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Note

On Convex Approximation by Quadratic Splines

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$\omega_3(f, n^{-1})$. The knots of the spline are basically equally spaced. In this paper we give a simple construction of such a spline with equally spaced knots.

In a recent paper [1] Hu proves that for any convex function f there is a C^1 convex quadratic spline s with n knots that approximates f at the rate of $\omega_3(f, n^{-1})$. The knots of the spline constructed in [1] are "basically equally spaced". We give here a simple construction of such a spline with equally spaced knots.

THEOREM. *Let $f \in C[0, 1]$ be convex and let n be natural. There is a C^1 convex quadratic spline s with knots at i/n, such that*

$$\|f - s\| \leqslant c\omega_3(f, n^{-1}), \tag{1}$$

where c is an absolute constant, $\|\cdot\|$ and ω_3 are the uniform norm and the third uniform modulus of smoothness in $[0, 1]$ respectively.

Proof. Set $h = n^{-1}$, $m = [n/3] - 1$ and $x_i = 3ih$ for $i = 0, 1, ..., m+1$. For $i = 1, 2, ..., m$ let $P_i(x) = A_i x^2 + B_i x + C_i$ be the parabola interpolating f at the points x_{i-1}, x_i, x_{i+1}, i.e.,

$$P_i(x_k) = f_k := f(x_k) \qquad \text{for} \quad k = i-1,\ i,\ i+1. \tag{2}$$

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From (2) and the convexity of f we have

$$A_i = \tfrac{1}{18} n^2 (f_{i+1} - 2f_i + f_{i-1}), \qquad A_i \geqslant 0. \tag{3}$$

Set $g(x) = P_1(x)$ for $x \in [0, x_1]$, $g(x) = P_m(x)$ for $x \in [x_m, 1]$. For $i = 2, 3, ..., m$ set

$$g(x) = \begin{cases} P_i(x) & \text{if } A_i \leqslant A_{i-1}; \\ P_{i-1}(x) & \text{if } A_{i-1} \leqslant A_i \end{cases} \qquad \text{for} \quad x \in [x_{i-1}, x_i]. \tag{4}$$

The function g is a continuous quadratic spline interpolating f at the knots $\{x_i\}$. For the distance between f and g we have from Whitney's theorem [3]

$$\|f - g\| \leqslant c_1 \omega_3(f, n^{-1}). \tag{5}$$

An important observation is that g is convex (see Remark 1). The only property, which g lacks to satisfy the theorem, is the discontinuity of g' at the knots.

The function $ax_+^2 + bx_+$ can be smoothed to C^1 spline with knots -1, 0, 1 by

$$\sigma(a, b; x) = ax_+^2 + bx_+ + \tfrac{1}{4} b(1 - |x|)_+^2. \tag{6}$$

In order to apply (6) we calculate the differences between the parabolas interpolating f at x_i. From (2) we obtain

$$\begin{aligned} P_i(x) - P_{i-1}(x) &= (A_i - A_{i-1})(x - x_i)(x - x_{i-1}) \\ &= h^2 (A_i - A_{i-1})\, n(x - x_i)(n(x - x_i) + 3), \end{aligned} \tag{7}$$

$$P_{i+1}(x) - P_i(x) = h^2 (A_{i+1} - A_i)\, n(x - x_i)(n(x - x_i) - 3). \tag{8}$$

Adding (7) and (8) we get

$$\begin{aligned} &P_{i+1}(x) - P_{i-1}(x) \\ &\quad = h^2 (A_{i+1} - A_{i-1})\, n^2 (x - x_i)^2 + 3h^2 (2A_i - A_{i+1} - A_{i-1})\, n(x - x_i). \end{aligned} \tag{9}$$

Now we change g on the intervals $[x_i - h, \; x_i + h]$ to a smoother function s in order to get a convex approximant to f. Define $s(x) = g(x)$ for $x \notin \bigcup_{i=1}^m [x_i - h, \; x_i + h]$. For every $[x_i - h, \; x_i + h]$, $i = 1, 2, ..., m$, we define s as follows ($A_0 = A_1$, $A_{m+1} = A_m$):

Case 1. $A_{i-1} \geqslant A_i \leqslant A_{i+1}$. We set $s(x) = P_i(x) (= g(x))$.

Case 2. $A_{i-1} < A_i \leqslant A_{i+1}$. Then $g = P_{i-1}$ in $[x_{i-1}, x_i]$ and $g = P_i$ in $[x_i, x_{i+1}]$. Having in mind (6) and (7) we set

$$s(x) = P_{i-1}(x) + h^2 \sigma(A_i - A_{i-1}, 3A_i - 3A_{i-1}; n(x - x_i)). \tag{10}$$

From (6) and (10) we get $s(x)-g(x)=\frac{3}{4}(A_i-A_{i-1})\,h^2(1-n\,|x-x_i|)^2$ and hence

$$0\leqslant s(x)-g(x)\leqslant \tfrac{3}{4}h^2(A_i-A_{i-1})$$
$$=\tfrac{1}{24}(f_{i+1}-3f_i+3f_{i-1}-f_{i-2})\leqslant \tfrac{1}{24}\omega_3(f,3n^{-1})\leqslant \tfrac{9}{8}\omega_3(f,n^{-1}). \quad (11)$$

Case 3. $A_{i-1}\geqslant A_i>A_{i+1}$. Having in mind (6) and (8) we set

$$s(x)=P_i(x)+h^2\sigma(-A_i+A_{i+1},3A_i-3A_{i+1};n(x-x_i)). \quad (12)$$

From (6) and (12) we get

$$0\leqslant s(x)-g(x)\leqslant \tfrac{3}{4}h^2(A_i-A_{i+1})\leqslant \tfrac{9}{8}\omega_3(f,n^{-1}). \quad (13)$$

Case 4. $A_{i-1}<A_i>A_{i+1}$. Having in mind (6) and (9) we set

$$s(x)=P_{i-1}(x)+h^2\sigma(A_{i+1}-A_{i-1},3(2A_i-A_{i+1}-A_{i-1});n(x-x_i)). \quad (14)$$

From (6) and (14)

$$0\leqslant s(x)-g(x)\leqslant \tfrac{3}{4}h^2(2A_i-A_{i+1}-A_{i-1})$$
$$=\tfrac{1}{24}(-f_{i+2}+4f_{i+1}-6f_i+4f_{i-1}-f_{i-2})\leqslant \tfrac{1}{12}\omega_3(f,3n^{-1})\leqslant \tfrac{9}{4}\omega_3(f,n^{-1}). \quad (15)$$

From (11), (13) and (15) we get

$$\|s-g\|\leqslant \tfrac{9}{4}\omega_3(f,n^{-1}),$$

which together with (5) implies (1).

The convexity of s will follow from the non-negativity of the leading coefficients of its parabolic components. Outside $\bigcup_{i=1}^{m}[x_i-h,\ x_i+h]$ these coefficients are non-negative in view of (3). In $[x_i-h,x_i]$ and $[x_i,x_{i+h}]$, $i=1,2,...,m$, the coefficients are (see (6), (10), (12) and (14))

$$\begin{aligned}
&A_{i-1}+\tfrac{3}{4}(A_i-A_{i-1})\geqslant 0\\
&\qquad\text{and } A_i+\tfrac{3}{4}(A_i-A_{i-1})\geqslant 0 \text{ in Case 2;}\\
&A_i+\tfrac{3}{4}(A_i-A_{i+1})\geqslant 0\\
&\qquad\text{and } A_{i+1}+\tfrac{3}{4}(A_i-A_{i+1})\geqslant 0 \text{ in Case 3;} \qquad (16)\\
&A_{i-1}+\tfrac{3}{4}(2A_i-A_{i+1}-A_{i-1})\geqslant 0\\
&\qquad\text{and } A_{i+1}+\tfrac{3}{4}(2A_i-A_{i+1}-A_{i-1})\geqslant 0 \text{ in Case 4;}
\end{aligned}$$

respectively. This completes the proof. ∎

Remark 1. A basic step in the proof is the simple (although non-linear) construction in (4) of the *convex* continuous quadratic spline g. The convexity follows from the convexity of the parabolic components and the positive jumps of g' at x_i, which are $3A_i - 3A_{i-1}$, $3A_i - 3A_{i+1}$ and $3(2A_i - A_{i-1} - A_{i-1})$ in cases 2, 3 and 4 respectively.

Remark 2. Hu, Leviatan and Yu show in [2, Theorem 3] that for any convex function f there is a C^2 convex *cubic* spline S with $O(n)$ equally spaced knots, such that

$$\|f - S\| \leqslant c\omega_3(f, n^{-1}), \qquad \|S'''\| \leqslant cn^3\omega_3(f, n^{-1}). \tag{17}$$

One can easily construct such a cubic spline by smooting the spline

$$s(x) = \alpha + \beta x + \sum_{j=0}^{n-1} \gamma_j (x - jh)_+^2 \tag{18}$$

from Theorem 1, where coefficients γ_j satisfy (see (16) and (3))

$$|\gamma_j| \leqslant \tfrac{5}{2} \max_i |A_i - A_{i-1}| \leqslant \tfrac{15}{4} n^2 \omega_3(f, n^{-1}). \tag{19}$$

Simply set

$$S(x) = s(x) + \sum_{j=1}^{n-1} \left(\frac{1}{4n}\right)^2 |\gamma_j| \, \eta(4n(x - jh) \operatorname{sign} \gamma_j),$$

$$\eta(x) = \theta(x) - x_+^2, \qquad \theta(x) := \frac{1}{9}(x+1)_+^3 + \frac{1}{6} x_+^3 - \frac{1}{3}(x-1)_+^3 + \frac{1}{18}(x-2)_+^3.$$

Note that S is a cubic spline with $4n$ equally spaced knots, $S^{(k)}(jh - h/2) = s^{(k)}(jh - h/2)$, $k = 0, 1, 2$, $j = 1, 2, ..., n$. (17) follows from (1) and (19). Finally, the convexity of S can be verify as follows. In $x \in [jh, jh + h]$, $j = 0, 1, ..., n-1$, write s as the parabola Q_j and note its convexity in view of the convexity of s. From (18) the difference between two consecutive parabolas is $Q_j(x) - Q_{j-1}(x) = \gamma_j (x - jh)_+^2$. Then in $[jh - h/2, jh + h/2]$, $j = 1, 2, ..., n-1$, we have

$$S(x) = Q_{j-1}(x) + \left(\frac{1}{4n}\right)^2 \gamma_j \theta(4n(x - jh)) \qquad \text{if} \quad \gamma_j \geqslant 0;$$

$$S(x) = Q_j(x) + \left(\frac{1}{4n}\right)^2 |\gamma_j| \, \theta(-4n(x - jh)) \qquad \text{if} \quad \gamma_j < 0.$$

This representation implies the convexity of S because θ and Q_j are convex.

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