On minimally circular-imperfect graphs☆

Baogang Xu

School of Mathematics and Computer Science, Nanjing Normal University, 122 Ninghai Road, Nanjing 210097, China

Received 7 March 2006; received in revised form 18 January 2007; accepted 26 June 2007
Available online 6 August 2007

Abstract

A circular-perfect graph is a graph of which each induced subgraph has the same circular chromatic number as its circular clique number. In this paper, (1) we prove a lower bound on the order of minimally circular-imperfect graphs, and characterize those that attain the bound; (2) we prove that if $G$ is a claw-free minimally circular-imperfect graph such that $\omega_c(G - x) > \omega(G - x)$ for some $x \in V(G)$, then $G = K_{(2k+1)/2} + x$ for an integer $k$; and (3) we also characterize all minimally circular-imperfect line graphs.

© 2007 Elsevier B.V. All rights reserved.

MSC: 05c15; 05c78
Keywords: Circular coloring; Circular clique; Circular-perfect

1. Introduction

All graphs considered are finite and simple, i.e., finite graphs without multiedges and loops. Undefined concepts and terminologies are from [5].

Let $G = (V, E)$ be a graph, where $V$ and $E$ denote the vertex set and edge set of $G$, respectively. Two vertices $u$ and $v$ are adjacent, denoted by $uv \in E(G)$, if there is an edge in $E(G)$ joining them. A proper subgraph of $G$ is a subgraph which is not $G$ itself. A subgraph $H$ of $G$ is called an induced subgraph if $E(H) = \{uv | u \in V(H), v \in V(H), uv \in E(G)\}$. Let $S \subset V$ be a subset of vertices. We use $G[S]$ to denote the subgraph of $G$ induced by $S$.

A graph $G$ is called a perfect graph if every induced subgraph $H$ of $G$ has the same chromatic number $\chi(H)$ as its clique number $\omega(H)$. A minimally imperfect graph is an imperfect graph of which each proper induced subgraph is perfect. An odd hole is an odd circuit of length at least five. The famous Perfect Graph Conjecture [3] was proved by Chudnovsky et al. in [6].

**Theorem 1 (Chudnovsky et al. [6], Strong Perfect Graph Theorem).** The only minimally imperfect graphs are the odd holes and their complements.

Let $k$ and $d$ be positive integers with $k \geq 2d$. A $(k, d)$-circular coloring of a graph $G$ is a mapping $\psi : V(G) \rightarrow \{0, 1, 2, \ldots, k - 1\}$ such that $d \leq |\psi(u) - \psi(v)| \leq k - d$ whenever $uv \in E(G)$. A graph $G$ is called $k/d$-circular...
colorable if it admits a \((k, d)\)-circular coloring. The \textit{circular chromatic number} of \(G\), denoted by \(\chi_c(G)\), is defined as \(\chi_c(G) = \inf\{k/d | G \text{ is a } k/d\text{-circular colorable graph}\}\).

The concept of circular coloring was first introduced in 1988 by Vince \cite{9} with the name \textit{star coloring}, and it got the current name from Zhu \cite{15}. It was proved elsewhere \cite{4,9} that \(\chi_c(G)\) is always attained at rational number and

\[
\chi(G) - 1 < \chi_c(G) \leq \chi(G) \quad \text{for any graph } G.
\]  

2. Circular-perfect graphs

A \((k, d)\)-\textit{partition} of \(G\) is a partition \((V_0, V_1, V_2, \ldots, V_{k-1})\) of \(V(G)\) such that for each \(i\), \(0 \leq i \leq k - 1\), \(V_i \cup V_{i+1} \cup \cdots \cup V_{i+d-1}\) is an independent set of \(G\), where the addition of indices is taken mod \(k\) (\(V_i = \emptyset\) for some \(i\) is permitted).

It is easy to see that a \((k, d)\)-partition of \(G\) is simply the color classes of a \((k, d)\)-coloring of \(G\). Below is a theorem from \cite{7}.

\textbf{Theorem 2 (Fan \cite{7}).} A graph \(G\) has a \((k, d)\)-circular coloring iff it has a \((k, d)\)-partition. Furthermore, \(\chi_c(G) = k/d\) iff \(G\) is \((k/d)\)-circular colorable and for every \((k, d)\)-partition \(V_0, V_1, \ldots, V_{k-1}\) of \(G\), \(V_i \neq \emptyset\) for every \(i\).

Section 2 is devoted to the concept of circular-perfect graphs and some examples of minimally circular-imperfect graphs. In Section 3, we present a lower bound on the order of minimally circular-imperfect graphs, and characterize those that attain the bound. In Section 4, we characterize the claw-free minimally circular-imperfect graphs with the property that \(\omega_c(G - x) > \omega(G - x)\) for some \(x \in V(G)\). In the last section, we characterize all minimally circular-imperfect line graphs.

2. Circular-perfect graphs

Given two positive integers \(k\) and \(d\) with \(k \geq 2d\), let \(K_{k/d}\) be a graph with \(V(K_{k/d}) = \{v_0, v_1, v_2, \ldots, v_{k-1}\}\) and \(E(K_{k/d}) = \{v_i v_j | d \leq |j - i| \leq k - d\}\). While \(d = 1\), \(K_{k/1}\) is simply the complete graph \(K_k\) of order \(k\).

It was proved that \(\chi_c(K_{k/d}) = k/d\) \cite{4,9}. So, if a graph \(G\) contains a subgraph \(H\) isomorphic to \(K_{k/d}\) (we simply denote it by \(H = K_{k/d}\)) for some \(k\) and \(d\), then \(\chi_c(G) \geq k/d\). Unless otherwise specified, \(\{v_0, v_1, v_2, \ldots, v_{k-1}\}\) and \(\{v_i v_j | d \leq |j - i| \leq k - d\}\) always refer to the vertex set and edge set of \(K_{k/d}\), respectively.

The \textit{circular clique number} of \(G\) (first introduced by Zhu in \cite{16}), denoted by \(\omega_c(G)\), is defined as the maximum fractional \(k/d\) such that \(K_{k/d}\) admits a homomorphism to \(G\). Let \(\gcd(k, d)\) be the greatest common divisor of integers \(k\) and \(d\). Zhu proved in \cite{16} that

\textbf{Theorem 3 (Zhu \cite{16}).} For any graph \(G\),

\[
\omega(G) \leq \omega_c(G) < \omega(G) + 1
\]

(2)

and \(\omega_c(G) = k/d\) for some \(k\) and \(d\) with \(\gcd(k, d) = 1\) indicates that \(G\) contains an induced subgraph isomorphic to \(K_{k/d}\).

A graph \(G\) is called \textit{circular-perfect} if \(\omega_c(H) = \chi_c(H)\) for each induced subgraph \(H\) of \(G\) \cite{16}. Up to now, we do not know too much on the structure of circular-perfect graphs. Some sufficient conditions and necessary conditions for a graph to be circular-perfect were discussed in \cite{11,16}. Bang-Jensen and Huang presented in \cite{1} a family of circular-perfect graphs, they called them \textit{convex-round graphs}, which is a super-family of \(K_{k/d}\’s\).

\textbf{Theorem 4 (Bang-Jensen and Huang \cite{1}, Zhu \cite{16}).} For any integers \(k \geq 2d\), \(K_{k/d}\) is circular-perfect.

A \textit{circular-imperfect} graph is a graph that is not circular-perfect, and a \textit{minimally circular-imperfect graph} is a circular-imperfect graph of which each proper induced subgraph is circular-perfect. The Strong Perfect Graph Theorem and Theorem 4 tell us that every minimally imperfect graph is in fact circular-perfect.

To study the circular-perfect graphs, a natural approach is to characterize the minimally circular-imperfect graphs. It seems that the structure of minimally circular-imperfect graphs is much more complicated than that of
minimally imperfect graphs. Below are two minimally circular-imperfect graphs obtained from the Petersen graph [11].

$H_2$ is obtained from the complement $G$ of the Petersen graph by removing three vertices that induce a path in the Petersen graph.

A major vertex in a graph $G$ is a vertex of degree $|V(G)| - 1$. In [12], we characterized all of the minimally circular-imperfect graphs that have a major vertex. Given graph $H$ and a vertex $u \notin V(H)$, we use $H + u$ to denote the graph obtained by joining $u$ to every vertex of $H$.

**Theorem 5 (Xu [12]).** Let $G$ be a minimally circular-imperfect graph with a major vertex $u$. Then, there exists an integer $n \geq 2$ such that $G = K(2n + 1)/n + u$, or $G = K(2n + 1)/2 + u$.

Motivated by $H_1$ of Fig. 1, we find a family of plane graphs that are minimally circular-imperfect. For integer $n \geq 2$, let $C_{2n + 1} = x_0x_1 \ldots x_{2n}x_0$ and $C'_{2n + 1} = y_0y_1 \ldots y_{2n}y_0$ be two circuits of length $2n + 1$, and let $F_n$ be the graph obtained from $C_{2n + 1}$ and $C'_{2n + 1}$ by identifying $x_0x_1$ with $y_0y_1$, and adding edge $x_{n+1}y_{n+1}$. $F_2$ is just the graph $H_1$ (Fig. 1).

**Theorem 6.** For every $n \geq 2$, $F_n$ is a minimally circular-imperfect graph.

**Proof.** First we show that $F_n$ is circular-imperfect. It is easy to check that $F_n$ contains exactly four induced circuits of length $2n + 1$, and contains no circuits of shorter length. That is, $\omega_c(F_n) = 2 + 1/n$.

In any $(2n + 1, n)$-coloring $f$ of $K(2n+1)/n = C_{2n+1}$, if $f(v_0) = 0$ and $f(v_{2n}) = 2n$, then every other vertex $v_i$ just receives a color $f(v_i) = i$, $i = 1, 2, \ldots, 2n - 1$ (recall that $V(K(2n+1)/n) = \{v_0, v_1, \ldots, v_{2n}\}$ as claimed). Therefore, $x_{n+1}$ and $y_{n+1}$ must receive the same color in any $(2n + 1, n)$-coloring of $F_n - x_{n+1}y_{n+1}$, and hence $\chi_c(F_n) > (2n + 1)/n = \omega_c(F_n)$, i.e., $F_n$ is not circular-perfect.

Since for an arbitrary induced proper subgraph $B$ of $F_n$, the unique 2-connected component of $B$ consists of either a circuit of length $2n + 1$, or two circuits of the same length $2n + 1$ that share some edges in common. In either case, $\chi_c(B) = \omega_c(B)$. So, $F_n$ is minimally circular-imperfect. □

A family of minimally circular-imperfect series-parallel graphs were constructed in [13]. These minimally circular-imperfect graphs listed above look so different, it seems that a general structural characterization is not easy to reach for them. The remaining of this paper is about some conditional characterizations of the minimally circular-imperfect graphs.
3. A lower bound on the order of minimally circular-imperfect graphs

By the Strong Perfect Graph Theorem, every minimally imperfect graph \( G \) satisfies \( |V(G)| \geq 2\omega(G) + 1 \), and half of the minimally imperfect graphs reach the bound. Following two theorems give a lower bound on the order of minimally circular-imperfect graphs.

**Theorem 7.** Let \( G \) be a minimally circular-imperfect graph. If \( \omega_c(G − x) = \omega(G − x) \) for every vertex \( x \) of \( G \), then \( |V(G)| \geq 2\omega(G) + 1 \).

**Proof.** Assume to the contrary that the theorem is not true. Let \( G \) be a minimally circular-imperfect graph with \( \omega(G) = r \), \( \omega_c(G − x) = \omega(G − x) \) for every vertex \( x \), and \( |V(G)| \leq 2r \). Below is a direct consequence of (1) and (2):

\[
\chi(G) \geq \chi_c(G) > \omega_c(G) = \omega(G) = r. \tag{3}
\]

If \( \omega_c(G) \neq r \), there must exist integers \( k \geq 2 \) and \( 0 < l < k \), and an induced subgraph \( H \) of \( G \) such that \( H = K_{(kr+l)/k} \).

Since \( K_{(kr+l)/k} \) is circular-perfect, \( G \neq K_{(kr+l)/k} \), and hence \( |V(G)| > kr + l \geq 2r + 1 \). Therefore, \( \omega_c(G) = r \).

Let \( u \) be a vertex of \( G \). Since \( G − u \) is circular-perfect, and since \( \omega_c(G − u) = \omega_c(G − u) \) is an integer, if \( \omega_c(G − u) = r − 1 \) then \( \chi_c(G − u) = \chi_c(G − u) = \omega_c(G − u) = r − 1 \) that provides \( \chi(G) \leq r \), thus \( \omega_c(G − u) = r \) and \( \chi(G − u) = \chi_c(G − u) = r \).

Then, \( V(G − u) \) can be partitioned into \( r \) independent sets. Choose \( V_0, V_1, \ldots, V_{r−1} \) to be such a partition that

\[
\sum_{0 \leq i < j \leq r−1} (|V_i| − |V_j|) \text{ is minimum.}
\]

Since \( |V(G − u)| \leq 2r − 1 \), there must be a set among \( V_0, V_1, \ldots, V_{r−1} \) that has cardinality one. Assume by symmetry that \( V_0 = \{u_0\} \). It is certain that \( uu_0 \in E(G) \) for otherwise \( G \) will be \( r \)-colorable.

Assume that there exists an \( i_0 \) and a vertex \( x \in V_{i_0} \) such that \( |V_{i_0}| \geq 3 \) and \( uu_0 \notin E(G) \). Let \( V'_0 = \{u_0, x\} \), \( V'_0 = V_{i_0} \setminus \{x\} \), and let \( V'_i = V_i \) for \( i \neq 0, i_0 \). Then, \( \{V'_i ∣ 0 \leq i < r − 1\} \) is a partition of \( V(G − u) \) into \( r \) independent sets with

\[
\sum_{0 \leq i < j \leq r−1} (|V'_i| − |V'_j|) < \sum_{0 \leq i < j \leq r−1} (|V_i| − |V_j|) \text{. Therefore,}
\]

\[
\text{if } |V_i| \geq 3 \text{, then } uu_0 \in E(G) \text{ for each } x \in V_i \text{, } 1 \leq i < r − 1. \tag{4}
\]

Let \( F \) be a complete subgraph of \( G − u \) on vertex set \( V(F) = \{u_0, u_1, \ldots, u_{r−1}\} \). It is obvious that \( u_i \in V_i \). Let \( G' = G − u_0 \) and let \( H \) be a complete subgraph of order \( G' \) (as depicted in Fig. 3) such that \( |V(H) \cap V(F)| \) is maximum. Certainly, \( u \in V(H) \) and \( V(H) \cap V_i \neq \emptyset \) while \( i > 0 \). If \( uu_0 \in E(G) \) for each \( x \in V(H) \), then \( G[V(H) \cup \{u_0\}] \cong K_{r+1}. \) So, we have

\[
uu_0 \notin E(G) \text{ for some } x \in V(H) \setminus V(F). \tag{5}
\]

If \( |V_i| = 1 \), then \( V_i = \{u_i\} \). If \( |V_i| \geq 2 \) and \( uu_0 \notin V(H) \), we choose \( x_i \) to be the vertex in \( V(H) \cap V_i \). Otherwise, let \( x_i \neq u_i \) be an arbitrary vertex in \( V_i \).

Since \( \omega(G) = r \) and \( \chi(G) > r \), \( uu_0 \notin E(G) \) for some \( 1 \leq i_0 \leq r − 1 \), and \( u \) must be adjacent to some vertices of \( V_i \) for every \( i \). Furthermore, we have that for every \( i \),

\[
\text{if } V_i = \{u_i, x_i\} \text{ and } uu_0 \notin E(G), \text{ then } uu_0x_i \in E(G). \tag{6}
\]

To prove (6), let \( V_{i_0} = \{u_{i_0}, x_{i_0}\} \) such that \( uu_{i_0} \notin E(G) \). If \( uu_{i_0}x_{i_0} \notin E(G) \), then \( G \) admits an \( r \)-coloring \( f \) with \( f(y) = i \) for \( y \in V_i \) with \( i \neq 0, i_0 \), \( f(u_0) = f(x_{i_0}) = 0 \) and \( f(u_{i_0}) = f(u) = i_0 \). If \( uu_{i_0} \notin E(G) \), then \( G \) admits an \( r \)-coloring \( f' \) with \( f'(y) = i \) for \( y \in V_i \) with \( i \neq i_0 \), and \( f'(u) = f'(x_{i_0}) = f'(u_{i_0}) = i_0 \). Both are contradictions.

![Fig. 3.](image-url)
Let $A = V(H - u) \setminus V(F) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\}$, where $s \geq 1$. Let $A_1 = \{x_i | x_i \in A \text{ and } uu_i \notin E(G)\}$, and let $A_2 = A \setminus A_1$. Recall that $V_h$ is an independent set among the selected partition of $V(G - u)$.

By (4) and (6), $u_0x \in E(G)$ for each $x \in A_1$. By (5), there must be an $x \in A_2$ such that $u_0x \notin E(G)$. We proceed to show that either $G'$ contains a subgraph $H' = K_r$ with $|V(H') \cap V(F)| > |V(H) \cap V(F)|$ that contradicts the choice of $H$, or $G$ admits an $r$-coloring that contradicts (3). These contradictions will end our proof.

Let $I_1 = \{i | x_i \in A_1\}$, and let $I_2 = \{i | x_i \in A_2\}$. We will partition $A_2$ and $I_2$ into subsets, respectively, as follows. Let $A_{2,0} = \{x \in A_2 | uu_0x \notin E(G)\}$, and let $I_{2,0} = \{i | x_i \in A_{2,0}\}$. For $i \geq 1$, we iteratively define

$$A_{2,i} = \left\{ x | x \in A_2 \setminus \left( \bigcup_{j=0}^{i-1} A_{2,j} \right), xu_i \notin E(G) \text{ for some } l \in I_{2,i-1} \right\},$$

and

$$I_{2,i} = \{l | x_l \in A_{2,i}\}.$$  

By (5), $A_{2,0} \neq \emptyset$. Since $A_2$ is finite, there must be an integer $q$ such that $A_{2,q} \neq \emptyset$ and $A_{2,q+1} = \emptyset$. By the definition, for $i = q, q - 1, \ldots, 2, 1$, every vertex in $A_{2,i}$ has a nonadjacent vertex $u_j \in V_j$ for some $j \in I_{2,i-1}$, and

$$x_iu_k \in E(G) \text{ for all } k \in \bigcup_{h=0}^{i} I_{2,h}, \text{ and } l \in I_2 \setminus \left( \bigcup_{h=0}^{i} I_{2,h} \right).$$  

(7)

By (4), $u_0$ is adjacent to every vertex of $V_h$ if $|V_h| \geq 3$. So, $|V_i| = 2$ for every $i \in I_{2,0}$. We will show that

$$|V_i| = 2 \text{ for each } i \in \bigcup_{h=0}^{q} I_{2,h}.$$  

(8)

Assume to the contrary that $a > 0$ is the smallest index such that $|V_b| \geq 3$ for some $b \in I_{2,a}$. Let $d_a = b$. By the definition of $A_{2,i}$’s, for $m = a, a - 1, \ldots, 2, 1$, there exists integer $d_{m-1} \in I_{2,m-1}$ such that $x_{d_m}u_{d_{m-1}} \notin E(G)$. Let $V_0' = \{u_0, x_{d_0}\}, V_{d_1}' = \{u_{d_1}, x_{d_{d_1}+1}\}, i = 0, 1, \ldots, a - 1, V_b' = V_b \setminus \{x_b\}$, and let $V_i' = V_i$ for $i \notin \{0, d_0, \ldots, d_{a-1}, b\}$. Then, $\{V_i' | 0 \leq i \leq r - 1\}$ is a partition of $V(G - u)$ into $r$ independent sets with $\sum_{0 \leq i < j \leq r-1} |V_i'| - |V_j'| < 0 \leq i < j \leq r-1 |V_i| - |V_j||$, that contradicts to the choice of $(V_i')$’s.

Let $x_b$ be an arbitrary vertex in $A_{2,0}$. Assume that there exists a $t \in I_1$ with the property that $u_bx_i \notin E(G)$. If $|V_i| = 2$, then $G$ admits an $r$-coloring $f$ with $f(y) = i$ for $y \in V_i$ and $i \notin \{0, t, h\}$, $f(u_0) = f(x_h) = f(x_i) = h$ and $f(u_t) = f(u) = t$ (recall that $uu_i \notin E(G)$), that contradicts to (3). If $|V_i| \geq 3$, let $V_i' = V_i \setminus \{x_i\}$, $V_h = \{u_h, x_i\}$, $V_0 = \{u_0, x_b\}$, and let $V_i' = V_i$ for $i \notin \{0, t, h\}$, then $\{V_i' | 0 \leq i \leq r - 1\}$ is a partition of $V(G - u)$ into $r$ independent sets with $\sum_{0 \leq i < j \leq r-1} |V_i'| - |V_j'| < 0 \leq i < j \leq r-1 |V_i| - |V_j||$, that contradicts to the choice of $(V_i')$’s. So, we have $u_ix_j \in E(G)$ for all $i \in I_{2,0}$ and $j \in I_1$. By using the same arguments, we will prove that

$$u_ix_j \in E(G) \text{ for all } i \in I_{2,0} \text{ and } j \in I_1.$$  

(9)

If it is not the case, let $a$ be an integer, $0 < a \leq q$, such that for some $b \in I_{2,a}$ and $c \in I_1$, $u_bx_c \notin E(G)$ (note here that $x_b \in A_{2,a}$, and $V_b = \{u_b, x_b\}$ by (8)). By the definition of $(A_{2,i})$’s, for $m = a, a - 1, \ldots, 2, 1$, there exists integer $d_{m-1} \in I_{2,m-1}$ such that $x_{d_m}u_{d_{m-1}} \notin E(G)$, where $d_a = b$. Using the same arguments as used in the last paragraph, we have either $|V_b| \geq 3$ and hence $V(G - u)$ admits a new partition $\{V_i' | 0 \leq i \leq r - 1\}$ into $r$ independent sets with $\sum_{0 \leq i < j \leq r-1} |V_i'| - |V_j'| < 0 \leq i < j \leq r-1 |V_i| - |V_j||$, or $V_i = \{u_c, x_c\}$ and hence $G$ admits an $r$-coloring $f$ with $f(w) = i$ for $w \in V_i$ and $i \notin \{d_0, d_1, \ldots, d_{a-1}, b, c\}$, $f(u_0) = f(x_d) = 0$, $f(u_{d_j}) = f(x_{d_{j+1}}) = d_j$ for $j = 1, \ldots, a - 1$, $f(u_b) = f(x_c) = b$ and $f(u_c) = f(x_d) = c$, both are contradictions.

Combine (9) and (7), $G'(V(H) \setminus \cup_{h=0}^{q} A_{2,h})) \cup \{u_i | i \in \cup_{h=0}^{q} I_{2,h}\} = K_r$ that contains at least one more vertex of $V(F)$ than $H$, a contradiction to the choice of $H$. The proof is completed. □

By Theorem 5, $(K_{2k+1}/2 + u | k \geq 2)$ is a family of minimally circular-imperfect graphs with $|V(K_{2k+1}/2 + u)| = 2\omega(K_{2k+1}/2 + u)$. Furthermore, we have the following theorem.
Theorem 8. Let $G$ be a minimally circular-imperfect graph. Then, $|V(G)| \geq 2\omega(G)$ and $\{K_{(2k+1)/2} + u | k \geq 2\}$ is the unique family of minimally circular-imperfect graphs with $|V(G)| = 2\omega(G)$.

Proof. It is easy to check that no minimally circular-imperfect graph has less than five vertices.

Let $r \geq 3$ be an integer and let $G$ be a minimally circular-imperfect graph with $|V(G)| \leq 2\omega(G) = 2r$. By Theorem 7, there exists a vertex $u \in V(G)$ such that $\omega_c(G - u) > \omega(G - u)$. It is certain that $\omega_c(G - u)$ cannot be integer. Let $\omega_c(G - u) = k/d$, where gcd$(k, d) = 1$. Then, $d \geq 2$, and $k/d > r - 1$.

If $d > 2$, then $d \geq 2 + 2/(r - 1) = 2r/(r - 1)$ that gives $d(r - 1) \geq 2r$, and hence $2r \geq |V(G)| \geq k + 1 > d(r - 1) + 1 \geq 2r + 1$. Therefore, $d = 2$, $\omega_c(G - u) = (2(r - 1) + 1)/2$, and $G - u = K_{(2r - 1)/2}$.

Recall that $V(K_{(2r - 1)/2}) = \{v_0, v_1, \ldots, v_{2r - 2}\}$. If $G \neq K_{(2r - 1)/2} + u$, we may assume by symmetry that $uv_0 \notin E(G)$. Since $K_{(2r - 1)/2} - v_0$ is $(r - 1)$-colorable, $\chi(G) = r$, and hence $r = \omega(G) = \omega_c(G) = \chi_c(G) = \chi(G) = r$. This contradicts the circular-imperfectness of $G$. Therefore, $G = K_{(2r - 1)/2} + u$. □

4. Minimally circular-imperfect claw-free graphs

A claw-free graph is a graph that contains no $K_{1,3}$ as an induced subgraph. Claw-free graphs is a family of dense graphs that possesses of many beautiful properties. Line graphs is a subfamily of claw-free graphs. As earlier as 1976, it was proved that the Perfect Graph Conjecture is valid on claw-free graphs [8].

Below is a characterization of claw-free graphs that have the same clique number as circular clique number [14]. We present its proof here for completeness.

Lemma 1 (Xu and Zhou [14]). Let $G$ be a connected non null claw-free graph. If $\omega_c(G) \neq \omega(G)$, then $G$ is either an odd hole, or $\omega_c(G) \geq 3$ and $\omega_c(G) = \omega(G) + \frac{1}{2}$.

Proof. Since $E(G)$ is nonempty, $\omega(G) \geq 2$. Since $K_{k/d}$ contains $K_{1,3}$ as an induced subgraph while $d \geq 3$ and $k \neq 2d + 1$, if $G$ contains an induced subgraph $F = K_{k/d}$ for some integers $k > 2d$ with gcd$(k, d) = 1$, then either $F$ is an odd circuit, or $d \leq 2$.

If $\omega(G) = 2$, then each vertex of $G$ has degree at most 2, and so $\omega_c(G) \neq \omega(G) = 2$ implies that $G$ is an odd hole.

If $\omega(G) \geq 3$ and $\omega_c(G) \neq \omega(G)$, then there exists an odd integer $k$ such that $\omega_c(G) = \frac{k}{2}$. By Theorem 3, $\omega_c(G) = \lceil k/2 \rceil$, that is, $\omega_c(G) = \omega(G) + \frac{1}{2}$. □

Given nonnegative integers $i$, $j$ and $k$, we use $I_{i,j}^k = \{i, i + 1, \ldots, j\}$ to denote a circular-integral-interval from $i$ to $j$ modulo $k$, where the summations are taken modulo $k$. An even circular-integral-interval is one containing even number of integers, and odd circular-integral-intervals are defined similarly.

Let $G$ be a graph obtained from $K_{(2r + 1)/2} + u$ by deleting some edges incident with $u$, say $uv_{i_1}, uv_{i_2}, \ldots, uv_{i_l}$. Let $I_h = \{2^h i + 1, 2^h i + 1, \ldots, 2^h l - 1\}$ for $1 \leq h \leq l - 1$, and $I_0 = \{2^1 i + 1, 2^1 i + 1, \ldots, 2^1 i + 1\}$ be the maximal circular-integral-intervals of $\{0, 1, \ldots, 2r\} \setminus \{i_1, i_2, \ldots, i_l\}$. It is not difficult to check that $\omega_c(G) = r$ if there are two even circular-integral-intervals among $I_1, I_2, \ldots, I_l$. The converse is true also. More precisely, we have the following lemma.

Lemma 2. Let $G$ be a claw-free graph obtained from $K_{(2r + 1)/2} + u$ by deleting edges $uv_{i_1}, uv_{i_2}, \ldots, uv_{i_l}$ ($l \geq 1$) such that $\omega(G) = 2$, and let $I_1, I_2, \ldots, I_l$ be the maximal circular-integral-intervals of $\{0, 1, \ldots, 2r\} \setminus \{i_1, i_2, \ldots, i_l\}$. Then, $\omega(G) = r$ if and only if there exist two even circular-integral-intervals among $I_1, I_2, \ldots, I_l$.

Proof. It is certain that $\omega(G) \geq \omega(K_{(2r + 1)/2}) = r$. Let $H = G[N(u)]$. Then, $\omega(H) = r$ if and only if $\omega(H) \leq r - 1$. Since $\omega(G) = 2$, $u$ is adjacent to at least one of $i_l$ and $v_{i_l + 1}$ for every $i$. Following fact comes from the structure of $K_{(2r + 1)/2}$ directly.

$$\omega(H) = r \quad \text{iff} \quad V(H) \supset \{v_{ih}, v_{ih + 2}, \ldots, v_{ih + 2r - 2}\} \quad \text{for an } i_H,$$

where the summations are taken modulo $2r + 1$.

Since $2r + 1 = \sum_{i=1}^l (|I_i| + 1)$, at least one item of the righthand side is odd, i.e.,

$$I_{i_0} \quad \text{is even for some } 1 \leq i_0 \leq l.$$

Without loss of generality, assume that $I_1 = \{0, 1, 2, \ldots, 2p - 1\} \quad (p \geq 1)$ is an even circular-integral-interval.
Sufficiency. Assume to the contrary that \( \omega(H) = r \), and assume that \( H' = H'[v_{ih}, v_{ih+2}, \ldots, v_{ih+2r-2}] = K_r \). Let \( I_1 \) and \( I_j = \{s, s+1, \ldots, t\} (1 < j \leq l) \) be two even circular-interval-intervals. Then, \( \{uv_{2r}, uv_{2p}, uv_{3s-1}, uv_{i+1}\} \cap E(G) = \emptyset \) (recall that if \( I_i \) is a maximal circular-interval-interval for every \( i \)).

If \( v_0 \notin V(H') \), then \( \{uv_{2r}, v_0\} \cap V(H') = \emptyset \), and hence \( V(H') = \{v_1, v_3, \ldots, v_{2r-1}\} \) by (10). If \( v_0 \in V(H') \) but \( v_{2r-1} \notin V(H') \), then \( V(H') = \{v_{2p-1}, v_{2p+3}, \ldots, v_{2r-1}, v_0\} \) since \( \{v_{2p-1}, v_{2p}\} \cap V(H') = \emptyset \). If \( \{v_0, v_{2p-1}\} \subset \ V(H') \), then there must exist an odd integer \( 1 \leq i_1 \leq 2p - 3 \) such that \( \{v_{i_1}, v_{i+1}\} \cap V(H') = \emptyset \) because that at most one of \( v_i \) and \( v_{i+1} \) is in \( V(H') \) for every \( i \), and hence \( V(H') = \{v_{i_1+2}, v_{i_1+4}, \ldots, v_{i_1+2r-2}\} \).

If \( I_1 \) is even then \( i_H \in \{1, 3, 5, \ldots, 2p+1\} \). (12)

By the same argument as used on \( I_1 \), if \( I_j \) is even then \( i_H \in \{1, 3, 5, \ldots, 2p\} \). The sufficiency follows from the fact that \( \{1, 3, 5, \ldots, 2p+1\} \cap \{1, 3, 5, \ldots, t+2\} = \emptyset \).

Necessity. Since \( \omega(G) = r \), \( \omega(H) \leq r - 1 \) and \( d(u) \leq 2r - 1 \), and there must be an even circular-interval-interval by (11). Assume that \( I_1 \) is the unique even circular-interval-interval. Since \( uv_{2p} \notin E(G), uv_{2p+1} \notin E(G) \), it is certain that \( uv_{2p+2} \notin E(G) \), then \( uv_{2p+3} \in E(G) \) for otherwise \( z(G) \geq 3 \). If \( uv_{2p+2} \in E(G) \), then again \( uv_{2p+3} \in E(G) \) for otherwise \( I_2 = \{v_{2p+1}, v_{2p+2}\} \) is even. Using the similar arguments, one gets \( \{v_{2p+1}, v_{2p+3}, \ldots, v_{2r-1}\} \subset N(u) \).

Then \( H'[\{v_0, v_2, \ldots, v_{2p-2}, v_{2p+1}, v_{2p+3}, \ldots, v_{2r-1}\}] = K_r \), a contradiction to \( \omega(H) \geq r - 1 \). □

**Theorem 9.** Let \( G \) be a claw-free minimally circular-imperfect graph. If \( G \) contains a vertex \( u \) such that \( \omega_c(G-u) = k/d \) with \( d \geq 2 \) and \( \text{gcd}(d, k) = 1 \), then \( d = 2 \) and \( G = K_{(2r+1)/2} + u \) for an integer \( r = [k/2] \).

**Proof.** Since \( \omega_c(G-u) = k/d \), \( G \) contains \( K_{k/d} \) as an induced subgraph. We claim that \( d = 2 \).

If it is not the case, assume that \( d \geq 3 \). If \( K_{k/d} \) is not an odd circuit, then \( K_{k/d} \) contains \( K_{1,3} \) as an induced subgraph. So, \( K_{k/d} \) must be an odd circuit that has length at least 7. Since \( G - u \) is circular-perfect, it contains no \( K_3 \), and thus the claw-free property provides that \( G - u = C \). Since \( \alpha(G) \geq 3 \), \( u \) cannot be adjacent to every vertex of \( C \). Assume that \( x \) is a vertex on \( C \) such that \( ux \notin E(G) \). Since \( G \) is claw-free, \( u \) has to be adjacent to at least one of the two neighbors of \( x \) on \( C \). This indicates that \( G \) contains \( K_3 \), and thus \( 3 = \omega(G) = \omega_c(G) = \chi_c(G) = \chi(G) = 3 \). This contradicts the circular-imperfectness of \( G \).

Therefore, \( d = 2 \) and hence \( k = 2r + 1 \) for an integer \( r \).

It is easy to verify that the wheel \( C_5 + x = K_{5/2} + x \) is the unique minimally circular-imperfect graph of order less than 7, we may suppose that \( r \geq 3 \).

First we show that \( G - u = K_{(2r+1)/2} \). Let us assume the contrary, then there exists a proper induced subgraph \( H \) of \( G \) that is isomorphic to \( K_{(2r+1)/2} \). Suppose that \( V(H) = \{v_0, v_1, \ldots, v_{2r}\} \).

Since \( G \) is minimally circular-imperfect, \( G - u \) is connected and admits a \((2r + 1, 2)\)-circular coloring. By Theorem 2, \( G - u \) admits a \((2r + 1, 2)\)-partition. We choose \( V_0, V_1, \ldots, V_{2r} \) to be such a partition with \( v_i \in V_i \). Since \( V(G - u) \neq V(H) \), we may choose a vertex \( x \in V(G - u) \setminus V(H) \) and assume by symmetry that \( x \in V_0 \) \( \in E(G) \). But then \( x \in V_0 \) for some \( 2 \leq i_0 \leq 2r - 1 \), and hence either \( G[\{v_0, v_{i_0}, v_{i_0-1}\}] \) or \( G[\{v_0, v_{i_0}, v_{i_0+1}\}] \) is an induced \( K_{1,3} \). This contradiction indicates that \( V(G - u) = V(H) \).

Now, we proceed to show that \( u \) is adjacent to every vertex of \( \{v_0, v_1, \ldots, v_{2r}\} \). If it is not the case, we may assume, by symmetry, that \( u \notin E(G) \).

If \( \omega(G) = r + 1 \), then \( \omega_c(G) = r + 1 \). Since the partition \( \{u, v_0\}, \{v_1, v_2\}, \ldots, \{v_{2r-1}, v_{2r}\} \) of \( V(G) \) gives an \((r + 1)\)-coloring of \( G, r + 1 \leq \omega_c(G) \leq \chi_c(G) \leq r + 1 \). This contradicts the circular-imperfectness of \( G \). So, we have

\[
\omega(G) = r .
\]

If there exists an \( i \) such that \( uv_i \notin E(G) \) and \( uv_{i+1} \notin E(G) \), assume (without loss of generality) that \( uv_{2r} \in E(G), uv_0 \notin E(G) \) and \( uv_{2r} \notin E(G) \). Since \( G[\{v_{2r}, v_1, v_2, u\}] \neq K_{1,3}, v_2u \in E(G) \). Since \( G[\{v_0, v_1, v_2, u\}] \neq K_{1,3}, uv_i \notin E(G) \) for \( i = 3, 4, \ldots, 2r - 1 \). Since \( r \geq 3 \) by our assumption, \( G \) has a \((2r + 1, 2)\)-partition \( \{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4, u\}, \{v_5\}, \ldots, \{v_{2r}\} \), that contradicts the choice of \( G \). Therefore,

\[
\{|uv_i, uv_{i+1}\} \cap E(G) \geq 1 \quad \text{for every } i,
\]

and hence \( \chi(G) = 2 \).
Let $I_1, I_2, \ldots, I_l$ be the maximal circular-integral-intervals of $\{0, 1, \ldots, 2r\} \setminus \{i | uv_i \notin E(G)\}$. By Lemma 2, at least two of these circular-integral-intervals are even. Since $2r + 1 = \sum_{l=1}^l (|I_l| + 1), l \geq 3$. Without loss of generality, assume that $I_1 = [0, 1, \ldots, 2p - 1]$ for an integer $p \geq 1$. Let $V' = I_1 \cup \{v_{2r}, v_{2p}, u\}$. Then, $G[V'] = K_{(2p+3)/2}$.

Let $I_b = \{v_i, v_{i+1}, \ldots, v_{j_b}\}$ be another even circular-integral-interval. Since $l \geq 3$, if $b = 2$ then $v_{j_b}$ is adjacent to each vertex of $V'$, and if $b \neq 2$ then $v_{j_b}$ is adjacent to each vertex of $V'$. Therefore, either $G[V' \cup \{v_{j_b}\}] = K_{(2p+3)/2} + v_{j_b}$, or $G[V' \cup \{v_{j_b}\}] = K_{(2p+3)/2} + v_{j_b}$. This is a contradiction because $K_{(2p+3)/2} + x$ is a minimally circular-imperfect graph. The proof is completed. \hfill \Box

The condition “claw-free” cannot be removed from Theorem 9. Following is an example. Let $G$ be the maximal circular-integral-intervals of $\{0, 1, \ldots, 5d + p, d\}$. Let $G$ be a graph obtained from $K_{(5d+p)/d}$ by adding a new vertex $w$ and joining it to $v_0, v_{2d-1}, v_{4d-2}, v_{4d}$ and $v_{5d}$ of $K_{(5d+p)/d}$. Then, $\omega(G) = 5$ and $\omega_c(G) = (5d + p)/d$. For any $j \in \{0, 1, \ldots, 5d + p - 1\}\{0, 2d - 1, 4d - 2, 4d, 5d\}$, there is an $i$ such that $i \leq j \leq i + d - 1$ and $w$ is adjacent to $v_i$, where the summation is taken modulo $5d + p$. Therefore, $G$ does not admit any $(5d + p, d)$-partition, and hence $\chi_c(G) > (5d + p)/d = \omega_c(G)$ by Theorem 2. Since $K_{(5d+p)/d}$ is circular-perfect, it is straightforward to check that $G$ is a minimally circular-imperfect graph, we leave the details to the readers.

5. Circular-imperfect line graphs

Given a graph $H$, the line graph $L(H)$ of $H$ is a graph on vertex set $E(H)$ and edge set $\{ef | e \in E(H), f \in E(H), \text{ and } e \neq f \text{ are adjacent in } H\}$. Line graphs is a special family of claw-free graphs. Let $F$ be the subgraph obtained from $K_4$ by inserting a new vertex into one of its edges. By a theorem of Beineke [2], $F$ is a forbidden subgraph of line graphs. Since for $k \geq 3$, $K_{(2k+1)/2}$ contains $F$ as an induced subgraph, by Lemma 1, we have

**Lemma 3.** If $\Delta(H) \geq 3$, then $\omega_c(L(H)) = \omega(L(H))$.

Given a graph $H$, we use $\Delta(H)$ and $\chi'(H)$ denote the maximum degree and the chromatic index of $H$, respectively. If $G = L(H)$ for a connected graph $H$, then $\omega(G) = \Delta(H)$ unless $H = K_3$, and there is a one-to-one correspondence between the coloring of $G$ and the edge coloring of $H$. By the famous Vizing’s theorem, $\Delta(H) \leq \chi'(H) \leq \Delta(H) + 1$. Therefore, $\chi'(L(H)) \leq \omega(L(H)) + 1$.

A graph $G$ is of Class Two if its chromatic index $\chi'(G)$ is $\Delta(G) + 1$, and is of Class One otherwise. A graph is called $\Delta(G)$-critical if $G$ is of Class Two and $\chi'(G - e) < \chi'(G)$ for every edge $e$ of $G$. Following theorem is due to Vizing.

**Theorem 10** (Vizing [10]). If $G$ is a graph of Class Two, then $G$ contains a $k$-critical subgraph for each $k$ satisfying $2 \leq k \leq \Delta(G)$.

**Lemma 4.** Let $G$ be a graph of Class Two. If $\Delta(G) \geq 3$, then $L(G)$ is circular-imperfect.

**Proof.** By Lemma 3, $\omega_c(L(G)) = \omega(L(G)) = \Delta(G)$ if $\Delta(G) \geq 3$.

Since $G$ is of Class Two, $\chi(L(G)) = \chi'(G) = \Delta(G) + 1 = \omega(L(G)) + 1$. Therefore, $\chi_c(L(G)) > \chi(L(G)) - 1 = \omega(L(G)) = \omega_c(L(G))$, that means $L(G)$ is circular-imperfect. \hfill \Box

Let $H$ be a 3-critical graph, and let $E' \subset E(H)$ be a nonempty subset of edges. If $\Delta(H - E') = 3$, then $\chi'(H - E') = 3$ by the criticality of $H$, and $\omega_c(L(H - E')) = \omega(L(H - E')) = 3$ by Lemma 3, and hence $\chi_c(L(H - E')) = \omega_c(L(H - E'))$. If $\Delta(H - E') \leq 2$, then every connected component of $H - E'$ is a path or a circuit, and hence $\chi_c(L(H - E')) = \omega_c(L(H - E'))$ also. By Lemma 4, $L(H)$ is minimally circular-imperfect.

Let $G = L(H)$ be a line graph. Since the only 2-critical graphs are the odd circuits $C_{2n+1}$ for $n \neq 1$, and since the line graph of $C_{2n+1}$ is still $C_{2n+1}$ that is circular-perfect for every $n \geq 1$, if $G$ is minimally circular-imperfect, then by Theorem 10 and Lemma 4, $H$ is of Class One. Therefore, we have the following theorem.

**Theorem 11.** Let $G = L(H)$. Then, $G$ is minimally circular-imperfect if $H$ is a 3-critical graph. In another words, $G$ is circular-perfect if every subgraph of $H$ except odd circuit is of Class One.
References