_2| = k-1$ and $|X_2| + |Y_2| \leq k-1$, and one additional vertex z, such that $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ are cliques in G, $N_G(z) = X_1 \cup Y_1$, and a vertex $x \in X$ is joined to a vertex $y \in Y$ by an edge in G if and only if $x \in X_2$ and $y \in Y_2$. Then G is strong k-critical. We call such a graph an *E_k-graph* and write $G = E_k(X_1, X_2, Y_1, Y_2, z)$.

Proof. Let $G = E_k(X, Y_1, Y_2, x_1, x_2)$ be a E_k-graph for $k \geq 3$. Obviously, G has $2k-1$ vertices and independence number 2. Hence $\chi(G) \geq k$. Furthermore, it is easy to check that $G - e$ is (k-1)-colorable for every edge e of G. Consequently, G is k-critical.

To complete the proof, consider a list assignment L of G such that $|L(v)| \geq k-1$ for all $v \in V(G)$ and G is not L-colorable. We show by induction on $k \geq 3$ that L is a constant $(k-1)$-assignment, i.e., there is a set of $k-1$ colors C such that $L(v) = C$ for all $v \in V(G)$. If $k = 3$, then $G = C_5$ and the statement is obvious.

Now assume that $k \geq 4$. If $|X_2| = 1$ or $|Y_2| = 1$, then G is a D_k-graph and the statement follows from Example 7. Otherwise, $|X_2| \geq 2$ and $|Y_2| \geq 2$. Since $|X_1| + |X_2| = k-1$ and $|X_2| + |Y_2| \leq k-1$ we have $|X_1| \geq 2$. If there is a vertex $x \in X_1$ and a vertex $y \in Y_2$ such that $L(x) \cap L(y) \neq \emptyset$, then we can repeat the argument from Case 1 in Example 7. Otherwise, we proceed as in Case 2 of Example 7. □

Let G_1 and G_2 be two disjoint graphs. The *join* $G_1 + G_2$ of these two graphs is obtained from their vertex-disjoint union by adding the edges joining each vertex of G_1 to each vertex of G_2. If $\chi(G_i) = k_i$ for $i = 1, 2$, then $\chi(G_1 + G_2) = k_1 + k_2$.

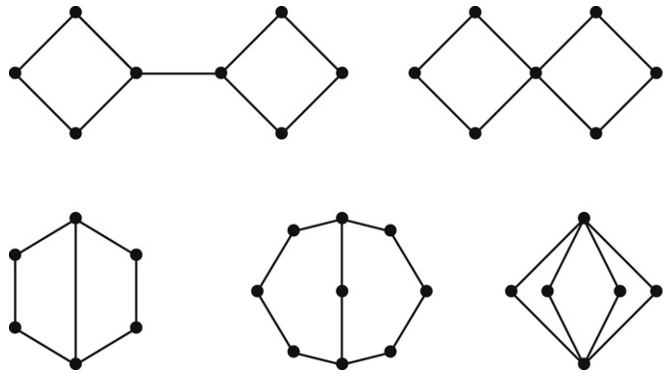

Fig. 1. Minimal 3-list critical bipartite graphs.

Furthermore, $G_1 + G_2$ is $(k_1 + k_2)$-critical if and only if G_i is k_i-critical for $i = 1, 2$. These facts were first used by Dirac [4] to show that the join of two odd cycles results in a 6-critical graph with many edges. Edgeless graphs have list chromatic number 1, but their joins can have an arbitrarily large list chromatic number. On the other hand, let G be the graph that consists of two vertices joined by three internally disjoint path with lengths 3, 3, and 1. Then Theorem 10 of the next subsection implies that $\chi_\ell(G) = 3$ and it is not difficult to prove that $\chi_\ell(G + K_1) = 3$, too.

Proposition 9. *Let $k, p \geq 1$ be integers. If G is a strong k-critical graph, then the graph $G' = G + K_p$ is strong $(k + p)$-critical.*

Proof. Clearly, it is sufficient to prove the statement for $p = 1$. Then $V(G') = V(G) \cup \{x\}$ and x is joined to every vertex of G by an edge in G'. Let L be a bad k-assignment for G'. Then $C = L(x)$ is a set of k colors. For a color $c \in C$ let L_c be the list assignment for G such that $L_c(v) = L(v) \setminus \{c\}$ for all $v \in V(G)$. Clearly, $|L_c(v)| \geq k - 1$ for all $V \in V(G)$. Since G' is not L-colorable, G is not L_c-colorable for every color $c \in C$. This implies that for every color $c \in C$ there is a set C_v of $k-1$ colors such that $L_c(v) = C_c$ for all $v \in V(G)$. Consequently, $L(v) = C$ for all $v \in V(G)$. This proves that L is a constant k-assignment. □

2.3. Basic properties of k-list critical graphs

Erdős, Rubin and Taylor [8] provide a characterization of 2-list colorable graphs, but not in terms of forbidden subgraphs. From the proof of their result, however, a description of all minimal 3-list critical graphs can also be derived, and in this way the following characterization of 2-list colorable graphs can be obtained (see Fig. 1).

Theorem 10. *A graph G is minimal 3-list critical if and only if G is a graph that belongs to one of the following six types:*

- *an odd cycle,*
- *two vertex-disjoint even cycles joined by a path,*
- *two even cycles having exactly one vertex in common,*
- *two vertices joined by three internally disjoint odd paths,*
- *two vertices joined by three internally disjoint even paths, where at least two of the paths are longer than 2,*
- *two vertices joined by four internally disjoint even paths, where the length of at least three of the paths is exactly 2.*

One cannot hope to find a good characterization for the class of all minimal k-list critical graphs if $k \geq 4$, but we can search for structural properties of critical graphs. One such property follows from Proposition 2, namely that if G is an L-critical graph, then G is equal to its L-core and, therefore, $d_G(x) \geq |L(x)|$ for every vertex x of G. In 1963, Gallai [9] published two fundamental papers dealing with structural properties of k-critical graphs. One of his results states that the vertices of degree $k - 1$ in a k-critical graph induce a subgraph all of whose blocks are complete graphs or odd cycles. This result can be generalized to L-critical graphs; for a short proof and an extension to hypergraphs, see [16].

Theorem 11. *Let G be an L-critical graph for a list assignment L. Then $d_G(x) \geq |L(x)|$ for every vertex x of G. Furthermore, every block of the subgraph $G[X]$ induced by $X = \{x \in V(G) \mid d_G(x) = |L(y)|\}$ is a complete graph or an odd cycle.*

The vertices of a k-list critical graph G whose degrees are equal to $k - 1$ are called the *low vertices* of G; and the others are called the *high vertices*. Theorem 11 implies that a k-list critical graph G with $k \geq 1$ has no high vertices if and only if either G is a complete graph of order k or $k = 3$ and G is an odd cycle.

Many known results about k-critical graphs are positive in the sense that they restrict the behavior of critical graphs. Some typical examples of such results are listed in the next theorem.

Theorem 12. *Let $G = (V, E)$ be a k-critical graph, where $k \geq 4$. Then the following properties hold:*

(a) *G is 2-connected and $(k - 1)$-edge connected.*
(b) *If X is any subset of low vertices of G with $\emptyset \neq X \neq V(G)$, then the subgraph $G[X]$ has at least as many components as the subgraph $G - X$.*

(c) $2|E| \geq ((k-1)+(k-3)/(k^2-2k-1))|V|$, *provided that G is not* K_k.
(d) $2|E| \geq (k-1)|V| + 2(k-3)$, *provided that G is neither* K_k *nor an* E_k*-graph.*
(e) *If* $|V| \leq 2k-2$, *then the complement of G is connected.*

The 2-connectivity of k-critical graphs follows from the simple fact that a graph is k-colorable whenever each of its blocks is k-colorable. The property of $(k-1)$-edge connectivity was proved by Dirac [6]. A proof of statement (b) was given in [22]. Using (b) and Theorem 11, Krivelevich [18] proved (c). The bound in (d) was established by Kostochka and Stiebitz [13]. Earlier, Dirac [7] proved that if $G = (V, E)$ is a k-critical graph ($k \geq 4$) such that G is not a K_k, then $2|E| \geq (k-1)|V| + (k-3)$, where equality holds if and only if G is a D_k-graph.

In 1963, Gallai [9] proved, by means of matching theory, that every k-critical graph with at most $2k-2$ vertices is the join of two non-empty graphs. Based on this deep result, he determined the minimum number of edges of k-critical graphs with at most $2k-1$ vertices and gave a complete description of the extremal cases.

Theorem 13 (*Gallai [9]*). *Let* k, p *be integers satisfying* $k \geq 4$ *and* $2 \leq p \leq k-1$. *If* $G = (V, E)$ *is a k-critical graph with* $k+p$ *vertices, then* $2|E| \geq (k-1)|V| + p(k-p) - 2$, *and equality holds if and only if G is the join of a* D_{p+2}*-graph and* K_{k-p-1}.

As pointed out by Dirac [4], for $k \geq 4$ there are k-critical graphs on n vertices for all $n \geq k$ except for $n = k+1$. Clearly, K_k is the only k-critical graph with k vertices. By Propositions 6 and 9 and Example 7, it follows that the extremal k-critical graphs in Theorem 13 are minimal k-list critical.

Let $f_k(n)$ denote the minimum number of edges of a k-critical graph with n vertices. Furthermore, let $g_k(n)$ denote the minimum number of edges in a minimal k-list critical graph with n vertices. Lower bounds for $g_k(n)$ can be found in [14,15]. In [14] it was proved that $2g_k(n) \geq (k-1)n + (k-3)$ for $n \geq k+2$. If $n = k+p$ with $k \geq 4$ and $2 \leq p \leq k-1$, then, by the above remark, $g_k(n) \leq f_k(n)$. The D_k-graphs show that $g_k(2k-1) = f_k(2k-1)$ for $k \geq 4$. But it is not known whether, for $k \geq 4$, there exists an integer n such that $g_k(n) < f_k(n)$.

On the other hand, there is also a large number of negative results showing that k-critical graphs may have a very unrestricted behavior. Thus, k-critical graphs for $k \geq 4$ may have many edges (Dirac [4]), many independent vertices (Brown and Moon [3], Toft [23], Lovász [19]), and high minimum degree (Simonovits [21], Toft [24]).

Minimal k-list critical graphs and k-critical graphs behave quite differently, even if they share some common properties. A typical example is the minimum degree. While the minimum degree of a k-critical graph for $k \geq 4$ can be arbitrarily large, the minimum degree of a minimal k-list critical graph is bounded above by a function of k (Alon [1]). The following example shows that a minimal k-list critical graph need not be 2-connected.

Example 14. Let $k \geq 3$ be an odd integer. Let G be the graph whose vertex set consists of four pairwise disjoint sets X_1, X_2, Y_1, Y_2, each of cardinality $k-2$, and three additional vertices x, y, z such that X_1, X_2, Y_1, Y_2 are cliques in G not joined by any edge, $N_G(x) = X_1 \cup X_2$, $N_G(y) = Y_1 \cup Y_2$, and $N_G(z) = X_1 \cup X_2 \cup Y_1 \cup X_2$. Then G is minimal k-list critical.

Proof. Let C_1, C_2, D_1, D_2 be four pairwise disjoint color sets each of cardinality $(k-1)/2$ and let L be the $(k-1)$-assignment for G defined by

$$L(v) = \begin{cases} C_1 \cup C_2 & \text{if } v = x, y \\ D_1 \cup D_2 & \text{if } v = z \\ D_1 \cup C_i & \text{if } v \in X_i,\ i = 1, 2 \\ D_2 \cup C_i & \text{if } v \in Y_i,\ i = 1, 2. \end{cases}$$

It is easy to check that in any L-coloring of the graph $G_1 = G[X_1 \cup X_2 \cup \{x, z\}]$ the vertex z receives a color from D_2, while in any L-coloring of the graph $G_2 = G[Y_1 \cup Y_2 \cup \{y, z\}]$ the vertex z receives a color from D_1. Hence G is not L-colorable showing that $\chi_\ell(G) \geq k$.

Next we claim that $G - e$ is (k-1)-list colorable for every edge e of G. By symmetry we need only to consider an edge e of G_2. Now let L be an arbitrary (k-1)-assignment for G. Then it is easy to see that G_1 is the L-core (that is the (k-1)-core) of G. Hence, by Proposition 2, it suffices to show that G_1 is L-colorable. First, assume that there is a color $c \in L(x) \cap L(z)$. Since $G[X_i]$ is a K_{k-2} for $i = 1, 2$ and $|L(v)| = k-1$ for every vertex v of G_1, there is an L-coloring of G_1 where both x and z receive the color c. Now, assume that $L(x) \cap L(z) = \emptyset$. If there exists two vertices $u, v \in X_i$ for some $i \in \{1, 2\}$ such that $L(x) \neq L(y)$, then choose a color $c \in L(u) \setminus L(v)$ and let L' be the list assignment for $G' = G - u$ such that $L'(w) = L(w) \setminus \{c\}$ for every vertex $w \in X_i \cup \{x, z\}$. Then $|L'(v)| = |L(v)| = k-1$ and the L'-core of G' is empty. From Proposition 2 it then follows that G' is L'-colorable and hence G_1 is L-colorable. Otherwise, for $i = 1, 2$, there is a sets C_i of k-1 colors such that $L(v) = C_i$ for every vertex $v \in X_i$. Then, since $L(x) \cap L(z) = \emptyset$, there are colors $c \in L(x)$ and $c' \in L(z)$ such that $|C_i \setminus \{c, c'\}| \geq k-2$ for $i = 1, 2$. Thus there is an L-coloring of G_1, where x has color c and z has color c'. This completes the proof of the claim that $\chi_\ell(G - e) \leq k-1$ for every edge e of G. Since $\chi_\ell(G) \geq k$, this shows that G is minimal k-list colorable. $\square$

Theorem 15. *For every* $k \geq p \geq 2$, *there is a minimal k-list critical graph G with chromatic number p.*

Proof. Let $\mathcal{K} = (V, \mathcal{E})$ be a complete $(k-p+1)$-partite hypergraph with k-1 vertices in each color class. Then the vertex set V is the disjoint union of $k-p+1$ sets $A_1, \ldots A_{k-p+1}$ each of cardinality $k-1$ and the edge set $\mathcal{E}$ consists of all subsets

E of V such that $|E \cap A_i| = 1$ for $i = 1, \dots k-p+1$. Now let $\mathcal{F}$ be a smallest set of (k-1)-element subsets of V that covers all hyperedges of $\mathcal{H}$ in the sense that for every hyperedge $E \in \mathcal{E}$ there is a set $F \in \mathcal{F}$ such that $E \subset F$.

Next we define a graph G as follows. Let K_F with $F \in \mathcal{F}$ be a collection of pairwise disjoint complete graphs each of order $p-1$. The graph G is obtained from the disjoint union of the complete graphs K_F with $F \in \mathcal{F}$ and a set $X = \{x_1, \dots, x_{k-p+1}\}$ of additional vertices by adding edges joining each vertex of X to each vertex of K_F with $F \in \mathcal{F}$. Clearly, $\chi(G) = p$ and we need only to show that G is minimal k-list colorable.

Let L be the list assignment for G such that $L(x_i) = A_i$ for $i = 1, \dots, k-p+1$ and $L(v) = F$ for every vertex v of K_F with $F \in \mathcal{F}$. Clearly, L is a (k-1)-assignment for G. Suppose that there is an L-coloring φ of G. Then $E = \{\varphi(x_1), \dots, \varphi(x_{k-p+1})\}$ is a hyperedge of $\mathcal{K}$ and, therefore, there is a set $F \in \mathcal{F}$ such that $E \subseteq F$. Now consider the complete subgraph K_F of G. For every vertex v of K_F we have $L(v) = F$ and, therefore, $|L(v) \setminus E| = |F \setminus E| = p-2$. Since K_F is a K_{p-1} and every vertex of X is joined to every vertex of K_F, this is a contradiction. Hence G is not L-colorable and, therefore, $\chi_\ell(G) \geq k$.

Now let e be an arbitrary edge of G. We claim that $G-e$ is (k-1)-list colorable. For the proof we consider an arbitrary (k-1)-assignment L of $G-e$ and show that $G-e$ is L-colorable. For the edge e, there exists a set $F' \in \mathcal{F}$ such that e is incident with a vertex of $K_{F'}$. Since $K_{F'}$ is a K_{p-1} and $d_G(v) = k-1$ for every vertex v of $K_{F'}$, it then follows that the L-core of $G-e$ is a subgraph of $G' = G - V(K_{F'})$. By Proposition 2, it suffices to show that G' is L-colorable.

First, consider the case that there are two vertices $x, x' \in X$ such that $L(x) \cap L(x')$ contains a color c. Then we color x and x' with c and each other vertex v of X with a color from the list $L(v)$. This gives a set C of at most $k+p-2$ colors. For each set $F \in \mathcal{F}$, K_F is a K_{p-1} and $L(v) \setminus C = F \setminus C$ is a set of at least $p-1$ colors for every vertex v of K_F. Hence φ can be extended to an L-coloring of G.

Now, consider the case that the lists $L(x)$ with $x \in X$ are pairwise disjoint. Let φ be an L-coloring of $G[X]$. Then $E = \{\varphi(x_1), \dots, \varphi(x_{k+p-1})\}$ is a set of $k+p-1$ colors and E forms a hyperedge of $\mathcal{K}$ and we obtain all hyperedges of $\mathcal{K}$ in this way. Our aim is to show that if we choose φ in an appropriate way, then φ can be extended to an L-coloring G. Let $\mathcal{F}' = \mathcal{F} \setminus \{F'\}$ and let $F \in \mathcal{F}'$. Let L' be the list assignment of $G' - X$ such that $L'(v) = L(v) \setminus E$ for all vertices v of $G' - X$. Clearly, φ can be extended to an L-coloring of G' if K_F is L'-colorable for every $F \in \mathcal{F}$. A complete graph K_F with $F \in \mathcal{F}'$ is called *good* if there are two vertices v and v' in K_F such that $L(v) \neq L(v')$; otherwise K_F is called *bad*. Let us first consider a good K_F. Then there are two vertices v and v' such that $L(v) \neq L(v')$. Then $|L'(w)| \geq p-2$ for every vertex w of K_F and $|\bigcup_{w \in V(K_F)} L(w)| \geq p-1$. Since K_F is a K_{p-1}, it then follows from Hall's theorem that there is an L'-coloring of K_F no matter how we choose φ.

Now, consider the set $\mathcal{F}^*$ of all sets of $F \in \mathcal{F}'$ such that K_F is bad. Then, for every $F \in \mathcal{F}^*$, there is a set C_F of k-1 colors such that $L(v) = C_F$ for every vertex v of K_F. By the definition of $\mathcal{F}$, there is a hyperedge E of $\mathcal{K}$ such that $E \not\subseteq C_F$ for all sets F of $\mathcal{F}^*$. Then there is an L-coloring φ of $G[X]$ such that $\varphi(x_i) = L(x) \cap E = A_i \cap E$ for $i = 1, \dots, k-p+1$. But then $|L'(v)| = |C_F \setminus E| \geq p-1$ for every vertex v of K_F with $F \in \mathcal{F}*$. Hence K_F is L'-colorable for every $F \in \mathcal{F}^*$. Consequently, G' is L-colorable.

Since $\chi_\ell(G) \geq k$ and $\chi_\ell(G-e) \leq k-1$ for every edge e of G, it follows that G is minimal k-list colorable. □

3. List critical complete graphs

The goal of this section is to analyze list criticality of complete graphs. A connection to extremal hypergraph theory will be pointed out, that yields an implicit characterization for 4-list criticality. For list sizes at least 5, the description is completely solved.

It is well-known that list colorings of complete graphs and Hall's theorem about systems of distinct representatives for finite set systems (see [11]) are closely related. We summarize some important facts of this kind as follows.

Proposition 16 (*Vizing [28]*). *Let K_n be a complete graph of order $n \geq 1$, let V be the vertex set of K_n, and let L be a list assignment for K_n with $L(x) = S_x$ for $x \in V$. Then the following statements hold:*

(a) *K_n is L-colorable if and only if the set system $(S_x)_{x\in V}$ satisfies Hall's condition, that is $|\bigcup_{x\in X} S_x| \geq |X|$ for all $X \subseteq V$.*
(b) *K_n is L-critical if and only if $|\bigcup_{x\in V} S_x| = n-1$ and for every two distinct vertices $x, y \in V$, there is a color $c \in S_x \cap S_y$ such that the set system $(S_v \setminus \{c\})_{v \in V\setminus\{x,y\}}$ satisfies Hall's condition.*

A set system $(S_x)_{x\in V}$ is called a *critical* (k, n)-*system* if the following conditions are satisfied:

(C1) $|V| = n$ and $S_x \subseteq [1, n-1]$ is a set with $|S_x| = k-1$ for every $x \in V$.
(C2) $\bigcup_{x\in V} S_x = [1, n-1]$.
(C3) For every two distinct elements $x, y \in V$ there is a color $c \in S_x \cap S_y$ such that $(S_v \setminus \{c\})_{v\in V\setminus\{x,y\}}$ satisfies Hall's condition.

As an immediate consequence of Proposition 16, we obtain the following result:

Proposition 17. *Let k, n be positive integers. The complete graph K_n is k-list critical if and only if there exists a critical (k, n)-system.*

Example 18. Let k, n be integers with $5 \le k \le n$. Let $V = [1, n]$ and let π be the cyclic permutation of the set $[1, n-2]$ with $\pi(1) = 2, \pi(2) = 3, \ldots, \pi(n-3) = n-2$, and $\pi(n-2) = 1$. For every $i \in V$, let

$$S_i' = \{\pi^{i-1}(j) \mid j = 1, \ldots, k-2\}$$

and $S_i = S_i' \cup \{n-1\}$. Then $(S_i)_{i\in V}$ is a critical (k, n)-system.

Proof. Clearly, the set system $\mathcal{S} = (S_i)_{i\in V}$ satisfies (C1) and (C2). To prove that $\mathcal{S}$ also satisfy (C3), it suffices to show that, for an arbitrary subset $X \subseteq V$ with $1 \le |X| \le n-2$, we have $|\bigcup_{i\in X} S_i'| \ge |X|$. To see this, let $P = \bigcup_{i\in X} S_i'$, $p = |P|$, $Q = [1, n-2] \setminus P$, and $q = |Q|$. If $p \ge n-2$, we are done. Hence $p \le n-3$ and, since $|X| \ge 1$, we have $p \ge k-2 \ge 3$ and, therefore, $1 \le q = n-2-p \le n-k$.

Now let $\mathcal{S}' = (S_i')_{i\in V}$ and let $\mathcal{S}''$ denote the set system consisting of all sets S' of $\mathcal{S}'$ that have a non-empty intersection with Q. Clearly, $s = |\mathcal{S}''| \le |\mathcal{S}'| - |X| = n - |X|$ and, therefore, $|X| \le n - s$. Next, we claim that $s \ge q+2$. Obviously, the set system $\mathcal{S}''$ contains the q sets S_i for $i \in Q$. Since $p \ge 3$ and $q \ge 1$, there are two distinct elements $x, y \in P$ such that, for $z \in \{x, y\}$, either $\pi(z)$ or $\pi^2(z)$ belongs to Q, and therefore, S_z belongs to $\mathcal{S}''$. This proves that $s \ge q+2$. Consequently, we have $|X| \le n - s \le n - q - 2 = p = |\bigcup_{i\in X} S_i'|$. This completes the proof. □

Corollary 19. *The complete graph K_n is k-list critical for all $5 \le k \le n$.*

For k larger than $n/2$, we describe a further construction as follows.

Example 20. Let k, n be positive integers with $\frac{n+1}{2} < k \le n$. Let $V = [1, n]$ and let π be the cyclic permutation of the set $[1, n-1]$ with $\pi(1) = 2, \pi(2) = 3, \ldots, \pi(n-2) = n-1$, and $\pi(n-1) = 1$. For every $i \in V$, let

$$S_i = \{\pi^{i-1}(j) \mid j = 1, \ldots, k-1\}.$$

Then $(S_i)_{i\in V}$ is a critical (k, n)-system.

Proof. Clearly, the set system $\mathcal{S} = (S_i)_{i\in V}$ satisfies (C1) and (C2). For the proof of (C3), let i_1, i_2 be two distinct elements of V. Since $|S_{i_1}| = |S_{i_2}| = k-1 > (n-1)/2$ and $k \le n$, we have $S_{i_1} \cap S_{i_2} \ne \emptyset$. By symmetry and since $S_1 = S_n$, we may assume that $1 \le i_1, i_2 \le n-1$ and $i_2 \in S_1$. Now let $S_i' = S_i \setminus \{i_2\}$ for $i \in V' = V \setminus \{i_1, i_2\}$. To show that $(S_i')_{i\in V'}$ satisfies Hall's condition, let $X \subseteq V'$ be an arbitrary set. Furthermore, let $P = \bigcup_{i\in X} S_i'$, $p = |P|$, $Q = [1, n-1] \setminus (P \cup \{i_2\})$, and $q = |Q|$. Since $|X| \le n-2$, we are done if $p \ge n-2$. Hence $p \le n-3$ and we have $p \ge k-2$ and $1 \le q = n-2-p \le n-k$.

Now let $\mathcal{S}' = (S_i')_{i\in V}$ and let $\mathcal{S}''$ denote the set system consisting of all sets S' of $\mathcal{S}'$ that have a non-empty intersection with Q. Clearly, $s = |\mathcal{S}''| \le |\mathcal{S}'| - |X| = n - |X|$ and, therefore, $|X| \le n - s$. Next, we claim that $s \ge q+2$. Obviously, the set system $\mathcal{S}''$ contains the q sets S_i for $i \in Q$. Since $p \ge k-2$ and $q \ge 1$, there are two distinct elements $x, y \in P$ such that, for $z \in \{x, y\}$, either $\pi(z)$ or $\pi^2(z)$ belongs to Q, and therefore, S_z belongs to $\mathcal{S}''$. This proves that $s \ge q+2$. Consequently, we have $|X| \le n - s \le n - q - 2 = p = |\bigcup_{i\in X} S_i'|$. This completes the proof. □

Let $\mathcal{S} = (S_x)_{x\in V}$ be a set system. For $c \in \bigcup_{x\in V} S_x$, let $d_{\mathcal{S}}(c)$ denote the number of sets from the set system $\mathcal{S}$ containing c.

Proposition 21. *Let $\mathcal{S} = (S_x)_{x\in V}$ be a critical (k, n)-system with $k \ge 3$. Then the following statements hold:*

(a) *$d_{\mathcal{S}}(c) \ge 2$ for every color $c \in [1, n-1]$.*
(b) *Let $c_1, c_2 \in [1, n-1]$ be two colors contained in some set of the set system $\mathcal{S}$. Then $d_{\mathcal{S}}(c_i) \ge 3$ for some $i \in \{1, 2\}$.*

Proof. We have $n \ge k$ by (C1) and $\bigcup_{x\in V} S_x = [1, n-1]$ by (C2). In order to prove (a), assume that there is a color $c \in [1, n-1]$ such that $d(c) \le 1$. Then, by (C2), we have $d(c) = 1$. Hence there is a vertex $x \in V$ such that $c \in S_x$. Since $|V| = n \ge k$, there is a vertex $y \in V \setminus \{x\}$. Then, by (C3), there is a color $c' \in S_x \cap S_y$. For every vertex $z \in V' = V \setminus \{x, y\}$, let $S_z' = S_z \setminus \{c'\}$. Then the set system $\mathcal{S}' = (S_v')_{v\in V'}$ consists of $n-2$ sets all contained in $[1, n-1] \setminus \{c, c'\}$, a contradiction to (C3). This proves (a).

In order to prove (b), assume that it is false. Then, by (a), $d(c_1) = d(c_2) = 2$. By assumption, there is a vertex $x \in V$ such that S_x contains both c_1 and c_2. Furthermore, for $i = 1, 2$, there is a vertex $y_i \ne x$ in V such that $c_i \in S_{y_i}$ and $c_i \notin S_z$ for $z \in V \setminus \{x, y_i\}$.

First consider the case that $y_1 = y_2 = y$. Choose a color $c \in S_x \cap S_y$. Then the set system $(S_z - \{c\})_{z\in V\setminus\{x,y\}}$ consists of $n-2$ sets all contained in $[1, n-1] - \{c_1, c_2, c\}$, a contradiction to (C3).

Now consider the case that $y_1 \ne y_2$. Choose a color $c \in S_{y_1} \cap S_{y_2}$. Since $d(c_1) = d(c_2) = 2$, we have $c \notin \{c_1, c_2\}$. Then $n \ge 4$ and the set system $(S_z - \{c\})_{z\in V\setminus\{x,y_1,y_2\}}$ consists of $n-3$ sets all contained in $[1, n-1] \setminus \{c_1, c_2, c\}$, a contradiction to (C3), too. This proves (b). □

If $\mathcal{S} = (S_x)_{x\in V}$ is a critical (n, k)-system with $n = k$, then it follows from (C1) that $S_x = [1, k-1]$ for all $x \in V$. The next proposition shows that for $k = 2$ this is the only critical (n, k)-system. For $k = 3$ there is one more possibility, namely $n = 4$.

Proposition 22. *Let $\mathcal{S} = (S_x)_{x\in V}$ be a critical (k, n)-system with $k \in \{2, 3\}$. Then $n = k$ and $S_x = \{1, k-1\}$ for all $x \in V$, or $k = 3$ and $n = 4$.*

Proof. By (C1) and (C2), we have $n \geq k$, $\bigcup_{x\in V} S_x = [1, n-1]$, $|V| = n$, and $|S_x| = k-1$ for all $x \in V$.

First, consider the case that $k = 2$. If $n = 2$, then $S_x = \{1\}$ for all $x \in V$ and we are done. Otherwise, $n \geq 3$ and from (C3) it follows that every two sets of the set system have a common color. Since S_x is a singleton, this implies that $S_x = \{c\}$ for all $x \in V$. Since $n \geq 3$, this gives a contradiction to (C2).

Next, consider the case that $k = 3$. If $n = 3$, then, clearly, $S_x = \{1, 2\}$ for all $x \in V$. Otherwise we have $n \geq 4$ and $|S_x| = 2$ for every $x \in V = [1, n]$. Let H be the multigraph whose vertex set is $[1, n-1]$ and whose edge set is the set system $(S_v)_{v\in V}$. Then H has $n-1$ vertices and n edges. Because of (C3), every pair of edges shares a common vertex. By Proposition 21, the minimum degree of H is 2. Then it is not difficult to check that $n = 4$ and the multigraph H is either a triangle with a double edge or a star with two double edges. □

In the proofs of our theorems on critical 3- and 4-assignments, we shall apply the following result on cross-intersecting set systems.

Theorem 23 (*Tuza [25]*). *Let s, t be positive integers. Then there exists an integer $b = b(s, t)$ such that the following statement holds: if $\mathcal{S}$ and $\mathcal{T}$ are two set systems such that, for every $S \in \mathcal{S}$ and every $T \in \mathcal{T}$, we have $|S| \leq s$, $|T| \leq t$, and $S \cap T \neq \emptyset$, then there is a set B such that $|B| \leq b$ and $S \cap T \cap B \neq \emptyset$ for every $S \in \mathcal{S}$ and every $T \in \mathcal{T}$.*

Theorem 24. *There is no critical $(4, n)$-system if n is large.*

Proof. For the proof consider a critical $(4, n)$-system $\mathcal{S} = (S_v)_{v\in V}$. Our aim is to arrive at a contradiction under the assumption that n is sufficiently large.

By (C1) and (C2), we have $\bigcup_{v\in V} S_v = [1, n-1]$, $|V| = n \geq 4$, and $|S_v| = 3$ for all $v \in V$. For a color $c \in [1, n-1]$, let $d(c) = d_{\mathcal{S}}(c)$. First, we claim that

$$\bigcap_{v\in V} S_v = \emptyset$$

provided that n is large. To prove the claim, suppose that there is a color c such that $c \in S_v$ for every vertex $v \in V$. Let H be the multigraph whose vertex set is $[1, n-1] \setminus \{c\}$ and whose edge set is the set system $(S_v \setminus \{c\})_{v\in V}$. Then H has $n-2$ vertices and n edges. Furthermore, it follows from Proposition 21 that H has minimum degree at least two and the vertices of degree 2 form an independent set in H. A simple counting argument shows that this gives a contradiction provided that n is sufficiently large. This proves the claim.

From (C3) and Theorem 23 we conclude that there is a constant b independent of n such that the following statement holds: there exists a set $X \subseteq [1, n-1]$ satisfying $|X| \leq b$ and $X \cap S_x \cap S_y \neq \emptyset$ for all $x, y \in V$.

Next, we claim that $|S_v \cap X| \geq 2$ for every vertex $v \in V$ provided that n is sufficiently large. Otherwise, there is a vertex $u \in V$ such that $S_u \cap X = \{c\}$ is a singleton. Then $c \in \bigcap_{v\in V} S_v$, a contradiction.

Let $|X| = p$, $V_1 = \{v \in V \mid S_v \subset X\}$, $V_2 = V \setminus V_1$ and $Y = [1, n-1] \setminus X$. Then $|Y| = n-p-1$. If $v \in V_2$, then $|S_v| = 3$ and the above claim implies that $|S_v \cap X| = 2$ and, therefore, $|S_v \cap Y| = 1$ (provided that n is sufficiently large). Consequently, we have

$$\sum_{c\in Y} d(c) \leq |\mathcal{S}| = n.$$

On the other hand, it follows from Proposition 21 that

$$\sum_{c\in Y} d(c) \geq 2|Y| = 2(n-p-1).$$

Combining these two inequalities, we obtain that $n \leq 2p + 2 \leq 2b + 2$ where b is a constant not depending on n. This completes the proof. □

Theorem 25. *Let $(S_v)_{v\in V}$ be a critical $(5, n)$-system. If n is sufficiently large, then $\bigcap_{v\in V} S_v \neq \emptyset$.*

Proof. For the proof consider a critical $(5, n)$-system $\mathcal{S} = (S_v)_{v\in V}$ such that $\bigcap_{v\in V} S_v = \emptyset$. Our aim is to arrive at a contradiction under the assumption that n is sufficiently large.

By (C1) and (C2), we have $\bigcup_{v\in V} S_v = [1, n-1]$, $|V| = n \geq 5$, and $|S_v| = 4$ for all $v \in V$. From (C3) and Theorem 23, we conclude that there is a constant b independent of n such that the following statement holds: there exists a set $X \subseteq [1, n-1]$ satisfying $|X| \leq b$ and $X \cap S_x \cap S_y \neq \emptyset$ for all $x, y \in V$.

Then $|S_v \cap X| \geq 2$ for every vertex $v \in V$. Otherwise, there is a vertex $u \in V$ such that $S_u \cap X = \{c\}$ is a singleton. Then $c \in \bigcap_{v\in V} S_v$, a contradiction.

Let $|X| = p$, $V_1 = \{v \in V \mid S_v \subset X\}$, $V_2 = V \setminus V_1$ and $Y = [1, n-1] \setminus X$. Then $|Y| = n-p-1$. If $v \in V_2$, then $|S_v| = 4$ and the above claim implies that $2 \leq |S_v \cap X| \leq 3$ and, therefore, $1 \leq |S_v \cap Y| \leq 2$. For $i = 1, 2$, let

$$W_i = \{v \in V_2 || S_v \cap Y| = i\}.$$

Furthermore, let $s = |W_1|$ and $e = |W_2|$. For a color $c \in [1, n-1]$, let $d(c) = d_{\mathcal{S}}(c)$. Clearly, $e + s \leq |\mathcal{S}| = n$ and we have

$$\sum_{c \in Y} d(c) \leq |\mathcal{S}| + e \leq n + e \leq 2n - s. \tag{1}$$

Let $Z = \{c \in Y \mid d(c) = 2\}$ and $U = \{c \in Y \mid d(c) \geq 3\}$. By Proposition 21, $Y = Z \cup U$ and, therefore, we have

$$\sum_{c \in Y} d(c) \geq 2|Z| + 3|U| = 2|Y| + |U| = 2(n - p - 1) + |U|. \tag{2}$$

Combining (1) and (2), we obtain that

$$|U| \leq 2(p+1) - s. \tag{3}$$

From Proposition 21 it follows that $|S_v \cap Z| \leq 1$ for every vertex $v \in W_2$. This implies that $e = |W_2| \geq 2|Z| - s$. Since $|Y| = |Z| + |U|$, this gives $e \geq 2(|Y| - |U|) - s = 2(n - p + 1) - 2|U| - s$. By (3), it then follows that $e \geq 2n - 6p - 2 + s$. On the other hand, $e \leq |\mathcal{S}| = n$. Consequently, we have $n \leq 6p + 2 - s \leq 6p + 2$. This completes the proof. $\square$

Example 18 tells us that the conclusion $\bigcap_{v \in V} S_v \neq \emptyset$ cannot be strengthened to $|\bigcap_{v \in V} S_v| \geq 2$. Furthermore the following example shows that the conclusion $\bigcap_{v \in V} S_v \neq \emptyset$ is not valid for $k \geq 6$.

Example 26. Let $n \geq 7$ be an integer. Let $V = [1, n]$ and let π be a permutation of the set $[1, n-1]$ consisting of two cycles with $\pi(1) = 2, \pi(2) = 3, \ldots, \pi(n-5) = n-4, \pi(n-4) = 1, \pi(n-3) = n-2, \pi(n-2) = n-1$ and $\pi(n-1) = n-3$. For every $i \in V$, let

$$S_i = \{\pi^{i-1}(j) \mid j = 1, 2, 3, n-3, n-2\}.$$

Then $(S_i)_{i \in V}$ is a critical $(6, n)$-system.

Proof. Clearly, the set system $\mathcal{S} = (S_i)_{i \in V}$ satisfies (C1) and (C2). For the proof of (C3), let i_1 and i_2 be two distinct elements of V. Then there is a color $c \in \{n-3, n-2, n-1\}$ such that $c \in S_{i_1} \cap S_{i_2}$. Now let $S'_i = S_i \setminus \{c\}$ for $i \in V' \setminus \{i_1, i_2\}$. To show that $(S'_i)_{i \in V'}$ satisfies Hall's condition, let $X \subseteq V'$ be an arbitrary set. Furthermore, let $P = \bigcup_{i \in X} S'_i$, $Q = [1, n-4] \setminus P$ and $R = \{n-3, n-2, n-1\} \setminus \{c\} \setminus P$. Obviously, $|P| + |Q| + |R| = n - 2$. Since $|X| \leq n - 2$, we are done if $|P| \geq n - 2$. Now assume that $|P| \leq n - 3$. Since $|X| \geq 1$, we obtain $|P| \geq 4$ and $0 \leq |R| \leq 1$.

First, let us consider the case that $|R| = 0$. Then $|Q| = n - 2 - |P| \geq 1$. Now let $\mathcal{S}'$ denote the set system consisting of all sets S of $\mathcal{S}$ that have a non-empty intersection with Q. Clearly, $s = |\mathcal{S}'| \leq |\mathcal{S}| - |X| = n - |X|$ and, therefore, $|X| \leq n - s$. Next, we claim that $s \geq q + 2$, where $q = |Q|$. Obviously, the set system $\mathcal{S}'$ contains the q sets S_i for $i \in Q$. Since $|P| \geq 4$ and $q \geq 1$, there are two distinct elements $x, y \in P$ such that, for $z \in \{x, y\}$, either $\pi(z)$ or $\pi^2(z)$ belongs to Q, and therefore, S_z belongs to $\mathcal{S}'$. This proves that $s \geq q + 2$. Consequently, we have $|X| \leq n - s \leq n - q - 2 = |P| = |\bigcup_{i \in X} S'_i|$.

Now, consider the case that $|R| = 1$. Then, $R' = \{n-3, n-2, n-1\} \setminus R$ is contained in $S_i = \{\pi^{i-1}(j) \mid j = 1, 2, 3, n-3, n-2\}$ for all $i \in X$. Hence, for $i, j \in X$ with $1 \leq i < j \leq n$ either $S_i \cap S_j = R'$ or $i \in \{1, 2, 3, 4\}$, where the latter case occurs at most twice. Then, for every $i \in X$, we can choose an element $c_i \in S'_i$ such that $c_i \neq c_j$ whenever $i \neq j$. This clearly implies that $|\bigcup_{i \in X} S'_i| \geq |X|$. $\square$

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