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A COURSE IN THE MATHEMATICS OF DESIGN

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Abstract—A project-oriented course on the Mathematics of Design taught for the past six years to freshman architecture students at the New Jersey Institute of Technology is described. The course uses mathematics as the organizing force linking scientific, artistic and cultural subject areas together. The sequence of topics is graph theory with application to planning a floor plan; polyhedra applied to Platonic solids; tilings of the plane with application to lattice designs; tiling of three-dimensional space and space-filling polyhedra; similarity, proportion and the golden mean with application to architectural design; transformations; mirrors and symmetry; and vectors applied to analysis of polyhedra and ruled surfaces. The mathematical elements of each topic lead students to carry out a two- or three-dimensional construction. Students are helped to focus on the ideas behind their work by writing a series of essays.

1. INTRODUCTION

Seven years ago several mathematicians, architects and computer scientists at the New Jersey Institute of Technology began to explore areas of common interest. The architects were trying to find ways to break out of their limited repertoire of forms and shapes and they wished their students to develop the mathematical skills necessary to make design tasks more operational. The mathematicians were trying to find interesting applications for otherwise abstract branches of mathematics such as group theory, topology and graph theory, and in addition they wanted to rediscover the geometrical roots of their subject. The computer scientists saw the architecture students as potential users of computers—in particular, of computer graphics. As a result of this interaction, the group decided that a new course on the Mathematics of Design was needed for the following reasons:

- Most disciplines have become overspecialized. It is important to rediscover the connections between subject areas.
- The computer has gained a preeminent position in all subject areas. It is important to develop courses dealing with computer-applicable mathematics.
- Most subject areas emphasize analysis at the expense of synthesis. Courses are needed to redress this imbalance.

A course addressing these needs would have to encompass a wide range of mathematical ideas and include topics applicable to disciplines not recognized as mathematical. The ideas would have to be made active; a mere survey of mathematics as a culture or use of mathematics as a tool for analysis would not suffice. The course would have to demonstrate how ideas originating in the realm of mathematics could lead to fresh approaches in nonmathematical areas such as art, architecture, biology and chemistry. I knew about a project-oriented course that responded well to these points that was taught by an engineer, artist and geometer, Professor Mary Blade at Cooper Union College. Professor Blade's work formed the early inspiration for the course that I will describe in this article.

The course is based on geometrical ideas. By geometry we do not mean the vehicle of axiomatics that conditions most students' first approach to the subject. Rather, we refer to geometry as the matrix of ideas contained within a study of symmetry, proportion, tilings of the plane and three dimensions, perspective and the theory of graphs.

The course prepares students to carry out a construction or project rather than just conveying information and perfecting computational technique. Within this format each topic has a tight mathematical structure. The pedagogical approach is to "tell the truth, nothing but the truth—but not the whole truth." Airtight mathematical language would soon lose an audience untrained in this arcane form of expression. A primary goal of the course is to present material in plain language.

In the process of teaching nonmathematically oriented students, the problem of math anxiety must be faced. Many students are not able to manipulate mathematical symbols or follow the narrow paths of lengthy mathematical arguments. Nevertheless, they are capable of understanding complex ideas and applying them to practical problems. Even though these students might not do well on exams emphasizing computation, they can gain satisfaction from applying mathematical ideas to design projects. We also feel that these students can be made more receptive to mathematical ways of thinking if mathematics is shown to be a key to understanding a wider realm of ideas. To stimulate this reappraisal of thought in the light of mathematical revelations, after each project the students write a one- or two-page essay exploring the higher-level ideas behind the project.

The course was originally organized as an experimental seminar, part of the third-year design studio of the School of Architecture at NJIT. Each week a faculty member gave a lecture focusing on the mathematical content of his subject. The architecture students and their professors then met to suggest a design project based on the lecture. The following week the lecturer returned and was presented a set of completed constructions. Often the students related to the original lecture in ways the lecturer had not imagined. There was always the element of surprise in this transaction between students and faculty. We knew at this point that we had the makings of an unusual course.

The new course was made a requirement for all students from the School of Architecture and would be taken by about 100 students in the second half of the freshman year. Now the task was more difficult. We no longer had the luxury of communicating with a highly motivated, self-selected group of upperclassmen, but the needs of wider and less experienced audience had to be met. As a result, some of the spontaneity and self-motivation of the experimental seminar was sacrificed in favor of the greater structure and formality.

Since the content of the course was spread through numerous reference books, the members of the group collaborated on a set of lecture notes. With the help of a grant from the Graham Foundation for Architecture and the Fine Arts, I was able to write the first draft of a text for the course[1].

The course has steadily evolved over the past six years. Due to the constructive nature of the course, every time a new project is carried out, it can be used to illustrate mathematical ideas that would otherwise be lifeless. Finally, after several cycles of teaching the course, one of the collaborators, Professor Alan Stewart, recognized that graph theory provided an underlying structure to the course that unified the topics. Since the theory of graphs constitutes the least constrained of all geometries and all other geometries gain their form and structure from graphical notions, it made sense to employ graph theory as the central element of the course and show how more constrained geometries arise by adding additional structure to graphs. Also, Prof. Stewart saw that the mathematical notions of duality, symmetry, combinatorial properties, space filling, vectors and transformations arose in several topics so that ideas appeared and reappeared throughout the course.

This article will describe the major topics of the course and illustrate them with examples of students' work. Not all of the topics in this survey can be covered in a three-credit course, but all the material has been covered at some time. The sequence of topics is graph theory with application to planning a floor plan; polyhedra applied to the Platonic solids; tilings of the plane with application to lattice designs; tiling of three-dimensional space and space-filling polyhedra; similarity, proportion and the golden mean with application to architectural design; transformations; mirrors and symmetry; and vectors applied to analysis of polyhedra and ruled surfaces.

2. GRAPH THEORY

Each topic of the course begins informally with constructive exercises, experiments, game playing or puzzle solving. The theory of graphs lends itself particularly well to this approach. True to the origin of this subject in 1736 by Leonhard Euler we consider the famous Konigsberg bridge problem and the Utilities problem[2]. The first of these problems makes it clear that graphs with different outward appearances are structurally identical so long as their "connectivities" are the same and that, in fact, graphs are completely defined in terms of their connectivities rather than length, angle or other familiar geometric properties. Thus we are justified in considering graphs to describe the least constrained of all geometries. In addition, the

Konigsberg bridge problem introduces the notion of Euler paths, i.e. a path through the graph that traverses each edge without retracing.

The lesson to be learned from the Utilities problem is that while some graphs, the planar graphs, can be redrawn while preserving their connectivities with no edge intersections, others cannot. Since the students are from the School of Architecture, we try to have them visualize a connected planar graph as a floor plan as shown in Fig. 1(a). By placing a vertex in each room and connecting two vertices by an edge if the rooms share all or part of a wall, we arrive at another graph known as the connectivity graph shown in Fig. 1(b). Later, we show that the floor plan is in some sense "dual" to the connectivity graph.

Problem. Draw a floor plan with four rooms so that each room borders the other three. Can you do this for a floor plan with five rooms? Why not? Draw connectivity graphs for each of these graphs. Are they planar?

The previous problem demonstrates one of the important ideas of the course, namely that spatial design is not as freewheeling as students imagine but is constrained by mathematical properties of space. In fact the connectivity graph for the five-room plan, the complete graph with five vertices, K_5, along with the utilities graph, the bipartite graph $K_{3,3}$, are in a sense described by a theorem of Kuratowski contained in all nonplanar graphs[3]. This problem also gives insight as to why four colors are sometimes needed to color a map where adjacent countries must have different colors, but five colors are never needed.

A final introductory problem, the "Handshake Lemma," introduces the subject of the combinatorial properties of graphs. The class is divided into groups of five students and the students are told to shake hands with whomever they wish from the group and draw a graph to illustrate their pattern of handshakes. They must verify the Lemma which states that the number of people in the group that shake hands an odd number of times is even.

At this point we have laid the groundwork for a more serious study of graphs. Notions of graph isomorphism, cycles, planar and nonplanar graphs, duality, map coloring and combinatorial properties have been introduced in an informal manner.

The principal application for this section of the course is the application of graphs to designing complex floor plans[4]. To carry out this program we must first specialize to connected nonplanar graphs called maps. Maps have well-defined sets of edges, vertices and faces where the faces are topologically equivalent to discs, possibly with pendant edges. It is important to our application for students to first think of the maps as being drawn on a sphere, in which case each face has finite extent. A map in the plane is then obtained by puncturing an arbitrary face, widening the hole, and deforming the map until it fills the plane, as shown in Fig. 2. The punctured face then becomes the exterior face of the map.

Again, combinatorial properties of maps are discussed and the students are asked to draw many maps in search of a relation between the number of edges (E), faces (F) and vertices (V). It is always a surprise for students to discover another constraint on space, the Euler–Poincare number:

$$\chi = F + V - E = 2.$$

The students then begin to explore maps on other surfaces such as cylinders, tori and Möbius strips and to discover the different values of χ for each of these surfaces. The Szilassi polyhedron shown in the student construction of Plate 1 shows that as many as seven colors

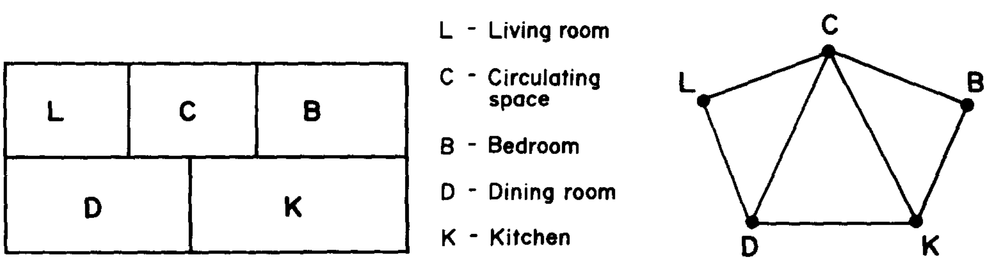


Fig. 1. A floor plan and its connectivity diagram. (a) Floor plan; (b) connectivity diagram.

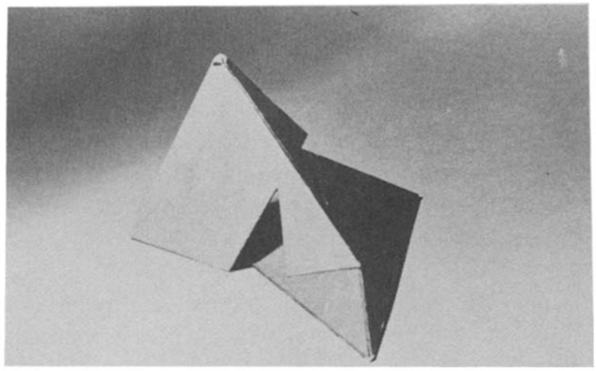


Plate 1. A Szilassi Polyhedron. Each face borders on the other six faces, demonstrating the need for seven colors to color a figure with the connectivity of a torus.

may be needed to color a map drawn on a surface with the connectivity of a torus[5]. Finally, we show students that graphs which are nonplanar in the plane and on a sphere may be planar when drawn on a torus, a fact that may prove useful for drawing floor plans for two-story houses.

By using the combinatorial properties of maps we are able to prove that there exist, aside from several trivial cases, only five regular maps, i.e. maps with vertices and faces surrounded identically by edges[3]. The students begin a search for these maps. This search will be borne out in the next section when the students encounter them as planar projections of the Platonic solids.

The notion of the dual map is introduced and students are now ready to carry out their first project:

> Design a one-story floor plan of a hypothetical house given a well-stated set of client constraints.

Here, constraints on the floor plan are imagined to be communicated verbally from client to architect. Based on these constraints the architect constructs a connectivity graph which includes partial information about the organization of rooms in the house. The connectivity graph is redrawn as a planar map. The dual of the connectivity map constitutes the first approximation to the floor plan. However, the exterior region to the house may appear in the dual map as an interior face at this stage of the design process. If this is so, the map is placed on a sphere and punctured so that the exterior region becomes the outside face. Finally the map is topologically deformed, preserving adjacencies until the desired floor plan obtains. This process is illustrated in Plate 2 by a student's project to design an office.

After completion of the design, the students are asked to write a one- or two-page essay in response to the following statement.

> *Writing Project*
>
> Christopher Alexander, in his book *Notes on the Synthesis of Form*, makes a case for there being a stage in the design process prior to the concrete planning stage, e.g. formal

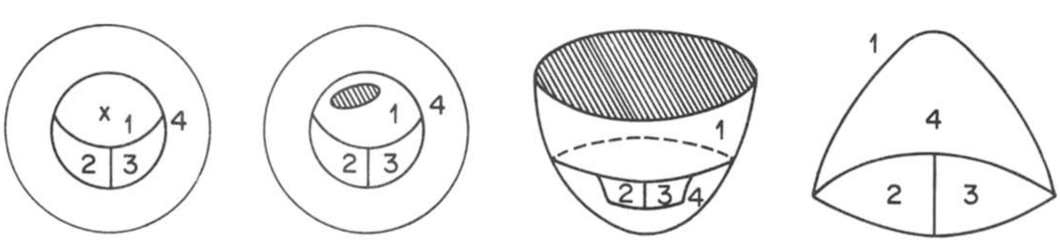


Fig. 2. Transformation of a map on a sphere to a map in the plane. (a) Map on the sphere with face 1 punctured. (b,c) The puncture is widened. (d) Map in the plane with face 1 external.

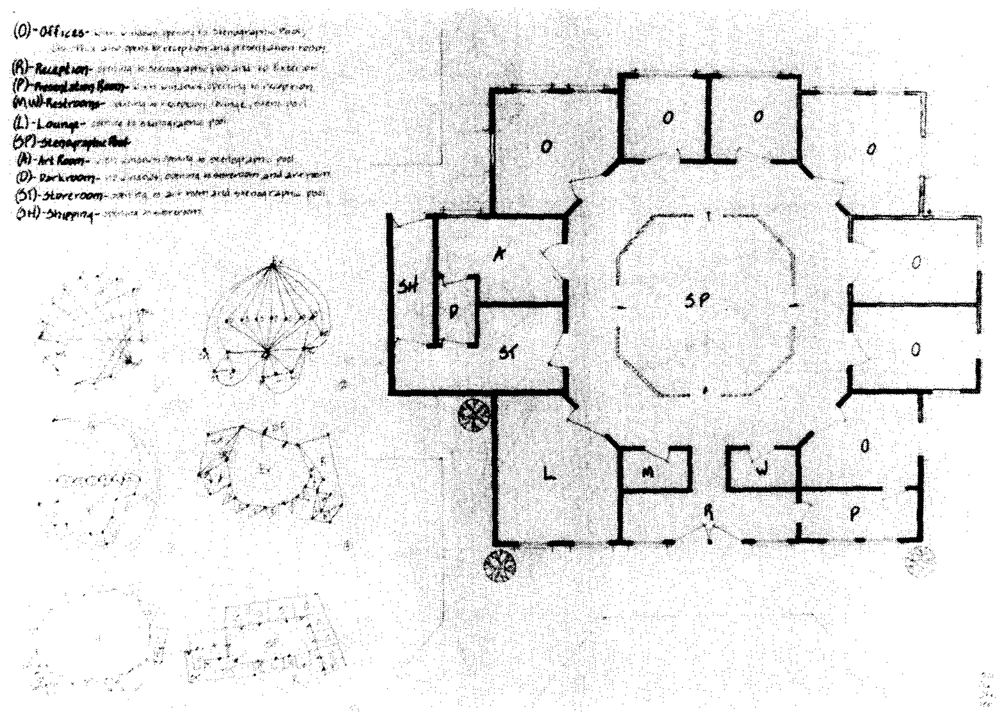


Plate 2. Design of a floor plan using graphs.

> presentation of a floor plan or community development project. In this primitive stage linkages or connections are drawn between the various components of the design.
>
> Whereas in primitive societies change occurred so slowly that this stage of the design was unconscious, in a more dynamic society such as our own, where design changes more radically, the process must be more self-conscious. Graphs are the appropriate tool to understand the linkages or connections.
>
> What do you think? Write a response to this statement in one or more pages. (A student response is included in the Appendix.)

Other topics covered in this section of the course have been application of directed graphs to planning a job by critical path analysis[6] and application of bipartite graphs to determining the minimum number of rods needed to brace a planar and three-dimensional space frame[4,7].

3. POLYHEDRA

We are now ready to apply some of the graph-theoretic results of the last section to three-dimensional structures. Polyhedra, and in particular the Platonic solids shown in Fig. 3, provide a source of interesting structures to study[8,9]. In fact, we began their study in the previous chapter when we attempted to find the five regular maps on the plane. Each of these maps can be drawn as the projection of one of the five Platonic solids onto the plane from a projection point above one of its faces, as shown in Fig. 3. Moreover, if the regular maps are placed on a sphere they result in surfaces topologically equivalent to the Platonic solids.

The Platonic solids also constitute a subject rich in connections to the worlds of art, architecture, chemistry and biology. For example, in Plato's *Timaeus* four of the solids were related to the four elements: earth, air, fire and water. The fifth solid, the dodecahedron, represented the cosmos. These solids also served as Kepler's model for the orbits of the planets, inspired the art of M. C. Escher[10], served as the basis of Buckminster Fuller's geodesic domes[11], serve as a starting point in the study of inorganic crystals[12] and the carbon–hydrogen bonds that make up the chemistry of life and finally arise as the form of microscopic organisms known as radiolaria[13], as shown in Fig. 4.

Again, we begin this section of the course in a constructive way. The students build polyhedral forms from miniature marshmallows and toothpicks. The resulting structures can be

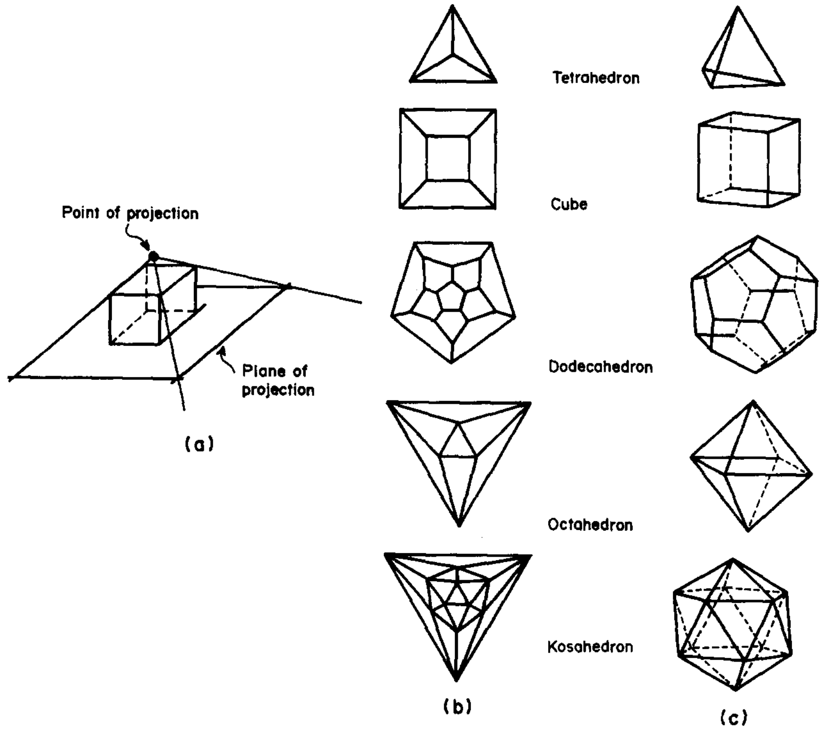


Fig. 3. The Platonic solids and their projections (Schlegel diagrams). (a) Schlegel diagram of cube formed by projection onto plane of bottom face (courtesy Dover[8]). (b) Schlegel diagram of Platonic solids. (c) Platonic solids.

thought of as three-dimensional graphs in which the marshmallows represent vertices and toothpicks play the role of edges, with the additional constraint that all edges have the same length. After some introductory puzzles which lead students to discover the tetrahedron and octahedron, the students create patterns by connecting tetrahedra vertex to vertex, edge to edge and face to face. This leads naturally to construction of octet space frames from combinations of tetrahedra and octahedra, as shown in the student constructions of Plate 3, where modulation of edge lengths results in curvature.

Another construction with marshmallows and toothpicks leads to the discovery of the Platonic solids. The students create regular polyhedra defined by the constraint that each vertex and face must have the same number of incident edges. In the process of constructing these polyhedra it becomes evident that whereas the tetrahedron, octahedron and icosahedron, with triangular faces, are all rigid, the cube and dodecahedron collapse. Thus students discover the importance of the triangle as a source of rigidity in structures.

By placing the octahedron inside the cube, the icosahedron within the dodecahedron and the tetrahedron inside another tetrahedron, the duality of these polyhedral pairs is made visually evident. Also, by tabulating F, E and V the Euler–Poincare number $\chi = 2$ is rediscovered.

The Platonic solids are considered to be a family because they relate to each other in many ways aside from duality. In fact, the golden mean, $\phi = (1 + \sqrt{5})/2$, is a number which ties this family together[14] as shown by Euclid in Book XIII of *The Elements*. The golden mean will play a major role in the portion of the course dealing with proportion. Here we demonstrate the remarkable internal structure of the icosahedron in which the vertices lie at the corners of three mutually orthogonal golden mean rectangles. Plate 4 shows a student construction of a tensegrity structure illustrating this fact.

After studying the combinatorial properties of the Platonic solids, it is natural to consider their metric properties. The cube constitutes the natural coordinate system in which to investigate metric properties since the Platonic solids can all be related to a cube. In fact, a complete characterization of the metric properties of any polyhedron related to a cube can be made in terms of the cube's three principal directions: the edge, face-diagonal and body-diagonal directions. We have found the "universal node system" of Peter Pearce[15] to be an excellent

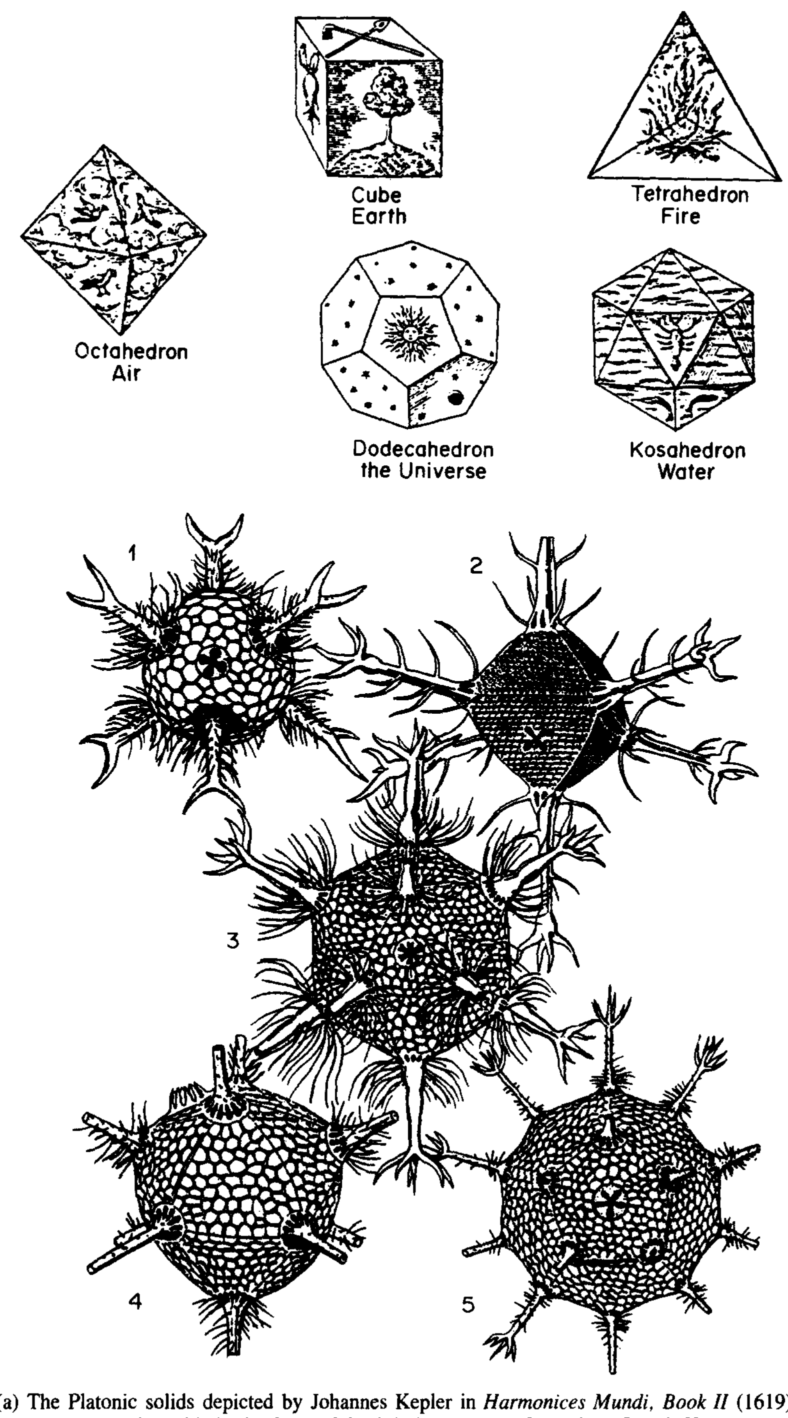
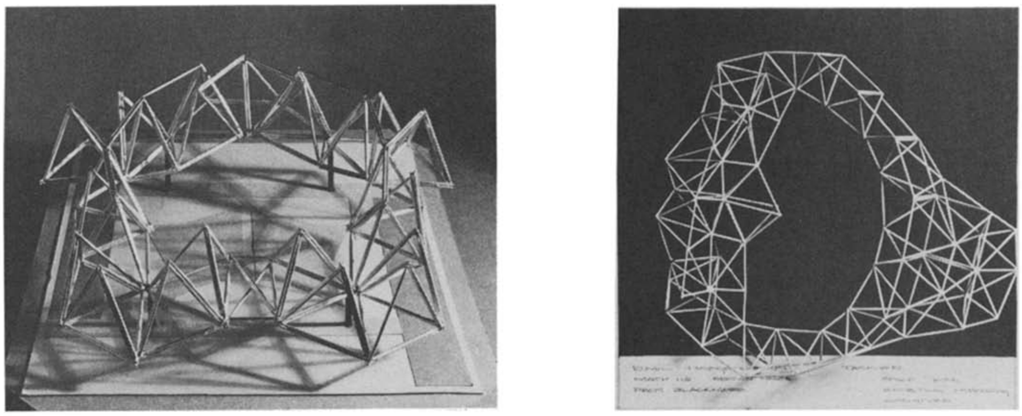


Fig. 4. (a) The Platonic solids depicted by Johannes Kepler in *Harmonices Mundi, Book II* (1619). (b) The Platonic solids in the form of Radiolaria (courtesy Cambridge Press[13]).



(a) (b)

Plate 3. Two examples of octet space frames. Curvature is the result of modulated edge lengths.

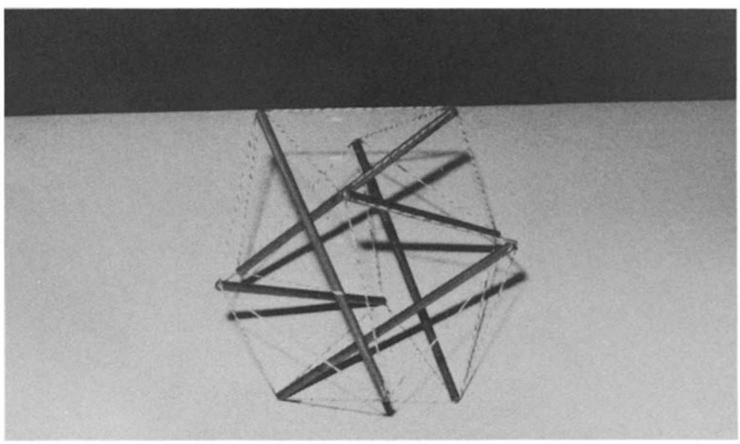
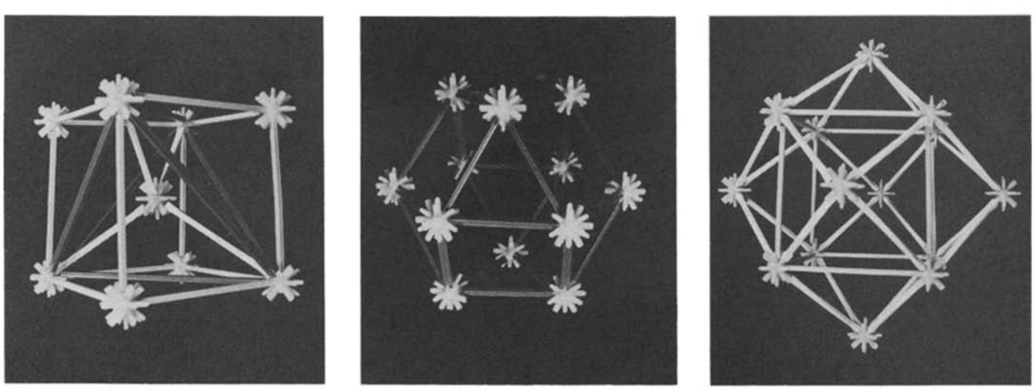


Plate 4. Tensegrity of an icosahedron illustrating inner structure of three intersecting golden mean triangles.



(a) (b) (c)

Plate 5. Three polyhedra constructed with the universal node system of Peter Pearce. (a) A cube showing principal directions. (b) Cuboctahedron constructed with face-diagonal directions. (c) Rhombic dodecahedron surrounding a cube constructed with body-diagonal directions.

tool for constructing and demonstrating at a glance the structure of polyhedra. In this system the edges are color and shape coded according to the principal directions of the cube and connect to equivalently shape-coded connectors. This enables polyhedra to be built with great ease. A cube with a tetrahedron embedded on its surface and four edges connecting the cube's center to each of the four vertices of the tetrahedron, constructed with the universal node system along with two polyhedra related to a cube, the cuboctahedron and rhombic dodecahedron, are shown in Plate 5. If we imagine a carbon atom to lie at the center of the cube and hydrogen atoms at the vertices of the tetrahedron, this system provides a model of the carbon–hydrogen bond found in organic molecules.

Symmetry will be discussed in great detail later in the course. In this section the topic is introduced by illustrating the 13 axes of rotational symmetry and the 9 planes of reflection symmetry of a cube using a cube constructed with the universal node system. The axes of rotational symmetry lie in the principal directions of the cube and can be detected by projecting the cube onto a plane perpendicular to the axes and then by finding the symmetry of the plane projections, as in Fig. 5.

The geodesic properties of the Platonic solids are introduced as the final topic of this section by posing the following problem:

Find a way to cut an orange into four congruent pieces other than the usual breakfast way.

The solution is found by projecting the tetrahedron to a circumscribed sphere from its centroid. This divides the sphere into four congruent solid angles. In the process the edges project to arcs of geodesics of the sphere. Of course, similar results obtain from other Platonic solids and their circumscribed spheres. This gives an opportunity to define geodesics and assign problems to compute shortest distances between points lying on a cube, parallelopiped, cylinder, torus, shell and cone.

Besides the circumscribed sphere, two other spheres are associated with the Platonic solids: the inscribed sphere tangent to polyhedral faces and the intersphere through the midpoints of the edges. These spheres are related to the symmetry of the Platonic solids in a striking way illustrated by the dihedral kaleidscope of the cube shown in Plate 6[16]. Here the 9 planes of reflective symmetry divide the cube into 48 congruent tetrahedra, known as orthoschemes, defined by the radii of the three spheres[8]. These tetrahedra also form an excellent set of building blocks from which to construct polyhedral sculptures as shown.

Three additional surfaces satisfying the criteria for Platonic solids were introduced by Coxeter[17] in 1937. They differ from the five already introduced by having an infinite number of faces. Two of them are duals since four hexagons surround each vertex in one while six squares surround a vertex in the other, as shown in the student project of Plate 7.

Finally, the three regular star polyhedra discovered by Kepler have been explored by several students and give entree to the visually fascinating area of star polyhedra[18].

4. TILINGS OF THE PLANE

Tiling a region of space is the concern of many disciplines. The architect fills open spaces with buildings and partitions the inside of a building with rooms. The artist subdivides a canvas into spaces in which to portray the subjects that make up a painting. The chemist and crystallographer deal with well-ordered patterns of molecules in the form of chemical compounds or crystals. The botanist studies regular orderings of stalks, or paristichies, of a plant. Electrical engineers consider breakdowns of space into close-packed spheres representing the coding of messages. In this section, we study the mathematics of tiling a plane and indicate how mathematics addresses the concerns of other subject areas. In the next section we consider tilings of three-dimensional space.

The previous section was devoted to a study of polyhedra, and in the introductory exercises students constructed octet space frames. Now these space frames are projected, by a light source, onto a plane, and triangles, squares and hexagons are observed in the resulting patterns. The

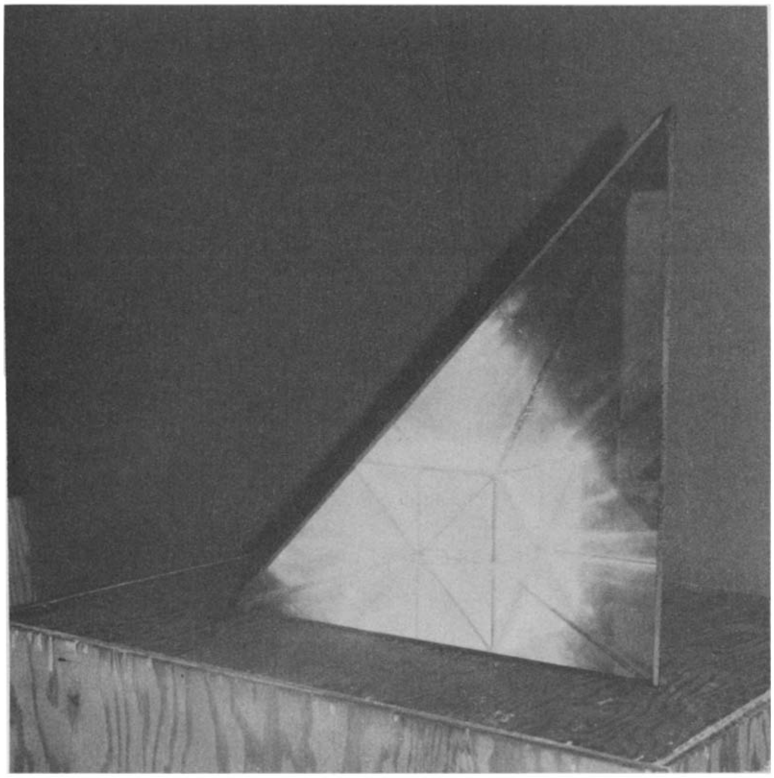


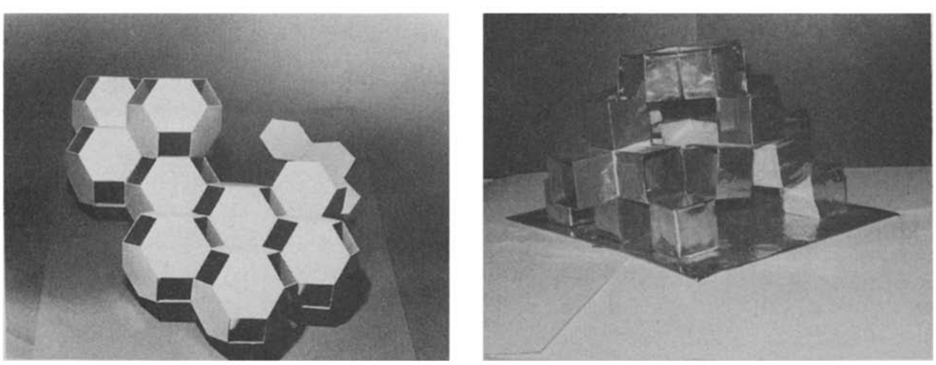
Plate 6. Dihedral kaleidoscope based on the symmetry of the cube.

mathematics of planar tilings begins with a study of these patterns of triangles, squares and hexagons.

As an introductory exercise, students are asked to observe patterns of triangles, parallelograms and hexagons formed by the grid lines of triangular-grid graph paper. The possibilities are noted for constructing unusual designs with this grid by circumscribing a circle about a large hexagon defined by the grid and shading the grid contained within the hexagon, as shown in Fig. 6.

After creating their own hexagonal design, the students construct designs using the entire grid, in a manner reminiscent of Islamic patterns[19]. A student project is shown in Plate 8.

Now that the students have had the opportunity to try their hand at some free-form planar designs, they learn that as a consequence of the combinatorial properties of graphs, triangles, squares and hexagons are the only regular maps that tile the plane with an infinity of faces[20]. Thus once again a mathematical property of space forbids other polygons, such as pentagons, from tiling regularly.



(a) (b)

Plate 7. Two additional Platonic solids discovered by Coxeter. They are duals.

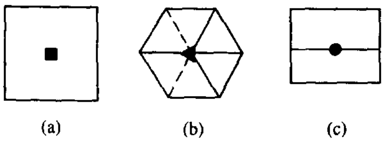


Fig. 5. Projection of a cube onto the plane along its axes of rotation. (a) Four-fold axis; (b) three-fold axis; (c) two-fold axis.

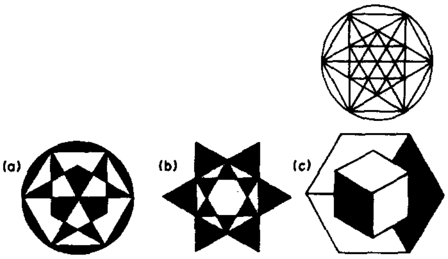


Fig. 6. A triangular grid with three hexagonal patterns (courtesy Creative Publications).

This result was proven for maps whose faces need not be congruent, or for that matter, have linear edges. If we now consider regular tilings made of congruent polygons, the three permissible tilings were already inherent in the triangular grid of the introductory exercise. However, any quadrilateral, convex or nonconvex, and hexagons with parallel opposite sides also tile the plane regularly.

Next we define semiregular tilings of the plane in which more than one polygon is used. For example, Fig. 7(a) shows a semiregular tiling with triangles and squares, five around a vertex. Next to it, in Fig. 7(b) is a tiling with pentagons. We try to get the students to explain why this tiling does not violate the proscriptions of regular pentagonal tilings. In fact, Fig. 7(a) is dual to Fig. 7(b) and brings up an important aspect of duals: although dual figures are structurally identical they are nevertheless visually quite different. Thus a single mathematical idea can serve as a carrier of a variety of visual patterns, a recurring theme in this course.

Parallelograms and hexagons with parallel opposite sides play a special role in tiling, namely they tile the plane by translation only. These polygons are members of a class of polygons known as zonagons[21], generated by stars of vectors. In order to study zonagons we first introduce the subject of vectors, which will find great utility throughout the course. The 3-zonagon is then defined by a vector star of three vectors as shown in Fig. 8.

The value of this construction to design is that vectors of the star can be altered in length to form zonagons changed in size and shape but with the same interior angles. Thus a space-filling array of hexagons continues to tile the plane, as shown by the student construction of Plate 9. From a design point of view, joints are the most difficult part of a structure to fabricate and zonagons enable structures to be "fitted" to their "function" in terms of size and shape without altering the joints.

In the regular grid tilings that introduced this section the focus was on edges and faces. Now we consider the vertices of those grids which constitute a planar lattice. Some exploratory exercises help students gain an understanding of the invariance of planar lattices under translations in two nonparallel directions characterized by vectors, and the notion of the fundamental domain of a lattice. It is not surprising that a structure as rich as the lattice in mathematical ideas is also a rich source of two- and three-dimensional tilings. A design idea suggested by William J. Gilbert[22] describes how patterns with lattice symmetry can be generated. The designs shown in Plate 10 illustrate some results of Gilbert's method. One of the tilings incorporates 90° rotations in addition to translation, while another applies Gilbert's ideas to three-dimensional lattice designs.

Up to now the plane has been tiled with polygons of identical size and shape. Now tilings

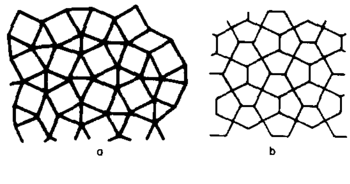


Fig. 7. A semiregular tiling and its dual (courtesy Addison Wesley[20]). (a) Semiregular tiling; (b) dual tiling.

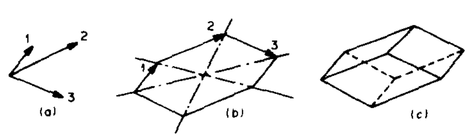


Fig. 8. Construction of a 3-zonagon. (a) 3-Vector star; (b) 3-zonagon (note: opposite sides are equal and parallel and zonagon is centrally symmetric); (c) 3-zonagon is the plane projection of a parallelopiped.

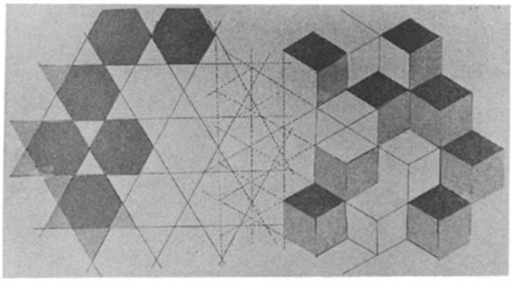


Plate 8. A design based on tiling the plane with triangles, squares and hexagons.

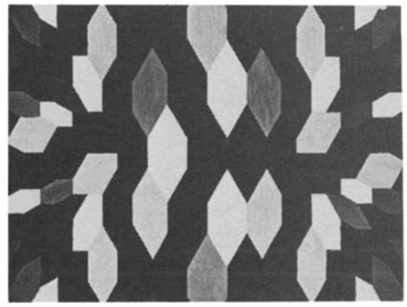


Plate 9. Tiling the plane with hexagons constructed from a 3-zonagon.

are considered with polygons that may be irregularly shaped, known as Dirichlet domains (DDs)[20]. Dirichlet domains have applications in biology, chemistry and architecture. They are generated by a set of points in which the regions of the tiling satisfy the following minimum principle: all points nearer to a given point of the generating set than any other point of the set belongs to the DD of the given point.

Subdivision of the plane into DDs can be carried out by compass-and-straightedge construction. For example, the boundary of the two points is clearly the perpendicular bisector of

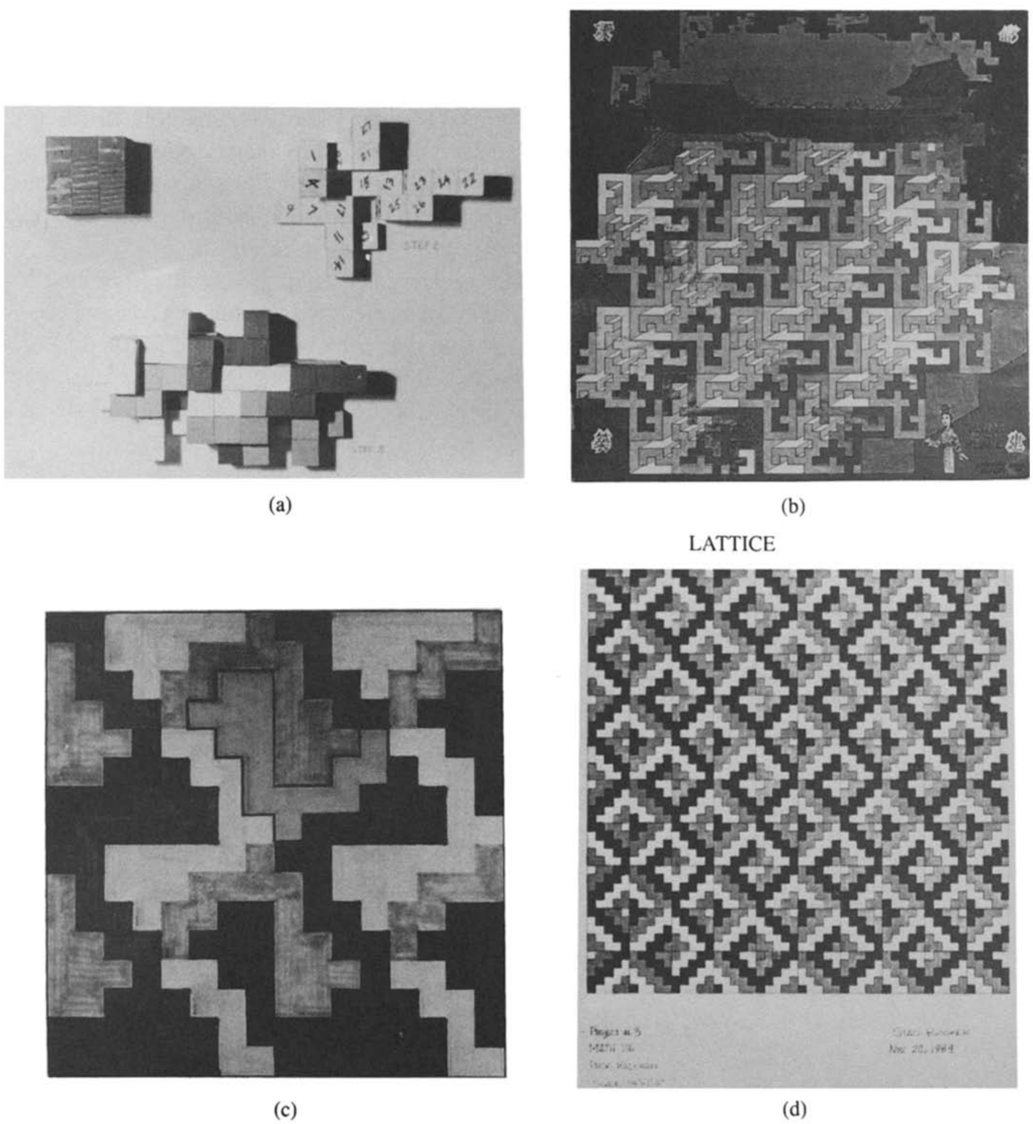
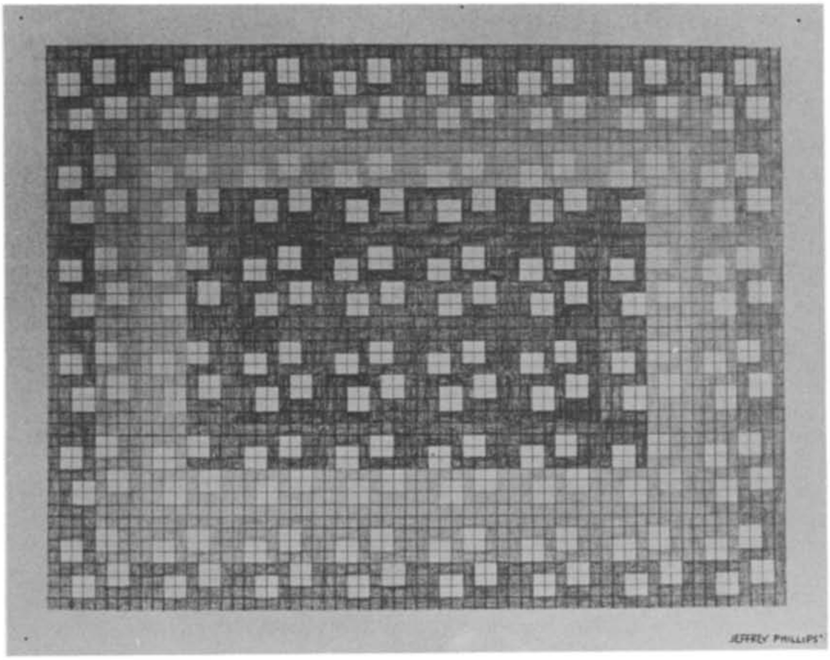
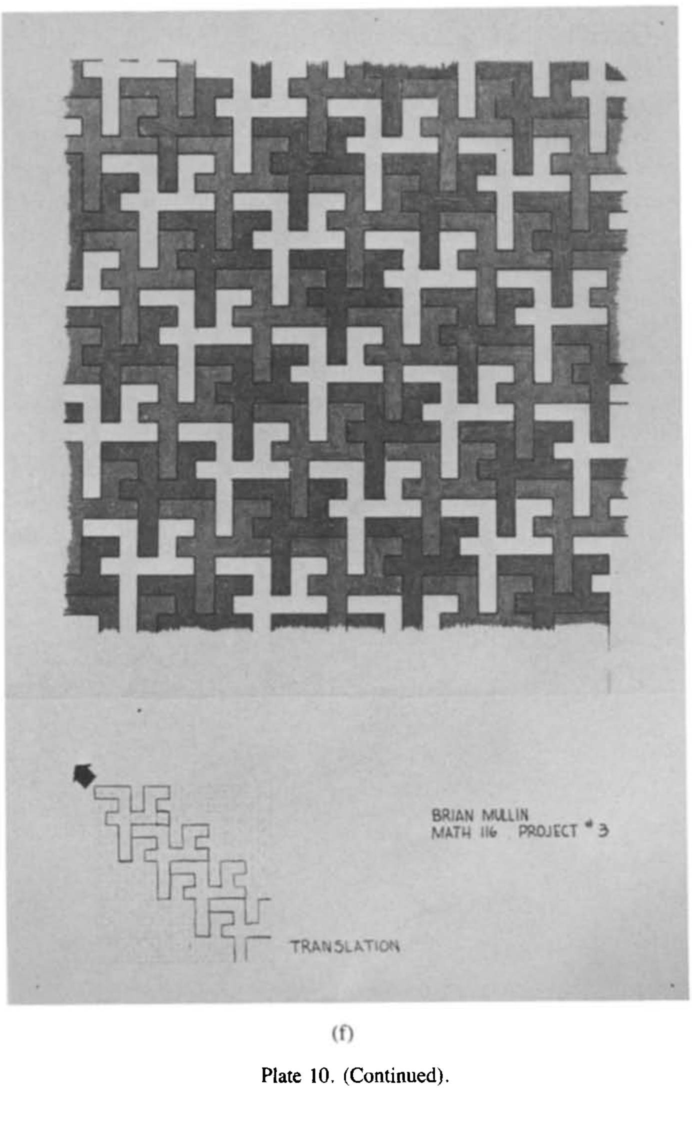


Plate 10. Some designs based on two- and three-dimensional lattices using Gilbert's method.



(e)



(f)

Plate 10. (Continued).

the line joining the two points. The DDs of three points are formed by the perpendicular bisectors of the sides of the triangle. The construction of the DDs of four points is more of a challenge and leads to an algorithm for constructing DDs for n points.

The purpose of introducing DDs is to exploit their connection to space filling in the plane, and more importantly in three-dimensional space, by congruent modules. These space-filling properties of DDs lie at the base of their applications to biology, chemistry and architecture.

The DDs of a plane lattice are either hexagons or rectangles. Coxeter[23] showed that the growth of plants can be studied by mapping the surface of the plant onto a planar lattice and identifying the stalks of the plant with the DDs of the lattice as shown in Fig. 9 for a pineapple.

Writing Project

In his paper entitled "Perception and Modular Coordination," Christopher Alexander suggests that we enjoy symmetric themes in design because our minds recoil at chaos but are put at ease by the repetition of a simple motif. We like to see things that look familiar, that we have seen before, and structure and order in art and architecture help us to feel secure and comfortable with our surroundings. On the other hand, people react adversely to mindless, monotonous order. To a great extent, it is the job of the artist and architect to supply, through their work, a solution to the problem of satisfying the needs of people for both order and novelty.

Comment on this statement. Do you agree or disagree? The Design Project on Lattices certainly satisfies the criterion of design based on order and repetition. Is it also capable of producing designs interesting enough to appeal to our need for surprise and novelty? (A student response is included in the Appendix.)

In the next section DDs of three-dimensional lattices will be shown to be space-filling polyhedra with opposite faces congruent and parallel, i.e., analogs in three dimensions of the zonagons known as zonahedra[24].

Additional material is included in the *Notes* describing an algebraic method of tiling a rectangle with noncongruent squares[25] and a class of nonperiodic tilings of the plane discovered by Roger Penrose[26]. Student projects illustrating these tilings are shown in Plate 11 and Plate 12. Also, following up on the star polyhedra introduced in the preceding section, the students explore regular star polygons, a source of both interesting patterns and number-theoretic results[27].

5. TILING OF THREE-DIMENSIONAL SPACE

This section extends the ideas of the last section to tilings of three-dimensional space. The section begins with two introductory exercises showing the relation between two- and three-dimensional tilings and the use of soap bubbles as a natural way to fill space with curvilinear polyhedra. A study of close packing of spheres leads to the subject of three-dimensional lattices, networks and dual networks[15]. The Archimedean solids are seen to be semiregular tilings of

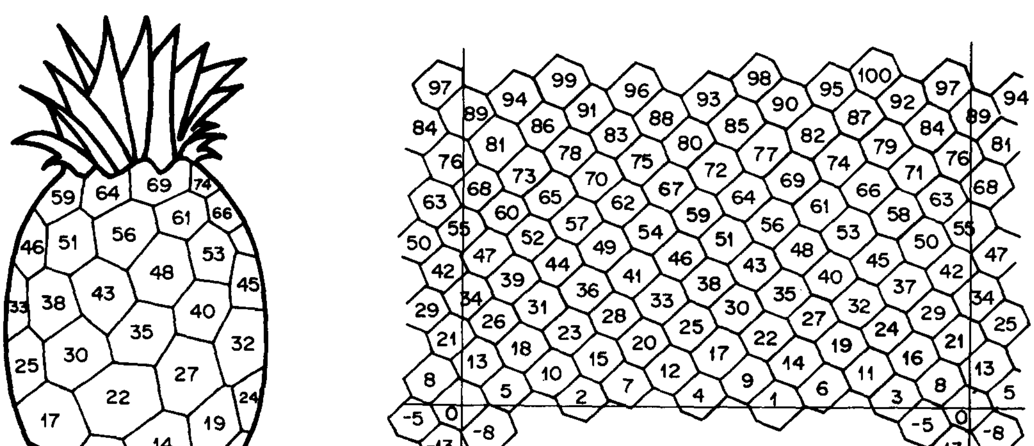
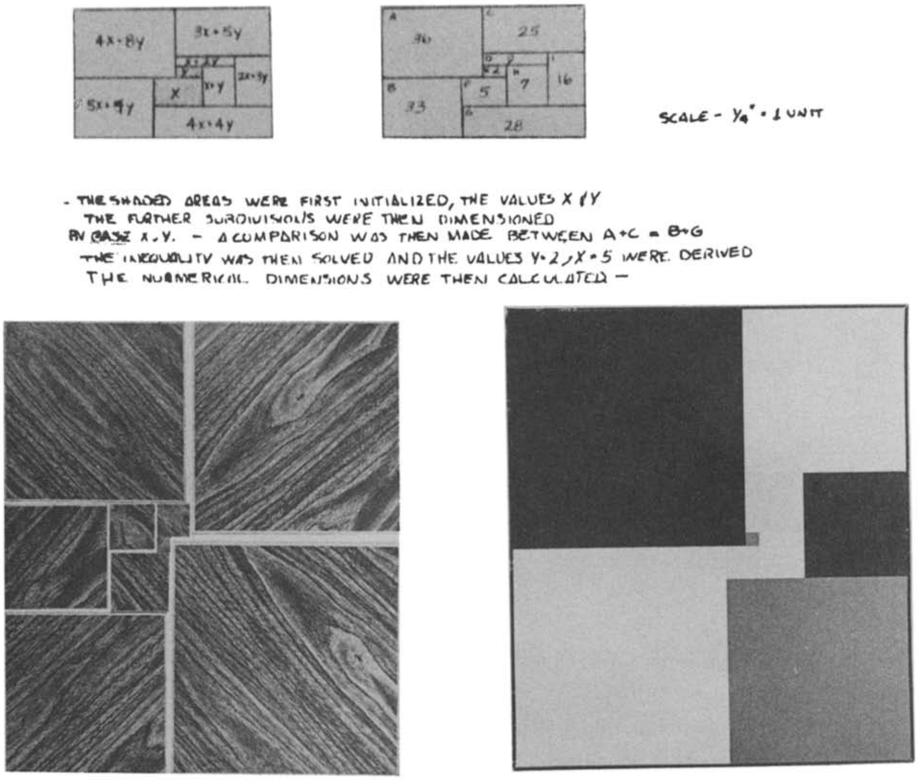


Fig. 9. Relation of pineapple phyllotaxis to a period lattice (courtesy Wiley[23]).

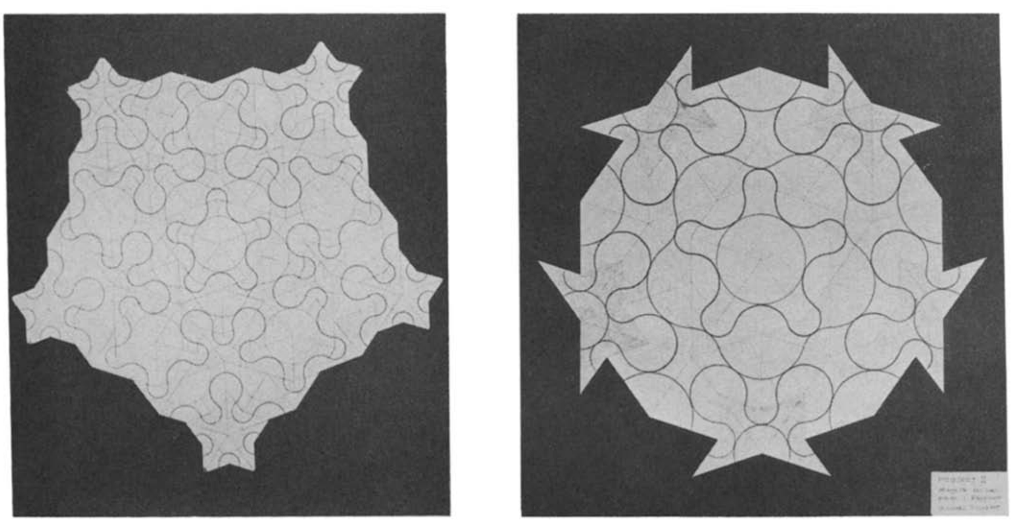


(a) (b)

Plate 11. Tiling a rectangle with noncongruent squares.

the sphere, and they lead to additional space-filling possibilities. Finally, prisms and antiprisms are studied and the latter are used as the basic module of a design with architectural applications.

The first introductory exercise involves using a soap solution to study what appears to be the structureless formation of bubbles making up a soap froth much like a three-dimensional graph drawn with curvilinear edges. In fact, closer study of the froth reveals a precise structure. Three bubbles always meet at an edge and four edges meet at each vertex. Also, the average number of faces in the polyhedra formed by the froth is approximately 14. Later we shall see that this state of affairs also prevails in the space-filling array of one of the Archimedean solids, the truncated octahedron.



(a) (b)

Plate 12. A nonperiodic tiling of the plane by kites and darts, based on the golden mean.

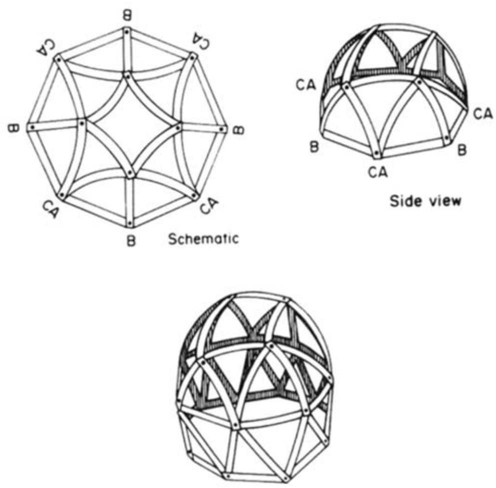


Fig. 10. Pattern for a dome constructed from paper strips using the method of Gerald Segal (courtesy Gerald Segal).

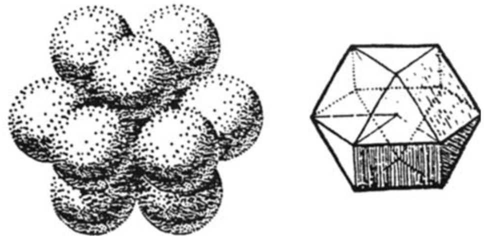


Fig. 11. The cuboctahedron as a figure of cubic close packing of spheres (courtesy Dover[8]).

Last semester I made use of a second constructive exercise, devised by Gerald Segal[28] to make the transition from tilings of the plane to polyhedra. Ninety-six heavy cardboard strips measuring 10 in. by $1\frac{1}{2}$ inches and paper connectors were distributed to groups of students. Their job was to place strips around a central square to make the square rigid. They realized that surrounding the square by triangles would do the job. At this point we presented them with the patterns shown in Fig. 10 to construct. The results are shown in Plate 13.

At a certain point in the construction, the two-dimensional pattern of strips is forced into the third dimension to form a dome. Once again the students are confronted with a mathematical property of space which forces a move from the second into the third dimension, namely, that the sum of the angles around a vertex is less than the 360° required to lie flat in a plane. The difference between 360° and this sum, known as the spherical deviation, is characteristic of all polyhedra.

Thinking back to the octet space frame of the section on polyhedra, we now visualize the marshmallows at the vertices to be spheres that expand to form a close-packed array of spheres.

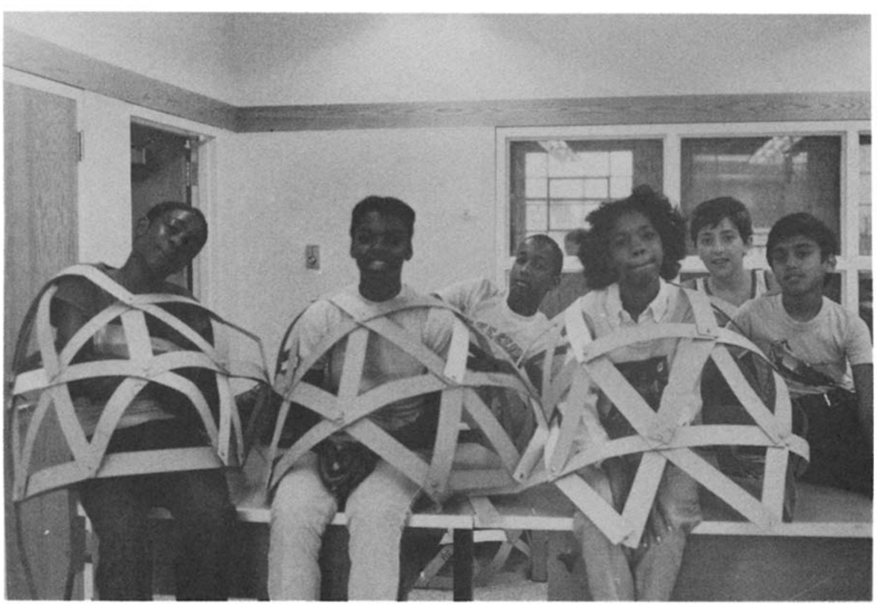


Plate 13. Some domes constructed with papers strips by Segal's method.

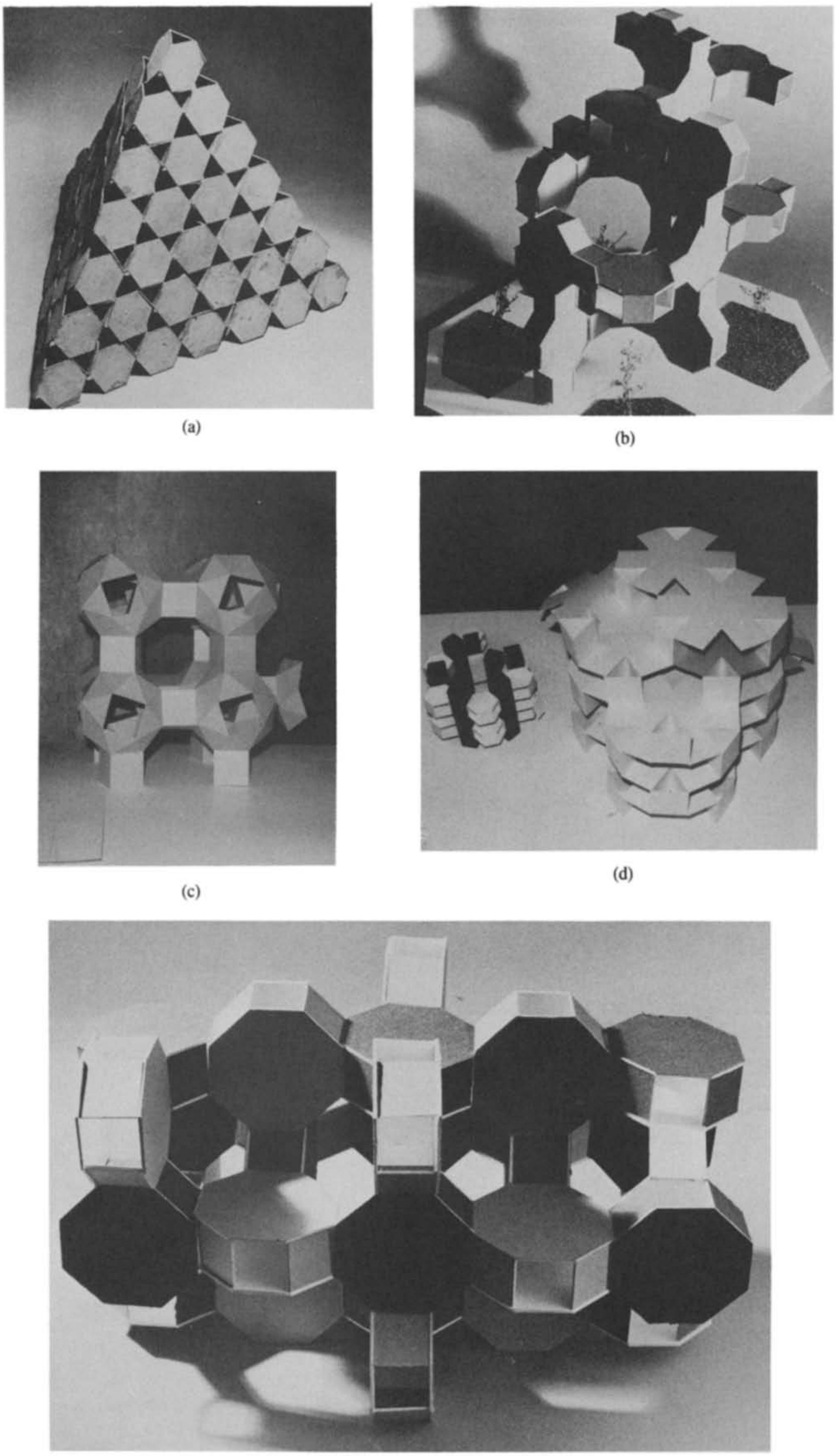
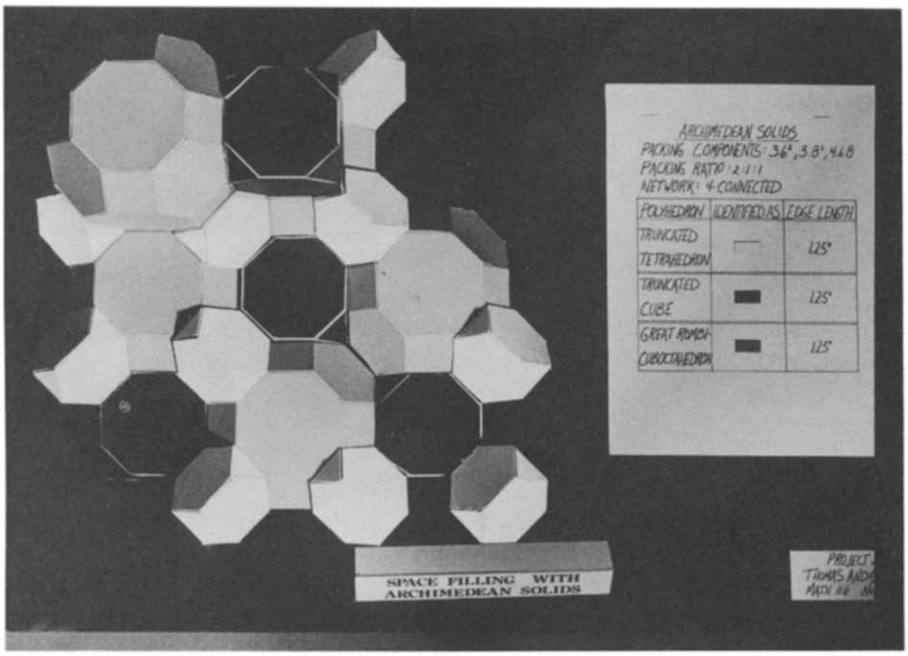


Plate 14. Some infinite regular surfaces based on a network and its dual, using Burt, Kleinmann and Wachman's method.

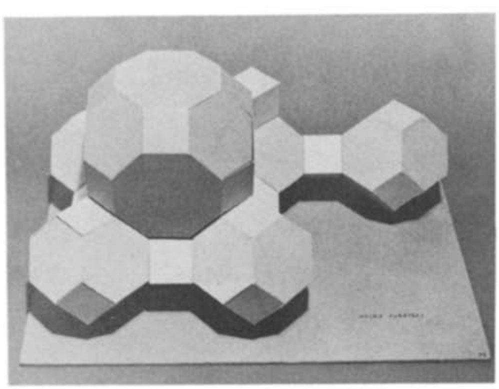
In this configuration, 12 spheres surround a central sphere, 6 in a plane, with 3 spheres lying in the interstices above and below with opposite orientations. The centers of these spheres lie at the vertices of another Archimedean solid known as the cuboctahedron (shown in Fig. 11). Also, the sphere centers of these close-packed spheres constitute one of the 14 Bravais lattices that make up the subject of crystallography, namely the face-centered cubic (FCC) lattice.

In addition to FCC we study two other lattices, the basic cubic lattice (C) and the body-centered cubic lattice (BCC), which are related, along with the FCC, to a cube. In particular, directions from point to point in these lattices occur in the edge (E), body diagonal (BD) and face diagonal (FD) directions respectively.

If lattice points are connected with edges, a network is formed made up of E, BD and FD directions. These edges also divide space into sets of space-filling polyhedra, namely, cubes, tetrahedra with curvilinear faces and octahedra and tetrahedra respectively. We can also define dual networks that connect the polyhedral centers of adjacent polyhedra through their common



(a)



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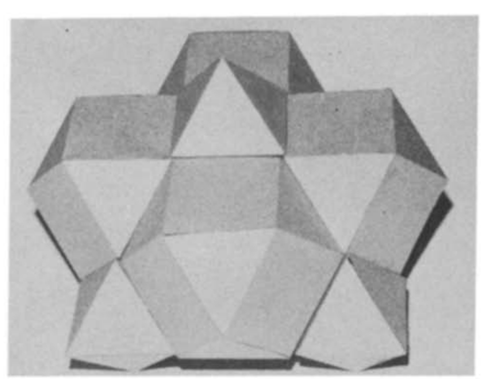


(c)

Plate 15. Two examples of space filling by combinations of Archimedean solids. (a) Cuboctahedra and octahedra; (b) great rhombicuboctahedron, truncated octahedron and cube; (c) truncated tetrahedron, truncated cube and great rhombic dodecahedron.

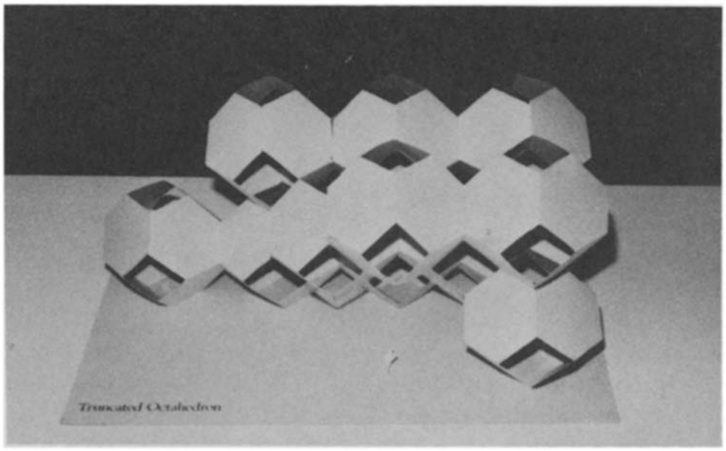


Plate 16. Space-filling truncated octahedra.

faces. Burt, Wachman and Kleinmann[17] have catalogued numerous examples of infinite rectangular surfaces with the structure of the network and its dual corresponding to a particular lattice. Student projects illustrating some of these structures are shown in Plate 14. The surface separates two connected labyrinths of tunnels[15,29].

Next we study Archimedean solids, which are the three-dimensional analogues of the semiregular tilings of the previous section. They each have more than one kind of polygonal face but surround vertices identically[11]. As for the Platonic solids, they can each be circumscribed by a sphere and through a projection from the center result in tilings of the sphere along arcs of geodesics. Several combinations of Archimedean polyhedra served as modules for the infinite regular surfaces. Many combinations of Archimedean solids combine to fill space. Two examples are shown in the student constructions of Plate 15.

In this section we focus on only the cuboctahedron shown in Plate 5(a) and the truncated octahedron shown in Plate 7(a), the former because of its relation to close-packed spheres and the latter because it is the only space filler by itself among the Archimedean solids and serves as a model for soap froths as shown in Fig. 11 and Plate 16. We also study the dual of the cuboctahedron, known as the rhombic dodecahedron and shown in Plate 5(c), because it too fills space by itself, serves as the structure of beehives[13] as shown in Fig. 12 and has possibilities as an architectural module to rival the parallelopiped[21]. In fact, the cube, rhombic dodecahedron and truncated octahedron are all zonahedra and constitute the Dirichlet domains of the C, FCC and BCC lattices—all of which connects this section strongly to the previous one. The lattices and their Dirichlet domains also have connections to the structure of metallic crystals, where two species of atoms lie at the lattice points and the vertices of the DDs[30–32].

We have used the prescription of Anthony Pugh[32] to construct tensegrity models of polyhedra. Tensegrities, discovered by the sculptor Mark Snelson, are described to the students as discrete analogues of the balloon in which the skin is tensed under the enclosed gas. They combine both tension and compression to an even degree, much like the body with its skeleton and tendons, and always result in light, airy strucctures. A tensegrity model of a cuboctahedron is shown in Plate 17.

Prisms and antiprisms are the final two classes of solids studied in this section. Since the lateral faces of the prism are parallelograms, prisms are not rigid. However, they can be made rigid to lateral forces by rotating the top face relative to the bottom and connecting top vertices to bottom ones to form antiprisms with triangulated lateral faces.

Many of the students have incorporated the hexagonal antiprism into models of prefabricated paper housing consisting of sequences of vaults, semidomes and domes[33]. These models are fabricated by folding paper into a pattern of congruent isosceles triangles. The vaults can be joined together by intersection structures which enable them to be continued to tile the plane. Student models are shown in Plate 18.

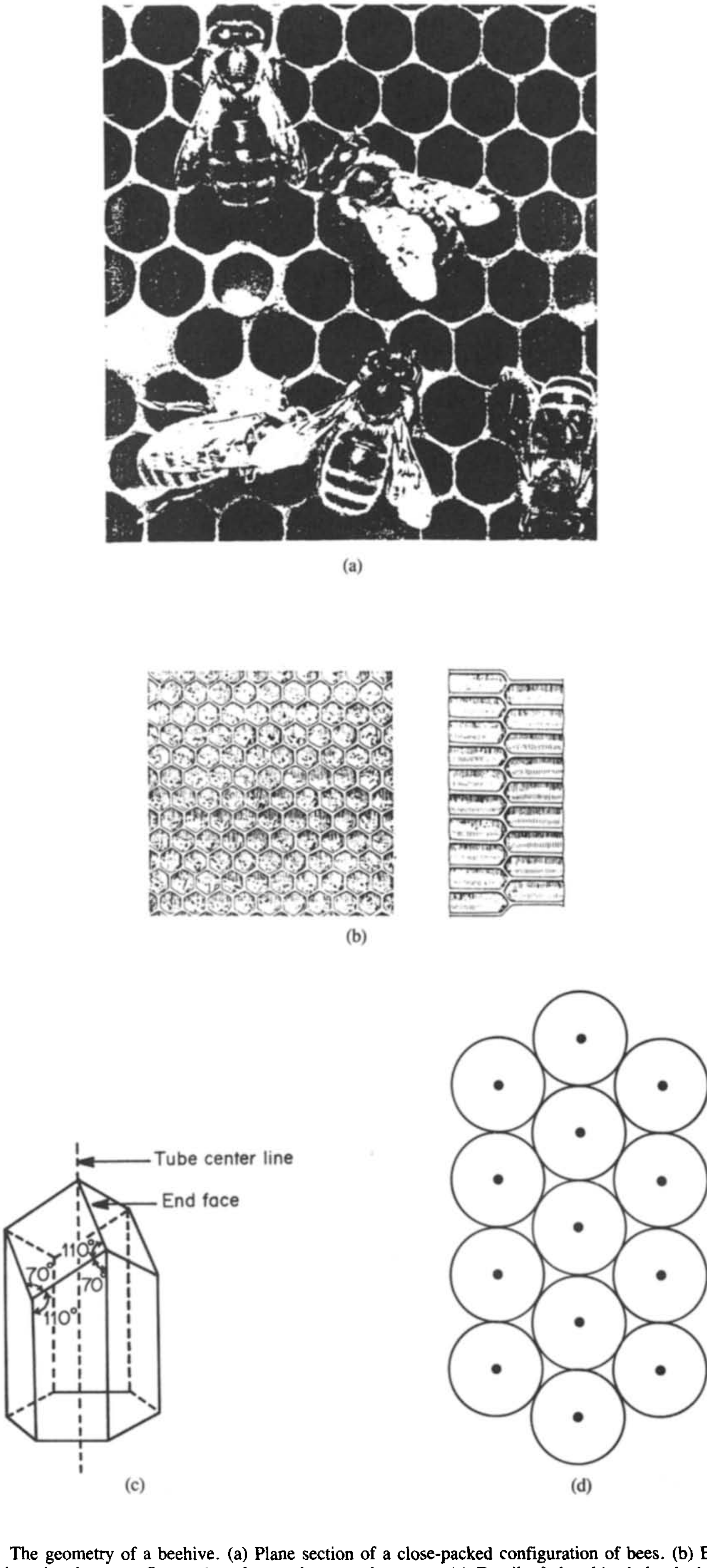
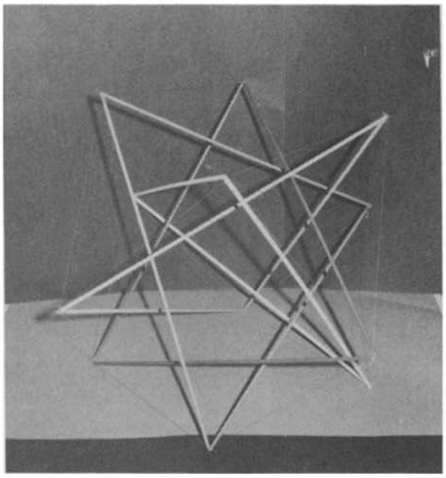
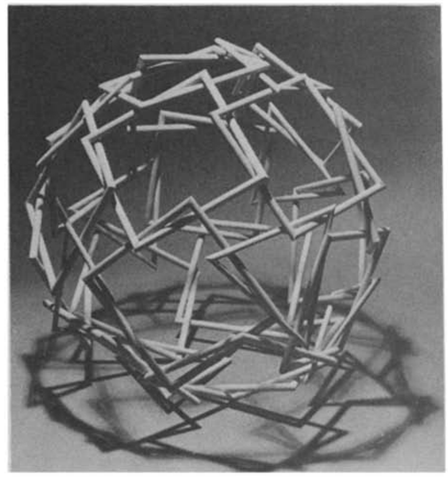


Fig. 12. The geometry of a beehive. (a) Plane section of a close-packed configuration of bees. (b) Edges of neighboring chambers are flattened to form a hexagonal pattern. (c) Detail of rhombic dodecahedron ends attached to hexagonal prisms. (d) Close packing of beehives. (Courtesy Cambridge Press[13]).



(a)



(b)

Plate 17. A tensegrity model of a cuboctahedron with inner structure of four interlocking equilateral triangles.

Writing Project

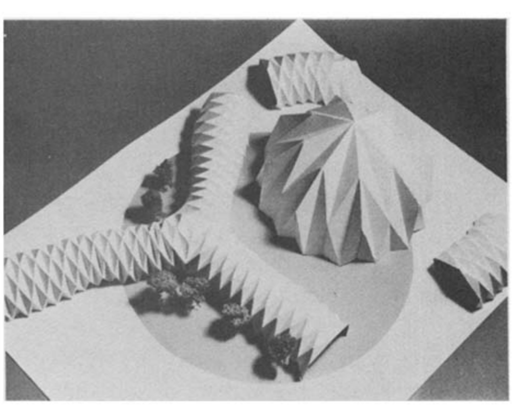
It is stated in the Kaballah, a book of Jewish mysticism, that there are actually two bibles or torahs handed to man by God: the one of the written words and the one made up of the space between those words.

Give your opinion as to whether the space left empty within a design has equal importance to the space that is filled. Use the example of infinite regular polyhedra in which space is divided into two congruent sets as an example of positive and negative space. (A student response is included in the Appendix).

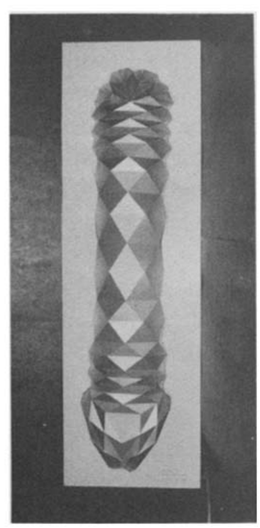
Additional material is included in the *Notes* on the structure of soap bubbles and curved surfaces in general[34,35]. In particular, we concentrate on surfaces of rotation and translation and ruled surfaces. We return to this material in the last section of the course and apply some of the ideas to constructing ruled surfaces. It has also served as an inspiration to the students to show a film at this point in the course by a master designer, Ron Resch, entitled *The Ron Resch Paper and Stick Thing Film*[36].

6. SIMILARITY, PROPORTION AND THE GOLDEN MEAN

The mathematical concept of similarity holds one of the keys to understanding processes of growth in the natural world. After all, as a member of a species grows to maturity it generally



(a)



(b)

Plate 18. Two folded-paper structures made of vaults, semidomes, domes and intersecting vaults.

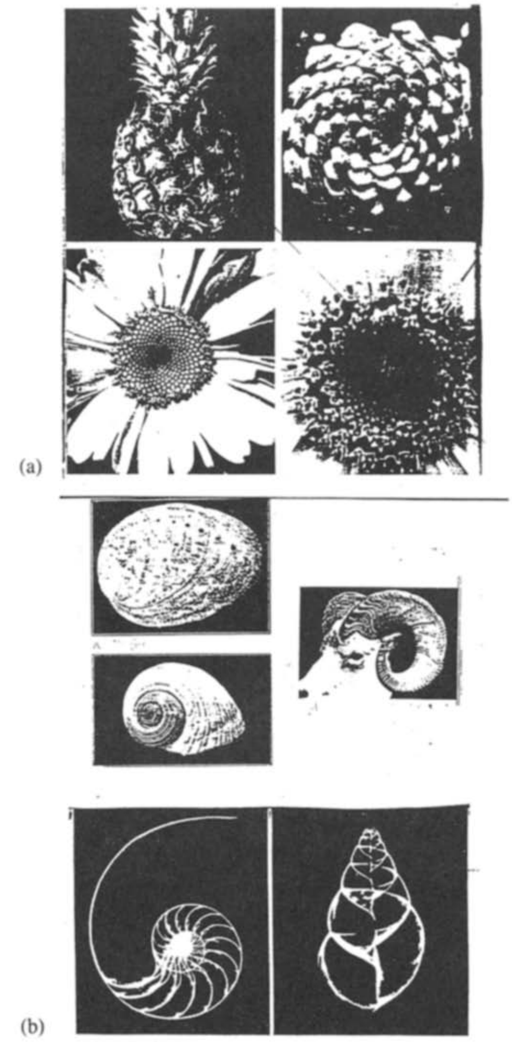


Fig. 13. The logarithmic spiral in nature. (a) Logarithmic spirals as they appear on the surface of a pineapple, pinecone and sunflowers (courtesy Dover[43]). (b) Logarithmic spirals in shells and horns (courtesy Little, Brown[37]).

transforms in such a way that its parts maintain approximately the same proportion with respect to each other. In this section of the course we show how shells, horns of horned animals and plants exhibit self-similar spiral growth[37]. In the case of plant growth, or phyllotaxis, the proportions are related to the golden mean, ϕ. The architect Le Corbusier took his cue from observations of plant growth to create a system of proportionality for architects known as the Modulor based on the golden mean. After mastering the mathematics behind the Modulor we apply it to creating designs[38,39].

As usual, the topic begins with a game, Fibonacci Nim[40]. Through this game students discover the Fibonacci series: 1,1,2,3,5,8, . . . , which is well known to lie at the heart of plant growth[41,42]. In fact, the number of spirals from the clockwise and counterclockwise sets of logarithmic spirals that appear on the surfaces of sunflowers, pine cones and pineapples are generally successive numbers from this series (as shown in Fig. 13(a)[37]) and the angular placement of stalks around the base of the plant is well known to depend on ϕ, the most frequent angle being $2\pi/\phi^2 = 136.5°$ shown in Fig. 9 of Section 3 for the pineapple. More discussion of the mathematics and mythology of the golden mean and its applications to art, architecture and biology is included in the course notes[43,44].

We begin the mathematical exposition of this subject by defining similarity and illustrating families of similar figures. We show the right triangle to be the embodiment of self-similarity by cutting a right triangle along its altitude to obtain the three similar triangles shown in Fig. 14.

From this dissection follow both the Pythagorean theorem and the mean proportionality of the altitude to the sections of the hypotenuse:

$$\frac{a}{b} = \frac{b}{c}. \tag{1}$$

Both of these theorems were considered by Kepler to be the most important truths of all geometry. From eqn (1) it follows that a geometric series of points on the logarithmic spiral can be constructed as vertices of a series of right triangles, as shown in Fig. 15. The remaining points can then be densely constructed with compass and straightedge using the growth principle: as the central angle doubles, the radius squares.

It is not surprising that the logarithmic spiral shares with the right triangle the property of self-similarity. In fact, all arcs subtending the same angle are similar. This is the same self-similarity that appears in the spiral structure of shells, horns, and other living forms shown in Fig. 13(b).

Next we show how eqn (1) governs the breakdown of rectangles into similar elements known as Gnomons and one unit similar to the original and tied together by a log spiral, as shown in Fig. 16[44]. As a result of this construction it follows that a square removed from a rectangle with golden mean proportions leaves another golden rectangle. We also discover that the golden mean ϕ forms a series

$$\ldots \frac{1}{\phi^2}, \frac{1}{\phi}, 1, \phi, \phi^2, \phi^3, \ldots$$

that is both double geometric and Fibonacci, i.e.

$$\frac{1}{\phi} + 1 = \phi, \quad 1 + \phi = \phi^2, \quad \phi + \phi^2 = \phi^3, \quad \text{etc.}$$

Artists have known that the golden mean modulates the parts of the human body. Figure 17 shows LeCorbusier's symbol of the Modulor, a 6 ft British policeman with arm outstretched and a Botticelli Venus with sections of the body modulated by powers of ϕ.

For ages architects have searched for systems of proportionality[45] to help them subdivide the inner space and facades of buildlings and the open sites upon which they placed buildings. A useful system of proportionality had to help the architect satisfy the following three design criteria. Good designs must

(i) be repetitive (made up of a small set of modules);
(ii) have parts that fit together;
(iii) be nonmonotonous (not completely predictable)[45,46].

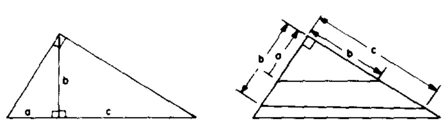


Fig. 14. Dissection of a right triangle into a family of three similar right triangles.

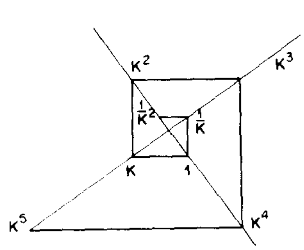


Fig. 15. Vertex points of a logarithmic spiral lie at a double geometric series of distances from the center.

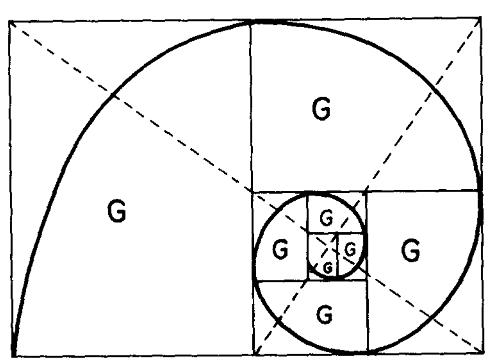
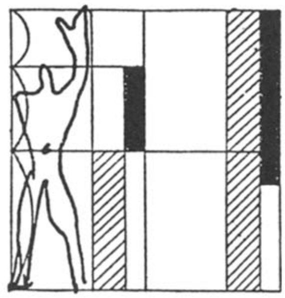
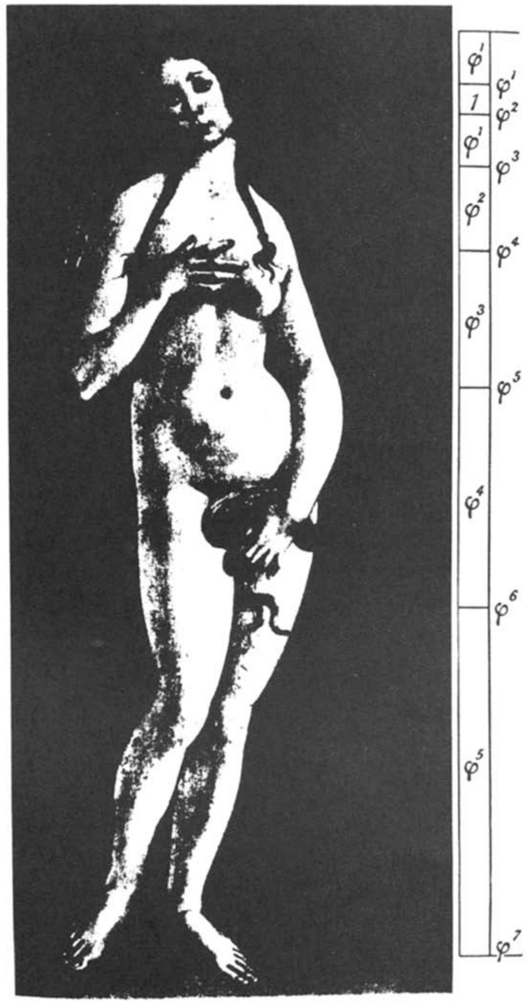


Fig. 16. Breakdown of a rectangle into a proportional unit and gnomons and spanned by a logarithmic spiral. In this example, the unit, U, has proportion $\sqrt{2}:1$ and the gnomon, G, has same proportions as the unit. For a golden mean rectangle, G is a square and U has proportion $\phi:1$.



(a)

The "trademark" of LeCorbusier's proportional system, the *Modulor*. "A man-with-arm-upraised provides, at the determining points of his occupation of space—foot, solar plexus, head, tips of fingers of the upraised arm—three intervals which give rise to a series of golden sections, called the Fibonacci series."



(b)

Fig. 17. (a) The "trademark" of LeCorbusier's proportional system, the Modulor. Three intervals give rise to a Fibonacci series of golden sections (courtesy M.I.T. Press[38]). (b) Analysis of a Botticelli Venus, using the golden mean (courtesy of Dover Press).

To satisfy these canons of architecture, Palladio used a system based on the proportions inherent in the musical scale[45,46]. Another system, used during the Renaissance, was based on geometric series. In this section we study the Modulor of Le Corbusier and show how it meets these architectural criteria.

The Modulor uses a double series of lengths known as the Red and Blue series. Each series is a geometric series with common ratio ϕ, in which for each element of the Red series there is an element of the Blue series twice as long (as shown in Fig. 18). Thus the elements of the two series complement each other by each filling in gaps between successive lengths of the other. In fact, each element of the Blue series divides the gap of two adjacent elements of the Red series in the golden section $\phi:1$. Finally, the ratio of the British policeman's upraised arm to his height in the Modulor symbol of Fig. 17(a) is $2:\phi$, a length taken from each series.

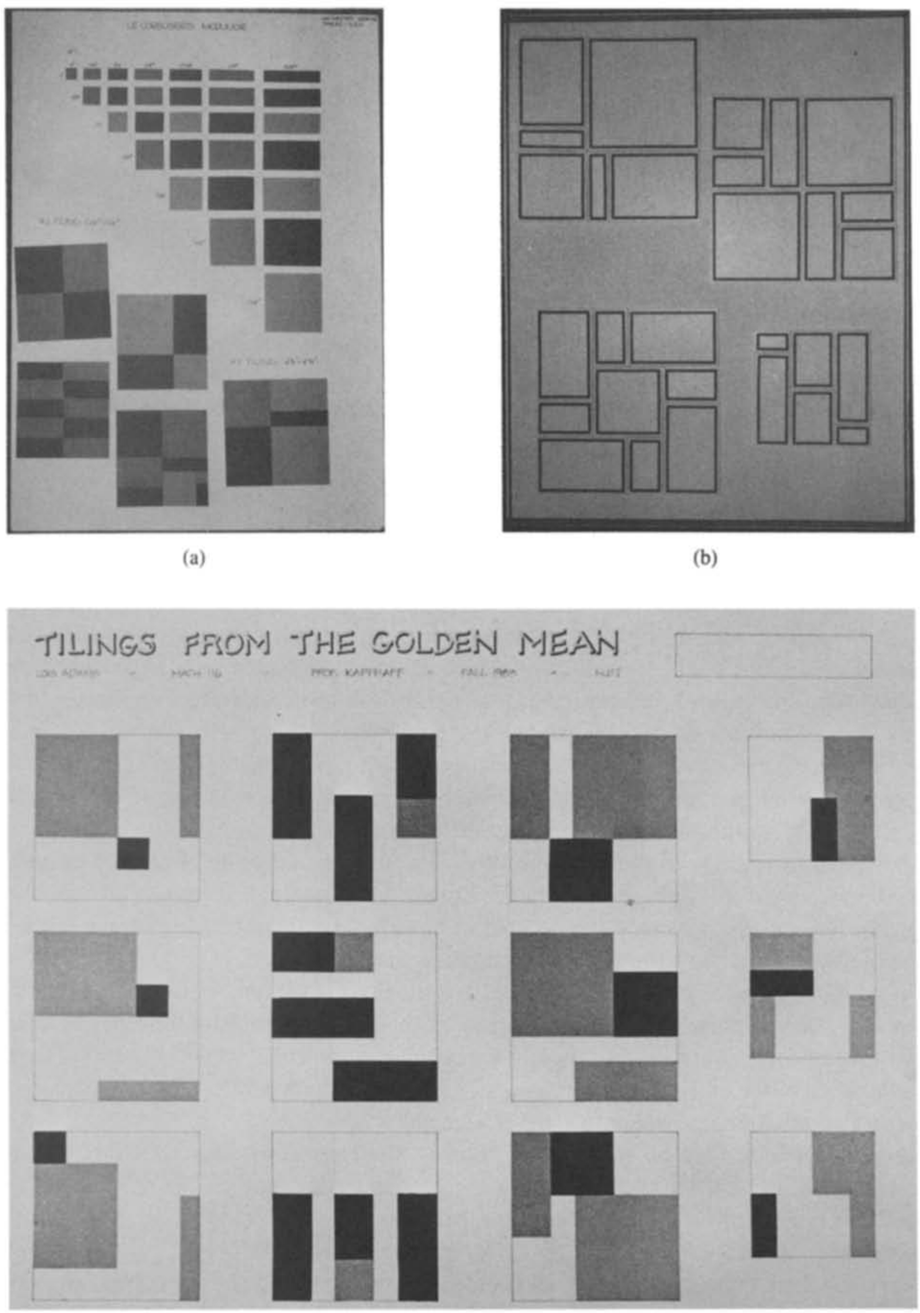
Le Corbusier used lengths from this double series to serve as dimensions of a set of rectangular tiles, as shown in Plate 19(a). The fact that any one of these tiles can be broken down into other tiles from the series by using the Fibonacci property enables tilings of a rectangle by Modulor tiles to be rearranged in many different ways to satisfy the interests of the architect. Plates 19(b) and (c) show two student projects exploring the capabilities of the Modulor system. In Plate 19(c) the same set of tiles is used to tile a rectangle three different ways.

The class of infinite self-similar curves known as fractals[47] has also been introduced in the *Notes*. Coastlines, lightning and many other forms from the natural world are fractals and these curves have potential for interesting designs (see The Geometry of Coastlines by J. Kappraff in this issue).

7. TRANSFORMATIONS

Children explore a new object by turning it first one way and then another, touching, smelling, tasting and examining it in different shadings of light. Similarly, scientists try to understand physical reality by mapping it onto abstract constructs that are easier to study and manipulate than the actual realities. Artists help others to understand the world by transforming familiar forms and objects so that the commonplace can be seen in new ways. On a more abstract plane, dancers use their bodies to transform both space and time, connecting to natural rhythms and forms inherent in the deeper levels of being. Finally, poets transform language and ideas, bringing to light deeper meanings and connections between things otherwise inaccessible to more mundane analysis.

As an introductory assignment to this section of the course, the students show through art



(a) (b)

(c)

Plate 19. Tiling of a square with the Modulor.

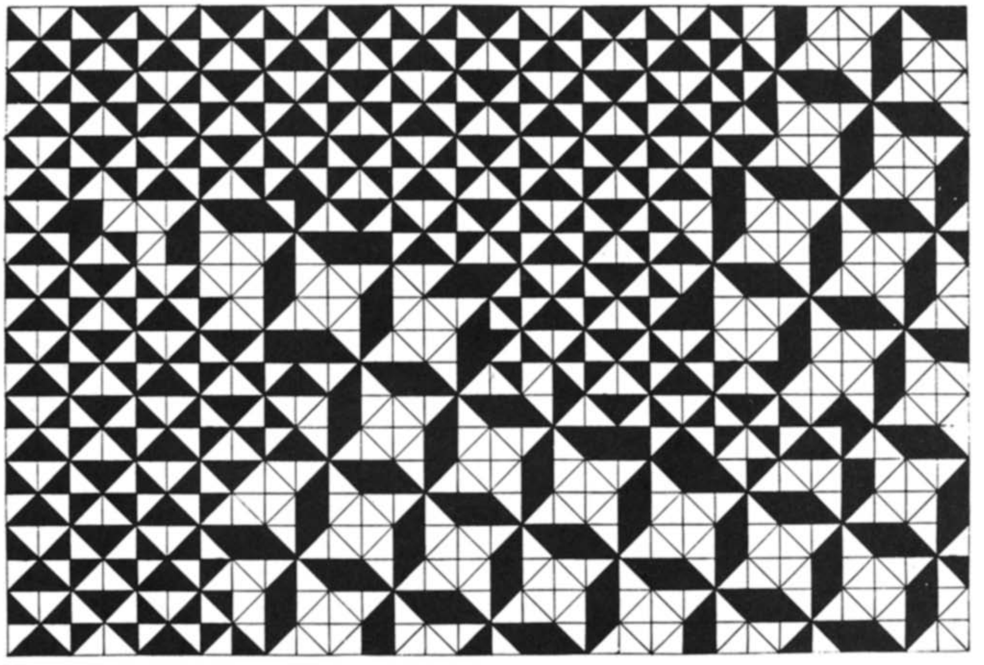


Plate 20. The panels of metamorphisis by M. C. Escher (courtesy the World of M. C. Escher[62]).

or photography how a familiar object in their everyday experience undergoes transformations. For example, they might show the tree in front of their house before and after its leaves fall or its appearance in morning and evening light, etc.

Transformation played an important but unstated role in previous sections of the course. Floor plans were transformed into graphs which were then easier to manipulate. Polyhedra were studied in planar projections. Lattice designs were constructed invariant under transformation by translation. Natural forms transformed themselves by self-similar growth. In this section, the mathematics of mappings and transformations is presented. Our immediate purpose is to use transformation to gain deeper insight into the structure of geometry and to lay the framework for studying the mathematics of symmetry in the next section. This section is organized into a hierarchy of ideas: sets, mappings, transformations in general, particular transformations associated with various kinds of projections and finally transformation by rubber-sheet topology. We show how the various geometries can be defined by invariance properties under different classes of transformations such as isometries, similarities, affinities, projectivities and topological transformations. An appendix to this section introduces matrices to carry out transformations. As usual, mathematical completeness and rigor play a subordinate role in showing how the ideas relate to concepts familiar to the students.

It is best to begin with the most primitive notion of the mapping of objects from one set to another, carried out with objects found around the classroom. Through this exercise the notions of one-to-one and many-to-one inverse mappings and transformations are described in a concrete way. Notation to represent mappings is also introduced.

It is well known to geometers that projective geometry is the most general geometry dealing with point, line and plane. It is also a natural way to show students how geometry is intimately related to transformations. In this section we specialize to planar transformations. Depending on the location of the object plane, image plane and point of projection, either projective, affine, similar transformations or isometries are produced.

For example, a road on the ground plane is transformed to a canvas from a projection point located at the artist's eye to render a scene as the artist sees it. The road which recedes in parallel lines to infinity converges on the canvas to a single point on the horizon line, (as shown in Fig. 19).

Besides demonstrating a principle of projective geometry, this example has the effect of making the elusive concept of infinity comprehensible to students. We also experiment with projecting objects by a flashlight to their shadows. In particular, conic sections are shown to

				$\frac{1}{\phi^2}$	$\frac{1}{\phi}$		1		ϕ		ϕ^2		
Red	1	1	1	1	1		1		1		1		1
Blue	1	1	1	1		1		1		1		1	
				$\frac{2}{\phi^3}$		$\frac{2}{\phi^2}$		$\frac{2}{\phi}$		2		2ϕ	

Fig. 18. The Red and Blue series.

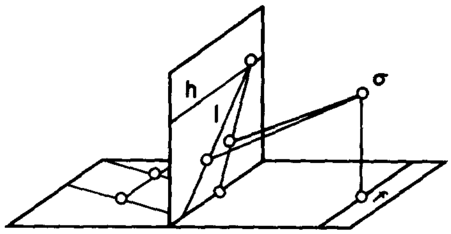


Fig. 19. A road receding to infinity depicted as converging to a point on the horizon line of an artist's canvas.

arise from circles. Finally, it is pointed out that although metric properties and even parallelism are not, in general, preserved under projective transformations, a quantity known as cross ratio is.

Projection from a point at infinity is within everyone's experience since it is the way the sun transforms objects to shadows. These so-called affine transformations generally do not preserve metric properties, unless the sun is directly overhead, but do preserve parallel lines.

If the sun is directly overhead, as it is two days per year between the tropics of Cancer and Capricorn, objects transform to their shadows under isometries preserving length and angle.

Similarity transformations, which were the subject of the previous section, map objects to enlarged or contracted similar images by means of a projection point that can be represented by the lens of a camera, overhead projector, etc. These similarity transformations preserve angle but not length.

A final class of transformations that played an important role in the design of the floor plans in Section 2 are topological transformations, which continue to be represented as transformations stretching on a rubber sheet without cutting. These transformations could be applied to design by distorting lattice tilings, such as the ones in Plate 10, by modulating the units on the coordinate axes. We have not yet tried this with our students, but the results should be reminiscent of some of Escher's prints as shown in Plate 20.

Thus we have defined a hierarchy of transformations: isometry $\rightarrow$ similarity $\rightarrow$ affinity $\rightarrow$ projectivity $\rightarrow$ topological. The corresponding geometries study the

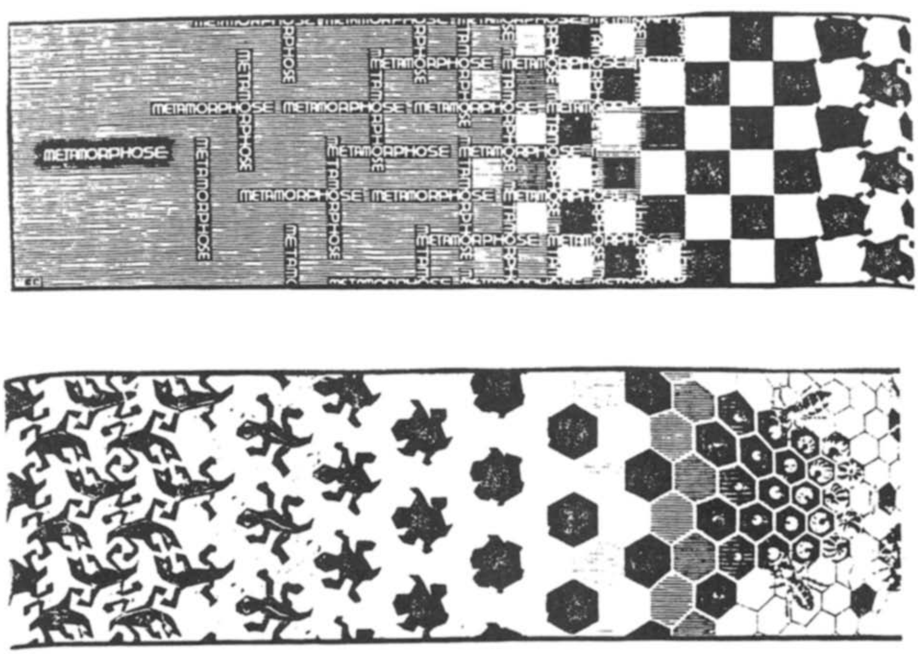


Plate 20. The panels of metamorphisis by M. C. Escher (courtesy the World of M. C. Escher[62]).

properties of figures that are invariant under these classes of transformations. Within each geometry, figures are considered ''equivalent'' or ''congruent'' if they can be transformed one to the other by transformations within that geometry. We find that even though it is beyond the scope of the course to study any one of these geometries in detail, it is nevertheless valuable for students to discover, through simple explanations, demonstrations and examples within their experience the overall structure of geometry.

The emphasis of this section of the course on projective transformations is an attempt to compensate for the complete absence of projective geometry in the educational background of students. Perspective drawing[48], an example of which is shown in Fig. 20, should be a precondition to studying geometry and should be introduced in the early grades. It creates the necessary links between what we observe in the real world and the abstractions of this world that make up the subject of Euclidean geometry[49].

The remainder of this section is devoted to studying the distance-preserving transformations or isometries. It is essential to think of isometries as rigid body movements which, in the plane, must be either translations, rotations, reflections or glide reflections. For application of isometries to the study of symmetry, it is important to classify these transformations as proper or improper. The proper transformations corresond to rigid-body motions entirely within the plane. Improper transformations require the transformed points to be removed from the plane, inverted in three-dimensional space and replaced in a manner similar to mirror images. In fact, in the next section we show that mirrors and isometries are intimately related subjects.

Writing Project

Transformation lies at the base of how people learn. For example, children learn about their world by manipulating or transforming the objects around them. Through the use of metaphor, poets and artists map the world of ideas onto their work, enabling the rest of us to sharpen our understanding of these ideas by seeing them in a different light.

Comment on this statement. Since mathematics deals primarily with transformations, state your opinion about whether mathematics can serve as a useful metaphor for architectural design. Cite ideas that you have been exposed to in Math 116 that may be applicable to architectural design. (A student response is included in the Appendix.)

Much of the material of this section is made concrete by a section in the *Notes* in which homogeneous coordinates and matrices are introduced to transform planar figures by rotation,

Fig. 20. An example of perspective in Renaissance art.

reflection, translation, similarity and projection of three-dimensional figures to a plane from a point representing the eye[50].

8. SYMMETRY

Symmetry is a concept that inspires the creative work of both artists and scientists and serves as the common root of artistic and scientific endeavors[51]. Considered naively, symmetry conjures up feelings of order, balance, harmony, and an organic relation between the whole and its parts. Artists and architects have a finely tuned sense of the symmetry of visual form without having consciously considered a precise definition of this concept from the mathematical standpoint. The objective of this section is to help students see greater possibilities for symmetry in design by exposing them to a mathematical treatment of the subject. In the process we show how the subject of symmetry is intimately connected to mirrors, as already illustrated in Section 3 with dihedral kaleidoscope. The key organizing factor for the mathematics of symmetry lies in the concept of a group of isometries[4,52].

The symmetry of a cube and the translational symmetry of lattices have already been considered in Sections 3 and 4. In this section we study bilateral symmetry, point or kaleidoscope symmetry, line or frieze symmetry and planar or wallpaper symmetry.

Before we begin to study the mathematics of symmetry it is important for students to develop an active awareness of the subject. The students are shown schematic representations and examples of the seven possible frieze patterns usable for ornamenting the edges of buildings and the patterns of wallpaper, including prints of M. C. Escher[52–54]. As an introductory exercise, the students are asked to collect as many different types of point-, line- and plane-symmetry patterns as they can from books or magazines and identify them by their symbols.

Mirrors are at once among the most familiar and puzzling of human artifacts. Why do mirrors reverse

S	but do not alter:	W	?
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Why do mirrors seem to reverse left and right, but not up and down? How do images appear in curved mirrors[55]? The students are asked to look at the following objects in a mirror and record their observations: a pear, banana, glove, box, spiral form, conical cup, etc.

The mystery behind mirrors was already hinted at in the previous section where reflections were shown to be improper transformations and thus constructible by rigid-body movements into a higher-dimensional space, a reversal and replacement to the lower-dimensional space. For a two-dimensional world, such a program can be physically implemented. However, in three dimensions the movement and reversal must take place in the fourth dimension. Such ideas do not reside in common experience. However, they are within the intellectual domain of mathematicians, artists and philosophers, and they have been described beautifully by E. A. Abbot[56] and Rudolph Rucker[57]. Thus mirrors give an entree to the subject of higher-dimensional space, although we have not yet explored this realm in the course.

For our purposes, mirrors are fundamental to an understanding of symmetry. In fact, the subject of symmetry begins with bilateral symmetry. Bilateral symmetry pervades natural forms as nature's response to the force of gravity, which distinguishes up from down but not left from right[58].

It gives valuable insight into the structure of isometries to learn how they are generated by mirrors[4,59]. In fact, the students learn that any isometry of the plane can be generated by one, two or three mirrors; intersecting mirrors generate rotations, parallel mirrors generate translations and three mutually intersecting mirrors generate glide reflections.

What the students learn about mirrors they apply to exploring the principle behind the kaleidoscope made up of two intersecting mirrors. They learn the relationship between the angle between the mirrors and the number of images of an object placed between the mirrors. Thus the region between the mirrors can be thought of as the fundamental region of the kaleidoscope symmetry. The students also construct kaleidoscope patterns by paper cutting.

After these concrete experiences with symmetry, the students are ready for a precise mathematical definition and treatment of the subject:

Definition. A symmetry of a figure is the group of isometries that keeps the figure invariant.

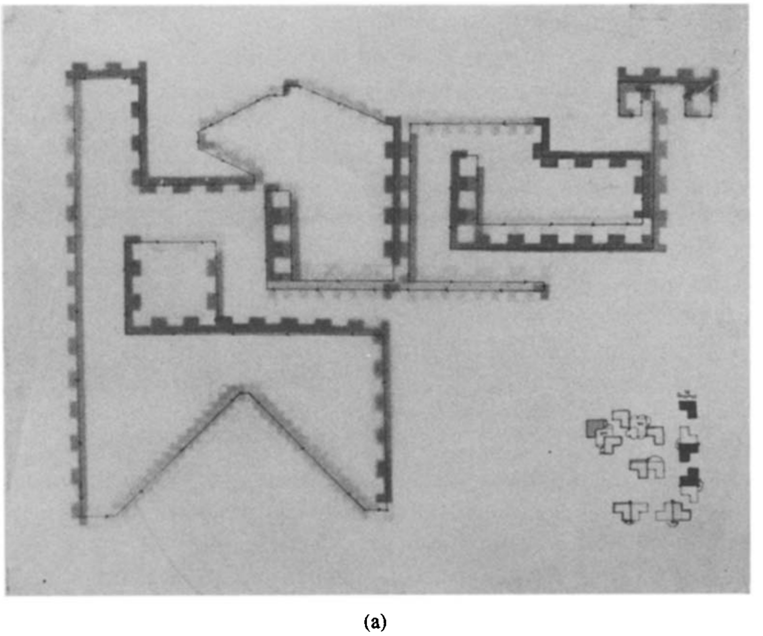
The definition is applied to several examples of point symmetry. In particular, the symmetry of the equilateral triangle, the dihedral group with three mirrors, D_3, is examined in great detail. In the process of this examination, the mathematical concept of a group is defined and applied to showing, algebraically, that all the isometries of the group can be generated by two mirrors (the kaleidoscope principle) and that symmetry patterns of D_3 are generated by transforming points from the fundamental domain between two mirrors by all the elements of the group. If the mirrors are removed from D_3, the subgroup C_3 remains. C_3 is a symmetry with rotations only that arises in floral patterns.

Finally, what we learn from studying the structure of point groups applies also to frieze and wallpaper symmetries. The fact that only two-, three-, four- and sixfold rotations occur in wallpaper patterns is the result of another constraint on the properties of space, the ''crystallographic restriction,'' which states that the images of any point of a symmetry pattern under all the transformations of the group do not accumulate at a point (i.e. there is a finite minimum distance separating them).

The emphasis of this section of the course is not on mastery of group concepts or complete cataloguing of the symmetries. An entire course could be based on this[60–62]. Rather, we are interested in conveying the idea that visually diverse symmetry patterns can have the same mathematical structure and that by understanding this structure the student can generate his or her own pattern. Some examples of student projects are shown in Plate 21.

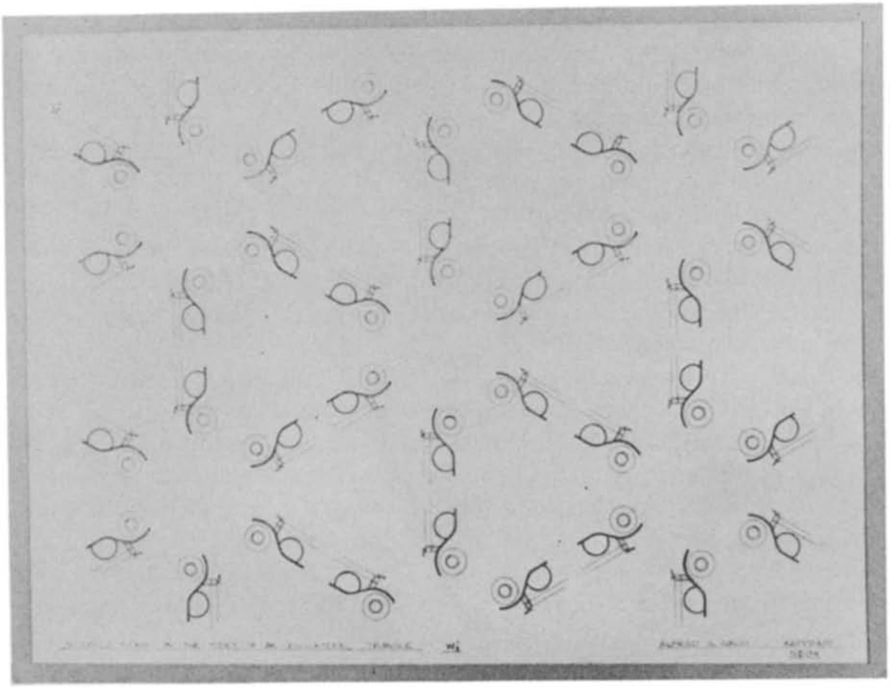
9. VECTORS

The subject of Euclidean geometry is primarily concerned with the mathematical properties of figures constructed from points, lines and planes. These elements are the abstract primitives upon which the axiomatic structure of geometry is built. Although structures of the natural world are irregular as far as we can see, points, lines and planes are nevertheless idealizations in the mind of the geometer of certain features of experience with the natural world. For example, two islands on the horizon may appear as points, while the horizon where sea meets sky is imagined to be a line. Yet we know that, unlike mathematical points, islands have extent while



(a)

Plate 21. Some examples of point, line and wallpaper symmetries.



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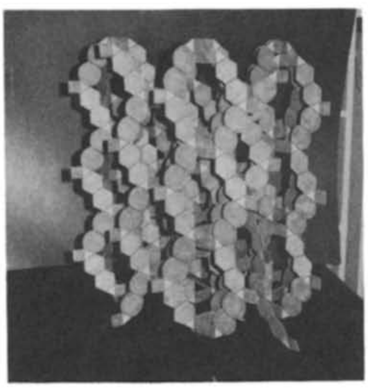

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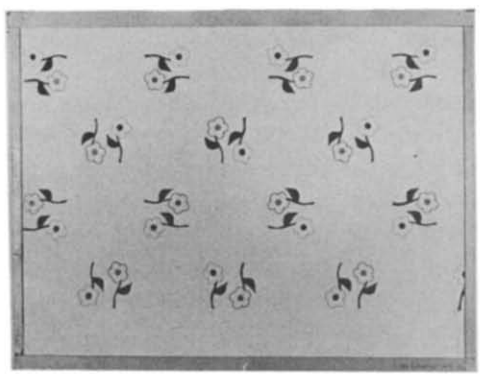

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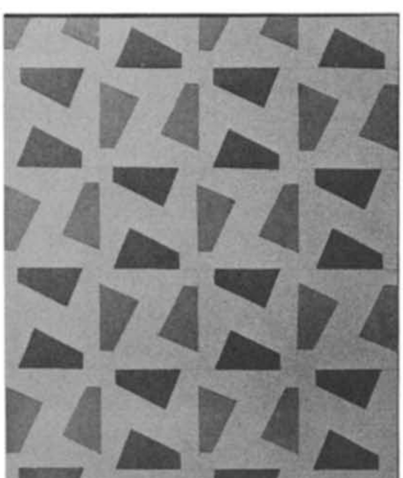

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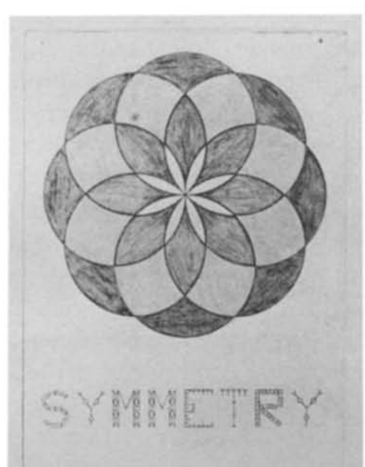


(f)

Plate 21. (Continued).

the horizon follows the curvature of the earth. Likewise, open prairies are not planes even though it is often convenient to imagine them to be. Beyond geometry, point, line and plane make up the fabric of civilization: they are the building blocks of cathedrals, skyscrapers, bridges, communication linkages, etc.

In this section we use the notion of vector to describe points, lines and planes mathematically and use these elements to study the geometry of polyhedra and a class of curved surfaces enveloped by lines known as ruled surfaces.

As an introductory exercise we ask students to find at least three examples each of configurations or objects from the natural world that can be described approximately by points, lines and planes. The students repeat the exercise for figures and objects from the world of civilization and human artifacts.

Geometrical vectors were introduced in Section 4 as vector stars to characterize the edge directions of zonagons. They were also used in that section to specify the translation directions of two- and three-dimensional lattices. In this section, since we wish to use vectors to analyze polyhedra, it is convenient to employ a unit cube as the basis of a three-dimensional cartesian coordinate system. In fact, a cube, built from the universal node system, with a tetrahedron embedded in it along with edges from the center of the cube to the vertices of the tetrahedron (as shown in Plate 5(b)) gives a sufficiently rich system of edges to begin analyzing the angles between pairs of edges and pairs of faces of polyhedra. There is educational value in using an actual cube at this stage of instruction. Any vertex of the cube can serve as the origin, while the three edge directions incident to this vertex correspond to the coordinate axes. Since the representation of a vector in this coordinate system depends only on the labeling of axes it is easily seen that representation is independent of the orientation of the axes, and that computation of angle and length are independent of any particular choice of a cartesian coordinate system, although we shall later see that the vector operation of cross product will require a choice, by convention, of a right-handed coordinate system. Finally, once an orientation of the cube is established, we represent vectors by the notation (a, b, c) in which the coordinate pairs $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$ are described by the dualisms front-back, left-right and up-down with respect to a viewer centered at the origin. I find that this approach to vectors makes representation natural and avoids the difficulty students have in comprehending the invariance properties of vectors under translation and various coordinate transformations.

Once students are able to represent vectors with confidence, they are taught the usual vector operations of addition, scalar multiplication, scalar product and cross product, with stress on the computational aspect of these operations. We have found it useful to have students compute scalar product and cross product between vectors (a_1, a_2, a_3) and (b_1, b_2, b_3) as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ and $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$ initially and then show how these computations can be made easier by introducing the $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ system.

The principal application of vectors in this section is the calculation of angles between edges and computation of the dihedral angles between faces of polyhedra related to a cube. Dihedral angles are particularly difficult for students to conceptualize, no less compute, even though the architecture students are familiar with the concept through their drafting experience. However, we have found that the vector approach makes the subject understandable. The importance of the dihedral angle was already shown in Section 5, where information about the dihedral angle between faces of the infinite regular surfaces (shown in Plate 14) were needed in order to score the paper properly in the construction. Also, as a necessary condition for polyhedra to fill space, it was shown that the sum of the dihedral angle around each edge of a space-filling array must sum to 360°. The students can now verify this condition for some of the space-filling polyhedra that have been previously mentioned.

An additional construction that requires sophisticated vector computation asks students to construct a sculpture of linear segments out of wooden dowels to form a closed cycle. The students must work out a procedure to cut the dowels so that adjacent segments match in cross section.

Finally, the representation of lines and planes in three dimensions using vectors is introduced. Three skew lines are used to generate two classes of ruled surfaces, the hyperboloid of revolution and the hyperbolic paraboloid, both of which have architectural applications[35,63,64]. These curved surfaces have already been introduced to the students in Section

5. Here we present students with practical ways to construct sculptures suggestive of architectural structures using ruled surfaces. Some student constructions are shown in Plate 22.

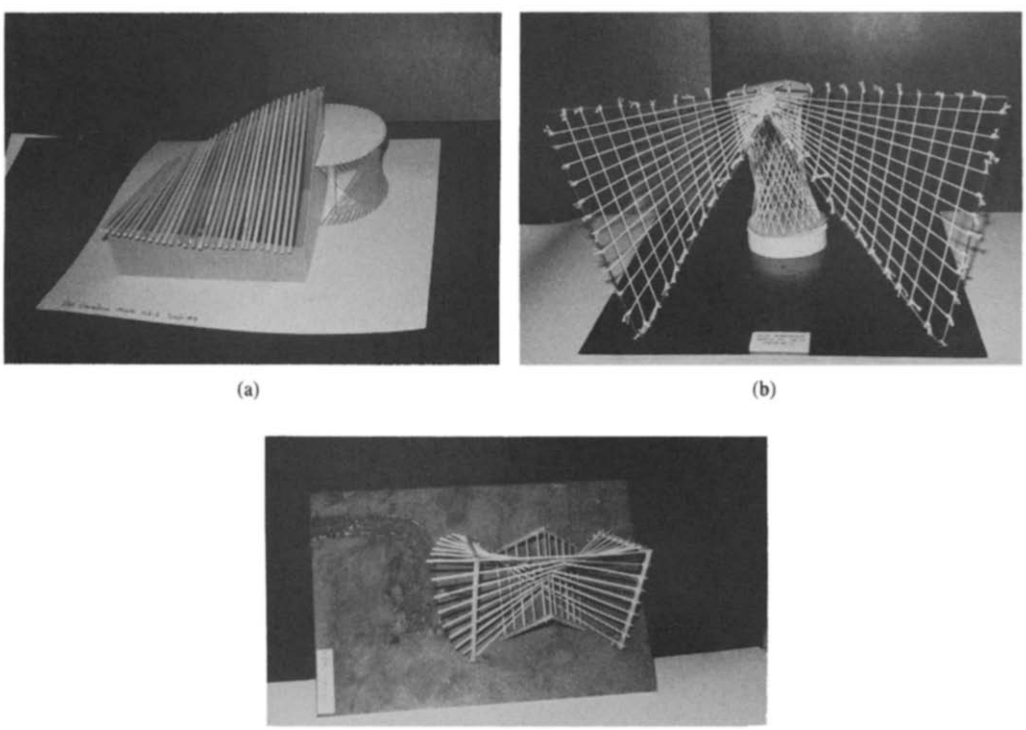
10. CONCLUSION

We feel that the course summarized in this article successfully fulfills the objectives set for it by the interdisciplinary group. We have been able to convey to students a sense of mathematics as the organizing force linking scientific, artistic and cultural subject areas. We have also made the course alive by involving students in the application of what they learn to constructing designs and projects and writing essays.

The results of teaching this course are always tangible. Each semester an exhibition of the students' best work is organized, and they share some of their writing with fellow students through school publications.

Perhaps more important to the life of this course is its steady growth in terms of subject matter and educational ideas. By no means do we wish to convey the idea that the sequence of topics in this article is either complete or the only natural ordering. Inevitably a course such as this must involve the instructor as an active participant in the formulation of curriculum. Actually, it is the creative process entered into by not just the students but the faculty that makes an exciting course possible. Much of the material of the course was unfamiliar to me when I began to work on this project. It was my own revelation that there lies rich untapped resources that has encouraged me to write this article as a suggestion of the possibilities. Beyond all the objectives that we set in organizing the course, it is most important to convey the idea that teaching a course like this is just plain fun for both teacher and students.

There are many avenues along which the work that we have begun can continue. First of all, work on a text for the course should be completed, since lack of such a book is the greatest impediment to replication of courses such as this by others. The course also should have a laboratory component in which portions now dealt with through lectures are conveyed by hands-on experience. Finally, we are giving thought to collaborating on a second course which develops computer applications for ideas generated by this course. After all, the technology and much



(a) (b) (c)

Plate 22. Ruled surface sculptures suggestive of architectural structures.

of the software is presently available to do graph-theory analysis; tile the plane; design polyhedra; transform figures by isometries and projection; experiment with symmetry; and utilize vectors, lines and planes in imaginative ways. We feel strongly, however, that unless a course such as ours is undertaken first, without computers, students will not fully appreciate the ways in which computers can enhance their design experiences.

Acknowledgments—I would like to acknowledge the vision of Professor Mary Blade who showed me by her example that courses such as this can be taught. Professor Alan Stewart helped to shape many of the ideas of the course and has been an able collaborator in teaching it each year. Professor Diana Bryant added her professional touch to the preparation of many of the photographs. Finally, Ms Rebeca Daniels provided her patience and skill to the preparation of the manuscript.

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APPENDIX
STUDENT RESPONSES TO WRITING PROJECTS

Graph Theory

In considering Christopher Alexander's thesis on the stages of design and the evolution of the design process from the unconscious instincts of primitive society to self-conscious decision making in a dynamic society, one comes to the realization that this is inevitable: the design process must have evolved parallel to all other stages of civilization's growth.

In an ideal, Hobbesian "state of nature" where man's only instinct is for survival, the concept of design must have been part of that unconscious instinct, appearing as a search for shelter. The beginnings of agricultural society and civilization were accompanied by complex needs for many types of shelter and settled communities whose designs required conscious planning with respect to function. It must be noted, however, that design of shelter, at this stage, was a skill of "everyman," passed down as a fundamental tool for survival.

With the evolution of specialization in crafts and professions in more complex economies, design probably began to emerge as a studied process. The use of architecture for the deification of both man and gods exemplifies this point of view. Inherent in the quest for more magnificent tributes to these deities is an increasing complexity of plan and structure followed by the emergence of design as a specialized art incorporating tools of math and engineering. The synthesis of complex demands and needs into a plan has become a structured "self-conscious" process.

Finally, in a dynamic technological society, the need for rapid design change is an effect of constant discovery and innovation in all facets of that society. Industrialization, space exploration, and socio-economic change heap new demands on the designer and engineer. The use of graphs as a tool for understanding linkages is certainly an appropriate step in the planning process as it facilitates and organizes the designer's thoughts at a crucial stage.

Rachel Stettler
Second-Year Architecture Student

Order and Symmetry

In his paper entitled "Perception and Modular Coordination," Christopher Alexander suggests that we enjoy symmetric themes in design because our minds recoil at chaos but are put at ease by the repetition of a simple motif. I think that this is true to a certain extent because human beings are very sensitive to their surroundings. Things like heat, lighting, color, smell and texture can have tremendous effects upon a human's mood.

In the areas of art and architecture, designs which are very intricate or haphazard cause mental stress because they demand more intense concentration. Although a high level of intricacy can cause mental stress, I feel that what makes design chaotic is a lack of cohesiveness as a whole. Intricacy is necessary to a certain extent in order to appease our appetite for new things. An artist or architect is faced with organizing and subduing his work while at the same time making it interesting. An artist does not want every part of his painting to jump out and demand equal attention. Similarly, the architect does not want his building to look like it was designed by more than one architect.

Design Project #2, which involved lattices, was a good example of repetition. I feel that there are many ways of making lattices appeal to our sense of surprise and novelty. Also, because they are on a small scale—that is, a picture or a model—I feel they can be very intricate. The level of intricacy can be increased because the viewer is not confined when viewing a picture. The level is more limited for three-dimensional lattices because they cause more mental stress in demanding that the viewer imagine three-dimensional objects twisted and intertwined.

Stephen Oliver
First-Year Architecture Student

Space Filling

Probably the most important feature of shape, the one that allows us to identify an object, is its contour, its general outline. Yet the perception of contour involves a differentiation of inside from outside, in front from behind, and, if necessity, figure from ground. In two dimensions, normally the figure stands out from the background because of a number of factors: convexity, position, texture, enclosure. The ground, whether because of lower energy or little contrast, blends into a continuous surface behind the figure. While our attention focuses on the figure, the ground is just as important because both are necessary to allow perception.

Figure is often quite different, then, in its visual qualities to make it stand out as figure. Since the artist or designer is creating both the figure and the ground, he must be aware of both to allow the differentiation to become clear. On the other hand, it is possible to create an ambiguity of figure and ground, when the ground becomes as important visually as the figure. This can be done by deliberately confusing some of the signals, the clues that we use to perceive figure as distinct from ground. The use of poché, for example, in architectural drawings, the blackening of the walls in plan, makes the thickness of the walls read as a kind of figure, when usually we think of the spaces between the walls as the figure.

In three dimensions, contour does not divide space into figure and ground, but into object and space. Sculptors, architects and dancers must all be aware of the effect their object, whether statue, building or body, has on the space around it. They are in a sense giving shape to space by placing their object in it.

Infinite polyhedra form a kind of contour, not a wiggly line as in two dimensions, but a wiggly plane separating space into two parts, inside and outside, two parts that happen to be congruent. Unlike the statue, the building or the body, there is no object which activates the space around it. There is simply the boundary between two spaces. Perhaps this is more analogous to the ambiguity it is possible to achieve in two dimensions when figure and ground can be made to have equal weight.

Allison Baxter
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Transformation

Architecture is the manipulation of forms and the creation of space through the use of those forms. Certainly, mathematics is always present in an architectural design; however, the emphasis placed on the mathematical relationships is, more often than not, secondary to the aesthetic considerations of a project. This fact is a sad one, for appreciation of the mathematical relationship within the forms and among the forms goes unnoticed. It is often true that the aesthetic choices are also the ones that offer the best mathematical metaphors, yet the aesthetic reasoning always receives the most emphasis.

While the idea of mathematics being a useful metaphor for architectural design is an intriguing one, few architects have practiced the theory to its fullest. Le Corbusier's Le Modulor epitomized the use of traditional mathematical relationships as architectural ideas while also allowing for aesthetic qualities of an outstanding calibre.

The metaphors of traditional mathematics are largely unnoticed in the current products of architecture, but the ideas of symmetry and graphing are more readily recognized as mathematical metaphors in architecture, though the field of mathematics from which they are generated is less understood by the populace than traditional mathematics. Perhaps the well-trained eye can search out and find the traditional mathematical relationships in a facade, such as ratios of window heights to the spaces between floors. It is the common eye, however, that can easily find the relationships of symmetry and graphs. These metaphors may be easily recognized, but seldom are they properly labeled. An untrained person may recognize symmetry and describe it as "the same on one side as it is on the other." The proper terminology may be lacking, but the mathematical condition known as symmetry is easily recognized by one and all.

Architects must use both traditional and nontraditional mathematical metahpors in their work; these mathematical ideas are a source of orientation and identification for users of architectural designed spaces. When done properly, the inclusion of these metaphors can create splendid architectural spaces and allow everyone an insight into the world of architecture; without mathematical metaphors, spaces become plain and lackluster.

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