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Cohomological dimension of Markov compacta

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Abstract

We rephrase Gromov's definition of Markov compacta, introduce a subclass of Markov compacta defined by one building block and study cohomological dimensions of these compacta. We show that for a Markov compactum X, $\dim_{\mathbf{Z}_{(p)}} X = \dim_{\mathbf{Q}} X$ for all but finitely many primes p where $\mathbf{Z}_{(p)}$ is the localization of $\mathbf{Z}$ at p. We construct Markov compacta of arbitrarily large dimension having $\dim_{\mathbf{Q}} X = 1$ as well as Markov compacta of arbitrary large rational dimension with $\dim_{\mathbf{Z}_p} X = 1$ for a given p.

MSC: primary 47A15; secondary 46A32, 47D20

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1. Introduction

1.1. Markov compacta

Let T be a rooted locally finite simplicial tree with the root $x_0 \in T$. For every vertex $x \in T$ by T_x we denote the subtree rooted at x, i.e. the tree with the vertices y such that the segment $[x_0, y]$ contains x. Gromov calls the tree T *Markov* [7] if there are only finitely many (say, k) isomorphism classes of rooted trees T_x. The name *Markov* is given since the $k \times k$ transition matrix $M = (m_{ij})$ defines a Markov chain where m_{ij} is the number of vertices of the type j neighboring the root in a tree of type i.

A rooted tree can be viewed as the telescope of an inverse sequence of finite spaces $S = \{K_i, \phi_i^{i+1}\}$ with $K_0 = x_0$, $|K_i| < \infty$. We call two points $x \in K_i$ and $y \in K_j$ equivalent if the inverse sequences

$$S_x = \{x \leftarrow (\phi_i^{i+1})^{-1}(x) \leftarrow (\phi_i^{i+2})^{-1}(x) \leftarrow \cdots\}$$

and

$$S_y = \{y \leftarrow (\phi_j^{j+1})^{-1}(y) \leftarrow (\phi_j^{j+2})^{-1}(y) \leftarrow \cdots\}$$

are isomorphic. Then Gromov's definition can be translated as follows: An inverse sequence of finite spaces $\{K_i, \phi_i^{i+1}\}$ is called *Markov* if this equivalence relation on $\coprod_i K_i$ has finitely many classes.

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This notion can be extend for sequences of higher dimensional polyhedra.

Definition 1.1. Let $S = \{K_i, \phi_i^{i+1}\}$ be an inverse sequence of simplicial complexes. We call this system *Markov* if the following equivalence relation on the set of all simplices in $\coprod K_i$ has finitely many classes: Two simplices $\sigma \subset K_i$ and $\sigma' \subset K_j$ are equivalent if the inverse sequences

$$S_\sigma = \left\{\sigma \leftarrow \left(\phi_i^{i+1}\right)^{-1}(\sigma) \leftarrow \left(\phi_i^{i+2}\right)^{-1}(\sigma) \leftarrow \cdots\right\}$$

and

$$S_{\sigma'} = \left\{\sigma' \leftarrow \left(\phi_j^{j+1}\right)^{-1}(\sigma') \leftarrow \left(\phi_j^{j+2}\right)^{-1}(\sigma') \leftarrow \cdots\right\}$$

are isomorphic.

A compactum X is called *Markov* if it can be presented as the limit of a Markov inverse system.

Note that Markov inverse system consists of complexes of uniformly bounded dimension. Hence Markov compacta are always finite-dimensional. Note that in a Markov system we have only finitely many homeomorphism types of preimages of simplices $(\phi_i^{i+1})^{-1}(\sigma)$, we call them *building blocks*. Informally, Gromov defined Markov compacta as those which can be built using finitely many building blocks. The Pontryagin surface Π_2 is a typical example of Markov compactum which is constructed from one building block. We recall the construction. Let $f : M \to \Delta$ be a map of the Möbius band M onto a 2-simplex which is the identity on the boundary. Fix a sufficiently small triangulation on M. Take a triangulation of a 2-sphere and replace all its 2-simplices σ by M by means of an identification $\partial\sigma \cong \partial M$. The resulting space is supplied with natural projection onto M glued out of maps f. Then apply this procedure to the resulting space and so on. We obtain an inverse system of polyhedra. The space Π_2 is the limit space of this inverse sequence. Here the building block is the Möbius band.

1.2. One building block compacta

Here we introduce a subclass of Markov compacta whose inverse sequences can be obtained from one building block in some uniform fashion.

A map $f : X \to Y$ is called *light* if the preimages of all points $f^{-1}(y)$ are at most 0-dimensional. We call a simplicial n-dimensional complex K *a complex over an* (*oriented*) *n-simplex* Δ^n if there is a light simplicial map $\chi : K \to \Delta^n$ (called a *characteristic map*). We denote by βK the barycentric subdivision of a simplicial complex K. Note that βK is a complex over Δ^n with the characteristic map $\chi : \beta K \to \Delta^n$ defined on the vertices of βK as follows: $\chi(b_\sigma) = e_{\dim\sigma}$ where b_σ denotes the barycenter of a simplex $\sigma \subset K$ and $e_0, \ldots, e_n$ are the vertices of Δ^n. The following proposition is obvious.

Proposition 1.2. *Suppose that in the pull-back diagram*

$$\begin{array}{ccc} K' & \longrightarrow & L \\ \phi\downarrow & & \downarrow f \\ K & \xrightarrow{\chi} & N \end{array}$$

the map f is simplicial and χ is light simplicial. Then K' is a simplicial complex and ϕ is simplicial map.

A triangulation τ of the simplex Δ^n is called *symmetric* if it is invariant under the natural symmetric group action on Δ^n. Note that a symmetric triangulation τ on Δ^n induces a triangulation τ_K for every complex K, $\chi : K \to \Delta^n$, over the simplex Δ^n.

Let $f : L \to \Delta^n$ be a simplicial map of a finite complex L onto the n-simplex Δ^n taken with a symmetric triangulation τ. Let K_0 be a complex over n-simplex with the characteristic map $\chi_0 : K_0 \to \Delta^n$. By induction we construct the following inverse sequence $\{K_i, \phi_i^{i+1}\}$ of simplicial complexes over Δ^n with simplicial bonding maps $\phi_i^{i+1} : K_{i+1} \to K_i$ with respect to some subdivision of the triangulation on K_i.

Assume that $\chi_i : K_i \to \Delta^n$ is constructed. We define K_{i+1} as the pull-back of the diagram

$$\begin{array}{ccc} K_{i+1} & \xrightarrow{\xi_{i+1}} & L \\ \phi^{i+1}\downarrow & & \downarrow f \\ K_i & \xrightarrow{\chi_i} & \Delta^n \end{array}$$

The map χ is simplicial with respect the triangulation τ on Δ^n and the induced triangulation τ_{K_i}. In view of Proposition 1.2 K_{i+1} is a simplicial complex and the map ϕ_i^{i+1} is simplicial with respect to the triangulation τ_{K_i} on K_i. We set the triangulation on K_{i+1} to be the first barycentric subdivision of K_{i+1}. Then there is a natural characteristic map $\chi_{i+1} : K_{i+1} \to \Delta^n$. The bonding map $\phi_i^{i+1} : K_{i+1} \to K_i$ is simplicial with respect to $\beta\tau_{K_i}$.

Definition 1.3. The limit space X of an inverse sequence $\{K_i, \phi_i^{i+1}\}$ of complexes over the n-simplex Δ^n defined above is called a *compactum defined by the building block* $f : L \to \Delta^n$.

Proposition 1.4. *Every compactum defined by one building block is Markov.*

Proof. Let X be the limit space of an inverse sequence $\{K_i, \phi_i^{i+1}\}$ of complexes over the n-simplex Δ^n from the definition of compactum with one building block and let $f : L \to \Delta^n$ be the building block. We note that simplices $\sigma_1 \subset K_i$ and $\sigma_2 \subset K_j$ are equivalent (see Definition 1.1) if $\chi_i(\sigma_1) = \chi_j(\sigma_2)$ where $\chi_i : K_i \to \Delta^n$ and $\chi_j : K_j \to \Delta^n$ are the characteristic maps. □

In the case of the Pontryagin surface Π_2 we take L to be the Möbius band viewed as the mapping cylinder M_g of a 2-fold covering map $g : S^1 \to S^1$. We present the domain of g as a 6-gon $S \simeq S^1$ and the range as a triangle $T \simeq S^1$. Then we take g simplicial. On the mapping cylinder of any simplicial map always there is a triangulation on with no extra vertices. We take such a triangulation on L and define a simplicial map $f : L \to \beta\Delta^2$ by an isomorphism taking S onto $\beta(\partial\Delta^2)$ and by collapsing T to the barycenter b_2 of Δ^2. Let K_0 be a 2-sphere with a structure of a complex over the 2-simplex. Then K_0 and $f : L \to \Delta^2$ define a compactum which is the Pontryagin surface Π_2.

We recall that the Pontryagin surface Π_2 is 2-dimensional with the rational dimension $\dim_{\mathbf{Q}} \Pi_2 = 1$. Mladen Bestvina asked me if there are Markov compacta of dimension n with rational dimension one for arbitrary large n. Here we answer his question and give an account of the cohomological dimension theory of Markov compacta.

1.3. Cohomological dimension

Here is the summary of the cohomological dimension theory of compacta (see [8,3,4]). The cohomological dimension of a space X with coefficient group G is defined as follows:

$$\dim_G X = \sup\{n \mid \check{H}^n(X, A; G) \neq 0 \text{ for some closed subset } A \subset X\}.$$

It is known for compact metric spaces that $\dim_G X \leqslant n$ if and only if the inclusion homomorphism $\check{H}^n(X; G) \to \check{H}^n(A; G)$ is an epimorphism for every closed subset $A \subset X$. The later is equivalent to the condition that for every closed subset $A \subset$, every continuous map $\phi : A \to K(G, n)$ to the Eilenberg–MacLane complex has a continuous extension $\bar{\phi} : X \to K(G, n)$. By Bockstein theorem to know the cohomological dimension of a compact space X with respect to any Abelian group it suffices to know it with respect to the so-called *Bockstein groups* which are $\mathbf{Q}$, $\mathbf{Z}_{(p)}$, $\mathbf{Z}_p$ and $\mathbf{Z}_{p^\infty}$ where p runs over all primes. Here $\mathbf{Z}_{(p)}$ is a localization of integers at p, $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$ and $\mathbf{Z}_{p^\infty} = \lim_{to} \mathbf{Z}_{p^k}$. In particular, $\dim_{\mathbf{Z}} X = \sup\{\dim_{\mathbf{Z}_{(p)}} X\}$. The cohomological dimension of compacta with respect to Bockstein's groups is subject to restriction given by Bockstein Inequalities:

$$\dim_{\mathbf{Z}_p} X - 1 \leqslant \dim_{\mathbf{Z}_{p^\infty}} X \leqslant \dim_{\mathbf{Z}_p} X;$$

$$\max\{\dim_{\mathbf{Q}} X, \dim_{\mathbf{Z}_p} X\} \leqslant \dim_{\mathbf{Z}_{(p)}} X \leqslant \max\{\dim_{\mathbf{Q}} X, \dim_{\mathbf{Z}_{p^\infty}} X + 1\};$$

$$\dim_{\mathbf{Z}_{p^\infty}} X \leqslant \max\{\dim_{\mathbf{Q}} X, \dim_{\mathbf{Z}_{(p)}} X - 1\}.$$

Let σ denote the set of all Bockstein groups. There is Realization Theorem [3,4], which states that for every function $\beta : \sigma \to \mathbf{N}$ satisfying the Bockstein inequalities there is a compact metric space X with $\dim_G X = \beta(G)$. A compactum X is called *p-regular* if

$$\dim_{\mathbf{Q}} X = \dim_{\mathbf{Z}_{p^\infty}} X = \dim_{\mathbf{Z}_p} X = \dim_{\mathbf{Z}_{(p)}} X.$$

For a p-singular compactum X the Bockstein inequalities split into the following equality

$$\dim_{\mathbf{Z}_{(p)}} X = \max\{\dim_{\mathbf{Q}} X, \dim_{\mathbf{Z}_{p^\infty}} X + 1\}$$

and the inequalities:

$$\dim_{\mathbf{Z}_p} X - 1 \leqslant \dim_{\mathbf{Z}_{p^\infty}} X \leqslant \dim_{\mathbf{Z}_p} X.$$

It is natural to suggest that Markov compacta could be p-singular for only finitely many p. In this paper we prove that $\dim_{\mathbf{Z}_{(p)}} X = \dim_{\mathbf{Q}} X$ for Markov compacta for all but finitely many primes p.

1.4. Operations

Clearly, the disjoint union and the product of two Markov compacta are Markov. It is not the case for compacta defined by one block. Nevertheless one can define these operation for such compacta.

Let X and Y be compacta defined by building blocks $f : L \to \Delta^n$ and $g : N \to \Delta^m$, $n \geqslant m$, such that there is an inclusion of the initial complexes $K_0 \supset K_0'$. Let $j : \Delta^m \to \Delta^n$ be the inclusion of the first m-face. Then one can define their "sum" $X \# Y$ as the compactum generated by the building block $f \cup (j \circ g) : L \amalg N \to \Delta^n$. This operation is most interesting when $n = m$ and $K_0 = K_0' = \Delta^n$.

The standard triangulation on the product of oriented simplices $\Delta^n \times \Delta^m$ turns $\Delta^n \times \Delta^m$ into the complex over Δ^{n+m}. Let $\chi : \Delta^n \times \Delta^m \to \Delta^{m+n}$ be its characteristic map. Then we define the "product" $X \tilde{\times} Y$ of compacta X and Y as the compactum defined by the building block $\chi \circ (f \times g) : L \times N \to \Delta^{n+m}$ and the initial complex $K_0 \times N_0$.

1.5. Open problems

Naturally, the compacta defined by one building block should have a fractal structure. We recall that a compact set $F \subset \mathbf{R}^N$ is called "self-similar" if there are finitely many similarities $h_i : \mathbf{R}^N \to \mathbf{R}^N$ with the similarity coefficients $r_i < 1$ such that $F = \bigcup_i h_i(F)$ [6]. Question: *Is every compactum defined by one building block homeomorphic to self-similar subsets of* $\mathbf{R}^N$?

For compacta X generated by one building block $f : L \to \Delta^n$ it would be nice to obtain a formula for cohomological dimension or even the formula for cohomology of X in terms of f in spirit of those for Coxeter groups in terms of the nerve of Coxeter system [1,5,2].

2. Cohomological dimension of one building block compacta

2.1. Restrictions on cohomological dimensions of Markov compacta

Let $\phi : K \to K'$ be and let $A \subset N \subset K'$. We consider the following condition:

$$(*)^m_G \quad \operatorname{im}(f|_{f^{-1}(A)})^* \subset \operatorname{im}\bigl\{H^m\bigl(f^{-1}(N); G\bigr) \to H^m\bigl(f^{-1}(A); G\bigr)\bigr\}.$$

We will use the notation $\phi \in (*)^m_G$ for saying that ϕ satisfies $(*)^m_G$ (for a certain pair (N, A)).

Easy diagram chasing yields the following.

Proposition 2.1. *Let $\phi : K \to \Delta^m$ be a map to the m-simplex. Then $\phi \in (*)^{m-1}_G$ for the pair $(\Delta^m, \partial\Delta^m)$ if and only if the homomorphism $\phi^* : H^m(\Delta^m, \partial\Delta^m; G) \to H^m(K, \phi^{-1}(\partial\Delta^m); G)$ is nonzero.*

Lemma 2.2. *Let $X = \varprojlim\{K_i; \phi_i^{i+1}\}$ be a Markov compactum with $\dim_G X \leqslant m$ for a principle ideal domain G. Then for every $l \in \mathbf{N} \cup \{0\}$ there is k such that the inclusion $(*)^m_G$ holds for the map ϕ_i^{i+k} for all $i \geqslant l$ for all pairs $(\phi^i_{i-l})^{-1}(\sigma, \partial\sigma)$ where σ is an arbitrary simplex in K_{i-l}.*

Proof. Let X be the limit of a Markov inverse sequence $\{K_i, \phi_i^{i+1}\}$. Let $\sigma_i \subset K_i$ and $\sigma_2 \subset K_j$ be two equivalent in the sense of Definition 1.1 simplices. Then we have homeomorphic pairs

$$\left(\left(\phi_i^{i+k}\right)^{-1}(\sigma_1), \left(\phi_i^{i+k}\right)^{-1}(\partial\sigma_1)\right) \quad \text{and} \quad \left(\left(\phi_j^{j+k}\right)^{-1}(\sigma_2), \left(\phi_j^{j+k}\right)^{-1}(\partial\sigma_2)\right)$$

for $k = 0, 1, \ldots, \infty$. We take one representative $\sigma \subset K_i$ for each equivalence class. By the definition of Markov compactum there are only finitely many of them. Since $A = (\phi_i^{i+1})^{-1}(\partial\sigma)$ is a finite complex, the G-module $H^m(A; G)$ is finitely generated. Let $\{a_1, \ldots, a_s\}$ be a generating set. Since $\dim_G X \leqslant m$, the inclusion $(\phi_{i+l}^{\infty})^{-1}(A) \subset (\phi_{i+l}^{\infty})^{-1}(N)$ induces an epimorphism for m-dimensional cohomology with coefficients in G where $N = (\phi_i^{i+1})^{-1}(\sigma)$. For every j there is an element $b_j \in \check{H}^m((\phi_{i+l}^{\infty})^{-1}(N); G)$ which goes to $(\phi_{i+l}^{\infty})^*(a_j)$ under this inclusion homomorphism. From the definition of Čech cohomology it follows that there is k_j such that $(\phi_{i+l}^{i+l+k_j})^*(a_j)$ lies in the image of the homomorphism induced by inclusion

$$\left(\phi_{i+l}^{i+l+k_j}\right)^{-1}(A) \subset \left(\phi_{i+l}^{i+l+k_j}\right)^{-1}(N).$$

We take k greater than every k_j for all equivalence classes. □

The converse to Lemma 2.2 is true in the following form.

Lemma 2.3. *Suppose that a compact X is presented as the inverse limit of the sequence of n-dimensional polyhedra $\{K_i, \phi_i^{i+1}\}$ supplied with triangulations τ_i such that for every j*

$$\lim_{i\to\infty} \operatorname{mesh}\left(\phi_j^{j+i}(\tau_{j+i})\right) = 0.$$

Assume that for every l there is k such that the inclusion $(*)_G^m$ holds for the maps ϕ_i^{i+k} for all $i \geqslant l$ for all pairs $(\phi_{i-l}^i)^{-1}(\sigma, \partial\sigma)$ where σ is a simplex in K_{i-l}. Then $\dim_G X \leqslant m$.

Proof. We show that given a continuous map $f : Y \to K(G, m)$ of a closed subset $Y \subset X$ there is a continuous extension $\bar{f} : X \to K(G, m)$. Since $K(G, m)$ is an ANE, there is i_0 and a map $f' : W \to K(G, m)$ of subcomplex $W \subset K_{i_0}$ which contains $\phi_{i_0}^{\infty}(A)$ such that the composition $f' \circ \phi_{i_0}^{\infty}|_A$ is homotopic to f. Here we used the condition that the mesh of triangulations on K_i tends to zero. In view of the Homotopy Extension Theorem it suffices to extend the map

$$g = f' \circ \phi_i^s|_{(\phi_i^s)^{-1}(W)} : \left(\phi_i^s\right)^{-1}(W) \to K(G, m)$$

to K_s for some s. Since $K(G, m)$ is $(m-1)$-connected, there is an extension $f_m : W \cup (K_{i_0})^{(m)} \to K(G, m)$. By induction on i we define a number n_i and construct a map

$$f_{m+i} : \left(\phi_{i_0}^{n_i}\right)^{-1}\left(W \cup (K_{i_0})^{m+i}\right) \to K(G, m)$$

such that $n_i \geqslant n_{i-1}$ and f_{m+i} extends the map

$$f_{m+i-1} \circ \phi_{n_{i-1}}^{n_i}|_{(\phi_{i_0}^{n_i})^{-1}(W \cup (K_{i_0})^{m+i-1})}.$$

Assume that f_{m+i-1} is already constructed. We take k for $l = n_{i-1} - i_0$ from the condition of lemma and define $n_i = n_{i-1} + k$. For every $(m+i)$-dimensional simplex σ in $K_{i_0} \setminus W$ we consider the pair $(\phi_{i_0}^{n_{i-1}})^{-1}(\sigma, \partial\sigma)$. By the condition $(*)_G^m$ there is an extension $\psi : (\phi_{i_0}^{n_i})^{-1}(\sigma) \to K(G, m)$ of the map $f_{m+i-1} \circ \phi_{n_{i-1}}^{n_i}|_A$. The union of these extensions for all σ together with

$$f_{m+i-1} \circ \phi_{n_{i-1}}^{n_i}|_{(\phi_{i_0}^{n_i})^{-1}(W \cup (K_{i_0})^{m+i-1})}$$

define f_{m+i}. Now the map f_n is an extension of the above map g (for some s). □

Theorem 2.4. *For every Markov compactum X there are only finitely many primes $p_1, \ldots, p_m$ such that* $\dim_{\mathbf{Z}_{(p_i)}} X \neq \dim_{\mathbf{Q}} X$.

Proof. Let $X = \varprojlim\{K_i, \phi_i^{i+1}\}$ be a presentation of X from the definition of Markov compacta and let $\dim_{\mathbf{Q}} X = n$. Let $k = k(l)$ be from Lemma 2.1. Thus, the condition $(*)^n_{\mathbf{Q}}$ holds for ϕ_i^{i+k} with $(\phi^i_{i-l})^{-1}(\sigma, \partial\sigma)$ for all simplices σ in K_{i-l} for all $i \geqslant l$. By the definition of Markov compacta there are finitely many isomorphism types of simplicial complexes in the family $(\phi^{i+k}_{i-l})^{-1}(\sigma)$, $i \in \mathbf{N} \cup \{0\}$, $\sigma \subset K_i$. Since all this complexes are finite, there is r_0 such that for every prime $p > r_0$ the condition $(*)^n_G$ holds for $G = \mathbf{Z}_{(p)}$ for the pair $(\phi^{i+k}_{i-l})^{-1}(\sigma, \partial\sigma)$ for every simplex σ in K_{i-l} for all $i \geqslant l$. Lemma 2.3 implies the inequality $\dim_{\mathbf{Z}_{(p)}} X \leqslant n$ for $p > r_0$. In view of Bockstein inequality $\dim_{\mathbf{Q}} \leqslant \dim_{\mathbf{Z}_{(p)}}$ we obtain $\dim_{\mathbf{Z}_{(p)}} X = n$ for $p > r_0$. □

2.2. Cohomological dimension of a complex over a simplex

Definition 2.5. Let $f : L \to \Delta^n$ be a map and let G be an Abelian group. We define the *cohomological dimension* $\mathrm{cd}_G f$ of a map f with respect to the coefficient group G to be the minimal m such that $f \in (*)^m_G$ for all pairs $(\sigma, \partial\sigma)$ where $\sigma \subset \Delta^n$ is a subsimplex. We define *the upper cohomological dimension* $\overline{\mathrm{cd}}_G f$ of a map f with respect to the coefficient group G to be the minimal m such that the inclusion homomorphism $H^m(f^{-1}(\sigma); G) \to H^m(f^{-1}(\partial\sigma); G)$ is an epimorphism.

Clearly, $\mathrm{cd}_G f \leqslant \overline{\mathrm{cd}}_G f$.

Proposition 2.1 implies the following.

Proposition 2.6. *Let $f : L \to \Delta^n$ be a map and let G be an Abelian group. Then $\mathrm{cd}_G f$ is the maximal k such that*

$$f^* : H^k(\sigma^k, \partial\sigma^k; G) \to H^k(f^{-1}(\sigma^k), f^{-1}(\partial\sigma^k); G)$$

is nonzero for some k-face $\sigma^k \subset \Delta^n$.

Theorem 2.7. *Let X be a compactum defined by the building block $f : L \to \Delta^n$. Then $\dim_G X \leqslant \overline{\mathrm{cd}}_G f$.*

Proof. The proof is similar to the proof of Lemma 2.3(1). Let $\overline{\mathrm{cd}}_G f = m$. Given a continuous map $\psi : Y \to K(G, m)$ of a closed subset $Y \subset X$, we construct a continuous extension. We may assume that there is i and a subcomplex $A_i \subset K_i$ together with a map $g : A_i \to K(G, m)$ such that $\phi_i^\infty(A) \subset A_i$ and $g \circ \phi_i^\infty|_A$ homotopic to f. Let

$$g' : A_i \cup (K_i)^{(m)} \to K(G, m)$$

be a continuous extension. By the condition $\overline{\mathrm{cd}}_G f \leqslant m$ we may assume that for every $(m+1)$-simplex σ in $K_i \setminus A_i$ there is an extension

$$g^{m+1}_\sigma : (\phi_i^{i+1})^{-1}(\sigma) \to K(G, m)$$

of the map

$$g' \circ \phi_i^{i+1}|_{(\phi_i^{i+1})^{-1}(\partial\sigma)} : (\phi_i^{i+1})^{-1}(\partial\sigma) \to K(G, m).$$

The union of $\bigcup_\sigma g^{m+1}_\sigma$ together with the composition $g' \circ (\phi_i^{i+1})^{-1}|_{(\phi_i^{i+1})^{-1}(A_i)}$ defines an extension $g^{m+1} : K^{m+1}_{i+1} \cup (\phi_i^{i+1})^{-1}(A_i) \to K(G, m)$ of the map

$$g \circ (\phi_i^{i+1})^{-1}|_{(\phi_i^{i+1})^{-1}(A_i)}.$$

Then for every $m+2$-simplex σ in K_i there is an extension

$$g^{m+2}_\sigma : (\phi_i^{i+1})^{-1}(\sigma) \to K(G, m)$$

of the map $g^{m+1}|_{(\phi_i^{i+1})^{-1}(\partial\sigma)}$. The union of $\bigcup_\sigma g^{m+2}_\sigma$ together with the map g^{m+1} defines a continuous map $g^{m+2} : K^{m+2}_{i+1} \cup (\phi_i^{i+1})^{-1}(A_i) \to K(G, m)$ extending $g \circ (\phi_i^{i+1})^{-1}|_{(\phi_i^{i+1})^{-1}(A_i)}$ and so on. Repeating this procedure we will obtain a map $g^n : K_{i+1} \to K(G, m)$ extending $g \circ (\phi_i^{i+1})^{-1}|_{(\phi_i^{i+1})^{-1}(A_i)}$. Hence $g^n \circ \phi^\infty_{i+1}$ is an extension of the map $g \circ \phi_i^\infty|_A$. By the Homotopy Extension Theorem the map ψ has an extension. □

Remark 2.8. The argument of Theorem 2.7 produces in fact a slightly better inequality:

$$\dim_G X \leqslant \max\{\mathrm{cd}_G f, \overline{\mathrm{cd}}_G f - 1\}.$$

2.3. Dimension over fields

Lemma 2.9. *Suppose that a map $f : L \to \Delta^n$ simplicial with respect to a symmetric triangulation τ on Δ^n induces an epimorphism*

$$f_* : H_n\big(L, f^{-1}(\partial\Delta^n); R\big) \to H_n\big(\Delta^n, \partial\Delta^n; R\big)$$

in the relative n-dimensional homology with the coefficients in a ring R with unit. Then for every light simplicial map $\chi : K \to \Delta^n$ and any subcomplex $A \subset K$ the induced homomorphism $\phi_* : H_n(Z, \phi^{-1}(A); R) \to H_n(K, A; R)$ is an epimorphism where Z is the pull-back in the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\xi} & L \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle f} \\ K & \xrightarrow{\chi} & \Delta^n \end{array}$$

Proof. We define homomorphisms $f!$ of the simplicial chain complexes with R-coefficients such that the diagram commutes $(*)$:

$$\begin{array}{ccc} C_n(L) & \xrightarrow{\partial} & C_{n-1}(L) \\ {\scriptstyle f!}\uparrow & & \uparrow{\scriptstyle f!} \\ C_n(\Delta^n) = R & \xrightarrow{\partial} & C_{n-1}(\partial\Delta^n) \end{array}$$

There is a relative n-cycle $z \in C_n(L)$ with the boundary $\partial z \in C_{n-1}(f^{-1}(\partial\Delta^n))$ such that $f_*([z]) = 1 \in R = H_n(\Delta^n, \partial\Delta^n; R)$ where $[z] \in H_n(L, f^{-1}(\partial\Delta^n); R)$ is the homology class of z. Note that on the chain level we have a homomorphism $f_* : C_n(L) \to C_n(\tau)$. We denote by $b : C_n(\Delta) \to C_n(\tau)$ the subdivision homomorphism. Since $f_*([z]) = [b(1 \cdot \Delta^n)]$, by dimensional reason there is only one element $b(1 \cdot \Delta^n)$ in the relative homology class $[b(1 \cdot \Delta^n)]$. Hence $f_*(z) = b(1 \cdot \Delta^n)$. We define $f!(\Delta) = z$. For an $(n-1)$-face $\sigma \subset \Delta^n$ we define $f!(\sigma) = \mathrm{pr}_\sigma(\partial z)$ where $\mathrm{pr}_\sigma : C_{n-1}(L) \to C_{n-1}(f^{-1}(\sigma))$ the natural projection. We define a homomorphisms $\phi! : C_*(K) \to C_*(Z)$ in dimensions n and $n-1$ on the generators $\sigma \subset K$ by the formula

$$\phi!(\sigma) = (\xi|_{\phi^{-1}(\sigma)})_*^{-1} f!\chi_*(\sigma).$$

The following diagram is commutative $(**)$:

$$\begin{array}{ccc} C_n(Z) & \xrightarrow{\partial} & C_{n-1}(Z) \\ {\scriptstyle \phi!}\uparrow & & \uparrow{\scriptstyle \phi!} \\ C_n(K) & \xrightarrow{\partial} & C_{n-1}(K) \end{array}$$

It suffices to check the commutativity on the generators, i.e. the equality $\partial\phi!(\sigma) = \phi!\partial(\sigma)$ for n-simplices $\sigma \subset K$ taken with the coefficient $1 \in R$. The equality holds since for every σ this diagram contains a copy of the commutative diagram $(*)$ with the identification $\sigma = \Delta^n$.

Let $v \in C_n(K)$ be a relative cycle, i.e., $\partial v \in C_{n-1}(A)$. From the commutativity of $(**)$ it follows that $\phi!(v)$ is a relative cycle with $\partial\phi!(v) \in C_{n-1}(\phi^{-1}(A))$. We note that $\phi_*(\phi!(v)) = b_K(v)$ where $b_K : C_n(K) \to C_n(\tau_K)$ is the subdivision homomorphism and τ_K is the triangulation on K induced from τ by means of the map χ. Then $\phi_*([\phi!(v)]) = [b_K(v)] = [v]$ for the relative homology classes. □

Lemma 2.10. *Let $X = \varprojlim\{K_i, \phi_i^{i+1}\}$ be a compactum defined by a building block $f : L \to \Delta^n$.*

(1) *Suppose the inequality $\mathrm{cd}_G f < n$ holds. Then $\dim_G X < n$.*
(2) *Let F be an additive group of a field and let $\mathrm{cd}_F f = n$. Then $\dim_F X = n$.*

Proof. We may assume that $K_0 = \Delta^n$.

(1) Follows from Remark 2.8.

(2) In view of Proposition 2.1 the homomorphism

$$f^*: H^n(\Delta^n, \partial\Delta^n; F) \to H^n(L, f^{-1}(L); F)$$

is nontrivial. Since F is a field, the dual homomorphism

$$f_*: H_n(L, f^{-1}(\partial\Delta^n); F) \to H_n(\Delta^n, \partial\Delta^n; F) = F$$

is nontrivial and hence, it is an epimorphism. Denote by $\partial X = (\phi_0^\infty)^{-1}(\partial\Delta^n)$. By induction using Lemma 2.9 we can construct a sequence

$$v_i \in H_n(K_i, (\phi_0^i)^{-1}(\partial\Delta^n); F)$$

such that $(\phi_{i-1}^i)_*(v_i) = v_{i-1}$ and $v_0 = 1 \in F = H_n(\Delta^n, \partial\Delta^n; F)$. Thus, we construct a nontrivial n-dimensional relative Čech F-homology class on $(X, \partial X)$. This implies that $H_n(X, \partial X; F) \neq 0$ for the Steenrod homology. By the Universal Coefficient Theorem over a field we obtain $\check{H}^n(X, \partial X; F) \neq 0$ for the Čech cohomology with F-coefficients. Hence $\dim_F X \geqslant n$, which contradicts to the assumption. □

We recall that for compact spaces X there are two possibilities for the dimension of the nth power:

$$\dim X^n = n \dim X \quad \text{for all } n \quad \text{or} \quad \dim X^n = (n-1)\dim X + 1.$$

We conjecture that all Markov compacta are of the first type. In the support of the conjecture we present the following.

Theorem 2.11. *Suppose that an n-dimensional compactum X is defined by a building block $f: L \to \Delta^n$ has dimension n. Then $\dim X^k = kn$ for all k.*

Proof. Let p be a prime such that $\dim_{\mathbf{Z}_{(p)}} X = n$. In view of Lemma 2.10(1) we have $\mathrm{cd}_{\mathbf{Z}_{(p)}} f = n$. This implies that

$$f^*: H^n(\Delta^n, \partial\Delta^n; \mathbf{Z}_{(p)}) \to H^n(L, f^{-1}(\partial\Delta^n); \mathbf{Z}_{(p)})$$

is nontrivial. If $\dim_{\mathbf{Q}} X = n$, the Theorem follows form the Kunneth formula over the field. So, we assume that $\dim_{\mathbf{Q}} X < n$. Then by Lemma 2.10(2) the inequality $\mathrm{cd}_{\mathbf{Q}} f < n$ holds. Hence

$$f^*: H^n(\Delta^n, \partial\Delta^n; \mathbf{Q}) \to H^n(L, f^{-1}(\partial\Delta^n); \mathbf{Q})$$

is a zero homomorphism. Hence the image of f^* with $\mathbf{Z}_{(p)}$-coefficient is a p-torsion group. By the Universal Coefficient Formula for $\mathbf{Z}_p$ as a module over $\mathbf{Z}_{(p)}$ we obtain that the homomorphism

$$f^*: H^n(\Delta^n, \partial\Delta^n; \mathbf{Z}_p) \to H^n(L, f^{-1}(\partial\Delta^n); \mathbf{Z}_p)$$

is nontrivial. Hence $\mathrm{cd}_{\mathbf{Z}_p} f = n$. By Lemma 2.10(2) $\dim_{\mathbf{Z}_p} X = n$. Then $kn = k\dim X \geqslant \dim X^k \geqslant \dim_{\mathbf{Z}_p} X^k = k\dim_{\mathbf{Z}_p} X = kn$. □

2.4. Symmetric building blocks

The group of all permutations on n elements is denoted by S_n. There is a natural action of S_{n+1} on the n-simplex Δ^n. A compactum defined by a building block $f: L \to \Delta^n$ is called *symmetric* if there is an action on L of the symmetric group S_{n+1} and the map f is S_{n+1}-equivariant.

Theorem 2.12. *Let X be a symmetric compactum with a building block $f: L \to \Delta^n$. Then for every field F there are the inequalities*

$$\mathrm{cd}_F f \leqslant \dim_F X \leqslant \overline{\mathrm{cd}}_F f.$$

Proof. In view of Theorem 2.7 it suffices to prove only the first inequality. Let $\mathrm{cd}_F\, f = m$. By Proposition 2.6 there is an m-face $\sigma \subset \Delta^n$ such that $\mathrm{cd}_F\, f|_{f^{-1}(\sigma)} = m$. Since f is symmetric, we may assume that σ is the first m-face $\Delta^m \subset \Delta^n$. Let Y be a compactum defined by the building block $f|_{f^{-1}(\sigma)} : f^{-1}(\sigma) \to \Delta^m$. We claim that there is an embedding $Y \subset X$. Let $\{K_i, \phi_i^{i+1}\}$ and $\{N_i, \psi_i^{i+1}\}$ be inverse systems for X and Y from the definition compacta generated by one building block. Without loss of generality we may assume that $K_0 = \Delta^n$ and $N_0 = \Delta^m$ with the identities as the characteristic maps. By induction we construct an embedding of inverse sequences $N_i \subset K_i$. The imbedding $\Delta^m \subset \Delta^n$ induces an imbedding $N_1 \subset K_1$. Since Δ^m is the first face and the characteristic maps on K_1 and N_1 are defined by means of the barycentric subdivision and the ordering of vertices of Δ^n and Δ^m, we have that the restriction $\chi_1|_{N_1}$ is the characteristic map for N_1. Therefore there is an embedding of $N_2 \subset K_2$ defined by the pull-back diagram from the definition of compacta defined by one building block, and so on.

By Lemma 2.10(2) $\dim_F Y = m$. Hence $\dim_F X \geqslant m$. □

Definition 2.13. Let $f : L \to \Delta^n$. A *symmetrization* of f is a map $\tilde{f} : L \times S_{n+1} \to \Delta^n$ defined by the formula: $\tilde{f}(x, s) = s(f(x))$.

It is easy to see that the map $\tilde{f}$ is S_{n+1}-equivariant with respect to the action on $L \times S_{n+1}$ generated by multiplication in S_{n+1} from the left and with natural action on Δ^n. The following is obvious.

Proposition 2.14. $\mathrm{cd}_G\, f = \mathrm{cd}_G\, \tilde{f}$ *and* $\overline{\mathrm{cd}}_G f = \overline{\mathrm{cd}}_G \tilde{f}$.

We note that the compactum $\tilde{X}$ obtained from the symmetrization $\tilde{f}$ of $f : L \to \Delta^n$ is homeomorphic to the sum

$$\#_{a \in S_{n+1}} X_a$$

where X_a is generated by $a \circ f$.

Proposition 2.15. *For every compactum X defined by a building block $f : L \to \Delta^n$ there is a symmetric compactum $\tilde{X}$ defined by the building block $\tilde{f} : L \times S_{n+1} \to \Delta^n$ that contains X as a subspace.*

Proof. The embedding $X \subset \tilde{X}$ is induced by the diagram

$$\begin{array}{ccc} L & \xrightarrow{x \mapsto (x,e)} & L \times S_{n+1} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ \Delta^n & \xrightarrow{=} & \Delta^n \end{array}$$

where e is the unit in S_{n+1}. □

3. Markov compacta with low rational dimension

The main results of this section is the following theorem:

Theorem 3.1. *For every $n \in \mathbf{N}$ and $k \leqslant n$, for every finite set of primes $\mathcal{L}$ there is a (symmetric) compactum X defined by one building block $f_n : L_n \to \Delta^n$ with dimensions* $\dim X = n$ *and* $\dim_{\mathbf{Z}[\frac{1}{p}]} X = k$ *for $p \in \mathcal{L}$ for every $k \leqslant n$.*

3.1. Rational dimension $\geqslant 2$

First we prove this theorem for $k > 1$.

Let $K_0 \xrightarrow{g_0} K_1 \xrightarrow{g_1} K_2 \xrightarrow{g_2} \cdots$ be a direct sequence. The telescope $T(\{g_i\})$ generated by this sequence is the quotient space $\coprod M_{g_i}/\sim$ where M_{g_i} is the mapping cylinder of the map $g_i : K_i \to K_{i+1}$ and the equivalence relation $\sim$ identifies $K_i \subset M_{g_i}$ with $K_i \subset M_{g_{i-1}}$.

Let $\mathcal{S}$ be a subset of the set $\mathcal{P}$ of all prime numbers. The standard construction of a localization $X_{(\mathcal{S})}$ of a space X at $\mathcal{S}$ uses the Postnikov tower. Sullivan's original construction of the localization for a simply connected CW complexes [10] defines $X_{(\mathcal{S})}$ as an infinite telescope $T(\{\nu_i\})$ of the direct sequence of simply connected complexes

$$K_0 \xrightarrow{\nu_0} K_1 \xrightarrow{\nu_1} K_2 \xrightarrow{\nu_2} \cdots$$

with $K_0 = X$, the localization map $l : X \to T(\{\nu_i\})$ equal to the inclusion, and $\dim K_i = \dim X$ for all i. In this case we say that an n-dimensional space X *admits a localization by means of a direct sequence of n-dimensional polyhedra*. Thus, the Sullivan's construction gives such a localization for every simply connected complex.

Proposition 3.2. *Let K be a finite simply connected simplicial complex of $\dim K = n$ and let $p \in \mathcal{P}$. Then there exists a finite simply connected n-dimensional simplicial complex K' and a map $g : K \to K'$, simplicial with respect to some iterated barycentric subdivision of K, such that the localization map $l : K \to K_{(\mathcal{P}\setminus\{p\})}$ is homotopically factored through f, $l \sim \xi \circ f$ and*

$$g_* : H_*(K; \mathbf{Z}_p) \to H_*(K'; \mathbf{Z}_p)$$

is zero homomorphism.

Proof. According to the above we may assume that $K_{(\mathcal{P}\setminus\{p\})} = T(\{\nu_i\})$ for a sequence of simply connected n-dimensional simplicial complexes

$$K_0 \xrightarrow{\nu_0} K_1 \xrightarrow{\nu_2} K_2 \xrightarrow{\nu_3} \cdots$$

with $K_0 = K$ and ν_i simplicial with respect to some iterated barycentric subdivision of K_i. First we note that $H_*(T(\{\nu_i\}); \mathbf{Z}_p) = 0$. Since

$$T\big(\{\nu_i\}_{i=0}^{\infty}\big) = \varinjlim T\big(\{\nu_i\}_{i=0}^{j}\big),$$

for every element $\alpha \in H_*(K; \mathbf{Z}_p)$ there is $j(\alpha)$ such that the image of α is zero in the finite telescope $T(\{\nu_i\}_{i=0}^{j(\alpha)})$. Since K is a finite complex, there is m such that the inclusion

$$l_m : K \to T\big(\{\nu_i\}_{i=0}^{m}\big)$$

induces zero homomorphism for the mod p homology. Note tat the telescope $T(\{\nu_i\}_{i=0}^{j(\alpha)})$ can be deformed to the space K_m. Let $r : T(\{\nu_i\}_{i=0}^{j(\alpha)}) \to K_m$ be the resulting retraction. We take $K' = K_m$ and $g = r \circ l_m$. Note that g is a simplicial map for s-iterated barycentric subdivision of K for sufficiently large s. □

Given a map $g : X \to Y$ we denote by M_g and C_g the mapping cylinder and the mapping cone respectively. By ΣX we denote the suspension over X and by CX the cone over X.

Proposition 3.3. *Let $g : K \to K'$ be as in Proposition 3.2, and let $q : C_g \to C_g/K' = \Sigma K'$ be the projection. Then*

$$q_* : H_{n+1}(C_g; \mathbf{Z}_p) \to H_{n+1}(\Sigma K'; \mathbf{Z}_p)$$

is an isomorphism.

Proof. Consider the diagram generated by the exact sequence of homology with coefficients in $\mathbf{Z}_p$ and the inclusions $(CK, K) \to (C_g, M_f) \leftarrow (C_g, K')$.

$$\begin{array}{ccccccc}
0 = H_{n+1}(K') & \longrightarrow & H_{n+1}(C_g) & \xrightarrow{q_*} & H_{n+1}(C_g, K') & \longrightarrow & H_n(K') \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow \\
0 = H_{n+1}(M_g) & \longrightarrow & H_{n+1}(C_g) & \xrightarrow{q_*} & H_{n+1}(C_g, M_g) & \xrightarrow{\partial} & H_n(M_g) \\
\uparrow & & \uparrow & & {\scriptstyle =}\uparrow & & {\scriptstyle g*}\uparrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & H_{n+1}(CK, K) & \xrightarrow{\partial'} & H_n(K)
\end{array}$$

By Proposition 3.2, $g_* = 0$. Hence $\partial = 0$ and the result follows. $\square$

Let us fix a prime p and a natural number $k > 1$. We define a collection of building blocks $\{f_n : L_n \to \Delta^n\}$, $n \geqslant k$ by induction on n such that each complex L_n is simply connected n-dimensional. We define the $L_k = \Delta^k$ and $f_k = \mathrm{id}_{\Delta^k}$. Assume that simply connected i-dimensional simplicial complexes L_i together with $f_i : L_i \to \Delta^i$ are defined for $i < n$ such that $\dim L_i = i$ and the maps f_i are simplicial with respect to a symmetric triangulation τ^i of Δ^i. Let $\chi_n : \beta(\partial\Delta^n) \to \Delta^{n-1}$ be the characteristic map. Denote by $\widetilde{\partial\Delta^n}$ the pull-back of the diagram

$$\begin{array}{ccc} \widetilde{\partial\Delta^n} & \longrightarrow & L_{n-1} \\ {\scriptstyle f'_{n-1}}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\ \partial\Delta^n & \xrightarrow{\chi} & \Delta^{n-1} \end{array}$$

Note that the space $\widetilde{\partial\Delta^n}$ is simply connected and $n-1$-dimensional. By Proposition 1.2 the map f'_{n-1} is simplicial with respect to the triangulation τ'_{n-1} on $\partial\Delta^n$ induced from τ^{n-1} by means the map χ. Let $g : \widetilde{\partial\Delta^n} \to K'$ be a map from Proposition 3.2 for the complex $K = \widetilde{\partial\Delta^n}$. It is simplicial with respect to the s-iterated barycentric subdivision $\beta^s K$ of K for some s. We define L_n as the mapping cylinder M_g where g is taken from Proposition 3.2 for the complex $\widetilde{\partial\Delta^n}$. We define the triangulation on Δ^n as the cone $\tau^n = \mathrm{cone}(\beta^s\tau'_{n-1})$ of the s-iterated barycentric subdivision of the triangulation τ'_{n-1} of the boundary $\partial\delta^n$. Note that it is symmetric. The mapping cylinder of a simplicial map admits a triangulation which coincides with the triangulation $\beta^s K$ on $\widetilde{\partial\Delta^n}$. We fix such triangulation on L_n and define $f_n : L_n \to \Delta^n$ as the simplicial map with respect to τ^n that takes all vertices from L_n which are not in $\widetilde{\partial\Delta^n}$ to the cone vertex and coincides with $\beta^s f'_{n-1}$ on $\widetilde{\partial\Delta^n}$. Note that the complex L_n is simply connected and n-dimensional.

Lemma 3.4. *For every n-dimensional simplicial complex K over Δ^n, $\chi : K \to \Delta^n$, the projection $\pi : \widetilde{K} \to K$ in the pull-back diagram*

$$\begin{array}{ccc} \widetilde{K} & \longrightarrow & L_n \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle f_n} \\ K & \xrightarrow{\chi} & \Delta^n \end{array}$$

induces an isomorphism $\pi_* : H_n(\widetilde{K}; \mathbf{Z}_p) \to H_n(K; \mathbf{Z}_p)$.

Proof. We prove it by induction on n.

Consider the diagram generated by the mod p homology and the mapping $\pi : (\widetilde{K}, \pi^{-1}(K^{(n-1)})) \to (K, K^{(n-1)})$:

$$\begin{array}{ccccccc} 0 \longrightarrow & H_n(\widetilde{K}) & \longrightarrow & H_n(\widetilde{K}, \pi^{-1}(K^{(n-1)})) & \longrightarrow & H_{n-1}(\pi^{-1}(K^{(n-1)})) \\ & {\scriptstyle \pi_*}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow \\ 0 \longrightarrow & H_n(K) & \longrightarrow & H_n(K, K^{(n-1)}) & \longrightarrow & H_{n-1}(K^{(n-1)}) \end{array}$$

By the construction we can identify $\pi^{-1}(K^{(n-1)})$ with the pull back of the diagram

$$\begin{array}{ccc} \widetilde{K^{(n-1)}} & \longrightarrow & L_{n-1} \\ {\scriptstyle \pi|_{\dots}}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\ K^{(n-1)} & \xrightarrow{\chi_n \circ \chi} & \Delta^{n-1} \end{array}$$

By induction assumption β is an isomorphism. We show that α is an isomorphism as well and apply the Five Lemma. We note that α is induced by the map $\bar{\pi} : \widetilde{K}/\widetilde{K^{(n-1)}} \to K/K^{(n-1)}$ which is the wedge of maps $\bar{f}_n : \widetilde{\Delta^n}/\widetilde{\partial\Delta^n} \to \Delta^n/\partial\Delta^n$ induced by f_n. By the construction $\widetilde{\Delta^n}/\widetilde{\partial\Delta^n} = C_g$ where $g : \widetilde{\partial\Delta^n} \to K'$ is from Proposition 3.2 and the projection $\bar{f}_n$ can be factored as

$$C_g \xrightarrow{q} \Sigma\widetilde{\partial\Delta^n} \xrightarrow{\Sigma(f'_{n-1})} \Sigma\partial\Delta^n.$$

By induction assumption $\Sigma f'_{n-1}$ induces isomorphism of n-dimensional homology. Then Proposition 3.3 implies that $\bar{f}_n$ induces an isomorphism. □

Corollary 3.5. *For every n, $\operatorname{cd}_{\mathbf{Z}_p} f_n = n$.*

Proof. Apply Lemma 3.4 to the diagram generated by the map $f_n : (L_n, f_n^{-1}(\partial\Delta^n)) \to (\Delta^n, \partial\Delta^n)$ to obtain that f_n induces nontrivial homomorphism of relative cohomology with coefficients in $\mathbf{Z}_p$. Hence, $\operatorname{cd}_{\mathbf{Z}_p} f_n \geqslant n$. □

Proposition 3.6. *For every simplex Δ^n the inclusion $\widetilde{\partial\Delta} \subset L_n$ induces an epimorphism of k-dimensional cohomology with coefficients in $\mathbf{Z}[\frac{1}{p}]$.*

Proof. This follows from the fact that the inclusion of a space X to its localization $X_{\mathcal{P}\setminus\{p\}}$ induces an isomorphism for cohomology with $\mathbf{Z}[\frac{1}{p}]$ coefficients. Since the localization map for $\widetilde{\partial\Delta}$ is homotopy factored through the inclusion $\widetilde{\partial\Delta} \subset \widetilde{\Delta}$ the required statement follows. □

Lemma 3.7. *For every n-dimensional simplicial complex K over Δ^n, $\chi : K \to \Delta^n$ and every subcomplex $N \subset K$, the inclusion $\widetilde{N} \subset \widetilde{K}$ induces an epimorphism of k-dimensional cohomology with coefficients in $\mathbf{Z}[\frac{1}{p}]$ where $\widetilde{K}$ is the pull-back in the diagram*

$$\begin{array}{ccc} \widetilde{K} & \longrightarrow & L_n \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle f_n} \\ K & \xrightarrow{\ \chi\ } & \Delta^n \end{array}$$

and $\widetilde{N} = \pi^{-1}(N)$.

Proof. By induction on n. Lemma is true for $n = k$ by the dimensional reason. Let $\phi : \widetilde{N} \to K(\mathbf{Z}[\frac{1}{p}], k)$ be a map. We construct an extension $\bar{\phi} : \widetilde{K} \to K(\mathbf{Z}[\frac{1}{p}], k)$. We note that $\pi^{-1}(K^{n-1})$ is the pull-back in the diagram

$$\begin{array}{ccc} \pi^{-1}(K^{(n-1)}) & \longrightarrow & L_{n-1} \\ {\scriptstyle\pi|_{\ldots}}\big\downarrow & & \big\downarrow{\scriptstyle f_{n-1}} \\ K^{(n-1)} & \xrightarrow{\ \chi_n \circ \chi|_{K^{(n-1)}}\ } & \Delta^{n-1} \end{array}$$

By induction assumption there is an extension $\phi' : \pi^{-1}(K^{n-1}) \to K(\mathbf{Z}[\frac{1}{p}], k)$ of the map

$$\phi|_{\pi^{-1}(N^{(n-1)})} : \pi^{-1}\big(N^{(n-1)}\big) \to K\left(\mathbf{Z}\left[\frac{1}{p}\right], k\right).$$

For every n-simplex $\sigma \subset K$ the pair $(\pi^{-1}(\sigma), \pi^{-1}(\partial\sigma))$ is homeomorphic to the pair $(L_n, f_n^{-1}(\Delta^n))$. By Proposition 3.6 there is an extension

$$\phi_\sigma : \pi^{-1}(\sigma) \to K\left(\mathbf{Z}\left[\frac{1}{p}\right], k\right)$$

of ϕ' restricted to $\pi^{-1}(\partial\sigma)$. The union of ϕ_σ for $\sigma \subset K \setminus \operatorname{Int}(N)$ together with ϕ gives us a required extension $\bar{\phi}$. □

The above construction can be summarized in the following.

Lemma 3.8. *Let p be a prime. Then for every $n \in \mathbf{N}$ and k with $2 \leqslant k \leqslant n$ there are an n-dimensional simplicial complex L_n and a map $f_n : L_n \to \Delta^n$ simplicial for some symmetric triangulation of Δ^n such that $\operatorname{cd}_{\mathbf{Z}_p} f_n = n$, $\operatorname{cd}_{\mathbf{Q}} f = k$ and $\overline{\operatorname{cd}}_{\mathbf{Z}[\frac{1}{p}]} f \leqslant k$.*

Moreover, if $\mathcal{L} = \{p_1, \ldots, p_s\}$ is a finite set of primes, then for the above k and n there is a map $f_n : L_n \to \Delta^n$ simplicial for some symmetric triangulation of Δ^n such that $\operatorname{cd}_{\mathbf{Z}_{p_1 \ldots p_s}} f_n = n$, $\operatorname{cd}_{\mathbf{Q}} f = k$ and $\overline{\operatorname{cd}}_{\mathbf{Z}[\frac{1}{p}]} f \leqslant k$ for all $p \in \mathcal{L}$.

Proof. The case of one p is presented above.

In the general case we replace p by the product $p_1 \dots p_s$. The construction and the proof remain the same. □

Proof of Theorem 3.1. $(k > 1)$. Let $\mathcal{L} = \{p_1, \dots, p_k\}$. By passing to the symmetrization, we may assume that the map $f_n : L_n \to \Delta^n$ in Lemma 3.8 is symmetric (see Proposition 2.14). Let X_n denote a compactum defined by $f_n : L_n \to \Delta^n$. By Theorem 2.12 $\dim_{\mathbf{Z}_p} X_n \geqslant n$, $p \in \mathcal{L}$, $\dim_{\mathbf{Q}} X_n \geqslant k$ and $\dim_{\mathbf{Z}[\frac{1}{\mathcal{L}}]} X_n \leqslant k$. The first inequality implies that $\dim X_n = n$. The other two inequalities together with the Bockstein inequalities imply $\dim_{\mathbf{Z}[\frac{1}{\mathcal{L}}]} X_n = k$. □

3.2. Rational dimension one

The following changes are needed to run the construction for the Theorem 3.1 for $k = 1$. First in the presence of the fundamental group the localization does not necessarily exists. So Proposition 3.2 must be changed. Still there is a localization for homology, i.e. a map $X \to \bar{X}$ such that $H_*(X) \to H_*(\bar{X})$ is the localization homomorphism. The problem here is that this localization is not necessarily given by the direct system of complexes of the same dimension ($= \dim X$). To make Proposition 3.2 working we map our complex to a complex of this type by a map that induced an epimorphism in 1-dimensional cohomology with the localized coefficient group.

Proposition 3.9. *Let p be a prime number. Let L denote a finite product $T^m \times \prod_{i=1}^{s} K(G_i, 1)$ of Eilenberg–MacLane complexes where T^m is the m-torus, $G_i = \mathbf{Z}_{q_i^{m_i}}$ where q_i is prime and $K(G_i, 1)$ is a complex with finite skeletons in all dimensions for all i. Then for every n the n-skeleton $L^{(n)}$ admits a homology localization at $\mathcal{P} \setminus \{p\}$ by means of a direct system of n-dimensional polyhedra.*

Proof. Let $p : S^1 \to S^1$ be a map of degree p and let $p^m : T^m \to T^m$ be the product of m copies of p. We define $\gamma_i : K(G_i, 1) \to K(G_i, 1)$ as follows. If $q_i \neq p$ we define $\gamma_i = \mathrm{id}$, if $q_i = p$ we take γ_i to be a map to a vertex in $K(G_i, 1)$. Consider the map

$$\gamma = p^m \times \prod_{i=1}^{s} \gamma_i : T^m \times \prod_{i=1}^{s} K(G_i, 1) \to T^m \times \prod_{i=1}^{s} K(G_i, 1).$$

Clearly, $\gamma(L^{(n)}) \subset L^{(n)}$. It is easy to check that the iteration of $\gamma|_{L^{(n)}}$ localizes the free part of the homology and the torsion part. □

Proposition 3.10. *Let K be a finite simplicial complex of* $\dim K = n > 1$. *Then there is a map $\psi : K \to K_0$ to an n-dimensional complex K_0 such that*

$$\psi^* : H^1\left(K_0; \mathbf{Z}\left[\frac{1}{p}\right]\right) \to H^1\left(K; \mathbf{Z}\left[\frac{1}{p}\right]\right)$$

is an epimorphism and K_0 admits a homology localization at $\mathcal{P} \setminus \{p\}$ by means of a direct sequence of finite n-dimensional polyhedra.

Proof. We attach finitely many 2-cells to K to make the fundamental group Abelian. Let N denote a new complex and let $j : K \to N$ be the inclusion. Clearly, $\dim N = n$. There is a map $\alpha : N \to L$ where L is as in Proposition 3.9 that induces an isomorphism of the fundamental groups. By the Universal Coefficient Theorem α induces an isomorphism of 1-cohomology with coefficients in $\mathbf{Z}[\frac{1}{p}]$. We may assume that α lands in $L^{(n)}$. Now take $K_0 = L^{(n)}$ and $\psi = \alpha \circ j$. □

The following is a modification of Proposition 3.2.

Proposition 3.11. *Let K be a finite simplicial complex of* $\dim K = n$ *and let $p \in \mathcal{P}$. Then there exists a finite n-dimensional simplicial complex K' and a map $g : K \to K'$, simplicial with respect to some iterated barycentric subdivision of K, such that*

$$g_* : H_*(K; \mathbf{Z}_p) \to H_*(K'; \mathbf{Z}_p) \tag{1}$$

is zero homomorphism and

$$g_*: H^1\left(K'; \mathbf{Z}\left[\frac{1}{p}\right]\right) \to H^1\left(K; Z\left[\frac{1}{p}\right]\right) \tag{2}$$

is an epimorphism.

Proof. Take $\psi: K \to K_0$ from Proposition 3.10 and consider a direct system

$$K_0 \xrightarrow{\nu_0} K_1 \xrightarrow{\nu_2} K_2 \xrightarrow{\nu_3} \cdots$$

that localizes homology of K_0. Then $\lim_{\to} H_*(K_i; \mathbf{Z}_p) = 0$. Take i such that $(\nu_i^0)_*: H_*(K_0; \mathbf{Z}_p) \to H_*(K_i; \mathbf{Z}_p)$ is zero homomorphism. Then take $K' = K_i$ and $g = \nu_i^0 \circ \psi$. Then (1) holds.

The homomorphism $(\nu_i^0)_*$ with coefficients in $\mathbf{Z}[\frac{1}{p}]$ is a monomorphism as a left devisor of the localization isomorphism. Since Ext term is zero in the Universal Coefficient Theorem over the ring $\mathbf{Z}[\frac{1}{p}]$ for 1-dimensional cohomology, we have that $(\nu_i^0)^*$ is an epimorphism for 1-dimensional cohomology with coefficients in $\mathbf{Z}[\frac{1}{p}]$.

We may assume that g is simplicial with respect to some iterated barycentric subdivision of K. □

The prove of the following proposition differs from the proof of Proposition 3.3 only by the reference to Proposition 3.11 instead of Proposition 3.2.

Proposition 3.12. *Let $g: K \to K'$ be as in Proposition* 3.11*, and let $q: C_g \to C_g/K' = \Sigma K'$ be the projection. Then*

$$q_*: H_{n+1}(C_g; \mathbf{Z}_p) \to H_{n+1}(\Sigma K; \mathbf{Z}_p)$$

is an isomorphism.

We have constructed a map g such that Lemma 3.8 holds for $k = 1$ with the same proof. The proof of Theorem 3.1 for $k = 1$ goes without changes.

4. Markov compacta with low mod p dimension

Theorem 4.1. *For every $n \in \mathbf{N}$ and $k \leqslant n$, for every finite set of primes $\mathcal{L}$ there is an n-dimensional compactum X defined by a building block $f_n: L_n \to \Delta^n$ with $\dim_{\mathbf{Z}_p} X = k$ for $p \in \mathcal{L}$ and every $k \leqslant n$.*

4.1. Mod p dimension $\geqslant 2$

We denote by $\mathcal{C}$ the class (Serre class) of torsion Abelian groups [9].

Lemma 4.2. *Let p be a prime. Then for every $n, k \in \mathbf{N}$ with $2 \leqslant k \leqslant n$ there are an n-dimensional simplicial complex L_n and a map $f_n: L_n \to \Delta^n$ simplicial with respect to some symmetric triangulation of Δ^n such that $\operatorname{cd}_{\mathbf{Z}_p} f_n = \overline{\operatorname{cd}}_{\mathbf{Z}_p} f_n = k$ and $\operatorname{cd}_{\mathbf{Q}} f_n = n$.*

Furthermore, for every finite set of primes $\mathcal{L} = \{p_1, \ldots, p_s\}$ for every $n, k \in \mathbf{N}$ with $2 \leqslant k \leqslant n$ there are an n-dimensional simplicial complex L_n and a map $f_n: L_n \to \Delta^n$ simplicial with respect to some symmetric triangulation of Δ^n such that $\operatorname{cd}_{\mathbf{Z}_p} f_n = \overline{\operatorname{cd}}_{\mathbf{Z}_p} f_n = k$ for every $p \in \mathcal{L}$ and $\operatorname{cd}_{\mathbf{Q}} f_n = n$.

Proof. In the case when $s > 1$ we set $p = p_1 \ldots p_s$. Let $k \geqslant 2$ be fixed. We use induction on n. We additionally assume the following:

(1) L_n is simply connected,

(2) for every complex K over the simplex Δ^n, $\chi: K \to \Delta^n$ the homomorphism $f'_*: H_*(\widetilde{K}; \mathbf{Q}) \to H_*(K; \mathbf{Q})$ induced by f' from the pull-back diagram

$$\begin{array}{ccc} \widetilde{K} & \xrightarrow{\chi'} & L_n \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f_n} \\ K & \xrightarrow{\chi} & \Delta^n \end{array}$$

is an isomorphism,

(3) for every complex K over the simplex Δ^n, $\chi : K \to \Delta^n$, and every subcomplex $A \subset K$, the inclusion homomorphism $j^* : H^k(\widetilde{K}; \mathbf{Z}_p) \to H^k(f^{-1}(A); \mathbf{Z}_p)$ is an epimorphism.

If $n = k$, we take $f_n = \mathrm{id}_{\Delta^n}$. Then all conditions are satisfied.

Now assume that $f_n : L_n \to \Delta^n$ is constructed and $n > k$. Let $\chi : \partial\Delta^{n+1} \to \Delta^n$ be a characteristic map of the barycentric subdivision of $\partial\Delta^{n+1}$. Let $\widetilde{\partial\Delta^{n+1}}$ be the pull-back in the diagram

$$\begin{array}{ccc} \widetilde{\partial\Delta^{n+1}} & \xrightarrow{\chi'} & L_n \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f_n} \\ \partial\Delta^{n+1} & \xrightarrow{\chi} & \Delta^n \end{array}$$

By induction assumption $f'_* : H_*(\widetilde{\partial\Delta^{n+1}}; \mathbf{Q}) \to H_*(\partial\Delta^{n+1}; \mathbf{Q})$ is an isomorphism. Hence with $\mathbf{Z}$ coefficients the induced homomorphism f'_* is a $\mathcal{C}$-isomorphism. Note that $\widetilde{\partial}\Delta^{n+1}$ is 1-connected. Then by the Mod$\mathcal{C}$ Hurewicz Theorem $f'_{\#} : \pi_n(\widetilde{\partial}\Delta^{n+1}) \to \pi_n(\partial^{n+1}) = \mathbf{Z}$ is a $\mathcal{C}$-isomorphism. Hence $\pi_n(\widetilde{\partial}\Delta^{n+1}) = \mathbf{Z} \oplus Tor$. Let $g : S^n \to \widetilde{\partial}\Delta^{n+1}$ be a map that represents an element $p \in \mathbf{Z}$. We define L_{n+1} as the mapping cone of g. There is a natural map $f_{n+1} : L_{n+1} \to \Delta^{n+1}$ which coincides with f' over $\partial\Delta^{n+1}$.

First we show that $\overline{\mathrm{cd}}_{\mathbf{Z}_p} f_{n+1} \leqslant k$. Since f_n is the identity above the k-skeleton of Δ^{n+1}, we have that $\overline{\mathrm{cd}}_{\mathbf{Z}_p} f_{n+1} \geqslant k$. When $n = k$, the space L_n is the mapping cone C_g of a map $g : S^k \to S^k$ of degree p and the inclusion $\partial\Delta^{n+1} \cong S^k \subset C_g$ induces a monomorphism of k-homologies with $\mathbf{Z}_p$ coefficients. Hence it induces an epimorphism of k-cohomologies with $\mathbf{Z}_p$ coefficients which implies that $\overline{\mathrm{cd}}_{\mathbf{Z}_p} f_{n+1} \leqslant k$. Let $n > k$ and let $\sigma \subset \Delta^{n+1}$ be a face. If $\sigma \subset \partial\Delta^{n+1}$, then the restriction of χ to σ makes it to be a complex over the simplex Δ^n. Then by the induction assumption (3), $H^k((f')^{-1}(\sigma); \mathbf{Z}_p) \to H^k((f')^{-1}(\partial\sigma); \mathbf{Z}_p)$ is an epimorphism. If $\sigma = \Delta^{n+1}$, the inclusion homomorphism

$$H^k(L_{n+1}; \mathbf{Z}_p) \to H^k\big(\widetilde{\partial\Delta^{n+1}}; \mathbf{Z}_p\big)$$

is an epimorphism, since L_{n+1} is obtained from $\widetilde{\partial\Delta^{n+1}}$ by attaching one cell of dimension $\geqslant k+1$ and hence the $(k+1)$-dimensional skeleton of L_{n+1} equals the $(k+1)$-dimensional skeleton of $\widetilde{\partial\Delta^{n+1}}$. Finally, we note that $\mathrm{cd}_{\mathbf{Z}_p} f_{n+1} \geqslant k$. Hence $\mathrm{cd}_{\mathbf{Z}_p} f_n = \overline{\mathrm{cd}}_{\mathbf{Z}_p} f_{n+1} = k$.

By construction, the inclusion $\widetilde{\partial\Delta^{n+1}} \subset L_{n+1}$ induces zero homomorphism in n-dimensional $\mathbf{Q}$-homology and hence in $\mathbf{Q}$-cohomology. This implies that $\mathrm{cd}_{\mathbf{Q}} f_{n+1} = n + 1$.

Next we verify the conditions (1)–(3).

(1) Using induction assumption it is not difficult to show that $\widetilde{\partial\Delta^{n+1}}$ is simply connected. Then L_{n+1} is simply connected by construction.

(2) Let $\nu : K \to \Delta^{n+1}$ be a light simplicial map. Then by the construction of L_{n+1}, the restriction

$$\nu' = \chi \circ \nu|_{K^{(n)}} : K^{(n)} \to \Delta^n$$

is a light simplicial map such that $(f')^{-1}(K^{(n)})$ is the pull-back of

$$K^{(n)} \xrightarrow{\nu'} \Delta^n \xleftarrow{f_n} L_n.$$

By the induction assumption

$$(f'|_{\dots})_* : H_*\big((f')^{-1}\big(K^{(n)}\big); \mathbf{Q}\big) \to H_*\big(K^{(n)}; \mathbf{Q}\big)$$

is an isomorphism. Consider the diagram generated by the exact sequences of pairs and the map $f' : (\widetilde{K}, (f')^{-1}(K^{(n)})) \to (K, K^{(n)})$.

$$\begin{array}{ccccc} H_i(\widetilde{K}; \mathbf{Q}) & \longrightarrow & H_i(\widetilde{K}, (f')^{-1}(K^{(n)}); \mathbf{Q}) & \longrightarrow & H_{i-1}((f')^{-1}(K^{(n)}); \mathbf{Q}) \\ {\scriptstyle f'_*}\downarrow & & {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle (f'|_{\dots})_*} \\ H_i(K; \mathbf{Q}) & \longrightarrow & H_i(K, K^{(n)}; \mathbf{Q}) & \longrightarrow & H_{i-1}(K^{(n)}; \mathbf{Q}) \end{array}$$

By the construction

$$\xi = (f_{n+1})_* : H_*\big(L_{n+1}, \widetilde{\partial \Delta^{n+1}}; \mathbf{Q}\big) \to H_*\big(\Delta^{n+1}, \partial \Delta^{n+1}; \mathbf{Q}\big)$$

is an isomorphism. Therefore, ψ is an isomorphism as the direct sum of ξ. By Five Lemma f'_* is an isomorphism.

(3) It suffices to show that every map $\phi : (f')^{-1}(A) \to K(\mathbf{Z}_p, k)$ has an extension $\bar{\phi} : \tilde{K} \to K(\mathbf{Z}_p, k)$. As in the proof of (2) we may use the induction assumption to construct an extension $\phi' : \widetilde{K^{(n)}} \to K(\mathbf{Z}_p, k)$ of the map $\phi|_{(f')^{-1}(A \cap K^{(n)})}$. Since the inclusion $\widetilde{\partial \Delta^{n+1}} \subset L_{n+1}$ induces an epimorphism in k-dimensional mod p cohomology, there is an extension $\bar{\phi} : \tilde{K} \to K(\mathbf{Z}_p, k)$ of the map $\phi \cup \phi' : (f')^{-1}(A) \cup \widetilde{K^{(n)}} \to K(\mathbf{Z}_p, k)$. □

Proof of Theorem 4.1. ($k > 1$). Let X_n be a (symmetric) compactum defined by the block $f_n : L_n \to \Delta^n$. By Theorem 2.12 implies that $\dim_{\mathbf{Z}_p} X_n = k$ and $\dim_{\mathbf{Q}} X_n \geqslant n$. Hence $\dim X_n = n$. In case of $p = p_1 \dots p_s$ we have $\dim_{\mathbf{Z}_{p_i}} X_n \leqslant \dim_{\mathbf{Z}_p} X_n \leqslant k$. Since the inequalities $\mathrm{cd}_G f_n \geqslant k$ for any G, we obtain that $\dim_{\mathbf{Z}_{p_i}} X_n = k$ for all i. □

4.2. Mod p dimension one

To prove Lemma 4.2 for $k = 1$ we need a sequence of results.

Let $\mathcal{C}_p$ denote the Serre class of p-torsion groups.

Proposition 4.3. *Let $X_0 \subset X_1 \subset \cdots \subset X_n$ be a sequence of cell complexes such that X_0 is finite and each X_{i+1} is obtained from X_i by attaching finitely many (possibly no) $(i+1)$-dimensional cells. Suppose that* $\operatorname{Tor} H_n(X_0) \in \mathcal{C}_p$ *and* $\dim X_0 \leqslant n$. *Then* $\operatorname{Tor} H_n(X_n) \in \mathcal{C}_p$.

Proof. Since $\dim(X_{n-1}/X_0) \leqslant n - 1$, the exact sequence of the pair (X_{n-1}, X_0) implies that $H_n(X_0) = H_n(X_{n-1})$. Consider the exact sequence of the pair (X_n, X_{n-1}):

$$0 = H_{n+1}(X_n, X_{n-1}) \to H_n(X_{n-1}) \xrightarrow{i} H_n(X_n) \to H_n(X_n, X_{n-1}) \to .$$

Note that $H_n(X_n, X_{n-1}) = H_n(\bigvee S^n) = \bigoplus \mathbf{Z}$. Then $H_n(X_n) \subset \operatorname{Im} i \cong (\bigoplus \mathbf{Z}) \oplus \operatorname{Tor} H_n(X_0)$. Thus, $\operatorname{Tor} H_n(X_n) \subset \operatorname{Tor} H_n(X_0)$. □

Let $q : X \to Y$ be the projection onto the orbit space of G-action for a finite group G. We denote by $\tau : H_*(Y) \to H^*(X)$ the homology transfer. Note that if the G-action is free, then $q_* \tau_*$ is multiplication by $|G|$.

Proposition 4.4. *Let X be a complex with $\pi_1(X) = \bigoplus_{i=1}^m \mathbf{Z}_p$, and $\pi_i(X) = 0$ for $2 \leqslant i \leqslant n$. Then $H_i(X) \in \mathcal{C}_p$ for $i \leqslant n$.*

Proof. Let $q : \bar{X} \to X$ be the universal cover. Then $q_* \tau_*$ is the multiplication by p^m. By the Hurewicz theorem $H_i(\bar{X}) = 0$ for $i \leqslant n$. Thus the homomorphism of multiplication by p^m in $H_i(X)$ is zero. It means that $H_i(X) \in \mathcal{C}_p$. □

Proposition 4.5. *Let X be an n-dimensional compact polyhedron such that $\pi_1(X) = \bigoplus_{i=1}^m \mathbf{Z}_p$, $\pi_i(X) = 0$ for $2 \leqslant i \leqslant n$ and* $\operatorname{Tor} H_n(X) \in \mathcal{C}_p$. *Then by attaching finitely many $n + 1$ cells to X, it is possible to construct a complex Y such that $H_i(Y) \in \mathcal{C}_p$ for all i.*

Proof. We chose a basis $a_1, \dots, a_k$ for $H_n(X)/\operatorname{Tor} H_n(X)$. Let $q : \bar{X} \to X$ be the universal cover of X and let τ be the transfer. We claim that $p^m a$ can be represented by a spherical cycle for every $a \in H_n(X)$. By the Hurewicz Theorem every element $H_n(\bar{X})$ can be represented by a spherical cycle. Note that $p^m a = q_*(\tau_*(a))$ and the claim follows. We attach $(n+1)$-cells along spherical cycles $p^m a_1, \dots, p^m a_k$ to obtain Y. We note $H_i(Y) = H_i(X) \in \mathcal{C}_p$ for $i < n$. By construction,

$$H_n(Y) = \operatorname{Tor} H_n(X) \oplus \left(\bigoplus_{i=1}^{k} \mathbf{Z}_{p^m} \right) \in \mathcal{C}_p.$$

We consider the homology exact sequence for the pair (Y, X):

$$0 \to H_{n+1}(Y) \to H_{n+1}(Y, X) \xrightarrow{\partial} H_n(X) \to H_n(Y) \in \mathcal{C}_p.$$

The group $H_{n+1}(Y, X)$ is free and by construction its rank is the same as that of $H_n(X)$. Since ∂ is a $\mathcal{C}_p$-epimorphism, it is $\mathcal{C}_p$-isomorphism. Thus, $H_{n+1} = 0$. $\square$

Proof of Lemma 4.2. ($k = 1$). As in the case $k > 1$ we will need three extra conditions (1)–(3) to run the induction. The condition (3) remains the same. The conditions (1)–(2) are changed to the following:

(1) *For every simply connected complex K over the simplex Δ^n, $\nu : K \to \Delta^n$, the* 1*st integral homology group of the pull-back $\tilde{K}$ is isomorphic to the direct sum $\bigoplus \mathbf{Z}_p$.*
(2) *for every complex K over the simplex Δ^n, $\chi : K \to \Delta^n$ the homomorphism $f'_* : H_*(\widetilde{K}; \mathbf{Z}[\frac{1}{p}]) \to H_*(K; \mathbf{Z}[\frac{1}{p}])$ induced by f' from the pull-back diagram*

$$\begin{array}{ccc} \widetilde{K} & \xrightarrow{\chi'} & L_n \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f_n} \\ K & \xrightarrow{\chi} & \Delta^n \end{array}$$

is an isomorphism.

For $n = 1$ we set $L_1 = \Delta^1$ and $f_1 = \mathrm{id}_{\Delta^1}$. For $n = 2$ we define L_2 as the mapping cone of a map $g : S^1 \to S^1$ of degree p. It is easy to check that all conditions are satisfied.

Assume that $f_n : L_n \to \Delta^n$ is constructed for $n \geqslant 2$. Let Δ be the standard $(n + 1)$-dimensional simplex and let $\chi : \beta\partial\Delta \to \Delta^n$ be the characteristic map of the barycentric subdivision. Consider $\widetilde{\partial\Delta}$, the pull-back of the map χ and $f_n : L_n \to \Delta^n$. Let $f' : \widetilde{\partial\Delta} \to \partial\Delta$ be the projection. First we attach finitely many 2-cells to $\widetilde{\partial\Delta}$ to obtain a complex Y with the Abelian fundamental group. We show that $\operatorname{Tor} H_n(Y) \in \mathcal{C}_p$. Indeed, from exact sequence of the pair $(Y, \widetilde{\partial\Delta})$ follows that $\operatorname{Tor} H_n(Y) = \operatorname{Tor} H_n(\widetilde{\partial\Delta})$. By condition (2) we have that $H_n(\widetilde{\partial\Delta}) \otimes \mathbf{Z}[\frac{1}{p}] \cong H_n(\partial\Delta) \otimes \mathbf{Z}[\frac{1}{p}] = \mathbf{Z}[\frac{1}{p}]$. Therefore $\operatorname{Tor} H_n(\widetilde{\partial\Delta}) \in \mathcal{C}_p$.

By condition (1) we have that $H_1(\widetilde{\partial\Delta}) = \pi_1(Y) = \bigoplus \mathbf{Z}_p$. The group $\pi_2(Y)$ is finitely generated since it is equal to the group $\pi_2(\bar{Y}) = H_2(\bar{Y})$ where $\bar{Y}$ is the universal cover. We attach 3-cells killing $\pi_2(Y)$ to $Y = X_0 = X_1 = X_2$ to obtain X_3. Similarly, the group $\pi_3(X_3)$ is finitely generated. We kill it by attaching 4-cells and so on. We construct a chain $X_0 \subset \cdots \subset X_n$ such that $\pi_1(X_n) = \bigoplus \mathbf{Z}_p$, $\pi_i(X_n) = 0$ for $2 \leqslant i < n$. Then by Proposition 4.3 $\operatorname{Tor} H_n(X_n) \in \mathcal{C}_p$. Then using Proposition 4.5 we attach finitely many $(n + 1)$-cells to X_n to obtain the complex L_{n+1}. We define a map $f_{n+1} : L_{n+1} \to \Delta$ by sending all new open cells in the interior of Δ by a map simplicial with respect to some symmetric subdivision of Δ.

We verify that $\operatorname{cd}_{\mathbf{Z}_p} f_{n+1} = \overline{\operatorname{cd}}_{\mathbf{Z}_p} f_{n+1} = 1$, $\operatorname{cd}_{\mathbf{Q}} f_{n+1} = n + 1$, and the conditions (1)–(3).

First we show that $\overline{\operatorname{cd}}_{\mathbf{Z}_p} f_{n+1} \leqslant 1$. Let $\sigma \subset \Delta^{n+1}$ be a face. If $\sigma \subset \partial\Delta^{n+1}$, then the restriction of χ to σ turns it into a complex over the simplex Δ^n. Then by the induction assumption (3), the inclusion homomorphism

$$H^1\big((f')^{-1}(\sigma); \mathbf{Z}_p\big) \to H^1\big((f')^{-1}(\partial\sigma); \mathbf{Z}_p\big)$$

is an epimorphism. If $\sigma = \Delta^{n+1}$, the inclusion homomorphism $H^1(L_{n+1}; \mathbf{Z}_p) \to H^1(\widetilde{\partial\Delta}; \mathbf{Z}_p)$ is an epimorphism as a composition of epimorphisms

$$H^1(L_{n+1}; \mathbf{Z}_p) \to H^1(X_n; \mathbf{Z}_p) \to \cdots \to H^1(X_3; \mathbf{Z}_p) \to H^1(Y; \mathbf{Z}_p) \to H^1(\widetilde{\partial\Delta}; \mathbf{Z}_p).$$

The last homomorphism in this chain is an epimorphism since it is dual to a monomorphism induced by the inclusion of a complex to its abelianization. All other homomorphisms are epimorphisms by the dimensional reason.

Clearly, $\operatorname{cd}_{\mathbf{Z}_p} f_{n+1} \geqslant 1$. Therefore, $\operatorname{cd}_{\mathbf{Z}_p} f_{n+1} = \overline{\operatorname{cd}}_{\mathbf{Z}_p} f_{n+1} = 1$.

By construction, $H_n(L_{n+1}) \in \mathcal{C}_p$. Hence the inclusion $\widetilde{\partial\Delta^{n+1}} \subset L_{n+1}$ induces zero homomorphism in n-dimensional $\mathbf{Q}$-homology and hence in $\mathbf{Q}$-cohomology. This implies that $\operatorname{cd}_{\mathbf{Q}} f_{n+1} = n + 1$.

(1) Let $\chi : K \to \Delta^{n+1}$ be a light simplicial map and K is simply connected. Since $n \geqslant 2$, the n-skeleton $K^{(n)}$ is simply connected. By induction assumption $H_1((f')^{-1}(K^{(n)})) = \bigoplus \mathbf{Z}_p$. Note that $\widetilde{K}/((f')^{-1}(K^{(n)})) = \bigvee (L_{n+1}/\widetilde{\partial\Delta})$

is the wedge of simply connected CW complexes. From exact sequence of the pair $(\widetilde{K}, (f')^{-1}(K^{(n)}))$ it follows that $H_1((f')^{-1}(K^{(n)})) \to H_1(\widetilde{K})$ is an epimorphism. Hence $H_1(\widetilde{K}) = \bigoplus \mathbf{Z}_p$.

(2) Let $\nu : K \to \Delta^{n+1}$ be a light simplicial map. By the induction assumption

$$(f'|_{\dots})_* : H_*\left((f')^{-1}\left(K^{(n)}\right); \mathbf{Z}\left[\frac{1}{p}\right]\right) \to H_*\left(K^{(n)}; \mathbf{Z}\left[\frac{1}{p}\right]\right)$$

is an isomorphism. Consider the diagram generated by the exact sequences of pairs and the map $f' : (\widetilde{K}, (f')^{-1}(K^{(n)})) \to (K, K^{(n)})$.

$$\begin{array}{ccccc}
H_i(\widetilde{K}; \mathbf{Z}[\frac{1}{p}]) & \longrightarrow & H_i(\widetilde{K}, (f')^{-1}(K^{(n)}); \mathbf{Z}[\frac{1}{p}]) & \longrightarrow & H_{i-1}((f')^{-1}(K^{(n)}); \mathbf{Z}[\frac{1}{p}]) \\
\downarrow {\scriptstyle f'_*} & & \downarrow {\scriptstyle \psi} & & \downarrow {\scriptstyle (f'|_{\dots})_*} \\
H_i(K; \mathbf{Z}[\frac{1}{p}]) & \longrightarrow & H_i(K, K^{(n)}; \mathbf{Z}[\frac{1}{p}]) & \longrightarrow & H_{i-1}(K^{(n)}; \mathbf{Z}[\frac{1}{p}])
\end{array}$$

By the construction

$$\xi = (f_{n+1})_* : H_*\left(L_{n+1}, \widetilde{\partial\Delta^{n+1}}; \mathbf{Z}\left[\frac{1}{p}\right]\right) \to H_*\left(\Delta^{n+1}, \partial\Delta^{n+1}; \mathbf{Z}\left[\frac{1}{p}\right]\right)$$

is an isomorphism. Therefore, ψ is an isomorphism as the direct sum of ξ. By Five Lemma f'_* is an isomorphism.

(3) It suffices to show that every map $\phi : (f')^{-1}(A) \to K(\mathbf{Z}_p, 1)$ has an extension $\bar{\phi} : \tilde{K} \to K(\mathbf{Z}_p, 1)$. As in the proof of (2) we may use the induction assumption to construct an extension $\phi' : \widetilde{K^{(n)}} \to K(\mathbf{Z}_p, 1)$ of the map $\phi|_{(f')^{-1}(A \cap K^{(n)})}$. Since the inclusion $\widetilde{\partial\Delta^{n+1}} \subset L_{n+1}$ induces an epimorphism in 1-dimensional mod p cohomology, there is an extension $\bar{\phi} : \tilde{K} \to K(\mathbf{Z}_p, 1)$ of the map $\phi \cup \phi' : (f')^{-1}(A) \cup \widetilde{K^{(n)}} \to K(\mathbf{Z}_p, 1)$. $\square$

The proof of Theorem 4.1 for $k = 1$ is the same as for $k > 1$.

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