Uniform asymptotic expansions of the Pollaczek polynomials

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Dedicated to Professor Roderick S.C. Wong on the occasion of his 60th birthday

Abstract

Two uniform asymptotic expansions are obtained for the Pollaczek polynomials $P_n(\cos \theta; a, b)$. One is for $\theta \in (0, \delta/\sqrt{n}]$, $0 < \delta < \sqrt{a + b}$, in terms of elementary functions and in descending powers of $\sqrt{n}$. The other is for $\theta \in [\delta/\sqrt{n}, \pi/2]$, in terms of a special function closely related to the modified parabolic cylinder functions, in descending powers of $n$. This interval contains a turning point and all possible zeros of $P_n(\cos \theta)$ in $\theta \in (0, \pi/2]$.

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1. Introduction

The Pollaczek polynomials $P_n(x; a, b)$ can be defined by the generating function

$$
(1 - we^{i\theta})^{-(1/2)+ih(\theta)}(1 - we^{-i\theta})^{-(1/2)-ih(\theta)} = \sum_{n=0}^{\infty} P_n(x; a, b)w^n,
$$

(1.1)

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where \( x = \cos \theta \) for \( \theta \in (0, \pi) \),
\[
h(\theta) = \frac{a \cos \theta + b}{2 \sin \theta}, \quad a > \pm b,
\]
and each of the factors in the generating function reduces to 1 for \( w = 0 \).

The Pollaczek polynomials show in many aspects a singular behavior, see [10, pp. 393–396]. In his 1954 thesis [7], A. Novikoff investigated the asymptotic behavior of these polynomials and their zeros. He extracted the asymptotic behavior of \( P_n(x; a, b) \) as \( n \to \infty \), where \( x = \cos(t/\sqrt{n}) \), and \( t > 0 \) is fixed.

He obtained the behavior in two intervals of \( t \) on both sides of \( t = \sqrt{a+b} \). Furthermore, if the zeros of these polynomials are denoted by \( \cos(\theta_{vn}) \), where \( 0 < \theta_{1n} < \ldots < \theta_{nn} < \pi \), then he showed that for any fixed \( v \),
\[
\lim_{n \to \infty} \sqrt{n} \theta_{vn} = \sqrt{a + b}.
\]

It was conjectured by R.A. Askey that the next term of the asymptotic expansion of \( \theta_{vn} \) would involve a certain transcendental function. Partially inspired by this idea, Ismail [6] and Bo and Wong [4] both derived two term asymptotic expansions for \( \theta_{vn} \). Our approach in this paper bears some impact of [4], and is actually in accordance with Bleistein [3], Chester et al. [5] and Wong [12, Chapter VII].

One of the main results of Bo and Wong [4] is the following expansion
\[
\theta_{vn} = \sqrt{\frac{a+b}{n}} + \frac{(a+b)^{1/6}(-a_v)}{2n^{5/6}} + O\left(\frac{1}{n^{7/6}}\right),
\]
where \( a_v \) is the \( v \)th negative zero of the Airy function. Since \( -a_v = O(n^{2/3}) \) as \( n \to \infty \) (cf. [10, p. 377]), we can see that (1.4) is unlikely to be applied to the largest zero. Indeed, (1.4) was derived based on a uniform asymptotic expansion for \( \theta \in [\delta/\sqrt{n}, \ M/\sqrt{n}] \). Despite the fact that this \( \theta \)-interval includes the transition point \( t = \sqrt{a+b} \), it includes only those extreme zeros. Hence it would be highly desirable to obtain a universal expansion in the whole interval \( \theta \in (0, \pi) \). This is one of our main motivations of this investigation.

The present research is also motivated by the uniform treatment of Darboux’s method. If we denote for short \( P_n(x) = P_n(x; a, b) \), then
\[
P_n(x) = \frac{1}{2\pi i} \int_C (1 - we^{i\theta})^{-(1/2)+ih(\theta)} (1 - we^{-i\theta})^{-(1/2)-ih(\theta)} w^{-n-1} dw,
\]
where \( x = \cos \theta \), \( C \) is a simple closed curve encircling the origin but not the branch points \( e^{\pm i\theta} \), positively oriented. It is readily seen that \( C \) can be deformed into the bold paths, still denoted by \( C \), as illustrated in Fig. 1. The integral in (1.5) seems closely related to Darboux’s method, as has been dealt with in a preceding paper [13]. But this time it is of more interest; with the appearance of \( h(\theta) \), along with the coalescing of a pair of branch points \( e^{\pm i\theta} \) on the circle of convergence as \( \theta \to 0 \), there is a singularity in the exponent since \( h(\theta) \sim \frac{a+b}{2\theta} \). Treating problems of this type is also of interest to us.

In view of the reflection formula [7, p. 7]
\[
P_n(x; a, b) = (-1)^n P_n(-x; a, -b),
\]
one may assume without loss of generality that $\theta \in (0, \pi/2]$. In this investigation, we divide the mentioned interval into two subintervals. In one of the subintervals, namely, $\theta \in (0, \delta/\sqrt{n}]$, where $\delta$ is a small positive constant, we obtain a uniform asymptotic expansion in descending powers of $\sqrt{n}$, in terms of elementary functions. While in the more important interval, that is, $\theta \in [\delta/\sqrt{n}, \pi/2]$, we introduce a two step procedure: firstly, we obtain an expansion in terms of a certain special function $I(\alpha, z)$, in descending powers of $n$, and, secondly, we show that in the interval of uniformity, $I(\alpha, z)$ can be uniformly approximated by the modified parabolic cylinder functions $W(\alpha, \pm z)$. In this case, we use a quite indirect method, via the Airy function, to estimate the error term in part of the interval.

The present paper is arranged as follows: In Section 2, we derive formally an expansion in terms of $I(\alpha, z)$. The asymptotic properties of $I(\alpha, z)$ are provided in Section 3. After that, in Section 4, we estimate the coefficients of the formal expansion obtained, and as a result the expansion can be modified to (4.8), in a more adaptable form. Section 5 is devoted to obtaining the error bounds in various cases for (4.8), and the latter is thus made a uniform asymptotic expansion for $\theta \in [\delta/\sqrt{n}, \pi/2]$, $\delta$ being a small positive constant. The remaining $\theta$-interval is treated in Section 6. A uniform asymptotic expansion for $\theta \in (0, \delta/\sqrt{n}]$, in terms of elementary functions, is provided there. In the last section, Section 7, some discussion is carried out. Remarks are given and some generalization is provided.
2. Formal derivation

From (1.5), we first separate the branch points \( w = e^{\pm i \theta} \) by applying the transformation

\[
w = e^{-s \theta},
\]

where the \( w \)-plane is cut along the negative real line. Accordingly, we have

\[
P_n(x) = \frac{1}{2\pi i} \int_{\Gamma} \Phi_0(s, \theta)(s - i)^{-(1/2) + ih(\theta)}(s + i)^{-(1/2) - ih(\theta)} e^{n0s} \, ds,
\]

where \( \Gamma \) is the image of \( C \) under the mapping \( w \leftrightarrow s \), the upper half of it is \( \overline{FEDC(C_1)G} \) and is accordingly oriented, as illustrated in Fig. 2, with \( F \) and \( G \) tending to infinity, and the lower half is symmetric with respect to the horizontal axis, \( x = \cos \theta \) and

\[
\Phi_0(s, \theta) = \theta \left( \frac{1 - w e^{i\theta}}{s - i} \right)^{-(1/2) + ih(\theta)} \left( \frac{1 - w e^{-i\theta}}{s + i} \right)^{-(1/2) - ih(\theta)}.
\]

\( \Phi_0 \) is determined by choosing

\[
\arg (s - i) \in [-\pi, \pi], \quad \arg (s + i) \in [-\pi, \pi].
\]

In view of (2.2), it is natural to expand \( P_n(\cos \theta) \) in terms of

\[
I(z, s) := \frac{1}{2\pi i} \int_{\Gamma} (s - i)^{-(1/2)+(z/2)i}(s + i)^{-(1/2)-(z/2)i} e^{(1/4)z^2s} \, ds
\]
and
\[
\frac{2}{z} I'(x, z) := \frac{1}{2\pi i} \int_{\Gamma} s(s - i)^{-(1/2) + (x/2)i}(s + i)^{-(1/2) - (x/2)i} e^{(1/4)z^2 s} \, ds,
\]
(2.5)
where \( x = 2h(\theta), z = 2\sqrt{n\theta} \), and the derivative for \( I \) is with respect to the second variable \( z \). We will show later that the function \( I(x, z) \) is closely related to the parabolic cylinder functions.

Next, we proceed to derive an expansion for \( P_n(\cos \theta) \) in terms of \( I \) and its derivative. To this end, introducing a uniform parameter
\[
t = \sqrt{n}\theta
\]
(2.6)
and denoting
\[
s_{\pm} = \begin{cases} 
\pm \sqrt{\frac{a+b}{t^2}} - 1, & t < \sqrt{a+b}, \\
\pm i\sqrt{1 - \frac{a+b}{t^2}}, & t > \sqrt{a+b}, 
\end{cases}
\]
(2.7)
and defining iteratively
\[
\phi_k(s, \theta) = z_k(\theta) + s\beta_k(\theta) + (s - s_+)(s - s_-)\psi_k(s, \theta)
\]
(2.8)
and
\[
\phi_{k+1}(s, \theta) = -(s - i)^{(1/2) - iH(\theta)}(s + i)^{(1/2) + iH(\theta)} \frac{d}{ds} \left[ (s - i)^{(1/2) - iH(\theta)}(s + i)^{(1/2) - iH(\theta)} \psi_k(s, \theta) \right] \\
= -(s^2 + 1) \frac{d\psi_k(s, \theta)}{ds} - (s - 2H(\theta))\psi_k(s, \theta)
\]
(2.9)
for \( k = 0, 1, 2, \ldots \), where
\[
H(\theta) = h(\theta) - \frac{a + b}{2\theta}
\]
is the regular part of \( h(\theta) \), formally we have
\[
P_n(\cos \theta) = I(x, z) \sum_{k=0}^{p-1} \frac{z_k(\theta)}{(z/2)^{2k}} + \frac{2}{z} I'(x, z) \sum_{k=0}^{p-1} \frac{\beta_k(\theta)}{(z/2)^{2k}} + \varepsilon_p(n, \theta)
\]
(2.10)
with
\[
\varepsilon_p(n, \theta) = \frac{1}{(z/2)^{2p}} \int_{\Gamma} \phi_p(s, \theta)(s - i)^{-(1/2) + (x/2)i}(s + i)^{-(1/2) - (x/2)i} e^{(1/4)z^2 s} \, ds
\]
(2.11)
for \( p = 1, 2, 3, \ldots \), in which the integral is of the same form as in (2.2). The coefficients are readily determined from (2.8) as follows.
\[
z_k(\theta) = \frac{\phi_k(s_+, \theta) + \phi_k(s_-, \theta)}{2}, \quad \beta_k(\theta) = \frac{\phi_k(s_+, \theta) - \phi_k(s_-, \theta)}{2s_+}
\]
(2.12)
for \( k = 0, 1, 2, \ldots \). Thus all \( \phi_k(s, \theta) \) and \( \psi_k(s, \theta) \) have the same domain of \( s \)-analyticity, namely,
\[
|\text{Im } s | < \pi/\theta.
\]
3. Asymptotic properties of $I(x, z)$

It is readily verified that

$$J'' + \left( \frac{1}{4}z^2 - x + \frac{1}{4}z^{-2} \right) J = 0,$$

where the derivatives are taken with respect to $z$,

$$J(x, z) = z^{1/2} I(x, z)$$

and $I(x, z)$ is defined in (2.4). From (2.4), we also have

$$J(x, z) \sim \tau(x)\sqrt{2/z} e^{i\theta} + \tau(z)\sqrt{2/z} e^{-i\theta}, \quad z \to +\infty$$

and

$$J'(x, z) \sim i\tau(x)\sqrt{z/2} e^{i\theta} - i\tau(z)\sqrt{z/2} e^{-i\theta}, \quad z \to +\infty,$$

where

$$\begin{align*}
\tau(x) &= -i\pi^{-1}e^{-(1/2)i\phi_2}2^{1/2}i\pi e^{(1/4)x^2} I\left(\frac{1}{2} + \frac{1}{2}ix\right) \cosh \left(\frac{1}{2}\pi^2\right), \\
\Theta &= \frac{1}{4}z^2 - x \ln z + \frac{1}{4}\pi + \frac{1}{2}\phi_2
\end{align*}$$

with $\phi_2 = \text{arg } I(\frac{1}{2} + ix)$. The value of $\text{arg } I(\frac{1}{2} + ix)$ is determined by the conditions that it is a continuous function of $x$ and equals to $x \ln x - x + O(x^{-1})$ as $x \to +\infty$. Straightforward verification shows that for large $x$,

$$\tau(x) = -\frac{i}{\sqrt{2\pi}} e^{(1/2)x^2} \left\{ 1 + O\left(\frac{1}{x}\right) \right\} = -\frac{i}{2\sqrt{\pi k}} \left\{ 1 + O\left(\frac{1}{x}\right) \right\}$$

with $k = k(x) = \sqrt{1 + e^{2\pi x} - e^{4\pi x}}$.

$I(x, z)$ and $J(x, z)$ defined above are closely related to the modified parabolic cylinder functions. Indeed, plugging $z = \sqrt{2}n^{1/4}\zeta$ and $x = \frac{1}{2}n^{1/2}\beta^2$ into (3.1) yields

$$\frac{d^2 J}{dx^2} = \left\{ n(\beta^2 - \zeta^2) - \frac{1}{4}\zeta^2 \right\} J,$$

which is in the form of the equation investigated by Olver [9, p. 168, (9.1)], with the obvious difference of the singularity at $\zeta = 0$. Modifying Olver’s proof by replacing the $\zeta$-intervals of integration $[0, \zeta]$ and $[\zeta, \xi_2]$, respectively, by $[\delta_1, \zeta]$ and $[\zeta, \infty)$ gives results similar to those stated in Theorem III of [9]. That is, we have solutions $w_1(n, \beta, \zeta)$ and $w_2(n, \beta, \zeta)$ of (3.5) such that

$$w_1(n, \beta, \zeta) = k(x)^{-1/2} W(x, z) + \varepsilon_1(n, \beta, \zeta),$$

$$w_2(n, \beta, \zeta) = k(x)^{1/2} W(x, -z) + \varepsilon_2(n, \beta, \zeta),$$
where \( W(x, \pm z) \) are the modified parabolic cylinder functions, i.e., the fundamental pair of real solutions of the equation:

\[
\frac{d^2W}{dz^2} + \left( \frac{1}{4}z^2 - x \right) W = 0.
\]

In (3.6) and (3.7), the error terms and their derivatives can be estimated as

\[
|\varepsilon_1(n, \beta, \zeta)| \leq E^{-1}(x, z) \left[ \exp \left\{ \frac{\ell_3(z)}{\sqrt{2}n^{1/4}} \varphi_{\zeta, \infty} \right\} - 1 \right]
\]

and

\[
|\varepsilon_2(n, \beta, \zeta)| \leq E(x, z) \left[ \exp \left\{ \frac{\ell_3(z)}{\sqrt{2}n^{1/4}} \varphi_{\delta_1, \zeta} \right\} - 1 \right],
\]

where

\[
\varphi_{\zeta_1, \zeta_2} = \int_{\zeta_1}^{\zeta_2} \frac{1}{4\xi^2 \Omega \left( \sqrt{2}n^{1/4} \xi \right)} \, d\xi,
\]

\[
\ell_3(z) = \sup_{z \in (0, \infty)} \{ \Omega(z)M^2(x, z) \}, \quad z > 0,
\]

\( \delta_1 > 0 \) is a constant, and \( \Omega(z) \) is to be specified later. Furthermore,

\[
E(x, z) = \left\{ \frac{k(z)W(x, -z)}{W(x, z)} \right\}^{1/2} \quad \text{for } 0 \leq z \leq \sigma(z), \quad E(x, z) = 1 \quad \text{for } z \geq \sigma(z),
\]

\( M(x, z) \) and \( N(x, z) \) are the modulus functions defined by

\[
k^{-(1/2)}(x)W(x, z) = E^{-1}(x, z)M(x, z) \sin(\theta(x, z)),
\]

\[
k^{1/2}(x)W(x, -z) = E(x, z)M(x, z) \cos(\theta(x, z))
\]

and

\[
k^{-(1/2)}(x)W'(x, z) = E^{-1}(x, z)N(x, z) \sin(\omega(x, z)),
\]

\[
k^{1/2}(x)W'(x, -z) = -E(x, z)N(x, z) \cos(\omega(x, z)).
\]

In Olver’s notation, \( \sigma(z) \) is the smallest positive zero of the equation

\[
k^{-(1/2)}(z)W(x, z) = k^{1/2}(z)W(x, -z).
\]

These estimates can be obtained by following exactly the steps in [9, Sec. 9]. Obviously the coefficient \(-\left(1/4\zeta^2 \right)\) in (3.5) is independent of \( n \) and \( \beta \). Employing the argument of [9, Sec. 9.3], taking \( \Omega(z) = z^{1/3} \), we have

\[
|\varepsilon_{\zeta, \infty}|, \quad |\varepsilon_{\delta_1, \zeta} \leq |\varepsilon_{\delta_1, \infty} = C_0n^{-(1/12)}
\]

for \( \zeta \geq \delta_1 \), where \( C_0 = 2^{13/6} \int_{\delta_1}^{\infty} \zeta^{-7/3} \, d\zeta \). With this choice of \( \Omega(z) \), we have \( \ell_3(z) \) bounded for arbitrary \( x \); cf. Olver [9, (9.7)]. Therefore, we can sharpen the results in Olver [9, Sec. 9.3] as follows:

\[
\varepsilon_1(n, \beta, \zeta) = E^{-1}(x, z)M(x, z)O(n^{-(1/3)}), \tag{3.8}
\]

\[
\varepsilon_2(n, \beta, \zeta) = E(x, z)M(x, z)O(n^{-(1/3)}) \tag{3.9}
\]
and
\[
\frac{\hat{v}_1(n, \beta, 2^{-1/2} n^{-1/4} z)}{\hat{z}} = E^{-1}(x, z) N(x, z) O(n^{-1/3}),
\]
\[
\frac{\hat{v}_2(n, \beta, 2^{-1/2} n^{-1/4} z)}{\hat{z}} = E(x, z) N(x, z) O(n^{-1/3}).
\]

In the previous discussion, we brought in the restriction of \( \zeta \) being kept away from 0. Since \( \zeta = \sqrt{2} \), requiring \( \zeta \geq \delta_1 \) is equivalent to assuming that \( \theta \in [\delta / \sqrt{n}, \pi / 2] \), \( \delta > 0 \), the error estimates (3.8)–(3.11) are valid uniformly on this \( \theta \)-interval. In view of the fact that the coefficient \( E(x, z) M(x, z) \) is of the same order of magnitude as \( k^{1/2}(x) W(x, -z) \), except of course in the neighborhoods of zeros of \( W(x, -z) \), and the facts that similar situations occur in the remaining cases of (3.8)–(3.11), we see that indeed (3.6) and (3.7) provide uniform approximations of the solutions to (3.5), and thus equivalently to (3.1).

For \( z \) large while \( x \) moderate, we have
\[
W(x, z) = \sqrt{2k / z} \cos \Theta + O \left( \frac{1}{z^{5/2}} \right), \quad W(x, -z) = \sqrt{2 / (kz)} \sin \Theta + O \left( \frac{1}{z^{5/2}} \right)
\]
and
\[
W'(x, z) = -\sqrt{kz / 2} \sin \Theta + O \left( \frac{1}{z^{3/2}} \right), \quad W'(x, -z) = -\sqrt{z / (2k)} \cos \Theta + O \left( \frac{1}{z^{3/2}} \right);
\]
compare [9, p. 164, (8.4)–(8.5)]. Matching the behavior of \( W(x, \pm z) \) for \( z \to +\infty \) with that of \( J \), as given in (3.3) and (3.4), we have \( J = 2 \Re \tau(x) w_1 - 2 \Im \tau(x) w_2 \). Hence
\[
I = 2k^{-(1/2)} \Re \tau(x) z^{-(1/2)} W(x, z) - 2k^{1/2} \Im \tau(x) z^{-(1/2)} W(x, -z) + \tilde{v}(x, z)
\]
with
\[
\tilde{v}(x, z) = (|\Re \tau(x)| z^{-(1/2)} E^{-1}(x, z) M(x, z) + |\Im \tau(x)| z^{-(1/2)} E(x, z) M(x, z)) O(n^{-1/3}).
\]

Since \( (2 / z) I'(x, z) = -z^{2} I(x, z) + 2z^{-(3/2)} J'(x, z) \), accordingly we have
\[
\frac{2}{z} I' = 4k^{-(1/2)} \Re \tau(z) z^{-(3/2)} W'(x, z) + 4k^{1/2} \Im \tau(z) z^{-(3/2)} W'(x, -z) + \tilde{v}(x, z),
\]
where
\[
\tilde{v}(x, z) = |I| O(n^{-1/2}) + z^{-(3/2)} N(x, z)(|\Re \tau(x)| E^{-1}(x, z) + |\Im \tau(x)| E(x, z)) O(n^{-1/3})
\]
for \( z \geq 2 \sqrt{\delta n}^{1/4} (\zeta \geq \delta_1) \).

Next, we provide some further asymptotic approximations of \( I \) for later use. Most of the results can be found in, e.g., [1,9].

For large \( z \), we have the following approximation in terms of the Airy functions
\[
I(x, z) \sim 2 \sqrt{\pi} (4x)^{-(1/4)} z^{-(1/2)} \left( \frac{\mu}{z^2 - 1} \right)^{1/4} \times [k^{-(1/2)} e^{-(1/2) \pi \mu} \Re \tau Bi (-\mu) - 2k^{1/2} e^{(1/2) \pi \mu} \Im \tau Ai (-\mu)],
\]
(3.16)
where \( \zeta = z/(2\sqrt{a}) \), \( \mu = (4z)^{2/3} \chi \), with

\[
\chi = -\left( \frac{3}{2} \vartheta_3 \right)^{2/3}, \quad \vartheta_3 = \frac{1}{2} \int_{-1}^{1} \sqrt{1-s^2} \, ds = \frac{1}{4} \arccos \zeta - \frac{1}{4} \sqrt{1-\zeta^2}, \quad \text{for } \zeta \leq 1
\]

and

\[
\chi = \left( \frac{3}{2} \vartheta_2 \right)^{2/3}, \quad \vartheta_2 = \frac{1}{2} \int_{1}^{\zeta} \sqrt{s^2-1} \, ds = \frac{1}{4} \zeta \sqrt{\zeta^2-1} - \frac{1}{4} \text{arccosh } \zeta, \quad \text{for } \zeta \geq 1.
\]

Since \( x = \frac{a+b}{\theta} + O(\theta) \) for \( x \) large, \( \theta \) is small in this case. It is worth pointing out that \( \chi \) is an analytic function in \( \zeta \) in a neighborhood of \( \zeta = 1 \). The parameters \( \zeta \) and \( \mu \) can then be approximated by

\[
-\mu = \eta[1 + O(x^{-4/3})], \quad \zeta = t/\sqrt{a + b[1 + O(x^{-2})]},
\]

where \( \eta = n^{1/3} B^2(t) \), as used in (5.14), and \( B(t) \) is defined in (5.7) and (5.8). The previous approximation now reads

\[
I(x, z) \sim 2\sqrt{\pi}(4a)^{-(1/4)}z^{-(1/2)} \left( \frac{\mu}{\zeta^2 - 1} \right)^{1/4} \
\times [k^{-1/2}e^{-(1/2)\pi\zeta} \text{Re } \tau \text{Bi}(\eta) - 2k^{1/2}e^{(1/2)\pi\zeta} \text{Im } \tau \text{Ai}(\eta)].
\]

We also note that this kind of approximation is the best possible in some sense in that attempts to seek expansions of \( J \) in terms of the parabolic cylinder functions will likely end up in failure. The reader is referred again to Olver [9] for a brief discussion of this point.

### 4. Estimates for the coefficients

To demonstrate the asymptotic nature of (2.10), we obtain firstly some estimates for the coefficients \( x_0(\theta) \) and \( \beta_0(\theta) \). As a preliminary step we examine \( \Phi_0(s, \theta) \) in the \( s \)-domain \( D_\Phi \), described as

\[
|\text{Im } s| \leq \frac{\pi}{\theta};
\]

cf. Fig. 2 for an illustration of the upper half of \( D_\Phi \).

Noticing that \( \Phi_0(s, \theta) \) is \( s \)-analytic in \( D_\Phi \), we can focus on the part \( D_{\Phi, f} \)

\[
-M_f/\theta \leq \text{Re } s \leq M_f/\theta, \quad |\text{Im } s| \leq \pi/\theta,
\]

where \( M_f \) is a large number, subject to change. The upper half of the boundary of \( D_{\Phi, f} \) is \( JTHGFA(A_1)K \), at this stage the points \( G \) and \( F \) are regarded as identical to each other; cf., Fig. 2. To estimate \( \Phi_0(s, \theta) \) in \( D_{\Phi, f} \) we need only estimate it along the boundary due to the \( s \)-analyticity of \( \Phi \). Indeed, we will have

\[
|\Phi(s, \theta)| \leq M_0(1 + \theta|s|)
\]

for \( s \in \partial D_{\Phi, f} \), where \( M_0 \) is a constant independent of \( s \) and \( \theta \), and is also independent of \( M_f \). Hence, the previous inequality holds for all \( s \in D_\Phi \). We provide some details as follows:
Take $J\bar{T}$ as an example. This portion of the boundary can be described as

$$\text{Re } s = M_f / \theta, \quad |\text{Im } s| \leq \pi / \theta.$$ 

Recalling that $w = e^{-s\theta}$ as defined in (2.1), and taking into account the facts that

$$|we^{\pm i\theta}| = e^{-M_f} \ll 1, \quad |s| \geq \frac{M_f}{\pi} \gg 1,$$

and

$$\arg (1 - we^{i\theta}) - \arg (1 - we^{-i\theta}) = O(\theta), \quad \arg (s - i) - \arg (s + i) = O(\theta),$$

a straightforward verification then shows that

$$|\Phi_0(s, \theta)| \leq C_1 |s|$$

for $s \in J\bar{T}$.

The situation for $I\bar{H}$ and $H\bar{G}$ are similar. In fact, we have

$$\text{Im } s = \pi / \theta \quad \text{and} \quad -M_f / \theta \leq \text{Re } s \leq M_f / \theta$$

along $I\bar{H}$. Consequently, we have

$$\left|1 - we^{\pm i\theta}\right| \geq \delta, \quad |s| \geq \frac{\pi}{\theta} \geq 2,$$

where $\delta$ is a positive constant and is independent of $M_f$. The same estimates as for $J\bar{T}$ also hold in this case. Thus, estimate (4.2) for $\Phi_0(s, \theta)$ in this portion follows accordingly. Similar justification applies to $H\bar{G}$ and estimate (4.2) keeps valid for $s \in H\bar{G}$, with $M_0$ independent of $M_f$.

The remaining case, i.e., $F\bar{K}$, is a little different. Attention has to be paid to the choices of branches. In fact, in this case we have

$$\arg (1 - we^{i\theta}) = -\pi + \theta + \arg (w - e^{-i\theta}), \quad \arg (1 - we^{-i\theta}) = \pi - \theta + \arg (w - e^{i\theta}),$$

and

$$\arg (s - i) - \arg (s + i) = -2\pi + O(\theta).$$

The previous estimates along the boundary still hold for $F\bar{K}$, and what is more, still hold for the lower half boundary. Hence, the estimate (4.2) is valid for the analytic function $\Phi_0(s, \theta)$ in the $s$-domain $D\Phi$, with $M_0$ independent of $s$, $\theta$ and $M_f$.

Associated with the iterative procedure (2.8) and (2.9), we define two classes of rational functions $\{A_k(s, \theta)\}$ and $\{B_k(s, \theta)\}$, (cf. [8]),

$$A_0 = \frac{s}{(s - s_+)(s - s_-)}, \quad A_k = \frac{1}{(s - s_+)(s - s_-)} \left[ \frac{dA_{k-1}}{ds} + (s + 2H)A_{k-1}(s, \theta) \right]$$

(4.3)

and

$$B_0 = \frac{1}{(s - s_+)(s - s_-)}, \quad B_k = \frac{1}{(s - s_+)(s - s_-)} \left[ \frac{dB_{k-1}}{ds} + (s + 2H)B_{k-1}(s, \theta) \right]$$

(4.4)
for \( k = 1, 2, 3, \ldots \). Then the coefficients \( a_k(\theta) \) and \( b_k(\theta) \) can be expressed in terms of these functions, as
\[
\beta_k(\theta) = \frac{1}{2\pi i} \oint \Phi_0(s, \theta) B_k(s, \theta) \, ds, \quad (4.5)
\]
and
\[
\alpha_k(\theta) = \frac{1}{2\pi i} \oint \Phi_0(s, \theta) A_k(s, \theta) \, ds, \quad (4.6)
\]
for \( k = 0, 1, 2, \ldots \). Here we have used the facts that \( A_k(s, \theta) \), \( B_k(s, \theta) = O(s^{-3}) \) for \( |s| \) large and \( k = 1, 2, 3, \ldots \). In (4.5) and (4.6), the integration path is a simple closed curve encircling both \( s = s_\pm \). Noticing the facts that \( \Phi_0(s, \theta) \) is analytic and uniformly bounded in \( |\text{Im} s| \leq \pi/\theta \), \( A_k(s, \theta) = O(s^{-k-1}) \) and \( B_k(s, \theta) = O(s^{-k-2}) \). Hence, by deforming the path of integration to the scale \( O(1/\theta) \), as long as \( s_\pm \) is kept a distance of \( O(1/\theta) \) away from the chosen integration path, as is always manageable, we have
\[
|\beta_k(\theta)| \leq M_k \theta^{k+1} \quad \text{and} \quad |\alpha_k(\theta)| \leq M_k \theta^k \quad (4.7)
\]
for \( k = 0, 1, 2, \ldots \). Hence, we can now write (2.10) in the form
\[
P_n(\cos \theta) = I(x, z) \sum_{k=0}^{p-1} a_k(\theta) n^k + 2 \varepsilon \sum_{k=0}^{p-1} \theta b_k(\theta) n^k + \varepsilon_p(n, \theta), \quad (4.8)
\]
where \( \varepsilon \) is given in (2.11), and
\[
a_k(\theta) = \theta^{-k} \alpha_k(\theta), \quad b_k(\theta) = \theta^{-(k-1)} \beta_k(\theta), \quad (4.9)
\]
satisfying
\[
|a_k(\theta)|, \quad |b_k(\theta)| \leq M_k, \quad (4.10)
\]
where \( M_k \) is a constant independent of \( \theta \) and \( n \).

5. Error estimates for \( \theta \geq \delta/\sqrt{n} \), \( \delta > 0 \)

As mentioned in Section 3, the function \( I(x, z) \) can be approximated uniformly by the modified parabolic cylinder functions for \( \theta \geq \delta/\sqrt{n} \), \( \delta > 0 \). In this section, we provide bounds for the error term in this \( \theta \)-interval. (4.8) is thus made a uniform asymptotic expansion for the Pollaczek polynomials in \( \theta \in [\delta/\sqrt{n}, \pi/2] \), in terms of \( I(x, z) \).

5.1. An estimate for \( \Phi_p(s, \theta) \)

We proceed to provide an estimate for \( \varepsilon_p \), expressed in (2.11). To this end, we estimate at first the function \( \Phi_p(s, \theta) \), analytic in \( |\text{Im} s| \leq \pi/\theta \). Again we appeal to a class of rational functions, say, \( \{Q_p(\zeta, s, \theta)\} \), defined as
\[
Q_0(\zeta, s, \theta) = \frac{1}{\zeta - s}, \quad Q_p(\zeta, s, \theta) = \frac{[\zeta^2 + 1](d Q_{p-1}(\zeta, s, \theta)/d\zeta) + (\zeta + 2H) Q_{p-1}(\zeta, s, \theta)]}{(\zeta - s_+)(\zeta - s_-)} \quad (5.1)
\]
for \( p = 1, 2, 3, \ldots \). In terms of this rational function class, it holds that

\[
\Phi_p(s, \theta) = \frac{1}{2\pi i} \oint \Phi_0(\zeta, \theta) Q_p(\zeta, s, \theta) d\zeta,
\]

(5.2)

where the integration path is a closed curve encircling the points \( \zeta = s \) and \( \zeta = s_\pm \). In obtaining (5.2), use has been made of the fact that \( Q_p(\zeta, s, \theta) = O(\zeta^{-3}) \) for \( \zeta \) large and \( p = 1, 2, 3, \ldots \).

It can be seen from (2.7) that \(|s_\pm| \leq \pi/(2\theta)\). Now the integration path in (5.2) can be deformed to the boundary of the rectangle \(|\text{Im } \zeta| \leq \pi/\theta, |\text{Re } \zeta| \leq M/\theta\). Paying attention to the facts that \( \Phi_0(s, \theta) \) is uniformly bounded in \( D_\theta \) by \( M_0(1 + |\theta|s|) \) and that

\[
Q_p(\zeta, s, \theta) = [\( \zeta - s_\pm \)(\zeta - s_-)]^{-2p+1} \sum_{l=1}^{p+1} \frac{R_{3p+l-3}(\zeta, \theta)}{(\zeta - s)^l},
\]

(5.3)

where \( R_{3p+l-3}(\zeta, \theta) \) is a polynomial in \( \zeta \) of degree \( 3p + l - 3 \) whose coefficients may depend on \( \theta \) but not on \( s \), we then have

\[
|\Phi_p(s, \theta)| \leq M_p \theta^p (1 + |\theta|s|)
\]

(5.4)

for \(|\text{Im } s| \leq (\pi - \varepsilon)/\theta \) and \(|\text{Re } s| \leq M/(2\theta)\), where \( M_p \) is a constant which may depend on \( \varepsilon \) and \( M \), but not on \( s \) or \( \theta \). We can also consider another case, namely, the case when \(|\text{Im } s| \leq (\pi - \varepsilon)/\theta \) and \(|\text{Re } s| > M/(2\theta)\) for \( M \) large but fixed. Taking the integration path in (5.2) to be of the size \( \zeta - s = O(1/\theta) \), and using (5.3) again, gives (5.4). By adjusting \( M \) and \( M \) we see that (5.4) holds for \(|\text{Im } s| \leq (\pi - \varepsilon)/\theta \), with \( M_p \) depending only on \( \varepsilon \) and \( p \).

We are now in a position to estimate \( \varepsilon_p \). We carry out the discussion case by case.

### 5.2. Error estimates for \( \theta > \sqrt{a + b}/\sqrt{n}, \theta \) small

In this case \( \alpha \) is large. This time the main idea is that the contribution to the asymptotic approximations of \( I(\alpha, z), I'(\alpha, z) \) and \( \varepsilon_p \) come from the points \( s = s_\pm \). Also \( s_\pm \) coalesce when \( \theta \to \sqrt{a + b}/n \). We estimate \( \varepsilon_p(n, \theta) \) in an indirect way, by comparing it with the Airy function.

To do so, we apply a transformation \( s \leftrightarrow Z \leftrightarrow u \) to (2.11), which can be established as

\[
F(s, t) := \frac{a + b}{2t} \ln \frac{s - i}{s + i} + ts = Z = \frac{1}{3}u^3 - B(t)^2u + C(t) =: \varphi(u)
\]

(5.5)

such that \( F(s_\pm, t) = \varphi(u_\pm) \), where \( u_\pm \), as previously given, and

\[
u_\pm = \pm B(t)
\]

(5.6)

are respectively the saddle points of \( F(s) \) and \( \varphi(u) \). The coefficients can be determined by putting \( s_\pm \) and \( u_\pm \) into (5.5). Indeed, we have

\[
C(t) = \frac{a + b}{2t} \pi, \quad B(t) = \begin{cases}
\frac{3}{2} \beta(t)^{1/3}, & t > \sqrt{a + b}, \\
\frac{3}{2} \beta(t)^{1/3}, & t < \sqrt{a + b},
\end{cases}
\]

(5.7)
where
\[ \beta(t) = \begin{cases} 
- \frac{a + b}{t} \arccosh \left( \frac{t}{\sqrt{a + b}} \right) + \sqrt{t^2 - (a + b)}, & t > \sqrt{a + b}, \\
\frac{a + b}{t} \arccos \left( \frac{t}{\sqrt{a + b}} \right) - \sqrt{(a + b) - t^2}, & t < \sqrt{a + b}.
\end{cases} \] (5.8)

It is readily seen that each of the two pairs of saddle points coincide when \( t \to \sqrt{a + b} \), and it is easily verified that \( \beta(t) > 0 \) for \( t \neq \sqrt{a + b} \).

The previously introduced notations are for general \( t \) and \( \theta \). In the rest of this subsection, however, we concentrate fully on the case when \( \theta \) is small but \( \theta > \sqrt{a + b}/\sqrt{n} \). Among the first steps, it can be shown via the intermediate variable \( Z \) and by using boundary correspondence that the mapping \( s \leftrightarrow u \) is one-to-one and analytic in the union of the domain \( JIHGC_{1}(C)DEFA(A_{1})KJ \) and the symmetrical domain on the lower half plane; as illustrated in Figs. 2–4. The curves \( ABLC \) in Figs. 2 and 4 are parts of the steepest descent paths of \( F(s, t) \) and \( \varphi(u) \), respectively. See also [11, Section 6.45], [12, p. 375] and [14] for justification of the conformal nature of the mappings.

Under transformation (5.5) we have
\[ \left( \frac{z}{2} \right)^{2p} \varepsilon_{p}(n, \theta) = \frac{1}{2\pi i} \int_{\Gamma} K_{p}(u)e^{\sqrt{n(t(1/3)u^3-B(t)^2u+C(t))}} \, du, \] (5.9)
where

\[ K_p(u) = K_p(u, \sqrt{n}, t) = \Phi_p(s, \theta)U(s, t) \]  

(5.10)

with

\[ U(s, t) = \frac{1}{t} (s - i)^{(1/2) + iH(0)} (s + i)^{(1/2) - iH(0)} \frac{(u - u_+) (u - u_-)}{(s - s_+) (s - s_-)}, \]  

(5.11)

and \( \Gamma \) is indeed the image of the original \( \Gamma \) under transformation (5.5). We still denote it by \( \Gamma \) for the sake of simplicity. This new \( \Gamma \) is illustrated in Fig. 4, of which \( ABLM \) is the upper half, with positive direction.

Now we turn to the function \( U(u, t) \). From the conformal nature of the mapping \( s \leftrightarrow u \), we can see that \( U(u, t) \) is \( u \)-analytic in the domain \( D_U \), consisting of the upper half domain \( JIHGC_1(C)DEFA(A_1)KJ \) and the symmetrical lower half; cf. Fig. 4 and the corresponding cut \( s \)-plane illustrated in Fig. 2. We intend to provide an estimate for \( U(u, t) \) for \( u \in D_U \). To this aim, again we turn to the boundary, and we restrict ourselves to the upper half boundary, namely, \( JIHGC_1(C)DEFA(A_1)K \), while the estimates for the lower half boundary can be obtained by symmetry.

Take the portion of boundary \( u \in \overline{DE} \) as an example. This part corresponds to the small circle around \( s = i \); cf. Fig. 2. To exclude the saddle point \( s = s_+ \), it is natural to demand at least \( |s - i| = O(1/t^2) \) for \( s \in \overline{DE} \). We purposely specify \( \overline{DE} \) as

\[ |s - i| = e^{-Mt^2}, \quad \arg(s - i) \in [-\pi, \pi], \]

where \( M \) is a large constant. Then it can be readily verified that \( \delta t \leq |Z - Z_+| \leq \Delta t \) also holds along this curve. In view of the formula

\[ Z - Z_+ = \frac{1}{3} (u - B)^3 + (u - B)^2 B, \]

where \( Z_+ = Z(u_+) \), and the fact \( |B(t)| \approx (\frac{3}{2}t)^{1/3} \), we have

\[ \delta t^{1/3} \leq |u - u_+| \leq \Delta t^{1/3}, \quad |s - i| = e^{-Mt^2}, \quad |s + i| \sim 2, \]

\[ |s - s_+| \sim (a + b)/(2t^2), \quad |s - s_-| \sim 2, \]

for \( u \in \overline{DE} \). Hence, from (5.5) we have

\[ |U(u, t)| \leq \Delta t^{-(1/3)}. \]  

(5.12)

The estimates along the curves \( JI, IH, HG \) and \( FA(A_1)K \) can be obtained similarly. Also the bounds can be made even sharper along these curves, which reads \( |U(u, t)| \leq M_0 n^{-1/6} \) for \( u \) belonging to the said boundary. The analysis for \( G\overline{C_1(C)} \overline{D} \) and \( \overline{EF} \) demonstrates some technique difficulties. For example, for \( s \in \overline{EF} \), it is convenient to write \( s = -\lambda + i \) and divide the discussion into three subcases: (i) \( \lambda \in [M/t^2, M_f/\theta] \), (ii) \( \lambda \in [\varepsilon/t^2, M/t^2] \) and (iii) \( \lambda \in [e^{-Mt^2}, \varepsilon/t^2] \). However, the same estimate (5.12) holds in each subcase. We thus conclude that (5.12) is valid for \( u \in \partial D_U \), and hence for \( u \in D_U \). Thus, we have

\[ |K_p(u)| \leq M_p \theta^p t^{-(1/3)} (1 + u^3/\sqrt{n}), \]  

(5.13)
the last factor appearing since the term \( \theta |s| \) in (5.4) is bounded for \( |s| = O(1/\theta) \), and \( ts \approx \frac{1}{3} u^3 \) for \( s \) (and hence \( u \)) large, as can be seen from (5.5).

A byproduct of the previous analysis is that within \( D_U \) we have a disk of size \( |u - B(t)| \leq \delta t^{-2/3} \). The parts of the boundary \( GC_1(C)D \) and \( EF \) restrict the radius from being enlarged. On the basis provided above, we can use the method and result employed in [4] to estimate the error term near the turning point, or, more precisely, for \( \theta \in (\sqrt{a + b}/\sqrt{n}, M/\sqrt{n}) \), and the classical steepest descent method for \( \theta \geq M/\sqrt{n} \) but small. In either case we have

\[
|\varepsilon_p| \leq M_p \left\{ \left| \frac{\text{Ai} (\eta)}{\eta^{1/2}} \right| + \left| \frac{\text{Ai}' (\eta)}{\eta} \right| \right\} \frac{1}{n^p} e^{\sqrt{n} C(t)},
\]

where \( \eta^{1/2} = n^{1/6} B(t) \), and \( M_p \) is a constant independent of \( n \) and \( \theta \). Approximating \( W(x, \pm z) \) and \( W'(x, \pm z) \) in (3.12) and (3.14) by the Airy functions and their derivatives for large \( x \), and comparing (5.14) with the approximation formulas obtained, such as (3.17), we have

\[
|\varepsilon_p| \leq \frac{M_p}{n^p} \left\{ |I(x, z)| + \frac{2}{z} |I'(x, z)| \right\}.
\]

In obtaining (5.15), we may have to consider two situations, namely (i) \( \xi \sim 1 + 0 \), and (ii) \( \xi \geq 1 + \delta_0 \) for \( \delta_0 \in (0, 2) \). In the first case, attention has to be paid to the facts that \( \frac{\mu}{\xi - 1} = O(x^{2/3}) \), \( \theta = O(1/\sqrt{n}) \), and \( \mu \sim -\eta \). In the second case, one need to keep in mind that \( \eta = O((n\theta)^{2/3}) \).

5.3. Error estimates for \( \theta \in [\delta/\sqrt{n}, \sqrt{a + b}/\sqrt{n}] \)

This case was actually covered by Bo and Wong [4]. So one treatment could be to use their result [4, (5.8)] directly. But we start in this paper from a slightly different integral expression and end with a different expansion. So it is more convenient to outline briefly our estimating process.

We resume in this case the same transformation (5.5). As in the \( t > \sqrt{a + b} \) case, it is readily verified by using boundary correspondence that the mapping \( s \leftrightarrow u \) is one-to-one and analytic in the domain \( J H G (R) S (L) K J \), as described after (4.1); cf., Figs. 5 and 7.

The deformation of the integration path is different from the previous case. Indeed, firstly we still have the expression (2.11) of the error term, where the \( s \)-integration path \( I \) can be chosen as \( L A B C G \) with \( G \) and \( L \) tending to infinity; see Fig. 5, and then we can deform it to \( W V B T U \), where \( W V \) and \( T U \) are horizontal with \( W \) and \( U \) tending to infinity; see Fig. 5 for an illustration of these curves. The mapping (5.5) actually has an image consisting of three \( Z \)-sheets, as illustrated in Figs. 6a–c. Obviously the multi-valued nature is caused by the saddle points \( s_{\pm} \). Cuts are chosen as the steepest descent path through \( s_+ \) and the steepest ascent path emanating from \( s_- \). Via the latter equality of (5.5), we can again map these three \( Z \)-sheets into a single \( u \)-domain; as illustrated in Fig. 7. The integration path is thus mapped into \( W V B T U \) in the \( u \)-plane; see also Fig. 7, which is the path \( I' \) in (5.9), corresponding to this case.

The error estimate for the case \( t < \sqrt{a + b} \) is similar, with only minor modifications, to that of the case \( t > \sqrt{a + b} \). As a result, we still have (5.14) for \( \theta \in [\delta/\sqrt{n}, \sqrt{a + b}/\sqrt{n}] \), and hence (5.15) follows accordingly.
Fig. 5. $s$-plane, $t < \sqrt{a + b}$.

Fig. 6. $Z$-plane, $t < \sqrt{a + b}$.

Fig. 7. $u$-plane, $t < \sqrt{a + b}$.
5.4. Error estimates for \( \theta \) moderate

In this case we apply Watson’s lemma to get the error bound. Also it is easily seen that the integrals in (2.11) and (2.4) have the same magnitude of order and (5.15) is valid in this case.

Putting all the results produced in this section together, we have obtained the error estimate (5.15) for \( \theta \in [\delta/\sqrt{n}, \pi/2] \).

6. Uniform asymptotic expansion for \( \theta \leq \delta/\sqrt{n} \)

Now we turn to the remaining case, \( \theta \in (0, \delta/\sqrt{n}] \). This time we use a modified version of (2.2) as our starting point. The current case is of less significance and is not so difficult. The reasons are that there are no zeros of \( P_n \) which belong to this interval and that the saddle points \( s_{\pm} \) in (2.7) are kept apart from each other, and away from the branch points \( s = \pm i \) as well. Hence, the classical steepest descent method provides a uniform asymptotic expansion. As described previously in Section 5.3, the integration path in (2.2) can be deformed such that it passes through only one relevant saddle, \( s_+ \). With the path so deformed, we can write (2.2) in the form

\[
P_n(x) = \frac{1}{2\pi i} \int_{\Gamma} \Psi(s, \theta)e^{1/\theta F_1(s,t)} ds,
\]

where

\[
F_1(s, t) = \frac{a + b}{2} \ln \frac{s - i}{s + i} + t^2 s,
\]

\[
\Psi(s, \theta) = \Phi_0(s, \theta)(s - i)^{-1/2} e^{iH(\theta)}(s + i)^{-1/2} e^{-iH(\theta)},
\]

and \( \Gamma \) as explained is the path through

\[
s_+ = \sqrt{\frac{a + b}{t^2}} - 1.
\]

Recalling that from (5.5), we have

\[
\frac{1}{\theta} F_1(s_+, t) = -\frac{2}{3} \eta^{3/2} + \frac{a + b}{2\theta} \pi
\]

with

\[
\eta^{1/2} = n^{1/6} B(t) \approx \frac{3(a + b)\pi}{4} 0^{-(1/3)},
\]

as can be seen from (5.7) and (5.8). Following the steps in Section 5, we have \( \Psi(s, \theta) \) is \( s \)-analytic in a neighborhood of \( \Gamma \). Up to a re-scaling of the form \( s = s_+ u \) the new saddle will be at \( u = 1 \) and both \( T \) and \( V \) are of \( O(1) \) in the \( u \)-coordinates. It can be justified following (4.2) that \( \Psi \) is uniformly bounded in a neighborhood of \( s_+ \) of size \( O(1/t) \). The uniformity is in the whole \( t \)- and \( \theta \)-range corresponding to \( t \in (0, \delta) \).
From (6.1), using the classical steepest descent method, we have the asymptotic expansion

\[ P_n(\cos \theta) = e^{[(a+b)/2] \pi e^{-(2/3)n^{1/2}B^3(\sqrt{n})}} \sum_{k=0}^{\infty} c_k \theta^k. \]  

(6.2)

One way to calculate the coefficients \( c_k \) is

\[ c_k = -\frac{I(k + 1/2)}{4\pi^2} \int_{s_+} \frac{\Psi(s, \theta) ds}{[F(s, t) - F(s, t)]^{k+1/2}}, \]

(6.3)

for \( k = 0, 1, 2, \ldots \), where \( \{s_+\} \) is a closed curve encircling \( s_+ \) in the positive direction, and the branch of the denominator is appropriately chosen; cf. e.g., Berry and Howls [2].

If truncated at a small term \( p \), the estimate of the remainder \( \varepsilon_p \) in this time follows from the steepest descent method, namely

\[ |\varepsilon_p| \leq M_p e^{[(a+b)/2] \pi e^{-(2/3)n^{1/2}B^3(\sqrt{n})} \theta^p + (1/2)} \leq M_p e^{[(a+b)/2] \pi e^{-(2/3)n^{1/2}B^3(\sqrt{n})} \theta^{p/2} - (1/4)}. \]  

(6.4)

The asymptotic formula (6.2) can also be written in the form

\[ P_n(\cos \theta) = e^{[(a+b)/2] \pi e^{-(2/3)n^{1/2}B^3(\sqrt{n})}} \sum_{k=0}^{\infty} d_k(\theta) \frac{\theta^k}{n^{k/2}}, \]

(6.5)

where

\[ d_k(\theta) = (\theta \sqrt{n})^k c_k(\theta). \]

The explicit expression of the remainder terms can also be provided in a different way, say like Berry and Howls [2], in a double integral, namely

\[ \varepsilon_p = \varepsilon_{s,p} + \varepsilon_{i,p}, \]

(6.6)

where the contribution from the saddle point can be expressed as

\[ \varepsilon_{s,p} = -\frac{\theta^p}{4\pi^2} \int_0^1 e^{-u \theta^p - (1/2)} \int_{\{VBT\}} \frac{\Psi(s, \theta) ds}{[F_1(s, t) - F_1(s, t)]^{p+1/2} \{1 - \theta u/[F_1(s, t) - F_1(s, t)]\}}, \]

where \( \{VBT\} \) is a simple closed curve encircling the curve \( VBT \), and the contribution from infinity is

\[ \varepsilon_{i,p} = \frac{1}{2\pi i} \int_{WV \cup T} \Psi(s, \theta) e^{(1/\theta)F_1(s, t)} ds. \]

The curves \( VBT, WV \) and \( T \) are illustrated in Fig. 7.

7. Conclusion and discussion

Up to now we have obtained uniform asymptotic expansions in two \( \theta \)-subintervals of \( (0, \pi/2) \). The expansion for \( \theta \in (0, \delta/\sqrt{n}] \), in terms of elementary functions, is given in (6.5). While the expansion for \( \theta \in [\delta/\sqrt{n}, \pi/2] \), in terms of function \( I(z, z) \), is provided in (4.8), with coefficients and error term
iteratively given, and rigorously estimated. Since $\delta$ is an arbitrary constant, these two intervals can be made overlapping with each other and thus cover the whole interval $(0, \pi/2]$. The function $I(\alpha, z)$ itself, as demonstrated in Section 3, is closely related to the modified parabolic cylinder functions.

One remark we must make is that the approach used here can be applied equally to polynomials introduced also by Pollacek, namely, the polynomials $P_n^\alpha(x; a, b)$, defined by the generating function

$$
(1 - we^{i\theta})^{-\lambda+i\phi}(1 - we^{-i\theta})^{-\lambda-i\phi}, \quad \lambda > -\frac{1}{2};
$$

compare [10, p. 394]. For $\lambda = \frac{1}{2}$, we get the Pollaczek polynomials. We notice with interest that $\lambda = \frac{3}{4}$ is superior to the other cases in that we have

$$
I'' + \left(\frac{1}{4}z^2 - x\right)I = 0
$$

for $\lambda = 3/4$, where $x$ and $z$ are the same as being used in Section 2, and $I = I(\alpha, z)$ is the approximator. The only difference of this function with that defined in (2.4) is that the powers $-\frac{1}{2}$ in (2.4) are now replaced by $-\frac{3}{4}$. Not surprisingly this $I(\alpha, z)$ is a linear combination of $W(\alpha, \pm z)$, as follows

$$
I(\alpha, z) = 2k^{-(1/2)}(x)\text{Re} \tau_1(\theta)W(\alpha, z) - 2k^{1/2}(x)\text{Im} \tau_1(\theta)W(\alpha, -z),
$$

where

$$
\tau_1(\theta) = 2^{(x/2)i-(3/4)}e^{(x\pi/4)}e^{-(1/8)i-(1/2)\phi_2}\left[i\Gamma\left(\frac{3}{4} - \frac{x}{2}i\right)\right]^{-1},
$$

and $k(\alpha)$, $\phi_2$ are as defined in Section 3. To us, this equality has actually inspired the approximation of $I(\alpha, z)$ in terms of the modified parabolic cylinder functions, carried out in Section 3.

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