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Fibration models and localisation of categories

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Abstract

We define a fibration model on the basis of a Thomason model and use it to analyse the localisation of a category. We show that the model suffices to prove existence of the localisation and to develop the basic homological tools, including derived functors, homotopy pullbacks and the unstable triangulated structure. The corresponding constructions in the case of Quillen, Baues and Thomason models can be recovered as special cases.

MSC: Primary: 18D; secondary: 18G55; 18G50

1. Introduction

The work of Gabriel and Zisman [5] has shown that homotopy can be described abstractly using localisation of categories. The construction of localisations, however, poses certain problems. For if $\mathbf{C}$ is a category belonging to a universe V and we localise it at a class of arrows $\mathscr{E}$, we obtain a category $\mathbf{C}_{\mathscr{E}}$ which may belong to a larger universe. This is not a desirable situation, if the aim is to develop homological algebra using an abstract version of stable homotopy.

A solution to the problem was provided by Quillen [12], based on the following observation. When localising categories of spaces, the class $\mathscr{E}$ usually fits into factorisation systems generated by fibrations and cofibrations, which allow to replace $\mathbf{C}$ with the full subcategory B$\mathbf{C}$ of bifibrant objects. The localisation of B$\mathbf{C}$ can be realised as a quotient by the homotopy congruence, and since every universe is closed under the formation of quotients, this suffices to prove the existence of $\mathbf{C}_{\mathscr{E}}$ inside V. Quillen models provide a framework which allows to recover this analysis in the setting of an abstract localisation.

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Generalisations of Quillen models have been explored in various directions. Among these, we are mainly interested in those described by Brown [3], Baues [1] and Thomason [13]. Brown models provide an abstract theory of fibrations by extracting, roughly speaking, half of the structure of the Quillen model. They are more general and more flexible than Quillen models, although they do not suffice to prove the existence of $\mathbf{C}_{\mathscr{E}}$ inside V. It was shown by Baues how to modify the axioms of Brown in order to achieve existence, thus defining a new model which is intermediate between those of Quillen and Brown. In a different direction, Thomason models retain the symmetric formulation of Quillen, weakening the factorisation axiom. Their aim is to prove transfer theorems for functor categories, as in Weibel [15] and [14]. Weakening the factorisation axiom, however, has the consequence that the analysis behind Quillen models does not apply anymore, because existence of the localisation cannot be reduced to the case of bifibrant objects.

What we propose in this article is the analysis of a fibration model which extracts half of the structure of the Thomason model, much like the Baues model does for the Quillen model. We show that this fibration model suffices to prove existence of the localisation and to develop the basic tools of homological algebra. Since the model is weaker than those of Baues, Quillen and Thomason—at least in their functorial form—the corresponding results for these models can be derived as special cases. The emphasis of the article, however, is not so much on the particular model. Rather, it is on the fact that what all these models have in common is the capability of proving existence theorems which provide a homological setting for localisations: existence of the localisation itself, of derived functors and, as a consequence, of the triangulated structure. All these existence theorems have a purely categorical content. From this point of view, the main object of investigation is the localisation and its intrinsic structure. It is for this reason that in the definition of a fibration model we regard the class of equivalences as a basic datum defining the localisation and we do not incorporate them in the model.

The structure of the article is the following. In Section 2 we recall the basic facts about localisations and we fix the notation. In Section 3 we describe the fibration model and in Section 4 we compare it with other models in the literature. In Section 5 we recall the elements of abstract homotopy theory needed to prove the localisation theorem and then describe the local structure of the localisation in Section 6. We develop derived functors in Section 7, apply them to the description of homotopy pullbacks in Section 8 and use these to define the unstable triangulated structure of the localisation in Section 9.

2. Localisation of equivalences

Let $\mathbf{C}$ be a category and $\mathscr{E} \subseteq \mathbf{C}$ a class of arrows. Recall that a *localisation* of $\mathbf{C}$ at $\mathscr{E}$ is assigned by a functor $\hat{\mathscr{E}} : \mathbf{C} \to \mathbf{C}_{\mathscr{E}}$ which is universal with respect to the property of making the arrows of $\mathscr{E}$ invertible. More briefly, we say that $\mathbf{C}_{\mathscr{E}}$ is a localisation of $\mathscr{E}$. The universal property determines $\mathbf{C}_{\mathscr{E}}$ uniquely up to isomorphism.

To obtain an explicit description of $\mathbf{C}_{\mathscr{E}}$, consider the class of arrows $\mathbf{C}+\mathscr{E}^{\mathrm{op}}$ and form all the words $\langle f_n \ldots f_1\rangle$ of composable arrows on this class. On these words consider the congruence generated by the relation which identifies a composable sequence in $\mathbf{C}$ with its composite, and which forces the elements of $\mathscr{E}^{\mathrm{op}}$ to be inverses of the corresponding elements in $\mathscr{E}$. The quotient class is the class of arrows of $\mathbf{C}_{\mathscr{E}}$. We refer the reader to Gabriel and Zisman [5, Section I.1] or to Borceux [2, Vol. I, Section 5.2], for more details.

When $\mathscr{E}$ admits a calculus of right fractions, as described in Gabriel and Zisman [5, Section I.2] or in Borceux [2, Vol. I, Definition 5.2.3], the description of $\mathbf{C}_{\mathscr{E}}$ in terms of words given above can be considerably simplified. For in this case an arrow $A \to B$ in $\mathbf{C}_{\mathscr{E}}$ can be represented by a *right fraction*, i.e. by the equivalence class in $\mathbf{C}$ of a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ {\scriptstyle e}\downarrow & & \\ A & & \end{array} \tag{1}$$

with $e \in \mathscr{E}$. More precisely, consider the filtering diagram

$$\hom(d_0\,\_\,, B) : (\mathscr{E}/A)^{\mathrm{op}} \to \mathbf{Sets}, \tag{2}$$

where $\mathscr{E}/A \subseteq \mathbf{C}/A$ is the full subcategory generated by the arrows of $\mathscr{E}$ with codomain A and $d_0 : \mathscr{E}/A \to \mathbf{C}$ is the domain functor. A right fraction from A to B is an element in the colimit of this diagram.

The problem with these constructions is their size. Recall that given a set-theoretical universe V, a V-category is a category $\mathbf{C}$ whose hom-objects belong to V. The elements of V are often called small sets, so that we also say that $\mathbf{C}$ is locally V-small. Reference to V is usually omitted when understood, and we simply say that $\mathbf{C}$ is locally small. For more details, the reader can consult Mac Lane [10, Section I.6]. If we now fix the universe V and consider a locally small category $\mathbf{C}$, the localisation $\mathbf{C}_{\mathscr{E}}$ need not be locally small. This happens because the words defining a hom-object in $\mathbf{C}_{\mathscr{E}}$ form a class, and the congruence relation does not usually suffice to reduce its size to a small set. It is true that when $\mathscr{E}$ is small, then $\mathbf{C}_{\mathscr{E}}$ is locally small; however, this condition is not satisfied in many cases. Even when a calculus of fractions is available, the diagram (2) need not be small, so that the colimit may belong to a larger universe. It is worth recalling, however, that if $(\mathscr{E}/A)^{\mathrm{op}}$ contains a small cofinal subcategory, then the colimit can be computed by restriction and is therefore small. Baues and Quillen models can be reduced to this case as will be shown in Section 6.

To summarise this discussion, we say that the localisation *exists* when $\mathbf{C}_{\mathscr{E}}$ is locally small. Thus, existence always means existence in the same universe of $\mathbf{C}$.

In addressing the question of the existence of $\mathbf{C}_{\mathscr{E}}$, we restrict to classes of arrows $\mathscr{E}$ satisfying the following conditions.

- Closure. $\mathscr{E}$ contains all the isomorphism of $\mathbf{C}$.

- Saturation. If the triangle

$$\begin{array}{ccc} A & \longrightarrow & C \\ & \searrow \quad \nearrow & \\ & B & \end{array} \tag{3}$$

  is commutative and any two of its arrows are in $\mathscr{E}$, so is the third.
- Retracts. $\mathscr{E}$ is closed under retracts in $\mathbf{C}^2$.

When $\mathscr{E}$ satisfies the axioms above, we say that it is a class of *equivalences*. To provide examples of equivalences, it is worth observing that the preimage of any class of equivalences along any functor is again a class of equivalences. Also, the class of isomorphisms of any category is a class of equivalences. Thus, the preimage of the class of isomorphisms along any functor is a class of equivalences. As an example, let $\mathbf{A}$ be an Abelian category and $\mathrm{Ch}(\mathbf{A})$ be the category of chain complexes of $\mathbf{A}$. The quasi-isomorphisms in $\mathrm{Ch}(\mathbf{A})$ are the preimage of the class of isomorphisms along the homology functor and therefore form a class of equivalences. The corresponding localisation $D(\mathbf{A}) = \mathrm{Ch}(\mathbf{A})_{\mathscr{E}}$ is the derived category of $\mathbf{A}$.

3. Fibration models

Let $\mathbf{C}$ be a category with finite limits and $\mathscr{E} \subseteq \mathbf{C}$ a class of equivalences. A *fibration model* for $\mathscr{E}$ is assigned by an isomorphism closed subcategory, which we identify with its class of arrows $\mathscr{F} \subseteq \mathbf{C}$ and call the class of *fibrations*. This class $\mathscr{F}$ is required to satisfy the following axioms; the terminology is explained below.

- Stability. Fibrations and acyclic fibrations are pullback stable.
- Factorisation. $(\mathscr{E}, \mathscr{F})$ is a (weak) factorisation system.
- Resolutions. $\mathbf{C}$ has $(\mathscr{E} \cap \mathscr{F})$-projective resolutions.

Factorisations and resolutions are assumed to be assigned *functorially*. To explain the terminology, recall from Quillen [12, I.1.3 Definition 2], that a fibration is *acyclic* or *trivial* if it is also an equivalence. The stability axiom means that given a pullback diagram

$$\begin{array}{ccc} C \times_A B & \longrightarrow & B \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ C & \longrightarrow & A \end{array} \tag{4}$$

if p is an (acyclic) fibration, so is q. By a factorisation system we always mean a weak one. Thus, the factorisation axiom means that every arrow in $\mathbf{C}$ factors as an equivalence followed by a fibration. Functoriality of the factorisation means that if we consider the domain and codomain functors $d_0, d_1 : \mathbf{C}^2 \rightrightarrows \mathbf{C}$ from the category of arrows and the tautological natural transformation $t : d_0 \to d_1$ defined by $t_f = f$, then

t admits a factorisation in $\mathbf{C}^2$ as in the diagram

$$\begin{array}{ccc} d_0 & \xrightarrow{\ t\ } & d_1 \\ & {}_{e}\searrow \quad \nearrow_{p} & \\ & F & \end{array} \tag{5}$$

where the components of e are equivalences and those of p are fibrations. Finally, an object $P \in \mathbf{C}$ is $(\mathscr{F} \cap \mathscr{E})$-*projective*—or simply projective, when this will not generate ambiguity—if it has the left lifting property with respect to $\mathscr{F} \cap \mathscr{E}$, that is if, given the solid part of the diagram

$$\begin{array}{ccc} & & X \\ & {}^{s}\nearrow & \downarrow p \\ P & \longrightarrow & Y \end{array} \tag{6}$$

with p an acyclic fibration, there exists a lifting s making the diagram commutative. Having projective resolutions means that for every object $A \in \mathbf{C}$ there exists an equivalence $P \to A$ with P projective. Resolutions are functorial if the assignment of the resolution is a functor $\mathbf{C} \to \mathbf{C}^2$.

For comparison with Baues models, we also introduce the following axiom

- Properness. Equivalences are pullback stable along fibrations

and say that a fibration model is *proper* if the properness axiom is satisfied. It should be observed that in presence of the factorisation axiom, stability of fibrations and properness imply stability of acyclic fibrations. Conversely, if all objects are fibrant, stability implies properness: see Brown [3, Lemma 2] or Baues [1, dual of Lemma I.1.4]. Thus, properness is slightly stronger than full stability. However, it is satisfied in a variety of concrete models.

From now on, we write $\circ \twoheadrightarrow \circ$ to indicate fibrations and $\circ \xrightarrow{\sim} \circ$ to indicate equivalences. The two notations will be combined in the case of acyclic fibrations. Finally, observe that the definitions given so far can be dualised to define a *cofibration model*.

4. Examples

This section is meant to provide generic examples and comparisons among some of the homotopy models in the literature. To prevent confusion in the terminology, the expression “fibration model” without further qualification will refer to a fibration model as defined in Section 3.

4.1. Brown models [3]. Brown models lie at the bottom of the hierarchy. Their axioms are similar to those of fibration models, with the following differences. First, every object is fibrant. Second, the axiom on resolutions is not satisfied. Finally, the factorisation axiom does not require functoriality. For the sake of comparison, call a Brow model functorial if the factorisation is. Then, every fibration model induces a functorial Brown model on the full subcategory of fibrant objects. Thus, Brown models are more general than fibration models. The problem is that Brown models do not suffice to prove existence of the localisation, so that some addition to the axioms is to be expected. Nevertheless, Brown models are fundamental because they provide a calculus of fractions for equivalences on a suitable quotient category.

4.2. Baues models [1]. This is the next step in the hierarchy. A *fibration category* in the sense of Baues is a proper fibration model without the functoriality assumptions and in which the resolution $P \to A$ of every object A is provided by a trivial fibration rather than by an equivalence. We refer to a fibration category without the properness axiom as a *Baues model* and say that the model is functorial if factorisations and resolutions are functorial. Thus, every functorial Baues model induces a fibration model. It is proved in Baues [1, Proposition II.3.6] that Baues models suffice to prove existence of the localisation. In fact, most of the theory for Quillen models can be deduced from Baues models. Note that the original definition of Baues uses *cofibrant* objects instead of projective objects. These are objects A such that every acyclic fibration $B \to A$ admits a section. In any case, cofibrant objects in the sense of Baues are the same as projective objects as defined above. In fact, using stability of acyclic fibrations it is immediate to prove the equivalence of the following statements.

1. P has the left lifting property with respect to $\mathscr{F} \cap \mathscr{E}$.
2. Every acyclic fibration to P splits.
3. $\hom(P, \_)$ takes acyclic fibrations to epimorphisms.

Condition 1 defines projectives in our sense and condition 2 defines cofibrant models in the sense of Baues.

4.3. Quillen models [12]. Quillen models lie at the top of the hierarchy. In fact, every Quillen model, standard [12, Definition I.1.1] or closed [12, Definition I.5.1], induces a Baues model with the same equivalences and fibrations. The stability and factorisation axiom for the Baues model follow immediately from the analogous axioms of the Quillen model. As to resolutions, suffices to observe that every cofibrant object in the Quillen model is projective, and that for every object $A \in \mathbf{C}$, the factorisation of the initial arrow $0 \to A$ as a cofibration followed by a trivial fibration provides a projective resolution for A in the sense of Baues. If the Quillen model is functorial (cf. Hovey [8, Section 1.1]), so is the induced Baues model. Although the functoriality assumption is not present in the original definition of Quillen, some of the most important examples of Quillen models are obtained from locally presentable factorisations—often called cofibrantly generated models—and are thus functorial.

4.4. Thomason models [14]. These do not fit precisely in the hierarchy we have outlined so far. There are numerous variants described in the original notes of Thomason [13]. We are interested in what is called a Thomason model category in [14]. This is essentially like a proper, functorial Quillen model. The main difference is in the factorisation axiom, which requires for every arrow $f \in \mathbf{C}$ the existence of a factorisation

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ & {\scriptstyle e}\searrow^{\sim} \quad \nearrow{\scriptstyle p} & \\ & B & \end{array} \tag{7}$$

as an equivalence followed by a fibration, and dually as a cofibration followed by an equivalence. We recall that in a Quillen model the equivalences in these two factorisations are required to be, respectively, a trivial cofibration and a trivial fibration. Every proper, functorial Quillen model induces such a Thomason model. However, it is not true that every Thomason model induces a Baues model, because the factorisation axiom is not strong enough to produce projective resolutions in the sense of Baues. What remains true is that every Thomason model still induces a fibration model. In this sense a fibration model extracts the fibration structure of a Thomason model, much like the Baues model does for the Quillen model. It is worth observing that in [14] there is also the definition of a *right* Thomason model, which essentially amounts to a Baues model without the axiom on resolutions.

In conclusion, the fibration model described in Section 3 lies between the functorial versions of the Brown and the Baues model, but also under the Thomason model. This explains its relative interest. Although we can deduce fibration models from Quillen or Baues models, the increased flexibility coming from the factorisation axiom allows some simplifications. For example, quasi-isomorphisms in the category of bounded below chain complexes of R-modules carry a fibration model, where $\mathscr{F}$ is the class of epimorphisms. This can be regarded as derived from the corresponding presentable Quillen model on unbounded chain complexes (cf. Hovey [8, Section 2.3]). However, the factorisation axiom in the fibration model can also be obtained from the usual factorisation via the mapping fiber P_f as in the diagram below,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle e}\searrow^{\sim} \quad \nearrow{\scriptstyle p} & \\ & P_f & \end{array} \tag{8}$$

whereas this is not the case for the Quillen model, simply because e is not a cofibration unless B is pointwise projective.

5. Homotopy

Fix a category $\mathbf{C}$ and a class of equivalences $\mathscr{E} \subseteq \mathbf{C}$; assume further that $\mathbf{C}$ carries a fibration model for $\mathscr{E}$. Define a *path object* functor $P : \mathbf{C} \to \mathbf{C}$ applying the functorial factorisation to the diagonal natural transformation, as in the diagram below.

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \cdot A \\ & {\scriptstyle e}\searrow^{\sim} \quad \nearrow{\scriptstyle p} & \\ & PA & \end{array} \tag{9}$$

More in detail, we associate to every object $A \in \mathbf{C}$ its diagonal $\triangle_A$ regarded as an object in the category of arrows $\mathbf{C}^2$ and then apply the factorisation axiom to obtain the path object PA; this is the object function of the functor P. Similarly, every morphism $f : A \to B$ in $\mathbf{C}$ induces a morphism between the corresponding diagonals and therefore a morphism $Pf : PA \to PB$ between the path objects. Given arrows $f, g : B \rightrightarrows A$, say that f is *homotopic* to g, in symbols $f \approx g$, if the two arrows admit a factorisation through the path object on A as shown in the diagram below, where the conjunction $f \wedge g$ indicates the unique arrow to the product whose components are f and g.

$$\begin{array}{ccc} & & PA \\ & {\scriptstyle h}\nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow[f \wedge g]{} & A \cdot A \end{array} \tag{10}$$

The homotopy relation is reflexive, symmetric and stable under composition, though not transitive in general. Note that our definition of path object is more restrictive than the one given in Quillen [12, Definition I.1.4], where any factorisation of the diagonal as an equivalence followed by a fibration is allowed. We refer to the object defined by Quillen as an *arbitrary* path object. As a consequence, our notion of homotopy is more restrictive than the one of Quillen. However, when B is projective and A is fibrant, all path objects on A define the same homotopy relation and this relation is an equivalence (Baues [1, dual of Proposition II.2.22]); hence

Proposition 5.1. *If P is projective and A fibrant, homotopy is an equivalence relation on* $\mathbf{C}(P, A)$.

In this case, we write $[P, A]$ for the set of homotopy classes. Our first aim for this section is to prove that projectives have a lifting property with respect to homotopy classes.

For this, restrict to the full subcategory $\mathrm{F}\mathbf{C} \subseteq \mathbf{C}$ of fibrant objects. Given arrows $f, g \in \mathrm{F}\mathbf{C}$ as in the diagram below, define their *homotopy pullback*

$$\begin{array}{ccc} P & \xrightarrow{p_B} & B \\ {\scriptstyle p_C}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & A \end{array} \tag{11}$$

via the ordinary pullback diagram

$$\begin{array}{ccc} P & \dashrightarrow & PA \\ {\scriptstyle p_B \wedge p_C}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ B \times C & \xrightarrow[f \times g]{} & A \times A \end{array} \tag{12}$$

The terminology will be justified in Section 8, where it will be shown that this definition is a special case of a more general one involving derived functors. Observe that $p_B \wedge p_C$ is a fibration by stability and so are the projections from $B \times C$; hence p_B and p_C are fibrations and P is fibrant. Observe also that the homotopy pullback diagram (11) commutes only up to homotopy and that (12) shows that it is universal with respect to this property.

This construction of homotopy pullbacks is in Quillen [12, I.3.1] and in Brown [3, p. 424], to which the reader is referred for comparison. However, we wish to clarify and make explicit some of its properties.

Proposition 5.2. *Consider, in* F**C**, *the diagram below.*

$$\begin{array}{ccccc} Q & \longrightarrow & P & \longrightarrow & B \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ D & \xrightarrow[h]{} & C & \xrightarrow[g]{} & A \end{array} \tag{13}$$

If the right square is a homotopy pullback and the left square is an ordinary pullback, the outer rectangle is a homotopy pullback.

Proof. This follows from the universal properties of the homotopy pullback and of the ordinary pullback. Equivalently, it can be made explicit as follows: consider the diagram

$$\begin{array}{ccccc} Q & \longrightarrow & P & \longrightarrow & PA \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ B \times D & \xrightarrow[1 \times h]{} & B \times C & \xrightarrow[f \times g]{} & A \times A \\ {\scriptstyle \mathrm{pr}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{pr}} & & \\ D & \xrightarrow[h]{} & C & & \end{array} \tag{14}$$

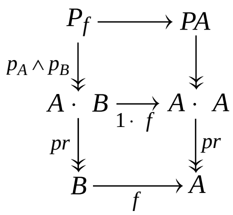
The composite vertical rectangle is a pullback by assumption. Since the bottom square is a pullback, the top left square is a pullback. But now, the right square is a pullback by assumption and therefore the composite horizontal rectangle is a pullback. This means that the outer rectangle in (13) is a homotopy pullback. □

Define the *mapping fiber* of $f \in$ F**C** to be the homotopy pullback of f along the identity, as in the diagram below.

$$\begin{array}{ccc} P_f & \overset{p_B}{\underset{\sim}{\twoheadrightarrow}} & B \\ {\scriptstyle p_A}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ A & = & A \end{array} \tag{15}$$

Lemma 5.3. *The projection* p_B *from the mapping fiber to the domain of* f *is an acyclic fibration.*

Proof. Consider the diagram

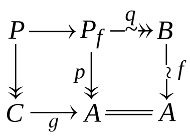


(16)

The top square is a pullback by definition of homotopy pullback and so is the bottom square, which is given by projections on the second component. By the pullback pasting lemma, the outer rectangle is a pullback. Now, the projections of the path object $PA \to A$ are acyclic fibrations, because A is fibrant (cf. Quillen [12, dual of lemma I.1.2]); hence p_B is acyclic by stability. □

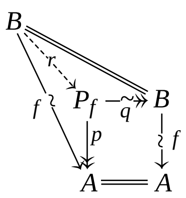
Proposition 5.4. *Equivalences in* **F**C *are stable under homotopy pullbacks.*

Proof. By Proposition 5.2, the homotopy pullback of f along g can be computed in stages



(17)

first forming the mapping fiber of f and then the ordinary pullback of p along g. Note that p is a fibration being a projection of the homotopy pullback. We will prove that if f is an equivalence then p is acyclic; the result then follows by stability. To see that p is acyclic, recall from Lemma 5.3 that q is an acyclic fibration and consider the diagram below.

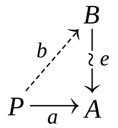


(18)

The outer part commutes strictly; hence, by the universal property of the homotopy pullback, there exists an arrow r making the two triangles commutative. Commutativity of the top triangle and saturation prove that r is an equivalence; and again saturation and commutativity of the left triangle prove that p is acyclic. □

We can now use the properties of homotopy pullbacks to extend the lifting properties of projectives. The following proposition should be compared with Quillen [12, Lemma I.1.7] and Baues [1, Proposition II.2.11].

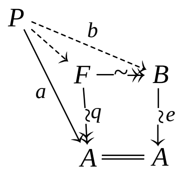
Proposition 5.5. *Given the solid part of the diagram*



(19)

with P projective and e an equivalence between fibrant objects, there exists a lifting b making the diagram commutative up to homotopy. Moreover, b is unique up to homotopy.

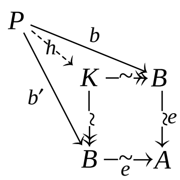
Proof. Existence. Form the mapping fiber F of e



(20)

and observe that by the universal property of homotopy pullbacks, liftings of a to B up to homotopy correspond to strict liftings of a to F. Since e is an equivalence, q is an acyclic fibration by 5.4 and projectivity of P implies the existence of the lifting.

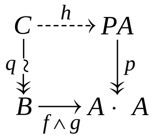
Uniqueness. Form the homotopy kernel pair of e, i.e. the homotopy pullback K of e along itself. If b and b' are liftings of a up to homotopy, the outer part of the diagram



(21)

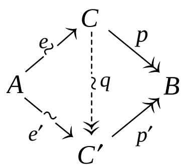
commutes up to homotopy. By the universal property of the homotopy pullback there exists an arrow h making the triangles strictly commutative. However, K with the two projections to B is a path object on B—this follows using the same diagram above replacing P with B and the diagonal arrows with identities. Since P is projective and K fibrant, homotopy can be realised on any path object. Thus $b \approx b'$. □

There is also a more general notion of homotopy which we need to discuss. Following Brown [3], we define on **FC** a *generalised homotopy* relation setting $f \sim g$ if there exists an acyclic fibration q such that $fq \approx gq$.



(22)

Observe that F**C** carries a functorial Brown model induced by the fibration model on **C**. This allows us to import the results of Brown [3, Section I.2], provided we show that generalised homotopy can be realised on any path object. To this aim, consider first two factorisations $pe = p'e'$ of the same arrow, as in the solid part of the diagram below.

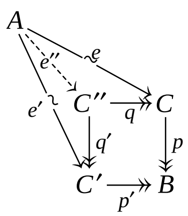


(23)

Say that the upper factorisation *refines* the lower one if there exists a trivial fibration q making the diagram commutative. Two factorisations are *equivalent* if they admit a common refinement.

Proposition 5.6. *Any two factorisations of the same arrow as an equivalence followed by a fibration are equivalent.*

Proof. Consider factorisations $pe = p'e'$ of the same arrow as shown in the diagram below and form the inner pullback square.



(24)

By stability, q and q' are fibrations, hence so is the diagonal of the square. Moreover, the universal property of the pullback provides an arrow e'' making the diagram commutative. In general e'' is not an equivalence, so we factor it as an equivalence followed by a fibration and relabel so that e'' is the equivalence and q, q' are now the composite fibrations. Now the square in (24) commutes, although it is not a pullback. By saturation both q and q' are trivial fibrations and $e''p''$ with $p'' = pq = p'q'$ is the desired refinement. □

Corollary 5.7. *Generalised homotopy in* F**C** *can be realised on any path object.*

Proof. With the same notation of diagram (22), suppose $f \sim g$ is realised on a path object PA as in the bottom row of the diagram below, with $ph = (f \wedge g)q$. Let $P'A$ be another path object on A.

$$\begin{array}{ccccc} C' & \xrightarrow{h'} & P''A & \overset{s}{\underset{\sim}{\twoheadrightarrow}} & P'A \\ {\scriptstyle r'}\downarrow\wr & & \wr\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle p'} \\ C & \xrightarrow[h]{} & PA & \xrightarrow[p]{} & A \cdot A \end{array} \tag{25}$$

Since path objects are obtained by factorisation of the diagonal of A, 5.6 provides a common refinement, hence an object $P''A$ and trivial fibrations r and s making the right square in (25) commutative. Now form the pullback square on the left and observe that sh' realises a generalised homotopy $f \sim g$ on $P'A$. □

Thus, our notion of generalised homotopy coincides with that of Brown. In particular, recalling that a congruence on a category is an equivalence relation stable under composition [10], we have from [3]

Proposition 5.8 (Brown [3]). *Generalised homotopy is a congruence on* F**C**.

6. Existence of the localisation

The aim of this section is to prove that if $\mathscr{E} \subseteq \mathbf{C}$ admits a fibration model, then the localisation $\mathbf{C}_{\mathscr{E}}$ exists. Consider first the full subcategory $\mathrm{F}\mathbf{C} \subseteq \mathbf{C}$ of fibrant objects with equivalences induced by the inclusion. We still write $\mathscr{E} \subseteq \mathrm{F}\mathbf{C}$ for the induced class of equivalences and observe that the inclusion $I : \mathrm{F}\mathbf{C} \to \mathbf{C}$ induces a functor $\partial I : \mathrm{F}\mathbf{C}_{\mathscr{E}} \to \mathbf{C}_{\mathscr{E}}$ by the universal property of the localisation.

Proposition 6.1. *If* **C** *carries a fibration model for* $\mathscr{E}$, *the induced functor* $\partial I : \mathrm{F}\mathbf{C}_{\mathscr{E}} \to \mathbf{C}_{\mathscr{E}}$ *is an equivalence.*

The proof depends only on the functorial factorisation axiom. It can be found in Hovey [8, Proposition 1.2.3] or later in this article in the more general context of (co)essential subcategories, in Section 7. Proposition 6.1 reduces the existence of $\mathbf{C}_{\mathscr{E}}$ to that of $\mathrm{F}\mathbf{C}_{\mathscr{E}}$. To analyse the local structure of $\mathrm{F}\mathbf{C}_{\mathscr{E}}$ we use the functorial Brown model on F**C** induced by the fibration model on **C**. From Brown [3, Section I.2] we have the following results.

Proposition 6.2 (Brown [3]). *Every arrow in* F**C** *admits a factorisation*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle e}\searrow\wr \quad \nearrow\!\!\!\!\nearrow{\scriptstyle p} & \\ & P & \end{array} \tag{26}$$

where e *admits an acyclic fibration as a retraction.*

Theorem 6.3 (Brown [3]). *Acyclic fibrations admit a calculus of right fractions in* $\mathrm{F}\mathbf{C}/\sim$.

Corollary 6.4 (Brown [3]). *Any arrow* $A \to B$ *in* $\mathrm{F}\mathbf{C}_{\mathscr{E}}$ *can be written as a right fraction*

$$\begin{array}{ccc} X & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow\wr & & \\ A & & \end{array} \tag{27}$$

and any two arrows $f, g : A \rightrightarrows B$ *in* $\mathrm{F}\mathbf{C}$ *are identified in the localisation if and only if* $f \sim g$.

If we write $\mathrm{F}\mathbf{C}/\sim$ for the quotient of fibrant objects by generalised homotopy and $\mathscr{F}_{\mathscr{E}}$ for the quotient class of acyclic fibrations, then $\mathrm{F}\mathbf{C}_{\mathscr{E}}$ can be identified with the localisation of $\mathrm{F}\mathbf{C}/\sim$ at $\mathscr{F}_{\mathscr{E}}$. By 6.3, $\mathscr{F}_{\mathscr{E}}$ admits a calculus of right fractions, hence $\hom_{\mathbf{C}_{\mathscr{E}}}(A,B)$ can be computed as the colimit of the functor

$$\hom(d_0\,\_\,, B) : (\mathscr{F}_{\mathscr{E}}/A)^{\mathrm{op}} \to \mathbf{Sets} \tag{28}$$

as in (2). As remarked in Section 4, the Brown model cannot guarantee the existence of this colimit, as the diagram need not be small. Suppose however that $\mathbf{C}$ carries a Baues model as defined in Section 4. Then every object A admits a projective resolution $p : P \to A$ which is a trivial fibration, and the lifting criterion 5.5 shows that any such p is an initial object in $\mathscr{F}_{\mathscr{E}}/A$, hence a terminal object in the opposite category. The inclusion of a terminal object is cofinal, so that the colimit of (28) exists in this case and is simply evaluation at p, giving $\hom_{\mathbf{C}_{\mathscr{E}}}(A,B) = \hom(P,B)/\sim$. Since P is projective, the latter hom-object can be identified with the set $[P,B]$ of ordinary homotopy classes as defined in Section 5. The same argument works for Quillen models, as every such model induces a Baues model. It does not work, however, for fibration models, because in this case the resolution $P \to A$ is only an equivalence, hence not an object of $\mathscr{F}_{\mathscr{E}}/A$. Nevertheless, it can be adapted as follows. For P projective and B fibrant, consider the diagram

$$\begin{array}{ccc} \mathbf{C}(P,B) & \longrightarrow & \mathbf{C}\mathscr{E}(P,B) \\ \downarrow & \nearrow_{j} & \\ [P,B] & & \end{array} \tag{29}$$

where the horizontal arrow is the component of the localisation functor and the vertical arrow is the projection on the quotient. Since homotopic arrows are identified in the localisation (cf. Quillen [12, Lemma I.1.8 (i)]) there is an induced arrow j making the triangle commutative.

Proposition 6.5. *If* P *is projective and* B *fibrant*,

$$j : [P,B] \to \mathbf{C}_{\mathscr{E}}(P,B) \tag{30}$$

is an isomorphism.

Proof. Surjectivity. Factor the terminal arrow $P \to 1$ to obtain a fibrant replacement $e : P \to FP$, the equivalence e inducing an isomorphism

$$\mathbf{C}_{\mathscr{E}}(FP, B) \to \mathbf{C}_{\mathscr{E}}(P, B). \tag{31}$$

Since both FP and B are now fibrant, an application of 6.4 shows that any morphism $f : P \to B$ in the localisation can be represented by a composite path as in the solid part of the diagram below.

$$\begin{array}{ccc} & & X & \overset{q}{\twoheadrightarrow} & B \\ & \overset{s}{\nearrow} & \downarrow {\scriptstyle \wr p} & & \\ P & \underset{e}{\xrightarrow{\sim}} & FP & & \end{array} \tag{32}$$

By projectivity of P, there exists a lifting s making the triangle commutative; thus, $qs = f$ in the localisation, and j is surjective.

Injectivity. Let $f, g : P \rightrightarrows B$ be arrows in $\mathbf{C}$ which are identified in $\mathbf{C}_{\mathscr{E}}$. Consider their fibrant replacements Ff and Fg, as in the diagram below.

$$\begin{array}{ccccc} & & P & \overset{f}{\underset{g}{\rightrightarrows}} & B \\ & \overset{s}{\swarrow} & {\scriptstyle e_A} \downarrow \wr & & \wr \downarrow {\scriptstyle e_B} \\ X & \underset{p}{\overset{\sim}{\twoheadrightarrow}} & F(P) & \overset{Ff}{\underset{Fg}{\rightrightarrows}} & F(B) \end{array} \tag{33}$$

Note that the vertical arrows are equivalences and that the rectangle commutes, in the sense that $e_B g = (Fg)e_A$ and $e_B f = (Ff)e_A$. Therefore, Ff and Fg are identified in $\mathbf{C}_{\mathscr{E}}$. By 6.4, they are equalised up to homotopy by an acyclic fibration p. Projectivity of P provides a lifting s making the triangle commutative. Thus, $Ff \circ e_A \approx Fg \circ e_A$, which implies $e_B \circ f \approx e_B \circ g$ and by 5.5, $f \approx g$. □

Corollary 6.6. *If $\mathscr{E} \subseteq \mathbf{C}$ admits a fibration model, the localisation $\mathbf{C}_{\mathscr{E}}$ exists.*

Proof. It suffices to observe that for any pair of objects $A, B \in \mathbf{C}$, the choice of a projective resolution $P \to A$ and of a fibrant replacement $B \to F$ induces isomorphisms

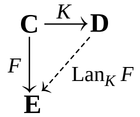
$$\mathbf{C}_{\mathscr{E}}(A, B) \simeq \mathbf{C}_{\mathscr{E}}(P, F) \simeq [P, F]. \tag{34}$$

□

The corollary shows that to analyse $\mathbf{C}_{\mathscr{E}}(A, B)$, the appropriate technique is to replace A with a projective object and B with a fibrant object.

7. Derived functors

The goal of this section is to analyse derived functors between localisations of categories. We follow Quillen [12, Section I.4], and Dwyer and Spalinski [4, Section 9], with the necessary modifications. In particular, we drop homotopy-theoretic arguments when not needed. Rather, our analysis is based on Kan extensions. We recall briefly that the left Kan extension of a functor F along a functor K as in the diagram



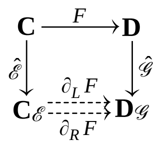
(35)

is a representation

$$\hom(F, (\_) \circ K) \simeq \hom(\mathrm{Lan}_K F, \_). \tag{36}$$

Right Kan extensions are defined dually. The reader is referred to Mac Lane [10] and to Kelly [9] for more details.

Given classes of equivalences $\mathscr{E} \subseteq \mathbf{C}$ and $\mathscr{G} \subseteq \mathbf{D}$, consider the solid part of the diagram



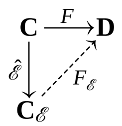
(37)

where the vertical arrows are localisations, and define the left and right *derived functors* of F setting

$$\partial_L F = \mathrm{Ran}_{\hat{\mathscr{E}}}(\hat{\mathscr{G}} F), \quad \partial_R F = \mathrm{Lan}_{\hat{\mathscr{E}}}(\hat{\mathscr{G}} F) \tag{38}$$

when these Kan extensions exist. Note that derived functors do not make the diagram commutative, in general. We analyse existence of derived functors on the basis of Quillen [12, Proposition I.4.1]. First, we consider Kan extensions in the framework of localisations. For brevity, we call *stable* a Kan extension which is preserved by all functors (cf. Mac Lane [10, Section X.5]). Similarly, a derived functor is *stable* if it is stable as a Kan extension.

Proposition 7.1. *Consider the diagram*



(39)

where $\hat{\mathscr{E}}$ is a localisation and F inverts the arrows of $\mathscr{E}$. The unique functor $F_{\mathscr{E}}$ determined by the universal property of the localisation and making the triangle commutative is both a left and right stable Kan extension of F along $\hat{\mathscr{E}}$.

Proof. From Gabriel and Zisman [5, Lemma I.1.2] we know that composition with $\hat{\mathscr{E}}$ induces an isomorphism of categories

$$\hat{\mathscr{E}}^* : \hom(\mathbf{C}_{\mathscr{E}}, \mathbf{D}) \to \hom'(\mathbf{C}, \mathbf{D}), \tag{40}$$

where $\hom'(\mathbf{C},\mathbf{D}) \subseteq \mathbf{D}^{\mathbf{C}}$ is the full subcategory generated by functors inverting $\mathscr{E}$. Thus, for every functor $X:\mathbf{C}_{\mathscr{E}} \to \mathbf{D}$ we have a natural bijection

$$\hom(F_{\mathscr{E}},X) \simeq \hom(F_{\mathscr{E}}\hat{\mathscr{E}},X\hat{\mathscr{E}}), \tag{41}$$

$$\simeq \hom(F,X\hat{\mathscr{E}}), \tag{42}$$

proving that $F_{\mathscr{E}} = \mathrm{Lan}_{\hat{\mathscr{E}}} F$. If now $G:\mathbf{D} \to \mathbf{E}$ is any functor, the universal property of the localisation gives $(GF)_{\mathscr{E}} = GF_{\mathscr{E}}$, proving that the Kan extension is stable. The proof in the case of the right Kan extension is similar. □

Thus, if a functor preserves equivalences, its derived functors are easily computed. The insight of Quillen (cf. [12, Proposition I.4.1]) is that in presence of a Quillen model, the general case can be reduced to this, restricting to a suitable subcategory. More precisely, consider the diagram

$$\begin{array}{ccccc} \mathbf{B} & \rightarrowtail & \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ & & {\scriptstyle K}\downarrow & \nearrow_{\mathrm{Ran}_K F} & \\ & & \mathbf{E} & & \end{array} \tag{43}$$

where $\mathbf{B} \subseteq \mathbf{C}$ is a full subcategory. We wish to know when $\mathrm{Ran}_K F$ can be computed by restriction to $\mathbf{B}$, i.e. when

$$\mathrm{Ran}_K F \simeq \mathrm{Ran}_{(K|\mathbf{B})}(F|\mathbf{B}). \tag{44}$$

Consider first what happens in the special case when $\mathbf{B} \subseteq \mathbf{C}$ is a full and representative subcategory. In this case the inclusion I admits a right adjoint R. The counit isomorphism $IR \to 1_{\mathbf{C}}$ and the adjunction $I \dashv R$ induce for every functor $X:\mathbf{E} \to \mathbf{D}$ natural isomorphisms

$$\hom(XK,F) \simeq \hom(XKIR,F) \simeq \hom(XKI,FI) = \hom(XK|\mathbf{B},F|\mathbf{B}) \tag{45}$$

proving the isomorphism (44), when either Kan extension exists. In fact, the only property used in this argument is that the functor I—not necessarily an inclusion—admits a full and faithful right adjoint. We now restrict to the case of an inclusion and show how the argument can be generalised.

Returning to diagram (43) and to the full subcategory $\mathbf{B} \subseteq \mathbf{C}$, assume that for every object $C \in \mathbf{C}$ there exists an object $B \in \mathbf{B}$ and an arrow $C \to B$ in $\mathbf{C}$ which is inverted by K. Assume further that the assignment of this arrow is functorial in C, i.e. that there is a functor $L:\mathbf{C} \to \mathbf{B}$ together with a natural transformation $1_{\mathbf{C}} \to IL$ which is pointwise inverted by K. In this case, we say that $\mathbf{B}$ is a K-*essential* subcategory of $\mathbf{C}$. If the arrow is in the opposite direction $B \to C$, so that we have a functor $R:\mathbf{C} \to \mathbf{B}$ together with a natural transformation $IR \to 1_{\mathbf{C}}$, we say that $\mathbf{B}$ is K-*coessential*. When K is the identity the two notions coincide and are equivalent to the assertion that the inclusion $\mathbf{B} \rightarrowtail \mathbf{C}$ is essentially surjective—hence an equivalence.

Proposition 7.2. *Let* $\mathbf{B} \subseteq \mathbf{C}$ *be a* K*-coessential subcategory. Then* $\mathrm{Ran}_K F$ *exists if and only if* $\mathrm{Ran}_{(K|\mathbf{B})}(F|\mathbf{B})$ *does. In this case the extensions are naturally isomorphic.*

Proof. It suffices to prove that given any functor $X:\mathbf{E}\to\mathbf{D}$, restriction to $\mathbf{B}$ induces an isomorphism

$$\text{res} : \hom(XK,F)\to\hom(XK|\mathbf{B},F|\mathbf{B}). \tag{46}$$

To see why this is true, let $e:IR\to 1$ be the natural transformation exhibiting $\mathbf{B}$ as a K-coessential subcategory of $\mathbf{C}$. Observe that for every natural $t:XK\to F$ and for every object $C\in\mathbf{C}$, the arrow $e_C:RC\to C$ induces a commutative square

$$\begin{array}{ccc} XKRC & \xrightarrow{t_{RC}} & FRC \\ {\scriptstyle XKe_C}\downarrow & & \downarrow{\scriptstyle Fe_C} \\ XKC & \xrightarrow[t_C]{} & FC \end{array} \tag{47}$$

where the left vertical arrow is an isomorphism. This shows that t is completely determined by its values on $\mathbf{B}$. Thus, given the components of t on $\mathbf{B}$ we can complete t to the whole $\mathbf{C}$ uniquely using diagram (47). Naturality of t on $\mathbf{C}$ follows from naturality of t on $\mathbf{B}$ and from functoriality of R. □

Note that the Proof of 7.2 also shows that the counit of $\mathrm{Ran}_{(K|\mathbf{B})}(F|\mathbf{B})$ is the restriction to $\mathbf{B}$ of the counit of $\mathrm{Ran}_K F$. It is also clear that there is a dual statement, involving K-essential subcategories and left Kan extensions.

Returning to localisations, let $\mathscr{E}\subseteq\mathbf{C}$ be a class of equivalences. We specialise the notion of (co)essential subcategory and say that a full subcategory $\mathbf{B}\subseteq\mathbf{C}$ is *$\mathscr{E}$-coessential* if there exists a functor $R:\mathbf{C}\to\mathbf{B}$ and a natural transformation $e:IR\to 1_{\mathbf{C}}$ whose components belong to $\mathscr{E}$. Note that if $\hat{\mathscr{E}}$ is the localisation functor associated to $\mathscr{E}$, then every $\mathscr{E}$-coessential subcategory is $\hat{\mathscr{E}}$-coessential. The converse is true if $\mathscr{E}$ is precisely the class of arrows inverted by $\hat{\mathscr{E}}$. $\mathscr{E}$-(co)essential subcategories have stronger properties than arbitrary (co)essential subcategories. In fact, given a full subcategory $\mathbf{B}\subseteq\mathbf{C}$, write $\mathscr{E}\subseteq\mathbf{B}$ for the class of equivalences inherited from $\mathbf{C}$. The inclusion functor I preserves equivalences and by 7.1 admits a derived functor ∂I as in the diagram below.

$$\begin{array}{ccc} \mathbf{B} & \underset{R}{\overset{I}{\rightleftarrows}} & \mathbf{C} \\ {\scriptstyle \hat{\mathscr{E}}_B}\downarrow & & \downarrow{\scriptstyle \hat{\mathscr{E}}} \\ \mathbf{B}_{\mathscr{E}} & \underset{\partial R}{\overset{\partial I}{\rightleftarrows}} & \mathbf{C}_{\mathscr{E}} \end{array} \tag{48}$$

If $\mathbf{B}$ is $\mathscr{E}$-coessential, we also have a functor R in the opposite direction. Now, observe the following.

1. R preserves equivalences, because if in the diagram below e is an equivalence, so is Re by saturation.

$$\begin{array}{ccc} RA & \xrightarrow[\sim]{e_A} & A \\ {\scriptstyle Re}\downarrow & & \downarrow{\scriptstyle \wr e} \\ RB & \xrightarrow[e_B]{\sim} & B \end{array} \tag{49}$$

2. There exists a natural transformation $RI \to 1$ with components in $\mathscr{E}$. In fact for every $B \in \mathbf{B}$, the component $e_B : R(B) \to B$ can be interpreted as that of a natural transformation $RI \to 1$.

By 1 and 7.1, R admits a derived functor ∂R. By 2 and the assumptions on e, it follows that $\partial I \dashv \partial R$ is an adjoint equivalence. The same kind of argument produces an adjoint equivalence $\partial L \dashv \partial I$ in the case of an $\mathscr{E}$-essential subcategory. This remark contains Proposition 6.1 as a special case, because the full subcategory $\mathrm{F}\mathbf{C} \subseteq \mathbf{C}$ is $\mathscr{E}$-essential by the factorisation axiom.

Proposition 7.3. *Let $\mathscr{E} \subseteq \mathbf{C}$ be a class of equivalences. Let F be a functor defined on $\mathbf{C}$, as in the diagram below.*

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ {\scriptstyle \hat{\mathscr{E}}}\downarrow & \nearrow_{\mathrm{Ran}_{\hat{\mathscr{E}}} F} & \\ \mathbf{C}_{\mathscr{E}} & & \end{array} \tag{50}$$

If there exists an $\mathscr{E}$-coessential subcategory $\mathbf{B} \subseteq \mathbf{C}$ such that $F|\mathbf{B}$ inverts equivalences, $\mathrm{Ran}_{\hat{\mathscr{E}}} F$ exists and is stable. Moreover, the counit of the Kan extension restricts to an isomorphism

$$(\mathrm{Ran}_{\hat{\mathscr{E}}} F \circ \hat{\mathscr{E}})|\mathbf{B} \simeq F|\mathbf{B}. \tag{51}$$

Proof. To prove the existence of $\mathrm{Ran}_{\hat{\mathscr{E}}} F$ it suffices, by 7.2, to prove the existence of $\mathrm{Ran}_{(\hat{\mathscr{E}}|\mathbf{B})}(F|\mathbf{B})$. This follows immediately from 7.1. To prove stability of the Kan extension observe that

$$\begin{aligned} \mathrm{Ran}_{(\hat{\mathscr{E}}|\mathbf{B})}(F|\mathbf{B}) &\simeq \mathrm{Ran}_{\partial I \hat{\mathscr{E}}_{\mathbf{B}}}(F|\mathbf{B}) \\ &\simeq \mathrm{Ran}_{\partial I}\, \mathrm{Ran}_{\hat{\mathscr{E}}_{\mathbf{B}}}(F|\mathbf{B}) \\ &\simeq \mathrm{Ran}_{\partial I}(F|\mathbf{B})_{\mathscr{E}} \\ &\simeq (F|\mathbf{B})_{\mathscr{E}}\, \partial R. \end{aligned} \tag{52}$$

The last expression shows that $\mathrm{Ran}_{(\hat{\mathscr{E}}|\mathbf{B})}(F|\mathbf{B})$ is stable. Since the counit of $\mathrm{Ran}_{\hat{\mathscr{E}}} F$ is the uniquely determined extension of the counit of $\mathrm{Ran}_{(\hat{\mathscr{E}}|\mathbf{B})}(F|\mathbf{B})$ as in the proof of Proposition 7.2, this too is stable.

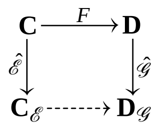
To prove the isomorphism (51), it suffices to show that the counit of $\mathrm{Ran}_{(\hat{\mathscr{E}}|\mathbf{B})}(F|\mathbf{B})$ is an isomorphism. Note that the latter is the top row in the commutative diagram

$$\begin{array}{ccc} \mathrm{Ran}_{(\hat{\mathscr{E}}|\mathbf{B})}(F|\mathbf{B}) \circ \hat{\mathscr{E}}|\mathbf{B} & \dashrightarrow & F|\mathbf{B} \\ \downarrow & & \| \\ (F|\mathbf{B})_{\mathscr{E}}\, \partial R \partial I \hat{\mathscr{E}}_{\mathbf{B}} & \xrightarrow[1\varepsilon 1]{} & (F|\mathbf{B})_{\mathscr{E}}\, \hat{\mathscr{E}}_{\mathbf{B}} \end{array} \tag{53}$$

The vertical arrow on the left is the restriction of the isomorphism (52). The bottom arrow is an isomorphism because $\partial I \dashv \partial R$ is an adjoint equivalence. Hence the counit is an isomorphism. $\square$

There is a dual statement asserting the existence of $\mathrm{Lan}_{\mathscr{E}} F$ whenever the restriction of F to an $\mathscr{E}$-essential subcategory inverts equivalences.

To apply these results to derived functors, consider the diagram



(54)

where $\mathscr{E} \subseteq \mathbf{C}$ and $\mathscr{G} \subseteq \mathbf{D}$ are classes of equivalences admitting fibration models and the vertical arrows are localisations.

Corollary 7.4. *If F preserves equivalences between projectives, $\partial_L F$ exists, is stable and is canonically isomorphic to F on projectives.*

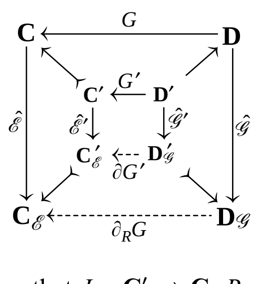
Proof. Let $\mathbf{B}$ be the full subcategory of projectives and observe that it is $\mathscr{E}$-coessential by the axiom on resolutions. □

Corollary 7.5. *If F preserves equivalences between fibrant objects, $\partial_R F$ exists, is stable and is canonically isomorphic to F on fibrant objects.*

Proof. Let $\mathbf{B}$ be the full subcategory of fibrant objects and observe that it is $\mathscr{E}$-essential by functoriality of the factorisation of the terminal arrow as an equivalence followed by a fibration. □

The verification of the hypotheses in 7.5 can sometimes be simplified by the Brown factorisation Lemma 6.2. For if F preserves acyclic fibrations between fibrant objects, then it preserves equivalences between fibrant objects. Note that if F preserves fibrations between fibrant objects, the two conditions are equivalent.

We illustrate the existence Theorem 7.5 in a special case which will appear in the treatment of homotopy pullbacks. To fix the notation, consider the diagram below where we are given a functor G and localisations $\hat{\mathscr{E}}$, $\hat{\mathscr{G}}$ and we wish to compute the derived functor $\partial_R G$.



(55)

Assume that $I_{\mathbf{C}} : \mathbf{C}' \rightleftarrows \mathbf{C} : R_{\mathbf{C}}$ and $I_{\mathbf{D}} : \mathbf{D}' \rightleftarrows \mathbf{D} : R_{\mathbf{D}}$ are essential subcategories and that G preserves their objects, so that it restricts to a functor G'. Assume finally that the restriction G' preserves equivalences, so that the unique functor $\partial G'$ making the inner square commutative is a left and right derived functor of G' by 7.1.

Corollary 7.6. *If $\mathscr{E}$ and $\mathscr{G}$ admit fibration models and G preserves fibrant objects and equivalences between them, then*

$$\partial_R G = \partial I_{\mathbf{C}} \circ \partial G' \circ \partial R_{\mathbf{D}}. \tag{56}$$

Proof. Let $\mathbf{C}'$ and $\mathbf{D}'$ be the subcategories of fibrant objects. Then

$$\partial_R G \simeq \mathrm{Ran}_{\hat{\mathscr{G}}|\mathbf{D}'}(\hat{\mathscr{E}} G|\mathbf{D}'), \tag{57}$$

$$\simeq \mathrm{Ran}_{\partial R_{\mathbf{D}}} \mathrm{Ran}_{\hat{\mathscr{G}}'}(\partial I_{\mathbf{C}} \hat{\mathscr{E}}' G'), \tag{58}$$

$$\simeq \partial I_{\mathbf{C}} \mathrm{Ran}_{\hat{\mathscr{G}}'}(\hat{\mathscr{E}}' G') \partial R_{\mathbf{D}}, \tag{59}$$

$$\simeq \partial I_{\mathbf{C}} \partial G' \partial R_{\mathbf{D}}, \tag{60}$$

where (57) follows from Proposition 7.3, (58) from the fact that Kan extension can be computed in stages (cf. [9]), and (59) from the adjunction $\partial I \dashv \partial R$ and from stability of the Kan extension. □

Finally, we analyse adjointness between derived functors. Consider the diagram

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ {\scriptstyle \hat{\mathscr{E}}}\downarrow & \nearrow_{\mathrm{Ran}_{\hat{\mathscr{E}}} F} & \\ \mathbf{C}_{\mathscr{E}} & & \\ {\scriptstyle X}\downarrow & & \\ \mathbf{E} & & \end{array} \tag{61}$$

where $\mathscr{E} \subseteq \mathbf{C}$ is a class of equivalences and $\hat{\mathscr{E}}$ is the corresponding localisation.

Lemma 7.7. *Assume* $\mathrm{Ran}_{\hat{\mathscr{E}}} F$ *exists and is stable. If* $\mathrm{Lan}_F(X\hat{\mathscr{E}})$ *exists, then*

$$\mathrm{Lan}_{(\mathrm{Ran}_{\hat{\mathscr{E}}} F)}(X) \simeq \mathrm{Lan}_F(X\hat{\mathscr{E}}). \tag{62}$$

Proof. For any functor $Y : \mathbf{D} \to \mathbf{E}$ there are natural bijections

$$\hom(X, Y\mathrm{Ran}_{\hat{\mathscr{E}}} F) \simeq \hom(X, \mathrm{Ran}_{\hat{\mathscr{E}}}(YF)), \tag{63}$$

$$\simeq \hom(X\hat{\mathscr{E}}, YF), \tag{64}$$

$$\simeq \hom(\mathrm{Lan}_F(X\hat{\mathscr{E}}), Y), \tag{65}$$

where (63) follows from stability of $\mathrm{Ran}_{\hat{\mathscr{E}}} F$, and (64), (65) from the definition of Kan extension. □

Here is the application to derived functors.

Proposition 7.8. *Consider the diagram*

$$\begin{array}{ccc} \mathbf{C} & \underset{G}{\overset{F}{\rightleftarrows}} & \mathbf{D} \\ {\scriptstyle \hat{\mathscr{E}}}\downarrow & & \downarrow{\scriptstyle \hat{G}} \\ \mathbf{C}_{\mathscr{E}} & \underset{\partial_R G}{\overset{\partial_L F}{\rightleftarrows}} & \mathbf{D}_G \end{array} \tag{66}$$

where $F \dashv G$, the vertical arrows are localisation and the derived functors exist. If both derived functors are stable, then $\partial_L F \dashv \partial_R G$.

Proof. Given any functor $X : \mathbf{C}_{\mathscr{E}} \to \mathbf{E}$, we have the following chain of isomorphisms

$$\mathrm{Lan}_{\partial_L F}(X) \simeq \mathrm{Lan}_{\hat{\mathscr{G}} F}(X\hat{\mathscr{E}}), \tag{67}$$

$$\simeq \mathrm{Lan}_{\hat{\mathscr{G}}}\,\mathrm{Lan}_F(X\hat{\mathscr{E}}), \tag{68}$$

$$\simeq \mathrm{Lan}_{\hat{\mathscr{G}}}(X\hat{\mathscr{E}}G), \tag{69}$$

$$\simeq X\mathrm{Lan}_{\hat{\mathscr{G}}}(\hat{\mathscr{E}}G), \tag{70}$$

$$\simeq X\partial_R G, \tag{71}$$

where the existence of every functor follows from the existence of its successor. More precisely, (67) follows from Lemma 7.7. The isomorphism (68) follows from the fact that Kan extensions can be computed in stages, see [9]; (69) from the adjointness $F \dashv G$, so that the left Kan extension can be computed by composing with the right adjoint, see Mac Lane [10, Theorem X.5.1]; (70) from stability of $\partial_R G$; (71) from the definition of $\partial_R G$. Note that stability of $\partial_L F$ is used in the reference to Proposition 7.7.

This implies that every functor from $\mathbf{C}_{\mathscr{E}}$ admits a stable left Kan extension along $\partial_L F$; in particular, this is true for the identity, whose Kan extension is the right adjoint $\partial_R G$ (cf. Mac Lane [10, Theorem X.7.2]). □

Note that the assumptions of Proposition 7.8 are satisfied if the existence of the derived functors is obtained from Proposition 7.3, as is usually the case.

8. Homotopy pullbacks

Let $\mathscr{E} \subseteq \mathbf{C}$ be a class of equivalences. For every small category $\mathbf{D}$, define the class $\mathscr{E}^{\mathbf{D}} \subseteq \mathbf{C}^{\mathbf{D}}$ of *pointwise equivalences* to be the class of natural transformations with components in $\mathscr{E}$. The terminal functor $! : \mathbf{D} \to 1$ induces by composition a constant functor $\mathbf{C}^!$

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\mathbf{C}^!} & \mathbf{C}^{\mathbf{D}} \\ {\scriptstyle \hat{\mathscr{E}}}\downarrow & & \downarrow{\scriptstyle \hat{\mathscr{E}}^{\mathbf{D}}} \\ \mathbf{C}_{\mathscr{E}} & \underset{\partial \mathbf{C}^!}{\dashrightarrow} & \mathbf{C}^{\mathbf{D}}{}_{\mathscr{E}^{\mathbf{D}}} \end{array} \tag{72}$$

which preserves equivalences and therefore admits by Proposition 7.1 a derived functor $\partial\mathbf{C}^!$ making diagram (72) commutative. If $\partial\mathbf{C}^!$ has a right adjoint, we say that $\mathbf{C}$ admits *homotopy limits* of type $\mathbf{D}$. When $\mathbf{D}$ is the category generated by the graph

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & A \end{array} \tag{73}$$

we write $\mathbf{C}^{\lrcorner}$ for the functor category and call homotopy limits of type $\mathbf{D}$ *homotopy pullbacks*. We will prove that whenever $\mathscr{E}$ admits a fibration model, homotopy pullbacks exist; the reader is referred to Dwyer and Spalinski [4, Section 10] for comparison with the case of Quillen models.

To prove existence of homotopy pullbacks, we first show that fibration models can be transferred to diagrams of pullback type. We recall from Baues ([1, dual of Lemma II, 1.5]) that the category of arrows $\mathbf{C}^2$ carries a Baues model whose fibrations are commutative diagrams

$$\begin{array}{ccccc} D & & & & \\ & \overset{r}{\searrow} & & \searrow & \\ & q\searrow & B \cdot_{A} C & \dashrightarrow & B \\ & & {\scriptstyle p'}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ & & C & \longrightarrow & A \end{array} \tag{74}$$

in which both p and r are fibrations. Note that this implies that also p' and hence q are fibrations. We refer to these diagrams as *fibration squares*. An arrow $p:F \to G$ in $\mathbf{C}^{\lrcorner}$ is a *strong fibration* if in the commutative diagram

$$\begin{array}{ccccc} GB & \xrightarrow{Gf} & GA & \xleftarrow{Gg} & GC \\ {\scriptstyle p_B}\Big\downarrow & & \Big\downarrow{\scriptstyle p_A} & & \Big\downarrow{\scriptstyle p_C} \\ FB & \xrightarrow[Ff]{} & FA & \xleftarrow[Fg]{} & FC \end{array} \tag{75}$$

both squares are fibration squares (cf. Dwyer and Spalinski [4], Heller [7, Section II.4], Goerss and Jardine [6], Weibel [14]). Note also that every strong fibration is a pointwise fibration.

Proposition 8.1. *If a class of equivalences* $\mathscr{E} \subseteq \mathbf{C}$ *admits a fibration model, then strong fibrations and pointwise projectives provide a fibration model for* $\mathscr{E}^{\lrcorner} \subseteq \mathbf{C}^{\lrcorner}$.

Proof. Stability. Consider in $\mathbf{C}^{\lrcorner}$ the pullback square

$$\begin{array}{ccc} G \cdot_{F} H & \overset{q'}{\dashrightarrow} & G \\ {\scriptstyle p'}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ H & \xrightarrow[q]{} & F \end{array} \tag{76}$$

with p a fibration. Then, in the diagram

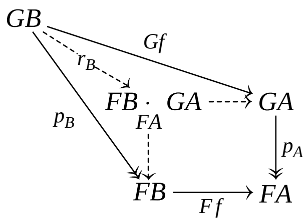
$$\begin{array}{ccccc} G \cdot_{F} H(B) & \longrightarrow & G \cdot_{F} H(A) & \xrightarrow{q'_A} & GA \\ {\scriptstyle p'_B}\Big\downarrow & & \Big\downarrow{\scriptstyle p'_A} & & \Big\downarrow{\scriptstyle p_A} \\ HB & \xrightarrow[Hf]{} & HA & \xrightarrow[q_A]{} & FA \end{array} \tag{77}$$

the right square is a pullback, so that p'_A is a fibration by stability. To prove that the left square is a fibration square it suffices to show that the outer rectangle is a fibration square. By naturality of q and q' this amounts to prove that the outer rectangle in the diagram

$$\begin{array}{ccccc} G\underset{F}{\times}H(B) & \xrightarrow{q'_B} & GB & \xrightarrow{Gf} & GA \\ {\scriptstyle p'_B}\downarrow & & \downarrow{\scriptstyle p_B} & & \downarrow{\scriptstyle p_A} \\ HB & \xrightarrow[q_B]{} & FB & \xrightarrow[Ff]{} & FA \end{array} \tag{78}$$

is a fibration square. But again, the left square being a pullback, it suffices to prove that the right square is a fibration square and this is true because p is a strong fibration. One argues similarly for C, thus proving that p' is a strong fibration. The case of acyclic fibrations is easily disposed of observing that if p is acyclic, all its components are acyclic fibrations; since p' is a pointwise pullback of p, its components are acyclic fibrations by stability, hence p' is acyclic.

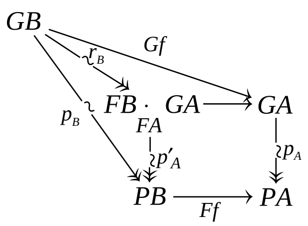
Factorisation. Consider an arrow $p : G \to F$. By functoriality of the fibration model, p can be factored as a pointwise equivalence followed by a pointwise fibration. Hence, to prove the factorisation axiom we may assume that p is a pointwise fibration. Consider the solid part of the diagram below.



(79)

Form the inner pullback, let r_B be the induced arrow and let $r_B = q_B \circ e_B$ be its functorial factorisation in $\mathbf{C}$. If we perform a similar construction for C, then e_B, 1_A and e_C are the components of an equivalence e factoring $p = p' \circ e$ with p' a strong fibration.

Resolutions. It suffices to prove that every acyclic fibration $p : G \to P$ in $\mathbf{C}^{\lrcorner}$ with P pointwise projective admits a section s; the existence of resolutions then follows from functoriality of the resolution assignment in $\mathbf{C}$. To construct s, consider the diagram below.



(80)

Let s_A be a section of p_A and s'_A be the section of p'_A induced by the universal property of the pullback; finally, let s_B be a lifting of s'_A along r_B. Construct s_C in a similar way. Then s_A, s_B and s_C are the components of a natural section for p. □

Proposition 8.2. *Let $\mathscr{E} \subseteq \mathbf{C}$ be a class of equivalences. If $\mathbf{C}$ carries a fibration model for $\mathscr{E}$, it admits homotopy pullbacks.*

Proof. The existence of a fibration model includes the assumption that $\mathbf{C}$ is finitely complete, so that the constant functor $\mathbf{C}^! : \mathbf{C} \to \mathbf{C}^{\lrcorner}$ admits the pullback functor as a right adjoint. The derived functor $\partial\mathbf{C}^!$ is stable by 7.1. Hence, by 7.8, it suffices to prove that the pullback functor admits a stable right derived functor. We use Lemma 7.5 and show that the restriction of pullbacks to fibrant objects preserves equivalences. Consider the commutative diagram

$$\begin{array}{ccccc} G(B) & \twoheadrightarrow & G(A) & \twoheadleftarrow & G(C) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ F(B) & \twoheadrightarrow & F(A) & \twoheadleftarrow & F(C) \end{array} \tag{81}$$

in which all objects are fibrant. The induced arrow between the pullbacks is an equivalence by Baues [1, dual of Lemma II.1.2(b)]. Note that Baues assumes that the model is proper; this is indeed is the case for fibrant objects, as follows from Brown [3, Lemma 4.2]. □

Since fibrant objects in $\mathbf{C}^{\lrcorner}$ are diagrams in which all objects are fibrant and all arrows are fibrations, the stability axiom implies that the pullback functor restricts to a functor on the subcategories of fibrant objects. Hence, homotopy pullbacks can be computed using Corollary 7.6. Explicitly, given arrows $f, g \in \mathbf{C}$ as in the diagram below, we can represent their homotopy pullback as a commutative square in $\mathbf{C}_{\mathscr{E}}$

$$\begin{array}{ccc} P & \overset{p_B}{\dashrightarrow} & B \\ {\scriptstyle p_C}\,\vdots & & \downarrow{\scriptstyle f} \\ C & \underset{g}{\longrightarrow} & A \end{array} \tag{82}$$

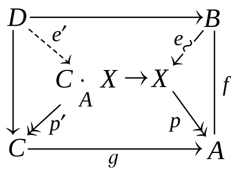
constructed as follows. Fit the solid part of (82) in the top row of the diagram below and take a fibrant replacement as the one in the bottom row.

$$\begin{array}{ccccc} C & \overset{f}{\longrightarrow} & A & \overset{g}{\longleftarrow} & B \\ {\scriptstyle e_C}\downarrow\wr & & \downarrow\wr\,{\scriptstyle e_A} & & \downarrow\wr\,{\scriptstyle e_B} \\ C' & \underset{f'}{\twoheadrightarrow} & A' & \underset{g'}{\twoheadleftarrow} & B' \end{array} \tag{83}$$

A fibrant replacement can be obtained by first factoring the terminal arrow $A \to 1$ as an equivalence e_A followed by a fibration, then factoring $e_A \circ f$ and $e_A \circ g$ as equivalences followed by fibrations as in (83). If P is the ordinary pullback of the bottom row with projections p'_B and p'_C, set $p_B = e_B^{-1} \circ p'_B$ and $p_C = e_C^{-1} \circ p'_C$ in $\mathbf{C}_{\mathscr{E}}$; these are the projections in (82).

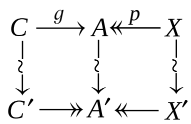
Although this procedure can always be applied to compute homotopy pullbacks, it is not necessarily the most efficient. In fact, the homotopy pullback of a diagram can be computed as the ordinary pullback of any fibrant replacement. For any two fibrant replacements are isomorphic in $\mathbf{C}^{\lrcorner}_{\mathscr{E}\lrcorner}$, hence their homotopy pullbacks are isomorphic in $\mathbf{C}_{\mathscr{E}}$. Here are two examples of how homotopy pullbacks can be computed in special cases.

1. Assume that the fibration model is proper. The outer rectangle in the diagram



(84)

   is a homotopy pullback if there exists a factorisation of f as in the right triangle, such that the induced arrow e' is an equivalence (cf. Baues [6, dual of Definition I.1.9], Goerss and Jardine [6, Section II.8]). The definition is meaningful because the fact that e' is an equivalence does not depend on the particular factorisation of f; in fact a factorisation of g can be used instead. To prove the claim observe that the outer rectangle in (84) is isomorphic in $\mathbf{C}_{\mathscr{E}}$ to the inner pullback diagram. Therefore, it suffices to prove that the ordinary pullback of a fibration p along an arbitrary arrow g is a homotopy pullback. To see this, use the diagram below:



(85)

   the bottom row provides a fibrant replacement of the original diagram in the top row. Again Baues [1, dual of Lemma II.1.2] proves that the pullback of the bottom row is equivalent to the pullback of the top row—here we need properness of the model. However, the pullback of the bottom row is the homotopy pullback as defined using derived functors.
2. If all the objects in the model are fibrant, the "fibrant" homotopy pullback defined by diagram (11) is a homotopy pullback in the present sense. To see this, assume first that f and g are fibrations, so that the ordinary pullback of f and g is also a

homotopy pullback. Now consider the diagram

$$\begin{array}{ccc} Q & \longrightarrow & A \\ {\scriptstyle e'}\downarrow & & \downarrow{\scriptstyle \wr\, e} \\ P & \xrightarrow[r]{} & PA \\ {\scriptstyle p\wedge q}\downarrow & & \downarrow \\ B\cdot C & \xrightarrow[f\cdot g]{} & A\cdot A \end{array} \tag{86}$$

where the composite vertical arrow on the right is the factorisation of the diagonal on A and both squares are pullbacks. The object P is the fibrant homotopy pullback of f and g and Q is their ordinary pullback. Observe that $f \times g = (f \times 1)(1 \times g)$ is a fibration by stability; hence r is a fibration again by stability and e' is an equivalence by properness. Therefore, $P \simeq Q$ in $\mathbf{C}_{\mathscr{E}}$ proving that P is a homotopy pullback. To deal with the general case, factor $f = p_f \circ e_f$ and $g = p_g \circ e_g$ as equivalences followed by fibrations and use the pullbacks

$$\begin{array}{ccccc} P & \xrightarrow{\sim} & P' & \longrightarrow & PA \\ {\scriptstyle p\wedge q}\downarrow & & \downarrow & & \downarrow \\ B\cdot C & \xrightarrow[e_f\cdot e_g]{\sim} & B'\cdot C' & \xrightarrow[p_f\cdot p_g]{} & A\cdot A \end{array} \tag{87}$$

with the left pullback showing that the fibrant homotopy pullback of f and g is isomorphic in $\mathbf{C}_{\mathscr{E}}$ to that of the associated fibrations. Since the homotopy pullback of f and g coincides with that of the associated fibrations, the claim follows.

To make effective use of homotopy pullbacks we need to extend the definition to the localisation. Note that the localisation does not have pullbacks in general, so that this extension can only define homotopy pullbacks by repleteness. To this aim, consider the diagram

$$\begin{array}{ccc} \mathbf{C}^{\lrcorner} & \longrightarrow & (\mathbf{C}\mathscr{E})^{\lrcorner} \\ \downarrow & \nearrow{\scriptstyle J} & \\ \mathbf{C}^{\lrcorner}_{\mathscr{E}^{\lrcorner}} & & \end{array} \tag{88}$$

where the top arrow is induced by composition with the localisation $\hat{\mathscr{E}} : \mathbf{C} \to \mathbf{C}_{\mathscr{E}}$ and the vertical arrow is the localisation at $\mathscr{E}^{\lrcorner}$. Since the equivalences of $\mathscr{E}^{\lrcorner}$ are pointwise, the top functor inverts them, hence it factors through the localisation via a uniquely determined functor J.

Proposition 8.3. *J is essentially surjective and full. Moreover, if two arrows are identified by J, they induce the same arrow between homotopy pullbacks.*

Proof. Essential surjectivity. Consider the diagram

$$\begin{array}{ccccc} B & \longrightarrow & A & \longleftarrow & C \\ \uparrow\wr & & \wr\downarrow & & \uparrow\wr \\ B' & \dashrightarrow & A' & \dashleftarrow & C' \end{array} \tag{89}$$

whose top row is in $\mathbf{C}_{\mathscr{E}}$. Let A' be a fibrant replacement of A and B', C' be projective replacements of B and C. Define the dashed arrows by composition in $\mathbf{C}_{\mathscr{E}}$ and observe that by 6.5 they can be realised in $\mathbf{C}$. Thus, the bottom row is in the image of J and is isomorphic to the top row in $(\mathbf{C}_{\mathscr{E}})^{\lrcorner}$.

Fullness. Suppose we have a commutative diagram in $\mathbf{C}_{\mathscr{E}}$

$$\begin{array}{ccccc} B & \xrightarrow{f} & A & \xleftarrow{g} & C \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle c} \\ B' & \xrightarrow[f']{} & A' & \xleftarrow[g']{} & C' \end{array} \tag{90}$$

with rows in $\mathbf{C}$. By the factorisation axiom we may assume that the objects on the bottom row are fibrant and by the axiom on resolutions we may also assume that the objects on the top row are projectives. An application of 6.5 shows that we can also assume that the vertical arrows are in $\mathbf{C}$ and the diagram commutes in $\mathbf{C}$ up to homotopy. Now consider the left square in (90) and let $h : af \approx f'b$ be a homotopy. Construct the diagram

$$\begin{array}{ccccc} B & \overset{f_1}{\dashrightarrow} & A \underset{A'}{\times} PA' & \overset{p_0'}{\underset{\sim}{\twoheadrightarrow}} & A \\ {\scriptstyle b_1}\vdots & \searrow^{h} & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a} \\ B \underset{A'}{\times} PA' & \xrightarrow[f_1']{} & PA' & \underset{p_0}{\overset{\sim}{\twoheadrightarrow}} & A' \\ {\scriptstyle p_1'}\wr\downarrow & & \wr\downarrow{\scriptstyle p_1} & & \\ B' & \xrightarrow[f']{} & A' & & \end{array} \tag{91}$$

as follows. First form the pullbacks of a and f' along the projections p_0 and p_1 of the cylinder PA'. Then use the homotopy h to factor $f = p_0' f_1$ and $b = p_1' b_1$ on the pullbacks. Do the same for the right square in diagram (90), to obtain a commutative diagram in $\mathbf{C}$

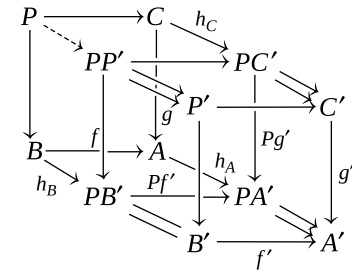
$$\begin{array}{ccccc} B & \xrightarrow{f_1} & A \underset{A'}{\times} PA' & \xleftarrow{g_1} & C \\ \downarrow{\scriptstyle b_1} & & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle c_1} \\ B \underset{A'}{\times} PA' & \xrightarrow[f_1']{} & PA' & \xleftarrow[g_1']{} & C' \underset{A'}{\times} PA' \end{array} \tag{92}$$

The triple $(1, p_0', 1)$ is an isomorphism in $\mathbf{C}_{\mathscr{E}}^{\lrcorner}$ from the top row of diagram (92) to the top row of diagram (90). Similarly, (p_1', p_1, p_1') is an isomorphism between the bottom rows. Since $p_0 = p_1$ in $\mathbf{C}_{\mathscr{E}}$, these isomorphism identify the vertical arrows of the two diagrams in $(\mathbf{C}_{\mathscr{E}})^{\lrcorner}$.

To prove the last claim, suppose we have arrows $s, t : G \rightrightarrows F$ in $\mathbf{C}_{\mathscr{E}}^{\lrcorner}$ which are identified by J. We do some reductions. First, using the functorial factorisation of the model, we may assume that F and G are fibrant. Second, we may assume $s, t \in \mathbf{C}^{\lrcorner}$. To see this, observe that the representation of s as a right fraction (27) provides a trivial fibration $H \overset{\sim}{\twoheadrightarrow} G$—the vertical arrow of the fraction—such that the composite $H \twoheadrightarrow F$ is in $\mathbf{C}^{\lrcorner}$. If $K \overset{\sim}{\twoheadrightarrow} G$ is the analogue for t, we can then use the composites $H \times_G K \overset{\sim}{\twoheadrightarrow} G \rightrightarrows F$ as replacements for s and t. With these reductions we can assume that our original data amounts to a commutative diagram in $\mathbf{C}$

$$\begin{array}{ccccc} B & \xrightarrow{f} & A & \xleftarrow{g} & C \\ {\scriptstyle s_B}\downarrow\downarrow{\scriptstyle t_B} & & {\scriptstyle s_A}\downarrow\downarrow{\scriptstyle t_A} & & {\scriptstyle s_C}\downarrow\downarrow{\scriptstyle t_C} \\ B' & \xrightarrow[f']{} & A' & \xleftarrow[g']{} & C' \end{array} \tag{93}$$

whose rows are fibrant objects in $\mathbf{C}^{\lrcorner}_{\mathscr{E}\lrcorner}$ and whose vertical parallel pairs are identified in $\mathbf{C}_{\mathscr{E}}$. By 6.4, every parallel pair is equalised up to homotopy by an acyclic fibration, and by stability we may patch these together to provide a pointwise acyclic fibration in $\mathbf{C}^{\lrcorner}$ which equalises the parallel pairs up to pointwise homotopy on the functorial path objects. Thus, we can assume that the parallel pairs in (93) are pointwise homotopic. Writing $h_{\_}$ for the homotopies, we can replace the previous diagram by



(94)

where the diagonal composites are the factorisations of the parallel pairs of (93) via the homotopies, and the vertical squares are homotopy pullbacks as defined in Section 5. These squares commute up to homotopy. The other faces of the front cube commute in $\mathbf{C}$, whereas the right and bottom faces of the back cube commute only in $\mathbf{C}_{\mathscr{E}}$. Now replace P by a projective using a resolution; the arrows from P to PA' via PC' and PB' coincide in the localisation, hence are homotopic by 6.5. By the universal property of the fibrant homotopy pullback, there exists a unique dashed arrow making the top and left faces of the back cube commutative. Uniqueness proves that the composites $P \to PP' \rightrightarrows P'$ are the arrows induced by s and t between the homotopy pullbacks. Since the parallel pairs in the front cube coincide in $\mathbf{C}^{\lrcorner}_{\mathscr{E}\lrcorner}$, the arrows $PP' \rightrightarrows P'$ coincide in $\mathbf{C}_{\mathscr{E}}$. Hence the induced arrows $P \rightrightarrows P'$ coincide in $\mathbf{C}_{\mathscr{E}}$. □

Let us return to the definition of the homotopy pullbacks in the localisation: call a commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & A \end{array} \tag{95}$$

in $\mathbf{C}_{\mathscr{E}}$ a *homotopy pullback* if it is isomorphic to a homotopy pullback diagram. Proposition 8.3 shows that f and g determine the homotopy pullback up to isomorphism in the localisation and that this becomes a functor when fibrant replacements in $\mathbf{C}^{\lrcorner}_{\mathscr{E}\lrcorner}$ are chosen. Note that this functor is not defined by an adjointness condition, so it does not define pullbacks in $\mathbf{C}_{\mathscr{E}}$.

Lemma 8.4 (The pasting lemma). *If both squares in the diagram below are homotopy pullbacks in* $\mathbf{C}_{\mathscr{E}}$, *so is the outer rectangle.*

$$\begin{array}{ccccc} Q & \longrightarrow & P & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & C & \longrightarrow & A \end{array} \tag{96}$$

Proof. Using the functorial factorisation, we may assume that all the objects in the diagram are fibrant. By definition of homotopy pullback in $\mathbf{C}_{\mathscr{E}}$, we may also assume that all the arrows in the right square are fibrations and that the square is an ordinary pullback. By 6.4, we may assume that $D \to C$ is a fibration. Since the left square is a homotopy pullback and so is $P \times_C D$, 8.3 provides an isomorphism $Q \simeq P \times_C D$ in $\mathbf{C}_{\mathscr{E}}$ commuting with the projections. But now, the pasting lemma for ordinary pullbacks shows that $Q \simeq B \times_A D$, which is a homotopy pullback. □

9. The unstable structure

In this section we consider a pointed category $\mathbf{C}$ equipped with a fibration model for a class of equivalences $\mathscr{E}$ and analyse the homological structure of the localisation $\mathbf{C}_{\mathscr{E}}$. In particular we describe fibration sequences and the induced triangulated structure.

Observe that the localisation is also pointed and define the loop functor on $\mathbf{C}_{\mathscr{E}}$ using the homotopy pullback

$$\begin{array}{ccc} \Omega A & \dashrightarrow & 1 \\ \vdots & & \downarrow \\ 1 & \longrightarrow & A \end{array} \tag{97}$$

When A is fibrant, this definition coincides with the one given in Brown [3, Theorem 4.3]. The corresponding results therefore hold also in our framework. A *fibration sequence* is a sequence $F \to E \to B$ in $\mathbf{C}_{\mathscr{E}}$ such that the diagram

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & B \end{array} \tag{98}$$

is a homotopy pullback.

Proposition 9.1. *Every fibration sequence* $F \to E \to B$ *induces a fibration sequence* $\Omega B \to F \to E$.

Proof. Consider the diagram

$$\begin{array}{ccccc} \Omega B & \dashrightarrow & F & \longrightarrow & 1 \\ \vdots\downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & E & \longrightarrow & B \end{array} \tag{99}$$

The right square is a homotopy pullback by assumption. Now form the homotopy pullback on the left. By the pasting Lemma 8.4, the outer rectangle is a homotopy pullback, hence the top left object can be identified with the loop object on B. By construction, the left square defines a fibration sequence. □

The construction can be iterated as shown in the diagram below.

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & & & \\ & & \downarrow & & \downarrow & & & & \\ \cdots & \longrightarrow & \Omega F & \longrightarrow & \Omega E & \longrightarrow & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & \longrightarrow & \Omega B & \longrightarrow & F & \longrightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & 1 & \longrightarrow & E & \longrightarrow & B \end{array} \tag{100}$$

Instead of using the structure given by fibration sequences it is possible, as already remarked by Quillen, to consider the induced triangulated structure. Since the axioms for a fibration model are weaker than the axioms for a Quillen model, we will only construct an unstable triangulated structure. The reader is referred to Margolis [11] for the relevant definitions. Define an unstable triangle in $\mathbf{C}_{\mathscr{E}}$ to be a diagram

$$\begin{array}{ccc} F & \longrightarrow & E \\ & \nwarrow \quad \swarrow & \\ & B & \end{array} \tag{101}$$

where the dashed arrow has degree -1 and the two composable pairs are fibration sequences. We also write

$$\Omega B \to F \to E \to B \tag{102}$$

to indicate an unstable triangle.

Proposition 9.2. *The unstable triangles provide $\mathbf{C}_{\mathscr{E}}$ with the structure of an unstable triangulation.*

Proof. Repleteness of unstable triangles follows from repleteness of homotopy pullbacks in $\mathbf{C}_{\mathscr{E}}$. Identities. For every object X, the triangle

$$1 \longrightarrow X = X \longrightarrow 1 \tag{103}$$

is an unstable triangle. This follows from the homotopy pullback diagrams

$$\begin{array}{ccccc} \Omega 1 & \longrightarrow & X & \longrightarrow & 1 \\ \downarrow & & \| & & \| \\ 1 & \longrightarrow & X & \longrightarrow & 1 \end{array} \tag{104}$$

and the observation that $\Omega 1 \simeq 1$. Rotation is an immediate consequence of Proposition 9.1. Extension. Given the solid part of the commutative diagram

$$\begin{array}{ccccccc} \Omega B' & \longrightarrow & F' & \longrightarrow & E' & \longrightarrow & B' \\ {\scriptstyle \Omega b}\downarrow & & \vdots {\scriptstyle f} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle b} \\ \Omega B & \longrightarrow & F & \longrightarrow & E & \longrightarrow & B \end{array} \tag{105}$$

define f using functoriality of the homotopy pullback on the localisation. Commutativity of the left square follows from commutativity of the corresponding fibrant replacements. □

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