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Narcissistic half-and-half power-sequence terraces for $\mathbb{Z}_n$ with $n = pq^t$

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Abstract

A power-sequence terrace for $\mathbb{Z}_n$ is a $\mathbb{Z}_n$ terrace that can be partitioned into segments one of which contains merely the zero element of $\mathbb{Z}_n$ whilst each other segment is either (a) a sequence of successive powers of an element of $\mathbb{Z}_n$, or (b) such a sequence multiplied throughout by a constant. If n is odd, a $\mathbb{Z}_n$ terrace $(a_1, a_2, \ldots, a_n)$ is a narcissistic half-and-half terrace if $a_i - a_{i-1} = a_{n+2-i} - a_{n+1-i}$ for $i = 2, 3, \ldots, (n+1)/2$. Constructions are provided for narcissistic half-and-half power-sequence terraces for $\mathbb{Z}_n$ with $n = pq^t$ where p and q are distinct odd primes and t is a positive integer. All the constructions are for terraces with as few segments as possible. Attention is restricted to constructions covering values of n with $n = pq^t$ and $n < 300$; terraces are provided for all such values except $n = 189$. Particularly elegant constructions are available for $n = 275$.

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Keywords: 2-sequencings; Euler's function; Matryoshka terraces; Power-sequence terraces; Primitive λ-roots; Primitive roots

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1. Introduction

1.1. Definitions and references

Let G be a finite group of order n with identity element e, let the group operation be multiplication, let $\mathbf{a}=(a_1,a_2,\ldots,a_n)$ be an arrangement of the elements of G, and let $\mathbf{b}=(b_1,b_2,\ldots,b_n)$ be the ordered sequence where $b_1=e$ and $b_i=a_{i-1}^{-1}a_i$ for $i=2,3,\ldots,n$. The arrangement $\mathbf{a}$ is a *terrace* [4] for G, and $\mathbf{b}$ is the corresponding 2-*sequencing* or *quasi-sequencing* for G, if $\mathbf{b}$ contains exactly one occurrence of each element $x\in G$ that satisfies $x=x^{-1}$, and if, for each $x\in G$ that satisfies $x\neq x^{-1}$, the sequence $\mathbf{b}$ contains exactly two occurrences of x but none of x^{-1}, or exactly two occurrences of x^{-1} but none of x, or exactly one occurrence of each of x and x^{-1}.

If G is $\mathbb{Z}_n$, with addition as the group operation, then x^{-1} in the above is replaced by $-x$, and the elements of the 2-sequencing are given by $b_1=0$ and $b_i=a_i-a_{i-1}$ ($i=2,3,\ldots,n$).

If G is a group of odd order n with $n=2m+1$, a terrace for G has the *half-and-half property* [2] if, for each non-zero element x of G, each of the sets $\{b_2,b_3,\ldots,b_{m+1}\}$ and $\{b_{m+2},b_{m+3},\ldots,b_n\}$ drawn from the terrace's 2-sequencing $(b_1,b_2,\ldots,b_n)$ contains either x or $-x$ exactly once. Such a terrace is *narcissistic* [2] if $b_i=b_{n+2-i}$ for all $i=2,3,\ldots,m+1$.

Anderson and Preece [3] provided general constructions for terraces for $\mathbb{Z}_n$ where n is an odd prime power, say $n=p^s$ with p an odd prime and s a positive integer. Their terraces are *power-sequence terraces* in the sense that the constructions are based on sequences of powers of elements in $\mathbb{Z}_n$. Each such terrace can be partitioned into segments one of which contains merely the zero element. Each other segment is either (a) a sequence of successive powers of an element of $\mathbb{Z}_n$, or (b) such a sequence multiplied throughout by a constant. Here the phrase "successive powers" covers index-sequences of the form $i, i+\alpha, i+2\alpha,\ldots$, where α may be any suitable positive or negative integer. For n prime, powers of primitive roots of n are used, or powers of the negatives of such primitive roots; for $n=p^s$, the elements whose powers are used are primitive roots of p^s for all s, or the negatives of such primitive roots. An example of a narcissistic half-and-half power-sequence terrace for $\mathbb{Z}_{11}$ is

$$9\ 7\ 3\ 6\ 1\ \mid\ 0\ \mid\ 10\ 5\ 8\ 4\ 2$$

i.e.

$$6^4\quad 6^3\quad 6^2\quad 6^1\quad 6^0\quad \mid\ 0\ \mid\ -6^0\ \ -6^1\ \ -6^2\ \ -6^3\ \ -6^4,$$

which uses the primitive root 6 of 11. Here, as elsewhere, we omit brackets and commas from our notation for a terrace, and we use vertical bars to separate segments. An example of a narcissistic half-and-half power-sequence terrace for $\mathbb{Z}_9$ is

$$4\ 2\ 1\ \mid\ 6\ \mid\ 0\ \mid\ 3\ \mid\ 8\ 7\ 5$$

i.e.

$$2^2\ \ 2^1\ \ 2^0\ \mid\ 6^1\ \mid\ 0\ \mid\qquad \text{negatives},$$

where, as subsequently, we omit elements following the zero, as they are merely, in reverse order, the negatives of the elements that precede the zero. This last terrace can also be written in power-sequence form as

$$5^4 \quad 5^5 \quad 5^0 \quad | \quad 6^1 \quad | \quad 0 \quad | \qquad \text{negatives}$$

as $5^6 = 5^0 \pmod 9$.

The present paper moves on from $n = p^s$ to provide general methods of construction for narcissistic half-and-half power-sequence terraces for $\mathbb{Z}_n$ with $n = pq^t$ where p and q are distinct odd primes and t is a positive integer. Structure in the ring $\mathbb{Z}_n$ with $n = pq^t$ was discussed in [1, Section 2].

We start in Section 2 with narcissistic half-and-half power-sequence terraces for $\mathbb{Z}_n$ with $n = 3p$, but first we need some results from number theory.

All our constructions are for $\mathbb{Z}_n$ terraces with as few segments as possible, and are therefore based on primitive λ-roots of n. Primitive λ-roots were introduced by Carmichael [7–9] as a generalisation of ordinary primitive roots, to cover composite positive integers n lacking primitive roots. To provide a basis for descriptions and proofs of our constructions, we now rehearse some details of both primitive roots and primitive λ-roots.

1.2. Primitive roots

If p is prime and $\omega \in \mathbb{Z}_p \setminus \{0\}$, then ω is a *primitive root* of p if $\mathrm{ord}_p(\omega) = p - 1$. All the elements of $\mathbb{Z}_p \setminus \{0\}$ are then given by $1, \omega, \omega^2, \ldots, \omega^{p-2}$. If ω is a primitive root of p, then the other primitive roots of p are precisely the elements ω^i where $\gcd(i, p-1) = 1$; thus there are $\phi(p-1)$ primitive roots of p, where $\phi(\cdot)$ is the *Euler function* (e.g. [5, p. 124; 16, p. 87; 17, p. 28]). Several of our constructions use the fact that if $p \equiv 1 \pmod 4$ then ω is a primitive root of p if and only if $-\omega$ is, whereas if $p \equiv 3 \pmod 4$ then $-\omega$ is a primitive root of p if and only if $\mathrm{ord}_p(\omega) = (p-1)/2$.

If a and b are given elements of $\mathbb{Z}_p \setminus \{0\}$, then, provided p is large enough, there exist primitive roots α and β of p such that $\alpha = a\beta + b$ [10–12]. The special cases $\alpha = \pm(2\beta - 2)$ are relevant to many of our constructions.

For any positive integer n, an integer a with $0 < a \leqslant n$ is a *unit* of $\mathbb{Z}_n$ if and only if $\gcd(a, n) = 1$. The set U_n of units in $\mathbb{Z}_n$ comprises $\phi(n)$ elements and forms a group under multiplication modulo n. If n is a prime power p^s (where p is prime and s is a positive integer), then an element ω from U_n is a primitive root of n if the order of $\omega \pmod{p^s}$ is $\phi(p^s) = p^{s-1}(p-1)$.

1.3. Primitive λ-roots

The least positive integer e such that $a^e = 1$ for all $a \in U_n$ is the *universal exponent* $e(n)$ of n [14, p. 116]. Sometimes (indeed perhaps more commonly), $e(n)$ is referred to merely as the *exponent* of n; often it is written $\lambda(n)$ (e.g. [5, p. 164; 10, Section 28; 17, p. 53]). If n takes any of the values 1, 2, 4, p^s or $2p^s$, where p is an odd prime and s is a positive integer, then $\phi(n) = e(n)$ and n has a primitive root [14, p. 99, 108]. For any other positive integer n, there is no primitive root, and $e(n) < \phi(n)$ with

$e(n) \mid \phi(n)$; then, if $x \in U_n$, the element x is a *primitive λ-root* of n if $\mathrm{ord}_n(x) = e(n)$ [9, Section 39]; [13, pp. 112–124]; [15, p. 55]. We write $\xi(n) = \phi(n)/e(n)$.

As the literature on primitive λ-roots is very sparse, detailed notes on them have been made available on the internet [6].

For values of n satisfying $\xi(n) > 1$ we define a primitive λ-root to be *negating* if it has -1 as a power, and to be *non-negating* otherwise. Similarly for values of n satisfying $\xi(n) > 1$ we define a primitive λ-root x of n to be *inward* if $x-1 \in U_n$ and to be *outward* otherwise; a primitive λ-root x of n is inward if and only if $\gcd(x-1, n)=1$. We define a primitive λ-root that is both non-negating and inward to be a *strong primitive λ-root.* The constructions in this paper are based on strong primitive λ-roots.

1.4. Primitive roots and λ-roots for $n = pq^t$

If $n = pq^t$ where p and q are distinct odd primes and t is a positive integer, then

$$\phi(n) = (p-1)q^{t-1}(q-1) = n(1-p^{-1})(1-q^{-1}),$$

$$e(n) = \mathrm{lcm}(\,(p-1), q^{t-1}(q-1)\,)$$

and

$$\xi(n) = \gcd(\,(p-1), q^{t-1}(q-1)\,)$$

with $\xi(n)$ even.

If $n = pq$ where p and q are distinct odd primes, the elements of $\mathbb{Z}_n$ consist of (i) zero, (ii) the set U_n of units of $\mathbb{Z}_n$, (iii) the set $W_{n,p}$ of non-zero multiples of p in $\mathbb{Z}_n$, and (iv) the set $W_{n,q}$ of non-zero multiples of q in $\mathbb{Z}_n$. If the element x from U_n has orders $\mathrm{ord}_p(x)$, $\mathrm{ord}_q(x)$ and $\mathrm{ord}_n(x)$ in $\mathbb{Z}_p$, $\mathbb{Z}_q$ and $\mathbb{Z}_n$, respectively, then

$$\mathrm{ord}_n(x) = \mathrm{lcm}(\mathrm{ord}_p(x), \mathrm{ord}_q(x)).$$

In particular, a common primitive root x of p and q is a primitive λ-root of n (see [15, p. 54]; [17, p. 109]) and may indeed be a strong primitive λ-root, e.g. $(n, p, q, x) = (15, 5, 3, 2)$. However, a strong primitive λ-root of n may be a primitive root of just one of p and q, or of neither. For example, as $\mathrm{ord}_3(2) = 2$ and $\mathrm{ord}_7(2) = 3$ we have $\mathrm{ord}_{21}(2) = 6$, so that 2 is a strong primitive λ-root of 21 and a primitive root of 3 but not a primitive root of 7. Also, as $\mathrm{ord}_7(18) = \mathrm{ord}_7(4) = 3$ and $\mathrm{ord}_{13}(18) = \mathrm{ord}_{13}(5) = 4$ we have $\mathrm{ord}_{91}(18) = 12$, so that 18, despite being a strong primitive λ-root of 91, is not a primitive root of either 7 or 13.

Still with $n = pq$, the members $p, 2p, \ldots, (q-1)p$ of $W_{n,p}$ constitute, under multiplication, a group whose identity element $I_{n,p}$ is the member that can be written in the form $iq + 1$ for some integer i. Likewise the members of $W_{n,q}$ constitute a group whose identity element $I_{n,q}$ can be written $jp + 1$ for some integer j. If $x \in W_{n,p}$, then the order of x in $W_{n,p}$ is the same as the order of x in $\mathbb{Z}_q$; thus $W_{n,p}$ contains as many distinct elements of order $q-1$ as there are distinct primitive roots of q. Likewise if $x \in W_{n,q}$, then the order of x in $W_{n,q}$ is the same as the order of x in $\mathbb{Z}_p$; thus $W_{n,q}$ contains as many distinct elements of order $p-1$ as there are distinct primitive roots of p.

If $n = pq^t$ where p and q are distinct odd primes and t is an integer with $t > 1$, a common primitive root x of p and q^t is a primitive λ-root of n and may indeed be a strong primitive λ-root, e.g. $(n, p, q, t, x) = (45, 5, 3, 2, 2)$. A strong primitive λ-root x of n may be a primitive root of p but not of q^t, e.g. $(n, p, q, t, x) = (117, 13, 3, 2, 71)$, or a primitive root of q^t but not of p, e.g. $(n, p, q, t, x) = (63, 7, 3, 2, 2)$, or a primitive root of neither, e.g. $(n, p, q, t, x) = (63, 7, 3, 2, 44)$.

2. Terraces with $n = 3p$

If $n = 3p$ where p is an odd prime with $p > 3$, then $\phi(n) = 2(p-1)$ and $e(n) = \operatorname{lcm}(2, p-1) = p-1$ so that $\xi(n) = 2$. Thus the units of $\mathbb{Z}_n$ can be segregated into (a) the powers of a primitive λ-root x of n, and (b) the remaining units. A primitive λ-root x of n may satisfy either $x \equiv 1$ or $x \equiv 2 \pmod 3$. If a strong primitive λ-root x is chosen so that the units in (b) are the negatives of those in (a), we have met one of the requirements for a narcissistic half-and-half power-sequence $\mathbb{Z}_n$ terrace in which, on each side of the central zero, there is only one segment for the units. The full set of requirements is met by the conditions imposed in the following theorem, which produces terraces where, on each side of the central zero, there is also only one segment for each of $W_{n,3}$ and $W_{n,p}$. The theorem is a special case of Theorem 3.1 in the next section, but is given here both because of its importance and for clarity of exposition.

Theorem 2.1. *Let p be a prime, $p \geqslant 3$, and let $\delta = 1$ or 2 according as $p \equiv 1$ or $5 \pmod 6$, respectively, so that δp is the identity element $I_{3p,p}$ of $W_{3p,p}$. Choose an element x satisfying $x \equiv 2 \pmod 3$, such that*

(a) *x is a primitive root of p if $p \equiv 1 \pmod 4$ or has order $(p-1)/2 \pmod p$ if $p \equiv 3 \pmod 4$,*

and suppose that there is an element y of $\mathbb{Z}_p$ satisfying $(2x-1)(y-1) \equiv \pm y \pmod p$ and such that

(b) *y is a primitive root of p if $p \equiv 1 \pmod 4$ and has order $p-1$ or $(p-1)/2 \pmod p$ if $p \equiv 3 \pmod 4$.*

Then x is a strong primitive λ-root of n, and the sequence

$$y^{(p-3)/2}(2x-1) \quad y^{(p-5)/2}(2x-1) \quad \dots \quad y(2x-1) \quad (2x-1) \quad | \quad x \quad x^2 \quad \dots \quad x^{p-2} \quad 1 \quad | \quad \delta p \quad | \quad 0 \quad | \quad \text{negatives}$$

is a narcissistic half-and-half power-sequence terrace for $\mathbb{Z}_{3p}$. Depending on whether y is of order $(p-1)/2$ or $p-1 \pmod p$, the first segment of the terrace can be written in the form

$$z^a \quad z^{a+2} \quad \dots \quad z^{a+(p-3)}$$

or

$$z^a \quad z^{a+1} \quad \dots \quad z^{a+(p-3)/2}$$

respectively, where z is an element of order $p-1$ from $W_{3p,3}$; if y is of order $(p-1)/2 \pmod p$ and a is even, then the first segment of the terrace can be written as

$$w^b \quad w^{b+1} \quad \dots \quad w^{b+(p-3)/2},$$

where w is an element of order $(p-1)/2$ from $W_{3p,3}$.

Proof. As $\mathrm{ord}_{3p}(x) = \mathrm{lcm}\,(\mathrm{ord}_p(x), 2) = p-1$ in all cases, x is a primitive λ-root of $3p$. To show that x is non-negating, we now prove that there is no integer i such that $x^i \equiv -1 \pmod{3p}$.

Suppose that $x^j \equiv -1 \pmod{3p}$; then (i) $x^j \equiv -1 \equiv 2 \pmod 3$, and (ii) $x^j \equiv -1 \pmod p$. By (i), j is odd. If $p \equiv 1 \pmod 4$ then, as x is a primitive root of p, (ii) requires $j \equiv (p-1)/2 \pmod{p-1}$ so that j is even; this gives us a contradiction. If $p \equiv 3 \pmod 4$, the element -1 is not a square [14, p. 126]. But if x has order $(p-1)/2$ then x is the square of a primitive root ω of p and so the element $-1 \equiv x^j \equiv \omega^{2j}$ is a square, again giving us a contradiction. Thus $\{1, x, \dots, x^{p-2}\} \cup \{-1, -x, \dots, -x^{p-2}\} = U_{3p}$.

As $x \equiv 2 \pmod 3$, $2x-1$ is a multiple of 3. Further, p does not divide $(2x-1)$, as otherwise the condition $(2x-1)(y-1) \equiv \pm y \pmod p$ would require $p|y$. So the elements in the first segment are members of $W_{3p,3}$. Suppose that $(2x-1)y^i \equiv \pm(2x-1)y^j \pmod{3p}$ for some integers i and j satisfying $0 \leqslant i < j < (p-1)/2$. Then, on putting $2x-1 = 3\mu$, we have $\mu y^i \equiv \pm \mu y^j \pmod p$, whence $y^{j-i} \equiv \pm 1 \pmod p$ where $0 < j-i < (p-1)/2$. This is impossible if y has order $p-1$, and can happen if y has order $(p-1)/2$ only when $j-i=(p-1)/4$; so it is impossible if $p \equiv 3 \pmod 4$. Thus no member of the initial sequence is equal to any other or its negative, and the proposed terrace does indeed contain each element of $\mathbb{Z}_{3p}$ exactly once.

Now consider the differences, noting that δ is chosen so that $\delta p - 1$ is a multiple of 3. The non-negating primitive λ-root x is inward, and therefore strong, as $x \not\equiv 1 \pmod 3$, and $x \not\equiv 1 \pmod p$ as otherwise x would have order less than $(p-1)/2 \pmod p$. The differences arising from the proposed terrace are ± the following:

$$x(x-1),\ x^2(x-1),\ \dots\ x^{p-2}(x-1);\ x-1;\ \delta p - 1;\ \delta p;$$
$$(2x-1)(y-1),\ (2x-1)(y-1)y,\ \dots\ (2x-1)(y-1)y^{(p-5)/2}.$$

To obtain all possible differences we require that

$$\begin{array}{rrcll} & \delta p - 1 & \equiv & \pm(2x-1)(y-1)y^{(p-3)/2} & \pmod{3p} \\ \text{i.e.} & \delta p - 1 & \equiv & \pm(2x-1)(y-1)y^{(p-3)/2} & \pmod p \\ \text{i.e.} & -1 & \equiv & \pm(2x-1)(y-1)y^{(p-3)/2} & \pmod p \\ \text{i.e.} & y & \equiv & \pm(2x-1)(y-1)y^{(p-1)/2} & \pmod p \\ \text{i.e.} & \pm y & \equiv & (2x-1)(y-1) & \pmod p. \end{array}$$

Thus, the conditions stated in the theorem ensure that the given sequence is a terrace.

If y is a primitive root of p, let z_1 be the inverse of y in $\mathbb{Z}_p$. Then z_1 is also a primitive root of p. Choose $z \in W_{3p,3}$ such that $z \equiv z_1 \pmod p$. Then z has order $p-1$

in $W_{3p,3}$. If we define a by $y^{(p-5)/2}(2x-1)=z^a$, the initial section of the terrace can be written as $z^a, z^{a+1}, \ldots, z^{a+(p-3)/2}$.

If y has order $(p-1)/2 \pmod p$, then z_1, again taken to be the inverse of y in $\mathbb{Z}_p$, also has order $(p-1)/2 \pmod p$. Thus $z_1 = \omega_1^2$ for some primitive root ω_1 of p. Choose z from $W_{3p,3}$ such that $z \equiv \omega_1 \pmod p$. Then z generates $W_{3p,3}$ and, if $y^{(p-3)/2}(2x-1)=z^a$, the first section of the terrace is $z^a, z^{a+2}, \ldots, z^{a+(p-3)}$. The choice $w=z^2$, taken in conjunction with $a=2b$, yields the final assertion of the theorem. □

Note on existence: The construction in Theorem 2.1 works provided that there exist primitive roots x and y of p such that

(i) if $p \equiv 1 \pmod 4$, then $(2x-1)(y-1) \equiv \pm y \pmod p$;
(ii) if $p \equiv 3 \pmod 4$, then $(-2x-1)(y-1) \equiv \pm y \pmod p$.

Now Cohen [11, p. 47] showed that, provided that p is sufficiently large, there exist primitive roots α and β of p such that $\alpha = 2\beta - 2$. In case (i) choose $x = -\alpha^{-1}$, $y = \beta$; then x and y are primitive roots of p and

$$\begin{aligned}(2x-1)(y-1) &= -(2\alpha^{-1}+1)(\beta-1)\\ &= -(2+\alpha)2^{-1}\cdot 2(\beta-1)\alpha^{-1} = -\beta = -y.\end{aligned}$$

In case (ii) choose $x = \alpha^{-1}$, $y = \beta$; then $(-2x-1)(y-1) = -y$.

Example 2.1. $\mathbb{Z}_{15}$ terrace with $p=5$, $x=y=2$; here y is of order $p-1=4 \pmod 5$, and $z=3$, $a=4$, and $z^5=3^5=3^1 \pmod{15}$:

$$\begin{aligned}&6\ \ 3 \mid 2\ \ 4\ \ 8\ \ 1 \mid 10 \mid 0 \mid 5 \mid 14\ \ 7\ \ 11\ \ 13 \mid 12\ \ 9\\ &\quad = 3^4\ \ 3^1 \mid 2^1\ \ 2^2\ \ 2^3\ \ 2^0 \mid 10^1 \mid 0 \mid \text{negatives.}\end{aligned}$$

Consistently with Theorem 1 of [17, p. 112], the elements in segments 2, 3, 4 and 5 together constitute a difference set (mod 15), as therefore do the elements in segments 3, 4, 5 and 6.

Example 2.2. $\mathbb{Z}_{33}$ terrace with $p=11$, $x=14$, $y=4$; here y is of order $(p-1)/2 = 5 \pmod{11}$, and $z=6$, $a=8$ (giving $w=3$, $b=4$); also $6^{12}=6^2 \pmod{33}$, i.e. $3^6 = 3^1 \pmod{33}$:

$$\begin{aligned}&15\ \ 12\ \ 3\ \ 9\ \ 27 \mid 14\ \ 31\ \ 5\ \ 4\ \ 23\ \ 25\ \ 20\ \ 16\ \ 26\ \ 1 \mid 22 \mid 0 \mid \text{negatives}\\ &\quad = 6^8\ \ 6^{10}\ \ 6^2\ \ 6^4\ \ 6^6 \mid 14^1\ \ 14^2\ \ \ldots\ \ 14^9\ \ 14^0 \mid 22^1 \mid 0 \mid \text{negatives}\\ &\quad = 3^4\ \ 3^5\ \ 3^1\ \ 3^2\ \ 3^3 \mid 14^1\ \ 14^2\ \ \ldots\ \ 14^9\ \ 14^0 \mid 22^1 \mid 0 \mid \text{negatives.}\end{aligned}$$

For $n=3p$ with $n<70$, the pairs of values (x,y) that satisfy Theorem 2.1 are as follows, where the two x-values within braces $\{\}$ are inverses of one another, and where a y-value marked with an asterisk is of order $(p-1)/2 \pmod p$, which can arise only if $p \equiv 3 \pmod 4$.

n	(x,y)
15	$(2,2)$
21	$(2,5)$
33	$\{(5,2 \text{ or } 8),\ (20,4^*)\},\ \{(14,4^*),\ (26,3^* \text{ or } 5^*)\}$
39	$\{(11,11),\ (32,2)\}$
51	$\{(5,6),\ (41,7 \text{ or } 11)\},\ \{(11,7 \text{ or } 11),\ (14,3)\},\ (20,14),\ (44,12)$
57	$\{(5,13),\ (23,17^*)\},\ \{(17,4^* \text{ or } 6^*),\ (47,2)\},\ \{(35,17^*),\ (44,3)\}$
69	$\{(8,6^* \text{ or } 11),\ (26,7 \text{ or } 20)\},\ \{(29,8^*),\ (50,5 \text{ or } 21)\},$ $\{(32,14 \text{ or } 15),\ (41,8^*)\},\ \{(59,2^* \text{ or } 16^*),\ (62,6^* \text{ or } 11)\},$ $(2,13^* \text{ or } 18^*)$.

For $70 < n < 300$, single or pairs of solutions (n,p,x,y) for each value of n are as follows, the pairs being given where $p \equiv 3 \pmod 4$, so as to provide one solution where y is of order $p-1 \pmod p$ and one where y (again asterisked) is of order $(p-1)/2 \pmod p$:

(n,p,x,y)	
$p \equiv 1 \pmod 6$	$p \equiv 5 \pmod 6$
$(93,31,80,22),\ (93,31,14,7^*)$	$(87,29,2,8)$
$(111,37,2,20)$	$(123,41,17,7)$
$(129,43,38,26),\ (129,43,14,24^*)$	$(141,47,8,45),\ (141,47,2,25^*)$
$(183,61,26,35)$	$(159,53,2,14)$
$(201,67,35,2),\ (201,67,23,33^*)$	$(177,59,5,38),\ (177,59,116,16^*)$
$(219,73,14,14)$	$(213,71,8,67),\ (213,71,29,12^*)$
$(237,79,2,60),\ (237,79,5,72^*)$	$(249,83,11,55),\ (249,83,23,10^*)$
$(291,97,14,57)$	$(267,89,23,30)$

3. Terraces with $n = pq$ and $\xi(n) = 2$

If $n = pq$ where p and q are distinct odd primes, we may have $\xi(n) = 2$, as for $n = 33, 35, 39, 51, 55, 57, 69, 77, 87, 93, 95, \ldots$, or $\xi(n) = 4$, as for $n = 65, 85, \ldots$, or $\xi(n) = 6$, as for $n = 91, \ldots$, etc. We now generalise Theorem 2.1 to cover completely the first of these possibilities.

As the restriction on $\xi(n)$ requires $\gcd(p-1, q-1) = 2$, it prevents us from having both p and q congruent to $1 \pmod 4$. For x to be a primitive λ-root of n we now require $\operatorname{lcm}(\operatorname{ord}_p(x), \operatorname{ord}_q(x)) = (p-1)(q-1)/2$, and so $\operatorname{ord}_p(x)$ must be $p-1$ or

$(p-1)/2$ and $\mathrm{ord}_q(x)$ must be $q-1$ or $(q-1)/2$. Further, we require

(i) if $p \equiv 1 \pmod 4$ then x is a primitive root of p;
(ii) if $q \equiv 1 \pmod 4$ then x is a primitive root of q;
(iii) if $p \equiv q \equiv 3 \pmod 4$ then x is a primitive root of at least one of p and q.

For x to be non-negating, (iii) must be replaced by

(iv) if $p \equiv q \equiv 3 \pmod 4$ then x is a primitive root of precisely one of p and q,

for if x were a primitive root of both p and q we would have

$$x^{((p-1)/2)((q-1)/2)} \equiv -1 \pmod{pq}.$$

Theorem 3.1. *Let n be a positive integer satisfying $n = pq$ where p and q are distinct odd primes, $q \not\equiv 1 \pmod 8$, such that $\xi(n) = 2$, and where 2 is a primitive root of q or, if $q \equiv 7 \pmod 8$, is of order $(q-1)/2 \pmod q$. Let x be a strong primitive λ-root of n with $2x \equiv 1 \pmod q$, and let y be an element from $\mathbb{Z}_p$ that satisfies $(2x-1)(y-1) \equiv \pm y \pmod p$. Then the sequence*

$$\begin{array}{c} y^{(p-3)/2}(2x-1) \quad y^{(p-5)/2}(2x-1) \quad \dots \quad y(2x-1) \quad (2x-1) \quad | \\ x \quad x^2 \quad \dots \quad x^{e(n)-1} \quad 1 \quad | \\ 2^0 I_{n,p} \quad 2^1 I_{n,p} \quad \dots \quad 2^{(q-3)/2} I_{n,p} \quad | \quad 0 \quad | \quad \text{negatives} \end{array}$$

is a narcissistic half-and-half power-sequence terrace for $\mathbb{Z}_n$ provided that y is a primitive root of p if $p \equiv 1 \pmod 4$, and has order $p-1$ or $(p-1)/2 \pmod p$ if $p \equiv 3 \pmod 4$. The first segment of the terrace can be written in power-sequence form exactly as described in Theorem 2.1, and the third segment can be written in power-sequence form similarly.

Proof. As in the proof of Theorem 2.1, we have to show that there is no integer i such that $x^i \equiv -1 \pmod{pq}$. For example, if $p \equiv q \equiv 3 \pmod 4$ and x has order $(q-1)/2 \pmod q$, then $x = \omega^2$ for some primitive root ω of q, so that the congruence $x^i \equiv -1 \pmod q$ would yield $(\omega^i)^2 \equiv -1 \pmod q$, contradicting the fact that -1 is not a square $\pmod q$ when $q \equiv 3 \pmod 4$.

As $2x \equiv 1 \pmod q$, we have $\mathrm{ord}_q(2) = \mathrm{ord}_q(x)$ and so the elements $I_{n,p}, 2I_{n,p}, \dots, 2^{(q-3)/2} I_{n,p}$ are distinct $\pmod q$. Further, if $2^i I_{n,p} \equiv -2^j I_{n,p} \pmod{pq}$ for some i, j satisfying $0 \leqslant i < j < (q-1)/2$, then $2^{j-i} \equiv -1 \pmod q$. This is impossible if $\mathrm{ord}_q(2) = q-1$, whereas if $\mathrm{ord}_q(2) = (q-1)/2$ it requires $j - i = (q-1)/4$, whence $q \equiv 1 \pmod 4$, contradicting $q \equiv 7 \pmod 8$.

Also, the first segment consists of multiples of q, and an argument exactly as for Theorem 2.1 of the present paper shows that the proposed sequence contains each multiple of q exactly once.

Finally, the differences arising from the proposed terrace constitute all the possible differences provided that

$$I_{n,p} - 1 \equiv \pm(2x-1)(y-1)y^{(p-3)/2} \pmod{pq}$$

i.e.

$$-1 \equiv \pm(2x-1)(y-1)y^{(p-3)/2} \pmod{p},$$

exactly as in the proof of Theorem 2.1. □

Note on existence: As for Theorem 2.1, the existence of appropriate x and y follows from [11, p. 47] for all sufficiently large p. The only case needing further consideration is when $p \equiv 3 \pmod 4$ and $q \equiv 7 \pmod 8$. Here, we need $\mathrm{ord}_q(2) = \mathrm{ord}_q(x) = (q-1)/2$ and so x has to be a primitive root of p. We now use the existence of primitive roots α and β such that $-\alpha = 2\beta - 2$. Taking $x = \alpha^{-1}$ and $y = \beta$ we obtain $(2x-1)(y-1) = -y$.

Example 3.1. $\mathbb{Z}_{35}$ terrace with $p=7$, $q=5$, $x=y=3$; here y is of order $p-1=6 \pmod 7$:

$$\begin{array}{l} 10\ 15\ 5 \mid 3\ 9\ 27\ 11\ 33\ 29\ 17\ 16\ 13\ 4\ 12\ 1 \mid 21\ 7 \mid 0 \mid \text{ negatives} \\ \quad = 5^5\ \ 5^6\ \ 5^1 \mid 3^1\ \ 3^2\ \ \ldots\ \ 3^{11}\ \ 3^0 \mid 7^4\ \ 7^1 \mid 0 \mid \text{ negatives.} \end{array}$$

Consistently with Theorem 1 in [17, p. 112], the elements in segments 2, 3, 4 and 5 together constitute a difference set (mod 35).

Example 3.2. $\mathbb{Z}_{35}$ terrace with $p = 7$, $q = 5$, $x = 3$, $y = 2$; here y is of order $(p-1)/2 = 3 \pmod 7$:

$$\begin{array}{l} 20\ 10\ 5 \mid 3\ 9\ 27\ 11\ 33\ 29\ 17\ 16\ 13\ 4\ 12\ 1 \mid 21\ 7 \mid 0 \mid \text{ negatives} \\ \quad = 5^3\ \ 5^5\ \ 5^1 \mid 3^1\ \ 3^2\ \ \ldots\ \ 3^{11}\ \ 3^0 \mid 7^4\ \ 7^1 \mid 0 \mid \text{ negatives.} \end{array}$$

The same comment applies as in Example 3.1.

Example 3.3. $\mathbb{Z}_{35}$ terrace with $p=5$, $q=7$, $x=32$, $y=2$; here y is of order $p-1=4 \pmod 5$:

$$\begin{array}{l} 21\ 28 \mid 32\ 9\ 8\ 11\ 2\ 29\ 18\ 16\ 22\ 4\ 23\ 1 \mid 15\ 30\ 25 \mid 0 \mid \text{negatives} \\ \quad = 28^4\ \ 28^1 \mid 32^1\ \ 32^2\ \ \ldots\ \ 32^{11}\ \ 32^0 \mid 30^3\ \ 30^1\ \ 30^2 \mid 0 \mid \text{negatives} \\ \quad = 28^4\ \ 28^1 \mid 32^1\ \ 32^2\ \ \ldots\ \ 32^{11}\ \ 32^0 \mid 10^6\ \ 10^2\ \ 10^4 \mid 0 \mid \text{negatives.} \end{array}$$

For $n < 300$ and $p, q > 3$, values of (p, q, x, y) for single or paired solutions for each value of n with $\xi(n) = 2$ are as follows; we adopt the same procedure and notation as in Section 2 to distinguish y-values of different orders, and we now use dashes to indicate where a solution does not exist despite the conditions on p and q being met:

n	(p, q, x, y)	
	$p > q$	$p < q$
35	$(7, 5, 3, 3)$, $(7, 5, 3, 2^*)$	$(5, 7, 32, 2)$
55	$(11, 5, 13, 7)$, $(11, 5, 13, 9^*)$	$(5, 11, 17, 2)$
77	$(11, 7, 46, 7)$, $(11, 7, 74, 5^*)$	$(7, 11, 72, 5)$, $(7, 11, —, —^*)$
95	$(19, 5, 53, 13)$, $(19, 5, 3, 4^*)$	$(5, 19, 67, 2)$
115	$(23, 5, 3, 7)$, $(23, 5, 48, 13^*)$	$(5, 23, 12, 2)$
119	$(17, 7, 116, 3)$	
143	$(13, 11, 6, 2)$	$(11, 13, 46, 7)$, $(11, 13, 46, 9^*)$
155	$(31, 5, 133, 13)$, $(31, 5, 103, 20^*)$	
161	$(23, 7, 158, 5)$, $(23, 7, 60, 9^*)$	$(7, 23, 150, 3)$, $(7, 23, 12, 2^*)$
187	$(17, 11, 105, 14)$	
203	$(29, 7, 11, 26)$	$(7, 29, 73, 3)$, $(7, 29, 73, 2^*)$
209	$(19, 11, 6, 3)$, $(19, 11, 61, 17^*)$	$(11, 19, 181, 8)$, $(11, 19, 48, 3^*)$
215	$(43, 5, 3, 12)$, $(43, 5, 13, 10^*)$	
235	$(47, 5, 3, 13)$, $(47, 5, 13, 3^*)$	$(5, 47, 212, 2)$
253	$(23, 11, 39, 11)$, $(23, 11, 6, 8^*)$	$(11, 23, 35, 7)$, $(11, 23, 35, 9^*)$
287	$(41, 7, 53, 29)$	
295	$(59, 5, 3, 50)$, $(59, 5, 3, 16^*)$	$(5, 59, 207, 2)$
299	$(23, 13, 7, 19)$, $(23, 13, 33, 9^*)$	$(13, 23, 58, 2)$

The gaps above for $(n, p, q) = (119, 7, 17)$, $(187, 11, 17)$, $(287, 7, 41)$ arise as 2 is a square $(\text{mod}\, q)$ if $q \equiv 1\,(\text{mod}\, 8)$ and so 2 cannot then be a primitive root of q. The gaps for $n = 155$ and 215 arise as 2 is not of order $15\,(\text{mod}\, 31)$ and is not a primitive root of 43.

As we have indicated for terraces obtainable from Theorem 3.1, the first and third segments, although each consisting of a multiple of a sequence of successive powers of an element, can each be rewritten merely as a sequence of successive powers. The same is true for analogous segments from terraces given throughout the rest of this paper. However, to avoid encumbering the text, we henceforth omit details of these alternative representations.

4. Terraces with $n = pq$ and $\xi(n) = 4$

If $n = pq$ where p and q are distinct odd primes, and $\xi(n) = 4$, then $p \equiv 1\,(\text{mod}\, 4)$ and $q \equiv 1\,(\text{mod}\, 4)$ but we cannot have p and q both congruent to $1\,(\text{mod}\, 8)$. For such a value of n, a narcissistic half-and-half power-sequence terrace for $\mathbb{Z}_n$ must have, on each side of the central zero, at least *two* segments for elements of U_n, as well as at least one segment for elements of $W_{n,p}$ and at least one segment for elements of $W_{n,q}$.

We have found three elegant constructions for achieving the minimum number of segments if 2 is a primitive root of q, which requires $q \equiv 5 \,(\mathrm{mod}\, 8)$. If 2 is a primitive root of *both* p and q, which requires both p and q to be congruent to $5\,(\mathrm{mod}\, 8)$, interchanging the roles of p and q enables these three constructions to give, in total, six patterns that *prima facie* may be tried for a particular n. Thus a goodly number of $\mathbb{Z}_n$ terraces can be produced.

To describe a $\mathbb{Z}_n$ terrace obtained from any one of the three constructions, we use m_1, m_2, m_3, m_4 for the differences "missing" from, respectively, the first four segments of the terrace, and f_1, f_2, f_3, f_4 for the differences arising at, respectively, the first four "fences" between segments of the terrace. With this notation, the constructions are as given in Theorems 4.1–4.3, as follows.

Theorem 4.1. *Let $n = pq$ where p, q are odd primes such that $\xi(n) = 4$, i.e.* $\gcd(p-1, q-1) = 4$, *and where 2 is a primitive root of q. Suppose that x is a strong primitive λ-root of n such that* $\gcd(2x-1, n) = 1$ *and k is a unit such that $(2x-1)k \equiv x \,(\mathrm{mod}\, n)$ where neither k nor $-k$ is a power of x. Write $v = k + x - 1$. If $v \equiv 0 \,(\mathrm{mod}\, q)$ and if an element y satisfying $y \equiv v(v \pm 1)^{-1} \,(\mathrm{mod}\, p)$ is a primitive root of p, then*

$$\begin{gathered}
y^{(p-3)/2}v \quad y^{(p-5)/2}v \quad \dots \quad yv \quad v \;| \\
k \quad kx^{e(n)-1} \quad kx^{e(n)-2} \quad \dots \quad kx^2 \quad kx \;|\; x \quad x^2 \quad \dots \quad x^{e(n)-1} \quad 1 \;| \\
2^0 I_{n,p} \quad 2^1 I_{n,p} \quad \dots \quad 2^{(q-3)/2} I_{n,p} \;|\; 0 \;|\; \text{negatives}
\end{gathered}$$

is a narcissistic half-and-half power-sequence terrace for which $m_1 = \pm f_3$, $m_2 = -f_2$, $m_3 = \pm f_1$.

Proof. The conditions on x and k ensure that

$$U_n = \{\pm 1, \pm x, \dots, \pm x^{e(n)-1}\} \cup \{\pm k, \pm kx, \dots, \pm kx^{e(n)-1}\}.$$

As y is a primitive root of p, we have $\{v, yv, \dots, y^{p-1}v\} = W_{n,q}$; as 2 is a primitive root of q, we have $\{I_{n,p}, 2I_{n,p}, \dots, 2^{q-1}I_{n,p}\} = W_{n,p}$.

Now $m_1 = v(y-1)y^{(p-3)/2}$, $m_2 = k(x-1)$, $m_3 = x - 1$, $f_1 = v - k$, $f_2 = x(k-1)$ and $f_3 = I_{n,p} - 1$. Thus $f_1 = m_3$ since $v = k + x - 1$, and $f_2 = -m_2$ since $(2x-1)k \equiv x \,(\mathrm{mod}\, n)$. Further, $f_1 = \pm m_1$ if and only if $v(y-1)y^{(p-3)/2} \equiv \pm(I_{n,p} - 1) \,(\mathrm{mod}\, pq)$ and, as both sides are divisible by q and as $I_{n,p}$ is a multiple of p, this condition is equivalent to $v(y-1)y^{(p-3)/2} \equiv \pm 1 \,(\mathrm{mod}\, p)$, i.e. $v(y-1) \equiv \pm y \,(\mathrm{mod}\, p)$, i.e. $y \equiv v(v \pm 1)^{-1} \,(\mathrm{mod}\, p)$. □

Note: (a) The construction fails for $p = 5$ as we must then have (x, k) equal to $(1, 1)$ or $(2, 4)$ or $(4, 2)$. The last two of these possibilities yield $v \equiv 0 \,(\mathrm{mod}\, 5)$, which is impossible, whereas the first gives only numbers that are congruent to 1 or 4, modulo 5, in U_n.

(b) If $q=5$, we can take x to be any primitive root of p such that $x\equiv 4\,(\mathrm{mod}\,5)$. Then $k\equiv 2\,(\mathrm{mod}\,5)$ and $\pm k$ cannot be a power of x. Further, $v=k+x-1$ is congruent to $0\,(\mathrm{mod}\,5)$. So the construction of Theorem 4.1 applies for $q=5$ provided $v(v\pm 1)^{-1}$ is a primitive root of p. Similarly, if $q=13$, we can take $x\equiv 3\,(\mathrm{mod}\,13)$; then $k\equiv 11\,(\mathrm{mod}\,13)$, so that $\pm k$ cannot be a power of x, and $v=k+x-1$ is congruent to $0\,(\mathrm{mod}\,13)$.

Example 4.1. A $\mathbb{Z}_{65}$ terrace from Theorem 4.1 with $(p,q)=(13,5)$, $x=19$, $k=62$, $v=15$, $y=2$:

25 45 55 60 30 15 | 62 58 27 63 17 18 42 33 12 28 22 8 |
19 36 34 61 54 51 59 16 44 56 24 1 | 52 | 0 | negatives.

Theorem 4.2. *Let* $n=pq$ *where* p, q *are primes with* $\gcd(p-1,q-1)=4$ *and with* 2 *a primitive root of* q. *Suppose that* x *is a strong primitive* λ*-root of* n *such that* $\gcd(2x-1,n)=1$, $x\equiv 2\,(\mathrm{mod}\,q)$ *and* k *is a unit such that* $kx\equiv 2x-1\,(\mathrm{mod}\,n)$ *where neither* k *nor* $-k$ *is a power of* x. *If* $v\equiv k(2-x)\,(\mathrm{mod}\,n)$ *and if an element* y *satisfying* $y\equiv v(v\pm 1)^{-1}\,(\mathrm{mod}\,p)$ *is a primitive root of* p, *then the* $\mathbb{Z}_n$ *sequence as printed in Theorem* 4.1 *is a narcissistic half-and-half power-sequence terrace for* $\mathbb{Z}_n$ *with* $m_1=\pm f_3$, $m_2=\pm f_1$, $m_3=-f_2$.

Proof. Similar to the proof of Theorem 4.1. □

Example 4.2. A $\mathbb{Z}_{65}$ terrace from Theorem 4.2 with $(p,q)=(5,13)$, $x=54$, $k=8$, $v=39$, $y=3$:

52 39 | 8 17 28 27 33 62 18 22 63 12 58 42 |
54 56 34 16 19 51 24 61 44 36 59 1 | 40 15 30 60 55 45 |
0 | negatives.

Theorem 4.3. *Let* $n=pq$ *where* p, q *are primes with* $\gcd(p-1,q-1)=4$ *and with* 2 *a primitive root of* q. *Write* $v=2x-1$ *where* x *is a strong primitive* λ*-root of* n *such that* $v\equiv 0\,(\mathrm{mod}\,q)$. *If an element* y *satisfying* $y\equiv v(v\pm 1)^{-1}\,(\mathrm{mod}\,p)$ *is a primitive root of* p *and if* $ky\equiv -1\,(\mathrm{mod}\,p)$ *for some* $k\in U_n$ *such that neither* k *nor* $-k$ *is a power of* x, *then*

$$\begin{array}{c} k\quad kx^{e(n)-1}\quad kx^{e(n)-2}\quad \ldots\quad kx^2\quad kx\quad | \\ y^{(p-3)/2}v\quad y^{(p-5)/2}v\quad \ldots\quad yv\quad v\ |\ x\quad x^2\quad \ldots\quad x^{e(n)-1}\quad 1\ | \\ 2^0 I_{n,p}\quad 2^1 I_{n,p}\quad \ldots\quad 2^{(q-3)/2} I_{n,p}\ |\ 0\ |\ \text{negatives} \end{array}$$

is a narcissistic half-and-half power-sequence terrace for $\mathbb{Z}_n$ *with* $m_1=\pm f_1$, $m_2=\pm f_3$, $m_3=-f_2$.

Proof. We have $m_1=k(x-1)$, $m_2=v(y-1)y^{(p-3)/2}$, $m_3=x-1$, $f_1=y^{(p-3)/2}v-kx$, $f_2=v-x$, $f_3=I_{n,p}-1$. As $v=2x-1$ we have $m_3=f_2$. The relationship $m_2=\pm f_3$ is dealt with as for Theorem 4.1. For $m_1=\pm f_1$ it suffices to have $k(x-1)=vy^{(p-3)/2}-kx$, i.e. $(2x-1)k \equiv vy^{(p-3)/2} \pmod{n}$. As q divides both sides and p does not divide v, we need $k \equiv y^{(p-3)/2} \pmod{p}$, i.e. $ky \equiv -1 \pmod{p}$. □

Note on alternative solutions: As k has only to satisfy $ky \equiv -1 \pmod{p}$ and must not be a multiple of q, there are potentially $q-1$ possible values of $k \pmod{pq}$. However, some of these may be in $\{\pm x^i\colon 0 \leqslant i < e(n)\}$.

If x is a primitive root of p, the power sequence $1, x, \ldots, x^{e(n)-1}$, when considered modulo p, consists of $(q-1)/4$ subsequences each comprising $1, x, \ldots, x^{p-2} \pmod{p}$. Each of these $(q-1)/4$ subsequences will contain one member that is congruent to $k \pmod{p}$ and one that is congruent to $-k \pmod{p}$. So each subsequence rules out two values of k that are in $\{\pm x^i\}$. Thus altogether $2 \times (q-1)/4 = (q-1)/2$ values of k are ruled out, leaving $(q-1)/2$ choices for k.

If $\mathrm{ord}_p(x) = (p-1)/2$ (which is even), the sequence $1, x, \ldots, x^{e(n)-1}$, considered modulo p, comprises $(q-1)/2$ identical subsequences $\pmod{p}$ of length $(p-1)/2$. If neither k nor $-k$ is in one of these, then no power of x is congruent to $\pm k \pmod{p}$ and there are $q-1$ choices for k. If k (and hence $-k$) occurs in the first subsequence, then $2 \times (q-1)/2$ choices of k are ruled out, so no choice of k is available for the construction.

Finally, if $\mathrm{ord}_p(x) = (p-1)/4$, which occurs only when $p \equiv 5 \pmod{8}$, then the power sequence of x is, when considered modulo p, made up of $q-1$ identical subsequences of length $(p-1)/4$. As $(p-1)/4$ is odd, if k is in the subsequence then $-k$ is not. If $\pm k$ is in the first subsequence then all of the $q-1$ choices of k are ruled out; if $\pm k$ is not in the subsequence then all $q-1$ choices of k can be used in the construction.

Note on existence: The primitive λ-root x is a primitive root of q and, as $v=2x-1$, we have $v(v+1)^{-1}=(2x-1)(2x)^{-1}$. In the case $p \equiv 1 \pmod{8}$, Cohen's results [11, p. 47] again show that the construction in the theorem works for all sufficiently large p. For, with $\alpha=2\beta-2$, take $x=-\alpha^{-1}$ and $y=\beta$. Then $(2x-1)(2x)^{-1}(2+\alpha)2^{-1}=\beta=y$ and, as x is a primitive root of p, the arguments in the previous note ensure that a suitable choice of k is available.

Example 4.3. A $\mathbb{Z}_{65}$ terrace from Theorem 4.3 with $(p,q)=(13,5)$, $x=3$, $k=7$, $v=5$, $y=11$:

7 24 8 46 37 34 33 11 47 59 63 21 | 35 15 25 20 55 5 |
3 9 27 16 48 14 42 61 53 29 22 1 | 26 52 | 0 | negatives.

With the other parameter values unchanged, k can also be 46, 33 or 59, each of which merely produces a different ordering of the elements in the first segment. Here $\mathrm{ord}_p(x) = \mathrm{ord}_{13}(3) = 3 = (p-1)/4$.

Example 4.4. A $\mathbb{Z}_{65}$ terrace from Theorem 4.3 with $(p,q)=(5,13)$, $x=59$, $k=12$, $v=52$, $y=2$:

$$12\ \ 63\ \ 22\ \ 18\ \ 62\ \ 33\ \ 27\ \ 28\ \ 17\ \ 8\ \ 42\ \ 58\ \mid\ 39\ \ 52\ \mid$$
$$59\ \ 36\ \ 44\ \ 61\ \ 24\ \ 51\ \ 19\ \ 16\ \ 34\ \ 56\ \ 54\ \ 1\ \mid$$
$$40\ \ 15\ \ 30\ \ 60\ \ 55\ \ 45\ \mid\ 0\ \mid\ \text{negatives.}$$

With the other parameter values unchanged, k can be any of the 1st, 3rd, 5th, … entries in the first segment above, and it can be the negative of any of the 2nd, 4th, 6th, … entries. Here $\mathrm{ord}_p(x)=\mathrm{ord}_5(59)=2=(p-1)/2$.

Example 4.5. A $\mathbb{Z}_{85}$ terrace from Theorem 4.3 with $(p,q)=(17,5)$, $x=3$, $k=6$, $v=5$, $y=14$:

$$6\ \ 2\ \ 29\ \ 38\ \ 41\ \ 42\ \ 14\ \ 33\quad 32\ \ 39\ \ 13\ \ 61\ \ 77\ \ 54\ \ 18\ \mid$$
$$30\ \ 75\ \ 60\ \ 65\ \ 35\ \ 45\ \ 70\ \ 5\ \mid$$
$$3\ \ 9\ \ 27\ \ 81\ \ 73\ \ 49\ \ 62\ \ 16\quad 59\ \ 7\ \ 21\ \ 63\ \ 19\ \ 57\ \ 1\ \mid$$
$$51\ \ 17\ \mid\ 0\ \mid\ \text{negatives.}$$

With the other parameter values unchanged, k can also be 74; the first segment of the terrace is then the negative of the first segment above, with its two halves swapped. Here x is a primitive root of p.

Within the range $n<300$, the values of n that are covered by the above theorems are 65, 85, 145, 185, 205, 221 and 265. None of these 7 values is covered by Theorem 4 of [17, p. 119], which produces difference sets when $n=pq$ and $\xi(n)=4$. Specimen parameter sets (x,k,v,y) that yield terraces from the above constructions are as follows.

n	(p,q)	(x,k,v,y)		
		Theorem 4.1	Theorem 4.2	Theorem 4.3
65	$(13,5)$	$(19,62,15,2)$	$(-,-,-,-)$	$(3,7,5,11)$
	$(5,13)$		$(54,8,39,3)$	$(59,12,52,2)$
85	$(17,5)$	$(29,2,30,7)$	$(7,14,15,12)$	$(3,6,5,14)$
145	$(29,5)$	$(84,17,100,3)$	$(132,69,20,27)$	$(23,66,45,18)$
	$(5,29)$		$(89,103,29,3)$	$(44,37,87,2)$
185	$(37,5)$	$(94,47,140,5)$	$(12,79,135,35)$	$(53,79,105,22)$
	$(5,37)$		$(39,168,74,3)$	$(19,2,37,2)$
205	$(41,5)$	$(19,67,85,11)$	$(17,14,200,35)$	$(13,104,25,13)$
221	$(17,13)$	$(29,206,13,7)$	$(54,47,208,11)$	$(7,12,13,7)$
265	$(53,5)$	$(14,187,200,5)$	$(17,189,80,18)$	$(13,141,25,3)$
	$(5,53)$		$(214,28,159,3)$	$(239,2,212,2)$

Each of the constructions from Theorems 4.1–4.3 can be generalised by replacing x by x^γ, where γ is an integer satisfying $0 < \gamma < e(n) - 1$ and $\gcd(\gamma, e(n)) = 1$, in the segment starting with the element k. We here do not discuss this possibility further. We merely cite, as an example, that a terrace for $(n, p, q) = (65, 13, 5)$ can be obtained from the generalised Theorem 4.2 construction by taking $(x, k, \gamma, v, y) = (42, 54, 5, 25, 7)$; this terrace compensates for the lack of a terrace obtainable from the ungeneralised Theorem 4.2 construction for $(n, p, q) = (65, 13, 5)$.

5. Terraces with $n = pq$ and $\xi(n) = 6$

We now move on to values of n such that constructions for $\mathbb{Z}_n$ terraces can use strong primitive λ-roots of n that are not primitive roots of any factor of n.

If $n = pq$ where p and q are distinct odd primes and $\xi(n) = 6$, then $p \equiv 1 \pmod 6$ and $q \equiv 1 \pmod 6$. Within the range $n < 300$ there are 5 such values of n, namely 91, 133, 217, 247 and 259. For such a value of n, a narcissistic half-and-half power-sequence terrace must have, on each side of the central zero, at least *three* segments for elements of U_n.

For such a value of n, let $H = \{\pm 1, \pm x, \ldots, \pm x^{e(n)-1}\}$ where x is a strong primitive λ-root of n and $e(n)$ now satisfies $e(n) = \frac{1}{6}\phi(n)$. Then the set H is a subgroup of U_n of index 3. Thus the quotient group U_n/H, being of order 3, is cyclic, whence U_n contains an element k such that $U_n = H \cup kH \cup k^2H$.

Theorem 5.1. *Let $n = pq$ where p and q are distinct odd primes such that $\xi(n) = 6$, and where 2 has order $q - 1 \pmod q$ if $q \equiv 1 \pmod{12}$ and has order $q - 1$ or $(q - 1)/2 \pmod q$ if $q \equiv 7 \pmod{12}$. Let x be a strong primitive λ-root of n such that $U_n = H \cup kH \cup k^2H$ where $H = \{\pm 1, \pm x, \ldots, \pm x^{e(n)-1}\}$ and k is an element of U_n such that $(2x - 1)k \equiv 1 \pmod n$. If the value v given by $v = k^2x \pm (x - 1)$ is a non-zero multiple of q and if the unit y, defined by $y = v(v \pm 1)^{-1} \pmod p$, has order $p - 1 \pmod p$ if $p \equiv 1 \pmod{12}$ and has order $p - 1$ or $(p - 1)/2$ if $p \equiv 7 \pmod{12}$, then*

$$y^{(p-3)/2}v \quad y^{(p-5)/2}v \quad \ldots \quad yv \quad v \quad |$$

$$k^2x^1 \quad k^2x^0 \quad k^2x^{e(n)-1} \quad k^2x^{e(n)-2} \quad \ldots \quad k^2x^2 \quad |$$

$$k^1x^1 \quad k^1x^0 \quad k^1x^{e(n)-1} \quad k^1x^{e(n)-2} \quad \ldots \quad k^1x^2 \quad |$$

$$x^1 \quad x^2 \quad \ldots \quad x^{e(n)-1} \quad x^0 \quad |$$

$$I_{n,p} \quad 2^1 I_{n,p} \quad \ldots \quad 2^{(q-3)/2} I_{n,p} \quad | \quad 0 \quad | \quad \text{negatives}$$

is a narcissistic half-and-half power-sequence terrace for $\mathbb{Z}_n$ with $m_1 = \pm f_4$, $m_2 = \pm f_2$, $m_3 = \pm f_3$, $m_4 = \pm f_1$.

Proof. We have $m_1 = vy^{(p-3)/2}(y-1)$, $m_2 = k^2x(x-1)$, $m_3 = kx(x-1)$ and $m_4 = x-1$. Also $f_1 = v - k^2x$, $f_2 = kx(kx-1)$, $f_3 = x(kx-1)$ and $f_4 = I_{n,p} - 1$. As $(2x-1)k \equiv 1 \pmod n$, we have $m_2 = -f_2$ and $m_3 = -f_3$. As $v = k^2x \pm (x-1)$ we have $m_4 = \pm f_1$. We further require $m_1 = \pm f_4$, i.e. $vy^{(p-3)/2}(y-1) \equiv \pm(I_{n,p}-1) \pmod{pq}$. Both sides are $0 \pmod q$, so we need $vy^{(p-3)/2}(y-1) \equiv \pm 1 \pmod p$, i.e. $v(y-1) \equiv \pm y \pmod p$ since $y^{(p-1)/2} \equiv \pm 1 \pmod p$.

Finally, the conditions on the orders of x, y and 2 ensure that no member of the left half of the terrace is equal to any other or its negative. □

Note: (a) A special case is obtained by requiring the first element of the second segment of the terrace to be 1 less than the last element of the second segment, which in turn is then 1 less than the first element of the third segment. We then require $k^2x^2 - k^2x = 1$ and $kx - k^2x^2 = 1$. Using the fact that $k(2x-1) = 1$, both of the required relationships reduce to $3(x - x^2) \equiv 1 \pmod n$ or, more elegantly, to $k^2 \equiv -3 \pmod n$.

(b) It follows from $k^2x \pm (x-1) \equiv 0 \pmod q$ and $k(2x-1) \equiv 1 \pmod q$ that $k^2(k+1) \mp (k-1) \equiv 0 \pmod q$. If $q = 7$ this requires $k \equiv 5 \pmod 7$ and hence $x \equiv 2 \pmod 7$. For $(n, p, q) = (91, 13, 7)$ and $(133, 19, 7)$ we can, as illustrated in Example 5.2, take $x = 2$, $k = 3^{-1}$, leading to $(x, k, v, y) = (2, 61, 70, 11)$ and $(2, 89, 14, 4$ or $6)$, respectively. But there are other possiblities. For $(n, p, q) = (91, 13, 7)$ we can take $(x, k, v, y) = (37, 5, 70, 11)$, $(44, 68, 28, 2)$ or $(86, 33, 21, 11)$; neither of the x-values 44 and 86 is a primitive root of either 7 or 13. For $(n, p, q) = (133, 19, 7)$ we can take $(x, k, v, y) = (51, 54, 105, 13)$, $(72, 40, 84, 3)$, $(79, 61, 84, 3)$ or $(128, 12, 84, 3)$; each of the x-values here is a primitive root of 19 but not of 7.

(c) If $q = 13$, the congruence $k^2(k+1) \mp (k-1) \equiv 0 \pmod q$ requires either (i) $k \equiv 6 \pmod{13}$ and hence $x \equiv 6 \pmod{13}$, or (ii) $k \equiv 2 \pmod{13}$ and hence $x \equiv 4 \pmod{13}$. However, a value x satisfying $x \equiv 4 \pmod{13}$ cannot be a primitive λ-root of $13p$, so only possibility (i) yields terraces.

(d) Similarly, if $q = 19$ the congruence requires $k = 3$ or $6 \pmod{19}$, giving $x = 7$ or $18 \pmod{19}$. No such choice of x is possible for a primitive λ-root. If $q = 37$, no value of k satisfies the congruence. So the construction in Theorem 5.1 fails to produce any terraces when $q = 19$ or $q = 37$.

(e) The requirement on $\mathrm{ord}_q(x)$ shows that Theorem 5.1 cannot produce terraces when $q = 31$.

Example 5.1. $\mathbb{Z}_{91}$ terrace for $(n, p, q) = (91, 7, 13)$ with $(x, k, v, y) = (6, 58, 78, 4)$ and thus with $3(x - x^2) \equiv 1 \pmod n$:

65 39 78 |

73 88 45 53 24 4 31 81 59 25 80 74 |

75 58 40 37 82 44 68 72 12 2 61 86 |

6 36 34 22 41 64 20 29 83 43 76 1 |

14 28 56 21 42 84 | 0 | negatives.

Example 5.2. $\mathbb{Z}_{91}$ terrace for $(n, p, q) = (91, 13, 7)$ with $(x, k, v, y) = (2, 61, 70, 11)$:

35 28 77 7 42 70 |

71 81 86 43 67 79 85 88 44 22 11 51 |

31 61 76 38 19 55 73 82 41 66 33 62 |

2 4 8 16 32 64 37 74 57 23 46 1 |

78 65 39 | 0 | negatives.

Specimen parameter sets that yield terraces obtainable from Theorem 5.1 are as in the following table, where a dagger indicates an x-value that is not a primitive root of either p or q, and an asterisk on a y-value has the same meaning as previously. Asterisked solutions can arise only if $p \equiv 7 \pmod{12}$; for such values of p the table has two rows, respectively, for solutions without and with asterisks:

n	(p,q)	(x,k,v,y)	
		$3(x-x^2) \equiv 1 \pmod{n}$	$3(x-x^2) \not\equiv 1 \pmod{n}$
91	$(13,7)$	$(86^\dagger,33,21,11)$	$(2,61,70,11)$
	$(7,13)$	$(-,-,-,-)$,	$(-,-,-,-)$
		$(6,58,78,4^*)$	$(-,-,-,-^*)$
133	$(19,7)$	$(79,61,84,3)$	$(128,12,84,3)$
		$(-,-,-,-^*)$	$(2,89,14,4^*)$
217	$(31,7)$	$(23^\dagger,82,126,11)$	$(33,207,77,21)$
		$(-,-,-,-^*)$	$(86,33,42,19^*)$
247	$(19,13)$	$(188,110,117,15)$	$(32,149,91,14)$
		$(136,175,221,17^*)$	$(6,45,52,6^*)$
259	$(37,7)$	$(-,-,-,-)$	$(163,208,77,20)$

6. Terraces with $n = 3^2 p$ and $\xi(n) = 2$

For $n = pq^t$, the number of possible constructions for narcissistic half-and-half power-sequence terraces can be expected to increase as t increases. A restriction to terraces with particularly elegant constructions therefore now seems natural, unless this should eliminate terraces with other appealing characteristics.

In all the terraces given so far in this paper, the three middle segments, taken together, constitute a multiple of another narcissistic half-and-half power-sequence terrace. For example, the three middle segments of the $\mathbb{Z}_{65}$ terrace from Example 4.2 are

40 15 30 60 55 45 | 0 | negatives,

which make up 5 times the $\mathbb{Z}_{13}$ terrace

8 3 6 12 11 9 | 0 | negatives.

This nesting of a multiple of one narcissistic half-and-half terrace in the centre of another resembles the enclosing of one wooden Russian doll inside another, and so we describe a narcissistic half-and-half terrace that has such nesting within it as a *matryoshka* terrace. In an obvious notation, the example just given is a $(65 \supset 13)$ matryoshka terrace for $\mathbb{Z}_{65}$.

A narcissistic half-and-half power-sequence terrace having as few segments as possible is not necessarily a matryoshka terrace. This is readily illustrated by the following $\mathbb{Z}_{45}$ terrace:

$$\begin{array}{l} 30 \mid 20^5 \;\; 20^6 \;\; 20^1 \mid 38^1 \;\; 38^2 \;\; \ldots \;\; 38^{11} \;\; 38^0 \mid \\ \qquad 27^2 \;\; 27^3 \mid 3.2^0 \;\; 3.2^1 \;\; 3.2^2 \;\; 3.2^3 \mid 0 \mid \text{negatives} \\ = 30 \mid 5 \;\; 10 \;\; 20 \mid 38 \;\; 4 \;\; 17 \;\; 16 \;\; 23 \;\; 19 \;\; 2 \;\; 31 \;\; 8 \;\; 34 \;\; 32 \;\; 1 \mid \\ \qquad 9 \;\; 18 \mid 3 \;\; 6 \;\; 12 \;\; 24 \mid 0 \mid \text{negatives.} \end{array}$$

Matryoshka power-sequence terraces for $\mathbb{Z}_n$ are, however, easy to construct when $n=3^2p$ and $p \equiv 5 \pmod 6$. Then $\phi(n)=6(p-1)$ and $e(n)=3(p-1)$ so that $\xi(n)=2$; also $\phi(3p)=2(p-1)$ and $e(3p)=p-1$, so that $\xi(3p)=2$. Thus each terrace needs, on each side of the central zero, just one segment for elements of U_n and just one for multiples of 3 that are not multiples of 9 or $3p$.

Within the range $n<300$ there are five such values of n, namely 45, 99, 153, 207 and 261. For each of these values, either of the following two schemes can be used to produce $(n \supset 9 \supset 3)$ matryoshka power-sequence terraces with $n=3^2p$. Each scheme gives terraces whose middle five segments constitute p times a $\mathbb{Z}_9$ terrace, and whose middle three segments constitute $3p$ times a $\mathbb{Z}_3$ terrace.

To avoid cumbersome notation, we now use I to denote the identity element of the group of elements from $\mathbb{Z}_n$ that are multiples of p but not of $3p$; we again use x for a strong primitive λ-root of n; we use y for a unit having order $p-1 \pmod{3p}$; we use w for a multiple of 9, and v for a multiple of 3 that is not a multiple of 9:

Scheme 6.1.

$$\begin{array}{l} z^{(p-3)/2}w \;\; z^{(p-5)/2}w \;\; \ldots \;\; zw \;\; w \mid y^{p-2}v \;\; y^{p-3}v \;\; \ldots \;\; yv \;\; v \mid \\ x \;\; x^2 \;\; \ldots \;\; x^{e(n)-1} \;\; 1 \mid I \;\; 2^{-1}I \;\; 2^{-2}I \mid 3p \mid 0 \mid \text{negatives} \end{array}$$

with $m_1=\pm f_3$, $m_2=\pm f_1$, $m_3=\pm f_2$, $m_4=\pm f_4$.

Scheme 6.2.

$$\begin{array}{l} y^{p-2}v \;\; y^{p-3}v \;\; \ldots \;\; yv \;\; v \mid z^{(p-3)/2}w \;\; z^{(p-5)/2}w \;\; \ldots \;\; zw \;\; w \mid \\ x \;\; x^2 \;\; \ldots \;\; x^{e(n)-1} \;\; 1 \mid I \;\; 2^{-1}I \;\; 2^{-2}I \mid 3p \mid 0 \mid \text{negatives} \end{array}$$

with $m_1=\pm f_1$, $m_2=\pm f_3$, $m_3=\pm f_2$, $m_4=\pm f_4$.

For the values $n=99$, 153, 207 and 261, and indeed more generally, these schemes can be used with $y=x$, but this is not possible for $n=45$. Accordingly, we first give examples for $n=45$; then we prove the theorems that establish the circumstances under which the schemes, with $y=x$, produce terraces of the desired form.

Example 6.1. A $(45 \supset 9 \supset 3)$ matryoshka terrace obtained from Scheme 6.1 by taking $(x, v, y, w, z) = (2, 3, 8, 9, 3)$:

$$27\ \ 9\ \mid\ 6\ \ 12\ \ 24\ \ 3\ \mid\ 2\ \ 4\ \ 8\ \ 16\ \ 32\ \ 19\ \ 38\ \ 31\ \ 17\ \ 34\ \ 23\ \ 1\ \mid$$
$$10\ \ 5\ \ 25\ \mid\ 15\ \mid\ 0\ \mid\ \text{negatives.}$$

Example 6.2. A $(45 \supset 9 \supset 3)$ matryoshka terrace obtained from Scheme 6.2 by taking $(x, v, y, w, z) = (32, 24, 2, 18, 2)$:

$$12\ \ 6\ \ 3\ \ 24\ \mid\ 36\ \ 18\ \mid\ 32\ \ 34\ \ 8\ \ 31\ \ 2\ \ 19\ \ 23\ \ 16\ \ 17\ \ 4\ \ 38\ \ 1\ \mid$$
$$10\ \ 5\ \ 25\ \mid\ 15\ \mid\ 0\ \mid\ \text{negatives.}$$

For $n = 45$, 99, 153, 207 and 261, parameter sets giving solutions with $y \neq x$ include the following:

n	(x, v, y, w, z)	
	Scheme 6.1	Scheme 6.2
45	$(2, 3, 8, 9, 3)$	$(32, 24$ or $39, 2, 18, 2)$
99	$(38, 75, 20, 81, 3)$	$(86, 21$ or $87, 5, 72, 4)$
153	$(29, 57, 5, 27, 3)$	$(5, 60$ or $111, 41, 9, 6)$
207	$(2, 3, 35, 9, 4)$	$(32, 66$ or $204, 2, 63, 14)$
261	$(2, 3, 44, 9, 27)$	$(14, 78$ or $165, 2, 27, 2)$

Theorem 6.1. *Let $n = 9p$ where p is a prime satisfying $p \equiv 5 \pmod 6$. Choose an element x from U_n that satisfies $x \equiv 2 \pmod 9$, $x \neq 2$, and is a primitive root of p if $p \equiv 5 \pmod{12}$ or has order $(p-1)/2 \pmod p$ if $p \equiv 11 \pmod{12}$. Then x is a strong primitive λ-root of n. Let $v = 2x - 1$ and $w = vx^{-1}(2 - x)$. If a value z satisfying $z \equiv w(w \pm 1)^{-1} \pmod p$ is a primitive root of p and if I denotes the multiple of p that is congruent to $1 \pmod 9$, then Scheme 6.1, with $y = x$, yields a narcissistic half-and-half matryoshka power-sequence terrace for $\mathbb{Z}_n$.*

Proof. We note first that $v \equiv 3 \pmod 9$ and $w \equiv 0 \pmod 9$, and that the conditions on x ensure that x is a strong primitive λ-root of $9p$.

We have $m_1 = z^{(p-3)/2}w(z-1)$, $m_2 = vy^{p-2}(p-1)$, $m_3 = x - 1$, $m_4 = 2^{-2}I - 2^{-3}I$, $f_1 = y^{p-2}v - w$, $f_2 = v - x$, $f_3 = I - 1$, $f_4 = 3p - 2^{-2}I$. As $v = 2x - 1$ we have $m_3 = -f_2$. For $m_2 = f_1$ we need $w \equiv 2vy^{p-2} - vy^{p-1} \pmod{9p}$. Modulo p, this is equivalent to $yw = 2v - vy \pmod p$, i.e. $w = vy^{-1}(2 - y)$, which is satisfied if $y = x$. Modulo 9, it is equivalent to $2v \equiv vy \pmod 9$, i.e. $y \equiv 2 \pmod 3$, which too is satisfied if $y = x$.

For $m_1 = \pm f_3$, we need $z^{(p-3)/2}w(z-1) \equiv \pm(I-1) \pmod{9p}$. This is true modulo 9 as $9|w$ and $I \equiv 1 \pmod 9$. So we need $z^{(p-3)/2}w(z-1) \equiv \pm 1 \pmod p$, i.e. $w(z-1) \equiv \pm z \pmod p$.

For $m_4 = f_4$ we need $2^{-2}I - 2^{-3}I \equiv 3p - 2^{-2}I \pmod{9p}$, i.e. $3I \equiv 24p \pmod{9p}$, i.e. $I \equiv 8p \pmod{3p}$. As $p|I$ this is true modulo p; as $I \equiv 1 \pmod 3$ we need $8p \equiv 1 \pmod 3$, i.e. $p \equiv 2 \pmod 3$, which is true.

As $\mathrm{ord}_9(2^{-1}){=}6$, no member of the fourth segment equals another or its inverse. □

Theorem 6.2. *Let n be of the form $n = 9p$ where p is a prime satisfying $p \equiv 5 \pmod 6$. For n, choose a strong primitive λ-root x satisfying $x \equiv 5 \pmod 9$ and such that x is a primitive root of p if $p \equiv 5 \pmod{12}$ and has order $(p-1)/2 \pmod p$ if $p \equiv 11 \pmod{12}$. Let $w = 2x - 1$. If the element z given by $z = w(w \pm 1)^{-1} \pmod p$ is a primitive root of p, choose $v \equiv -xz^{-1} \pmod p$ where v is a multiple of 3 but not of 9. Then Scheme 6.2, with $y = x$, yields a narcissistic half-and-half matryoshka power-sequence terrace for $\mathbb{Z}_n$.*

Proof. The condition $m_3 = \pm f_2$ is satisfied if $w = 2x - 1$. As $x \equiv 5 \pmod 9$, the element w is a multiple of 9. The condition $m_2 = \pm f_3$ requires $w(z-1) \equiv \pm z \pmod p$, as in Theorem 6.1. The condition $m_1 = \pm f_1$ will be satisfied if

$$\begin{aligned} y^{p-2}v(y-1) &\equiv z^{(p-3)/2}w - v &&\pmod{9p} \\ \text{i.e.}\quad z^{(p-3)/2}w &\equiv v\{1 + y^{p-2}(y-1)\} &&\pmod{9p}. \end{aligned}$$

Modulo 9 this requires $y^{p-2}(y-1) \equiv 2 \pmod 3$, so we need $y \equiv 2 \pmod 3$, and this will be satisfied if we choose $y = x$, as $x \equiv 5 \pmod 9$. We then need

$$\begin{aligned} -w &\equiv vz(2 - y^{p-2}) &&\pmod p \\ \text{i.e.}\quad -wy &\equiv 2vzy - vz &&\pmod p, \\ \text{i.e.}\quad wy &\equiv -vz(2y-1) &&\pmod p, \\ \text{i.e.}\quad wx &\equiv -vzw &&\pmod p, \\ \text{i.e.}\quad v &\equiv -xz^{-1} &&\pmod p. \qquad \square \end{aligned}$$

Note on alternative solutions: As Theorem 6.2 requires v to be a multiple of 3 but not of 9, the above congruence yields two values of $v \pmod{9p}$. For $p \equiv 11 \pmod{12}$, changing from one value to the other is equivalent to interchanging the two halves of the first segment of the terrace; for $p \equiv 5 \pmod{12}$, changing the value of v is equivalent to negating the first segment of the terrace and then interchanging its two halves.

Note on existence: We now show that it follows from [11, p. 47] that the construction in Scheme 6.2 works for all sufficiently large p. We chose x as a primitive root of p if $p \equiv 1 \pmod 4$ and as the negative of a primitive root of p if $p \equiv 3 \pmod 4$. The construction works provided that z satisfying $z \equiv w(w \pm 1)^{-1}$ is a primitive root of p. Assume that there are primitive roots α and β of p with $\alpha = 2\beta - 2$. If $p \equiv 1 \pmod 4$ choose $x = -\alpha^{-1}$ and $z = \beta$. Then $w(w+1)^{-1} = (2x-1)(2x)^{-1} = 1 + \alpha 2^{-1} = (2+\alpha)2^{-1} = \beta = z$. If $p \equiv 3 \pmod 4$, make the same choice of x and z, with x now the negative of a primitive root.

For $n = 99$, 153, 207 and 261, parameter sets giving solutions with $y = x$ are as follows:

n	(x, v, y, w, z)	
	Theorem 6.1	Theorem 6.2
99	$(20, 39, 20, 54, 6)$	$(5, 3$ or $69, 5, 9, 2)$
153	$(29, 57, 29, 63, 14)$	$(5, 87$ or $138, 5, 9, 6)$
207	$(29, 57, 29, 54, 11)$	$(32, 24$ or $93, 32, 63, 14)$
261	$(11, 21, 11, 54, 11)$	$(14, 51$ or $138, 14, 27, 2)$

7. Terraces with $n = 3^2p$ and $\xi(n) = 6$

The previous Section considered terraces with $n = 9p$ where the prime p satisfies $p \equiv 5 \pmod 6$. We now turn to $n = 9p$ with the prime p satisfying $p \equiv 1 \pmod 6$. We now have $\phi(n) = 6(p-1)$ and $e(n) = p-1$, so that $\xi(n) = 6$; also $\phi(3p) = 2(p-1)$ and $e(3p) = p-1$, so that $\xi(3p) = 2$. Thus a narcissistic half-and-half power-sequence terrace for $\mathbb{Z}_n$ now needs, on each side of the central zero, at least three segments for elements of U_n, but may need no more than one segment for multiples of 3 that are not multiples of 9 or $3p$.

Within the range $n < 300$, we are now dealing with $n = 63$, 117, 171 and 279.

We have found constructions producing $(n \supset 9 \supset 3)$ matryoshka power-sequence terraces based on a strong primitive λ-root x satisfying $x \equiv 2 \pmod 3$. With I denoting the identity element of the group of multiples of p that are not multiples of $3p$, the constructions are of the following forms, where w is a multiple of 9 and v is a multiple of 3 but not of 9:

Scheme 7.1.

$$k^2x^1 \;\; k^2x^0 \;\; k^2x^{p-2} \;\; k^2x^{p-3} \;\; \dots \;\; k^2x^2 \;\mid\; y^{p-2}v \;\; y^{p-3}v \;\; \dots \;\; yv \;\; v \;\mid$$

$$z^{(p-3)/2}w \;\; z^{(p-5)/2}w \;\; \dots \;\; z^0w \;\mid\; kx^1 \;\; kx^0 \;\; kx^{p-2} \;\; kx^{p-3} \;\; \dots \;\; kx^2 \;\mid$$

$$x^1 \;\; x^2 \;\; \dots \;\; x^{p-2} \;\; 1 \;\mid\; I \;\; 2^{-1}I \;\; 2^{-2}I \;\mid\; 6p \;\mid\; 0 \;\mid\; \text{negatives}$$

with $m_1 = \pm f_4$, $m_2 = \pm f_2$, $m_3 = \pm f_5$, $m_4 = \pm f_3$, $m_5 = \pm f_1$.

Example 7.1. A $(63 \supset 9 \supset 3)$ matryoshka terrace obtained using Scheme 7.1 with $(n, p) = (63, 7)$, $(x, k) = (11, 19)$, $(v, w) = (24, 9)$, $y = z = 11^{-1} \pmod{21}$:

$$2 \;\; 46 \;\; 50 \;\; 16 \;\; 53 \;\; 22 \;\mid\; 12 \;\; 6 \;\; 3 \;\; 33 \;\; 48 \;\; 24 \;\mid$$

$$36 \;\; 18 \;\; 9 \;\mid\; 20 \;\; 19 \;\; 59 \;\; 34 \;\; 26 \;\; 31 \;\mid$$

$$11 \;\; 58 \;\; 8 \;\; 25 \;\; 23 \;\; 1 \;\mid\; 28 \;\; 14 \;\; 7 \;\mid\; 42 \;\mid\; 0 \;\mid\; \text{negatives.}$$

In this example, the first segment may be reversed.

Scheme 7.2.

$$k^2x^1 \quad k^2x^0 \quad k^2x^{p-2} \quad k^2x^{p-3} \quad \ldots \quad k^2x^2 \mid z^{(p-3)/2}w \quad z^{(p-5)/2}w \quad \ldots \quad z^0w \mid$$

$$y^{p-2}v \quad y^{p-3}v \quad \ldots \quad yv \quad v \mid kx^1 \quad kx^0 \quad kx^{p-2} \quad kx^{p-3} \quad \ldots \quad kx^2 \mid$$

$$x^1 \quad x^2 \quad \ldots \quad x^{p-2} \quad 1 \mid I \quad 2^{-1}I \quad 2^{-2}I \mid 6p \mid 0 \mid \quad \text{negatives}$$

with $m_1 = \pm f_4$, $m_2 = \pm f_5$, $m_3 = \pm f_2$, $m_4 = \pm f_3$, $m_5 = \pm f_1$.

Example 7.2. A $(63 \supset 9 \supset 3)$ matryoshka terrace obtained using Scheme 7.2 with $(n, p) = (63, 7)$, $(x, k) = (53, 40)$, $(v, w) = (51, 27)$, $y = z = 11 \pmod{21}$:

$$2 \quad 25 \quad 29 \quad 16 \quad 11 \quad 43 \mid 54 \quad 45 \quad 27 \mid$$

$$39 \quad 15 \quad 30 \quad 60 \quad 57 \quad 51 \mid 41 \quad 40 \quad 59 \quad 13 \quad 5 \quad 31 \mid$$

$$53 \quad 37 \quad 8 \quad 46 \quad 44 \quad 1 \mid 28 \quad 14 \quad 7 \mid 42 \mid 0 \mid \text{negatives.}$$

In this example as in Example 7.1, the first segment may be reversed. Also, the 2nd and 3rd segments of Example 7.1 are the 13th and 14th segments of Example 7.2, and vice versa.

Parameter sets that yield terraces available from Schemes 7.1 and 7.2 include the following, where a dagger indicates a primitive λ-root that is a primitive root of 3 but is not a primitive root of either p or 9, where asterisks have their usual meaning, and where parameter sets marked with a hash $^{\#}$ yield terraces whose first segments are reversible:

(n, p)	Scheme	(x, k, v, w, y, z)	
		$k = (x-1)^{-1}$	$k \neq (x-1)^{-1}$
$(63, 7)$	7.1	$(11, 19, 24, 9, 2^*, 2^*)^{\#}$	$(11, 61, 3, 9, 2^*, 2^*)$
	7.2	$(53^{\dagger}, 40, 51, 27, 11^*, 11^*)^{\#}$	$(-, -, -, -, -, -)$
$(117, 13)$	7.1	$(-, -, -, -, -, -)$	$(-, -, -, -, -, -)$
	7.2	$(89, 4, 33, 90, 11^*, 7)$	$(32, 40, 93, 9, 11^*, 6)$
$(171, 19)$	7.1	$(137, 127, 15, 162, 5^*, 13)$	$(47, 88, 102, 99, 5^*, 16^*)$
	7.2	$(161^{\dagger}, 31, 42, 144, 17^*, 3)$	$(5, 58, 156, 18, 47^*, 10)$
$(279, 31)$	7.1	$(227, 100, 60, 153, 80^*, 11)$	$(38, 187, 147, 27, 20^*, 22)$
	7.2	$(71^{\dagger}, 4, 213, 108, 50^*, 21)$	$(71^{\dagger}, 94, 123, 198, 59^*, 18^*)$

8. Other terraces with $n = pq^2$ and $\xi(n) = 2$

If p and q are distinct odd primes satisfying $n = pq^2$, the only values of n with $n = pq^2$ within the range $n < 300$ that have not so far been considered in this paper are 75, 147, 175 and 245, i.e. 3.5^2, 3.7^2, 7.5^2 and 5.7^2, all of which have $\xi(n) = \xi(pq) = 2$. A single general construction is available that produces $(n \supset pq \supset p)$ matryoshka power-sequence terraces for each of these values. A further general construction produces $(n \supset q^2 \supset q)$ matryoshka power-sequence terraces for values that include $n = 175, 245$. The terraces obtained from the respective constructions have units from $\mathbb{Z}_n$ in, respectively, the first and third segments.

Construction 8.1. We start with a $\mathbb{Z}_{pq}$ terrace, as in Section 2 or Section 3 above, and multiply each entry by q. Let the first entry in the resulting sequence be apq. With $n = pq^2$, we aim for a $\mathbb{Z}_n$ terrace of the form

$$x \quad x^2 \; \ldots \; 1 \;\mid\; vy^{(q(q-1)/2)-1} \quad vy^{(q(q-1)/2)-2} \; \ldots \; vy \quad v \;\mid$$
$$apq \; \ldots\ldots \;\mid\; 0 \;\mid\; \text{negatives},$$

where p divides v but q does not divide v. Putting $v = rp$ we aim to satisfy the requirements $m_1 = -f_1$ and $m_2 = -f_2$. The first of these reduces to $(2y-1)r \equiv aqy \pmod{q^2}$, so we require $2y = 1 + \mu q$ and $\mu r \equiv ay \pmod{q}$. Choosing a suitable y so that $2y \equiv 1 \pmod{q}$, we solve $\mu r \equiv ay \pmod{q}$ for r and hence obtain v. Then, provided that $x = z - vy^{(q(q-1)/2)-1}$ is a strong primitive λ-root of pq^2, we satisfy $m_2 = -f_2$ too.

Example 8.1. A $(75 \supset 15 \supset 3)$ matryoshka terrace from Construction 8.1:

$$38^1 \quad 38^2 \;\ldots\; 38^{19} \quad 38^0 \;\mid\; 3^9.33 \quad 3^8.33 \;\ldots\; 3^0.33 \;\mid$$
$$2^1.15 \quad 2^0.15 \;\mid\; 2^1.5 \quad 2^2.5 \quad 2^3.5 \quad 2^0.5 \;\mid\; 2^0.50 \;\mid\; 0 \;\mid\; \text{negatives}$$
$$= 38 \quad 19 \quad 47 \;\ldots\; 4 \quad 2 \quad 1 \;\mid\; 39 \quad 63 \quad 21 \quad 57 \quad 69 \quad 48 \quad 66 \quad 72 \quad 24 \quad 33 \;\mid$$
$$30 \quad 15 \;\mid\; 10 \quad 20 \quad 40 \quad 5 \;\mid\; 50 \;\mid\; 0 \;\mid\; \text{negatives.}$$

The second segment has $3^0.33 = -3^{10}.33$.

Example 8.2. A $(147 \supset 21 \supset 3)$ matryoshka terrace from Construction 8.1:

$$74^1 \quad 74^2 \;\ldots\; 74^{41} \quad 74^0 \;\mid\; 81^{20}.48 \quad 81^{19}.48 \;\ldots\; 81^0.48 \;\mid$$
$$5^2.42 \quad 5^1.42 \quad 5^0.42 \;\mid\; 2^1.14 \quad 2^2.14 \;\ldots\; 2^5.14 \quad 2^0.14 \;\mid$$
$$2^0.98 \;\mid\; 0 \;\mid\; \text{negatives.}$$

The second segment has $81^0.48 = +81^{21}.48$.

Example 8.3. A $(175 \supset 35 \supset 7)$ matryoshka terrace from Construction 8.1:

$$58^1 \quad 58^2 \;\ldots\; 58^{59} \quad 58^0 \;\mid\; 8^9.98 \quad 8^8.98 \;\ldots\; 8^0.98 \;\mid$$
$$2^1.70 \quad 2^0.70 \;\mid\; 32^1.115 \quad 32^2.115 \;\ldots\; 32^{11}.115 \quad 32^0.115 \;\mid$$
$$2^0.150 \quad 2^1.150 \quad 2^2.150 \;\mid\; 0 \;\mid\; \text{negatives.}$$

The second segment has $8^0.98 = -8^{10}.98$.

Example 8.4. A $(245 \supset 35 \supset 5)$ matryoshka terrace from Construction 8.1:

$$137^1\ \ 137^2\ \ \dots\ \ 137^{83}\ \ 137^0 \ \mid\ 39^{20}.125\ \ 39^{19}.125\ \ \dots\ \ 39^0.125\ \mid$$

$$2^2.35\ \ 2^1.35\ \ 2^0.35\ \mid\ 3^1.7\ \ 3^2.7\ \ \dots\ \ 3^{11}.7\ \ 3^0.7\ \mid$$

$$2^0.147\ \ 2^1.147\ \mid\ 0\ \mid\ \text{negatives.}$$

The second segment has $39^0.125 = +39^{21}.125$.

Construction 8.2. We start with a $\mathbb{Z}_{q^2}$ terrace as in [3, Section 2], which we multiply throughout by a suitable multiple of p so that the first entry is ap where $ap \equiv 1 \pmod{q^2}$. We then aim for a terrace of the form

$$z^{(p-3)/2}q^2\ \ z^{(p-5)/2}q^2\ \ \dots\ \ zq^2\ \ q^2\ \mid$$

$$(2x-1)y^{((p-1)(q-1)/2)-1}\ \ (2x-1)y^{((p-1)(q-1)/2)-2}\ \ \dots\ \ (2x-1)y\ \ (2x-1)\ \mid$$

$$x\ \ x^2\ \ \dots\ \ 1\ \mid\ ap\ \ \dots\dots\ \mid\ 0\ \mid\ \text{negatives,}$$

where $2x \equiv 1 \pmod q$ and where we have to satisfy $m_1 = \pm f_3$ and $m_2 = \pm f_1$. (The condition $m_3 = \pm f_2$ is already satisfied.)

Example 8.5. Two $(175 \supset 25 \supset 5)$ matryoshka terraces from Construction 8.2:

(a) Constructed from a $\mathbb{Z}_{25}$ terrace from Theorem 2.2 of [3]:

$$12^2.25\ \ 12^1.25\ \ 12^0.25\ \mid\ 12^{11}.5\ \ 12^{10}.5\ \ \dots\ \ 12^0.5\ \mid$$

$$3^1\ \ 3^2\ \ \dots\ \ 3^{59}\ \ 3^0\ \mid\ 3^0.126\ \ 3^{-1}.126\ \ \dots\ \ 3^{-9}.126\ \mid$$

$$2^0.70\ \ 2^1.70\ \mid\ 0\ \mid\ \text{negatives.}$$

(b) Constructed from a $\mathbb{Z}_{25}$ terrace from Theorem 2.4 of [3]:

$$12^2.25\ \ 12^1.25\ \ 12^0.25\ \mid\ 12^{11}.5\ \ 12^{10}.5\ \ \dots\ \ 12^0.5\ \mid$$

$$3^1\ \ 3^2\ \ \dots\ \ 3^{59}\ \ 3^0\ \mid\ 12^0.126\ \ 12^1.126\ \ \dots\ \ 12^9.126\ \mid$$

$$2^0.105\ \ 2^1.105\ \mid\ 0\ \mid\ \text{negatives.}$$

Example 8.6. A $(245 \supset 49 \supset 7)$ matryoshka terrace from Construction 8.2; it is constructed from the $\mathbb{Z}_{49}$ terrace of Example 2.7 of [3]:

$$3^1.49\ \ 3^0.49\ \mid\ 23^{11}.28\ \ 23^{10}.28\ \ \dots\ \ 23^0.28\ \mid$$

$$137^1\ \ 137^2\ \ \dots\ \ 137^{83}\ \ 137^0\ \mid\ 46^{20}.95\ \ 46^{19}.95\ \ \dots\ \ 46^0.95\ \mid$$

$$46^2.42.95\ \ 46^1.42.95\ \ 46^0.42.95\ \mid\ 0\ \mid\ \text{negatives.}$$

9. Easily constructed terraces for *n* = 275

If n satisfies $n = pq^t$ where p and q are distinct odd primes and where t is a positive integer, the values of $\xi(n)$ that occur in the range $n < 300$ are 2, 4, 6 and 10. This last occurs just once in the range, namely for $n = pq^2 = 11.5^2 = 275$, for which $\xi(pq) = 2$. Six $(275 \supset 25 \supset 5)$ matryoshka power-sequence terraces are easily constructed for this value of n, which has 96 primitive λ-roots. They use the strong primitive λ-root 2, and the method of construction of the sequence of segments of units is strongly reminiscent of the construction in Theorem 5.1 of [3]. With I written for the identity element 176 of the group of multiples of 11 that are not multiples of 55, one of the terraces is as follows:

$$\begin{gathered}
2^4.200\quad 2^3.200\quad \ldots\quad 2^0.200 \mid 26^{19}.210\quad 26^{18}.210\quad \ldots\quad 26^0.210 \mid\\
3^{-4}.2^{19}\quad 3^{-4}.2^{18}\quad \ldots\quad 3^{-4}.2^0 \mid 3^{-3}.2^{19}\quad 3^{-3}.2^{18}\quad \ldots\quad 3^{-3}.2^0 \mid\\
3^{-2}.2^{19}\quad 3^{-2}.2^{18}\quad \ldots\quad 3^{-2}.2^0 \mid 3^{-1}.2^{19}\quad 3^{-1}.2^{18}\quad \ldots\quad 3^{-1}.2^0 \mid\\
2^{19}\quad 2^{18}\quad \ldots\quad 2^0 \mid I\quad 3^{-1}I\quad 3^{-2}I\quad \ldots\quad 3^{-9}I \mid\\
2^0.220\quad 2^1.220 \mid 0 \mid \quad \text{negatives}
\end{gathered}$$

$$\begin{gathered}
= 175\quad 225\quad 250\quad 125\quad 200 \mid 205\quad 115\quad \ldots\quad 195\quad 210 \mid\\
73\quad 174\quad \ldots\quad 17\quad 146 \mid 219\quad 247\quad \ldots\quad 51\quad 163 \mid\\
107\quad 191\quad \ldots\quad 153\quad 214 \mid 46\quad 23\quad \ldots\quad 184\quad 92 \mid\\
138\quad 69\quad \ldots\quad 2\quad 1 \mid 176\quad 242\quad \ldots\quad 66\quad 22 \mid\\
220\quad 165 \mid 0 \mid \quad \text{negatives.}
\end{gathered}$$

Another possibility is the same as this save that the powers of 2 in the first segment are replaced by the corresponding powers of 8. A third possibility is also the same save that it has the entries $3^i.125$ $(i = 4, 3, \ldots, 0)$ in the first segment and $7^j.210$ $(j = 19, 18, \ldots, 0)$ in the second. These three possibilities are all constructed from a $\mathbb{Z}_{25}$ terrace from Theorem 2.2 of [3]. A further three possibilities are obtained by using instead Theorem 2.4 of [3], so that segments 8 and 9 of the $\mathbb{Z}_{275}$ terrace become

$$I\quad 12^1I\quad 12^2I\quad \ldots\quad 12^9I \mid 2^0.55\quad 2^1.55$$

i.e.

$$176\quad 187\quad 44\quad 253\quad 11\quad 132\quad 209\quad 33\quad 121\quad 77 \mid 55\quad 110.$$

The value $n = 275$ is from a series of values $n = pq^2 = (4\kappa - 1)(2\kappa - 1)^2$ for which $4\kappa - 1$ and $2\kappa - 1$ are primes so that $\xi(n) = 2(2\kappa - 1)$ and $\xi(pq) = 2$. (With $4\kappa - 1$ and $2\kappa - 1$ both prime, $2\kappa - 1$ is a *Sophie Germain prime* [5, p. 184] and $4\kappa - 1$ is a *safe prime* [16, p. 249]. Whether there are infinitely many Sophie Germain primes is not known [5, p. 184].) After $n = 275$, the next value of n such that $2\kappa - 1 \equiv 1 \pmod 4$ is $n = 59.29^2$, each of whose prime factors does indeed have 2 as a primitive root. However, this value of n is so large that any attempt to generalise our results for $n = 275$ would be well beyond the remit of this paper.

10. Terraces with $n = 3^3 p$ and $\xi(n) = 2$

Amongst the values of n with $n = pq^3$ where p and q are distinct odd primes, the range $n < 300$ contains just two values with $\xi(n) = \xi(pq^2) = \xi(pq) = 2$, namely $n = 135$ and 297, i.e. $3^3.5$ and $3^3.11$. Matryoshka power-sequence terraces for $\mathbb{Z}_{135}$ and $\mathbb{Z}_{297}$ are easily obtained by extending ideas already presented in this paper, so we now present them without further ado; they use the strong primitive λ-root 2 of n, and are based on a $\mathbb{Z}_{27}$ terrace that is obtainable from Theorem 2.2 of [3]:

Example 10.1. A $(135 \supset 27 \supset 9 \supset 3)$ matryoshka power-sequence terrace for $\mathbb{Z}_{135}$, where element 55 is the identity element of the group of multiples of 5 that are not multiples of 15:

$$2^1.27 \quad 2^0.27 \quad | \quad 2^{-3}.9 \quad 2^{-2}.9 \quad 2^{-1}.9 \quad 2^0.9 \quad | \quad 2^{-11}.3 \quad 2^{-10}.3 \quad \ldots \quad 2^0.3 \quad |$$

$$2^1 \quad 2^2 \quad \ldots \quad 2^{35} \quad 2^0 \quad | \quad 2^0.55 \quad 2^{-1}.55 \quad \ldots \quad 2^{-8}.55 \quad |$$

$$2^0.105 \quad 2^{-1}.105 \quad 2^{-2}.105 \quad | \quad 90 \quad | \quad 0 \quad | \quad \text{negatives.}$$

Example 10.2. A $(297 \supset 27 \supset 9 \supset 3)$ matryoshka power-sequence terrace for $\mathbb{Z}_{297}$, where the element 55 is the identity element of the group of multiples of 11 that are not multiples of 33:

$$7^4.162 \quad 7^3.162 \quad \ldots \quad 7^0.162 \quad | \quad 5^{-9}.117 \quad 5^{-8}.117 \quad \ldots \quad 5^0.117 \quad |$$

$$5^{29}.3 \quad 5^{28}.3 \quad \ldots \quad 5^0.3 \quad |$$

$$2^1 \quad 2^2 \quad \ldots \quad 2^{89} \quad 2^0 \quad | \quad 2^0.55 \quad 2^{-1}.55 \quad \ldots \quad 2^{-8}.55 \quad |$$

$$2^0.132 \quad 2^{-1}.132 \quad 2^{-2}.132 \quad | \quad 198 \quad | \quad 0 \quad | \quad \text{negatives.}$$

11. Terraces with $n = 3^3 p$ and $\xi(n) = \xi(3^2 p) = 6$

Amongst the values of n satisfying $n = pq^t$ where p and q are distinct odd primes and t is a positive integer, there remains just one value to be considered from the range $n < 300$. This value is $n = 3^3.7 = 189$, which has $\phi(n) = 108$, $e(n) = 18$ and $\xi(n) = \xi(3^2.7) = 6$, and has 54 primitive λ-roots, of which only 12 are strong. A narcissisitic half-and-half power-sequence terrace for $\mathbb{Z}_n$ for this value of n, and for other values of n satisfying $n = 3^3 p$ with $\xi(n) = \xi(3^2 p) = 6$ and $\xi(3p) = 2$, must contain at least 23 segments. No such terrace has been found.

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